

Oscillating Fields, Emergent Gravity and Particle Traps

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Topics covered

● Dynamical stabilization

● *Main concept and its realizations:*

Kapitsa pendulum (1953), Paul traps (NP 1989), ...

● Theory of rapidly oscillating fields

(effective field theory in classical mechanics)

● *scale separation and averaging*

● *effective action and emergent gravity*

● *quantization*

● Dynamical magnetic confinement

● *new class of magnetic traps*

Based on

*A. Penin and A. Su, Phys.Rev.Lett. **132** (2024) 5, 051601*

A. Besharat and A. Penin, e-Print:2502.14035 [cond-mat.mes-hall]

Rapidly oscillating fields

- Rapidly oscillating force $\mathcal{F} \propto f(\mathbf{r}) \cos(\omega t)$

→ *small particle displacement for $\omega \rightarrow \infty$*

- Spatially homogeneous amplitude

→ *zero average effect*

- Inhomogeneous amplitude

→ *nonvanishing net effect due to the particle displacement*

- Time-average force

→ *proportional to the amplitude gradient*

$$\mathcal{F}_{eff} \propto f_i \partial_i f$$

→ *vanishes in the high frequency limit as $1/\omega^2$*

Theory of rapidly oscillating fields

● Main idea

- *separate fast and slow modes*
- *integrate out (average) fast modes*
- ➡ *effective action*
- *may change stability of dynamical systems (bifurcation)*
- ➡ *dynamical stabilization*

Kapitza pendulum

● Equation of motion $\ddot{\theta} + (\delta + \epsilon \cos t) \sin \theta = 0$ ($\delta \ll \epsilon \ll 1$)

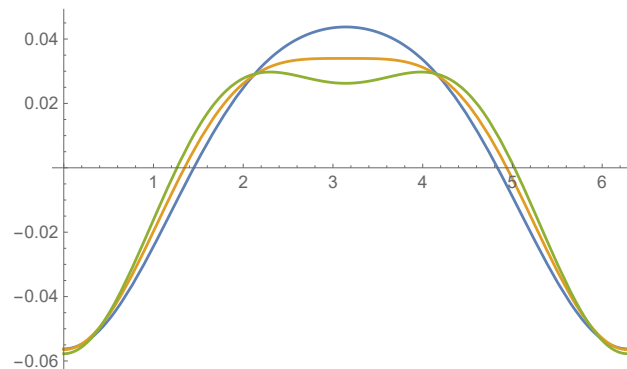
● *slow and fast modes* $\theta = \theta_s + a(\theta_s) \cos t$, $\dot{\theta}_s \ll 1$, $a(\theta_s) \ll 1$

● *equation of motion for fast mode* $(-a + \epsilon \sin \theta_s) \cos t = 0$

● *equation of motion for slow mode* $\ddot{\theta}_s + \delta \sin \theta_s + \frac{\epsilon^2}{2} \cos \theta_s \sin \theta_s = 0$

● Effective action $\mathcal{L} = \frac{\dot{\theta}_s^2}{2} - \left(-\delta \cos \theta_s - \frac{\epsilon^2}{8} \cos 2\theta_s \right)$ P. Kapitza (1953)

➡ for $\delta < \epsilon^2/2$ *new minimum at $\theta_s = \pi$* ➡ *dynamical stabilization*



Paul traps

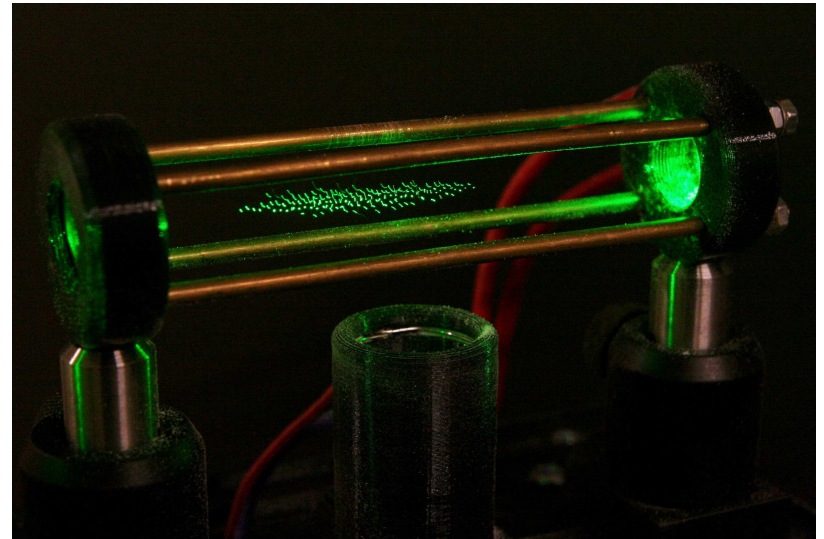
● Electric field in empty space

- *static potential* $\Delta V = 0$

- ➡ *no local minima*

- *oscillating field*

- ➡ *effective potential may have a local minimum*



● Quadrupole traps

- *theory still relies on Kapitsa analysis*

- ➡ *a country mile behind the theory of Penning traps*

Quest for general theory

● The theory should cover:

- *three dimensions*
- *arbitrary fields*
- *arbitrary precision (expansion parameter?)*
- *velocity-dependent forces (magnetic field)*
- *quantum systems*

Effective theory of rapidly oscillating fields

● Equation of motion $m\ddot{\mathbf{R}} + \mathbf{G}(\mathbf{R}) + \mathbf{F}(\mathbf{R}) \cos \omega t = 0$

● What does “rapidly” mean?

● *characteristic large-scale slow velocity* $v \sim (GL/m)^{1/2}$

● *reference velocity* $c = L\omega$

➡ *rapid oscillation:* $v/c \ll 1$

● Rescaled equation $\ddot{\mathbf{R}} + \mathbf{g}(\mathbf{R}) + \mathbf{f}(\mathbf{R}) \cos t = 0$

● *with* $\mathbf{g} = \mathbf{G}/(Lm\omega^2)$, $\mathbf{f} = \mathbf{F}/(Lm\omega^2)$, $c = 1$

➡ *power counting:* $\mathbf{g} = O(v^2)$, $\mathbf{f} = O(v)$

Effective theory of rapidly oscillating fields

● Bogoliubov-Krylov inspired technique

$$\mathbf{R} = \mathbf{r} + \sum_{n=1}^{\infty} [\mathbf{c}_n(\mathbf{r}) \cos(nt) + \mathbf{s}_n(\mathbf{r}) \sin(nt)]$$

● $\mathbf{r}$ describes the large-scale slow evolution, $\dot{\mathbf{r}} \equiv \mathbf{v} = \sum_n \mathbf{v}^{(n)}(\mathbf{r}) \ll 1$

● slow variable: path along the smeared trajectory

➡ $d/dt = \mathbf{v} \partial + \partial/\partial t$

● substitute, expand in velocity and in harmonics

● higher order derivatives are eliminated by lower order EOM

● $\mathbf{c}_n, \mathbf{s}_n, \mathbf{v}^{(n)} = \mathcal{O}(v^n)$

● averaging over fast oscillation period gives $\ddot{\mathbf{r}} + \mathcal{F}_{\text{eff}}(\mathbf{r}, \mathbf{v}) = 0$

➡ effective Lagrangian of slow evolution $\mathcal{L}_{\text{eff}}(v, r)$

Leading order effective action

$$\mathbf{R} = \mathbf{r} + \sum_{n=1}^{\infty} [\mathbf{c}_n(\mathbf{r}) \cos(nt) + \mathbf{s}_n(\mathbf{r}) \sin(nt)]$$

$$d/dt = \mathbf{v} \partial + \partial/\partial t$$

$$\ddot{\mathbf{R}} + \mathbf{g}(\mathbf{R}) + \mathbf{f}(\mathbf{R}) \cos t = 0$$

● *first harmonic*

$$-\mathbf{c}_1(\mathbf{r}) \cos t + \mathbf{f}(\mathbf{r}) \cos t = 0 \quad \Leftrightarrow \quad \mathbf{c}_1 = \mathbf{f}$$

● *averaged equation (zero harmonic)*

$$\mathbf{f}(\mathbf{R}) = \mathbf{f}(\mathbf{r}) + (\mathbf{c}_1 \cdot \partial) \mathbf{f}(\mathbf{r}) \cos t \quad \Leftrightarrow \quad \ddot{\mathbf{r}} + \mathbf{g} + \frac{1}{2} f_i \partial_i \mathbf{f} = 0$$

➡ $\mathcal{O}(v^2)$ *effective Lagrangian*

$$\mathcal{L} = \frac{\mathbf{v}^2}{2} - \left(V_g + \frac{\mathbf{f}^2}{4} \right) \quad (\mathbf{g} = \partial V_g)$$

Next-to-leading effective action

• Averaged force $\mathcal{F}_{\text{eff}} = \partial V_{\text{eff}} - \frac{3}{2} v_i v_j (\partial_i \partial_j f_k) \partial_k \mathbf{f}$

• $\mathcal{O}(v^4)$ effective Lagrangian

$$\mathcal{L}_{\text{eff}} = \frac{v_i v_j}{2} \left[\delta_{ij} - \frac{3}{2} \partial_i \mathbf{f} \partial_j \mathbf{f} \right] - V_{\text{eff}}$$

$$V_{\text{eff}} = V_g + \frac{\mathbf{f}^2}{4} + \frac{1}{4} f_i \mathbf{f} \partial_i \mathbf{g} + \frac{1}{64} f_i f_j (\partial_i \mathbf{f}) \partial_j \mathbf{f} + \frac{1}{16} f_i f_j \mathbf{f} \partial_i \partial_j \mathbf{f}$$

• Geodesic equation $\dot{v}^k + \Gamma_{ij}^k v^i v^j = 0$

➡ affine connection $\Gamma_{ij}^k = -\frac{3}{2} (\partial_i \partial_j \mathbf{f}) \partial^k \mathbf{f}$

Emergent gravity

- Spatial metric

$$\gamma_{ij} = \delta_{ij} - \frac{3}{2} \partial_i \mathbf{f} \partial_j \mathbf{f}$$

➡ *Riemann curvature for a harmonic field:* $R^{(3)} = \frac{3}{2} (\partial_i \partial_j \mathbf{f})^2$

- Velocity independent term

● *static gravitational potential:* $g_{00} = 1 + 2V_{\text{eff}}$

- Effective Lagrangian = post-Newtonian expansion of

$$\mathcal{L} = -(g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu)^{1/2}$$

with $x^\mu = (t, \mathbf{r}), \quad g_{00} = 1 + \mathbf{f}^2/2, \quad g_{0i} = 0, \quad g_{ij} = -\gamma_{ij}$

➡ *pseudo-Riemann curvature* $R^{(4)} = (\partial_i \mathbf{f})^2 - \frac{3}{2} (\partial_i \partial_j \mathbf{f})^2$

Next-to-next-to-leading effective action

● One spatial coordinate q

$$\begin{aligned}\mathcal{L}_{\text{eff}} = & \frac{\dot{q}^2}{2} \left[1 - \frac{3}{2} f'^2 - 10 f' f'' g - 10 f'^2 g' - 3 f f' g'' \right. \\ & - \frac{379}{128} f'^4 - \frac{691}{64} f f'^2 f'' - \frac{3}{128} f^2 f''^2 - \frac{9}{8} f^2 f' f''' \left. \right] \\ & + \frac{\dot{q}^4}{12} \left[-5 f''^2 + 10 f' f''' \right] - \left[V_g + \frac{f^2}{4} + \frac{f^2 g'}{4} + \frac{f^2 f'^2}{64} \right. \\ & + \frac{f^3 f''}{16} - \frac{5}{4} f'^2 g^2 - \frac{5}{4} f f'^3 g + \frac{f^2 g'^2}{4} + \frac{9}{256} f^2 f'^2 g' \\ & + \frac{3}{16} f^3 f'' g' + \frac{f^3 f' g''}{64} + \frac{f^4 g'''}{64} - \frac{1435}{4608} f^2 f'^4 \\ & \left. + \frac{65}{4608} f^3 f'^2 f'' + \frac{41}{1152} f^4 f''^2 + \frac{f^4 f' f'''}{192} + \frac{f^5 f''''}{384} \right]\end{aligned}$$

Dynamical stabilization

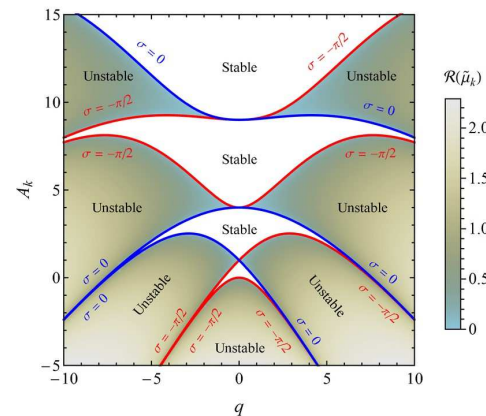
● Equilibrium point $g(q) = \delta q + \mathcal{O}(q^2), \quad f(q) = \epsilon q + \mathcal{O}(q^2)$

● Bifurcation (stability change) $\left. \frac{\partial \mathcal{F}(q, \dot{q})}{\partial q} \right|_{q=\dot{q}=0} = 0$

● Stability curve
$$\delta = -\frac{\epsilon^2}{2} + \frac{7}{32}\epsilon^4 - \frac{29}{144}\epsilon^6 + \mathcal{O}(\epsilon^8)$$

(M.Beneke et al. (2023) pushed it to $\mathcal{O}(\epsilon^{32})$ using FORM)

*agrees with
Floquet analysis of
Mathieu equation*



Quantization

● How it is done in CMT (weird way)

- *Floquet analysis of Schrödinger eq. with $\mathcal{H} = \hat{\mathbf{p}}^2/2 + V_g + V_f \cos t$*

➡ $\mathcal{H}_{\text{eff}} = \hat{U}^\dagger \mathcal{H} \hat{U} - i\hbar \hat{U}^\dagger \partial_t \hat{U}$

- *many variants: Brillouin-Wigner, van Vleck, Floquet-Magnus, ...*

● How it is done in particle physics (smart way)

- *effective theory = expansion of the action in scale ratio*

- *guiding line: nonrelativistic QED expansion in v^2/c^2*

- *basic object: Green's function*

$$\mathcal{G} = (i\hbar\partial_t - \mathcal{H} + i\varepsilon)^{-1}$$

High Frequency Effective Theory

● External field expansion

$$\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_0 + \tilde{\mathcal{G}}_0 \tilde{V}_g \tilde{\mathcal{G}}_0 + \tilde{\mathcal{G}}_0 \tilde{V}_f \tilde{\mathcal{G}}_0 \tilde{V}_f \tilde{\mathcal{G}}_0 + \dots = \tilde{\mathcal{G}}_0 + \tilde{\mathcal{G}}_0 \tilde{V}_{\text{eff}} \tilde{\mathcal{G}}_0 + \dots$$

with $\tilde{\mathcal{G}}_0(\mathbf{p}, E) = (E - \mathbf{p}^2/2 + i\varepsilon)^{-1}$

$$\begin{aligned} \text{double line} &= \text{single line} + \text{self-energy} \\ &+ \frac{1}{2!} \left(\text{two wavy lines} + \text{crossed wavy lines} \right) + \dots \end{aligned}$$

The diagram shows the expansion of a double line (two parallel horizontal lines) into a sum of terms. The first term is a single horizontal line. The second term is a horizontal line with a vertical dashed line (self-energy) attached to its midpoint. The third term is a horizontal line with two vertical wavy lines attached to its midpoint. The fourth term is a horizontal line with two crossed vertical wavy lines attached to its midpoint. The expansion is shown as a sum of these terms, with the first two terms being the leading order and the subsequent terms being higher order corrections.

High Frequency Effective Theory

● High frequency expansion

$$\tilde{\mathcal{G}}_0(\mathbf{p}, E + \hbar\omega) = \frac{1}{\hbar\omega} - \frac{E - \mathbf{p}^2/2}{(\hbar\omega)^2} + \dots$$

$$\frac{1}{2!} \left(\text{diagram 1} + \text{diagram 2} \right) \rightarrow \text{diagram 3}$$

● *leading power HFET seagull vertex*

$$\frac{\langle V_f(\hat{\mathbf{p}}^2/2 - E)V_f \rangle}{2(\hbar\omega)^2} = \frac{\mathbf{f}^2}{4\omega^2}$$

similar to NRQED seagull vertex due to off-shell positron

High Frequency Effective Theory

● Next-to-leading $\mathcal{O}(v^4)$ HFET Hamiltonian

$$\mathcal{H}_{\text{eff}} = \frac{1}{8} [\{\hat{p}_i \hat{p}_j, \gamma^{ij}\}_+ + 2\hat{p}_i \gamma^{ij} \hat{p}_j] + \frac{\hbar^2}{16} (\partial_i \partial_j \mathbf{f})^2 + V_{\text{eff}}$$

● *or for harmonic field*

$$\mathcal{H}_{\text{eff}} = \hat{p}_i \gamma^{ij} \hat{p}_j / 2 - \frac{\hbar^2}{12} R^{(3)} + V_{\text{eff}}$$

● *not the covariant Laplacian!*

➡ *one can distinguish emergent gravity in quantum phenomena!*

Oscillating magnetic field

● Equation of motion

$$\mathbf{F}(t, \mathbf{r}) = e (\mathbf{E}(t, \mathbf{r}) + \mathbf{v} \times \mathbf{B}(t, \mathbf{r}))$$

● *induced electric field circulation* $\partial \times \mathbf{E}(\mathbf{r}) = \omega \mathbf{B}(\mathbf{r})$

● Effective action

● *induced metric* $\gamma_{ij} = \delta_{ij} - \frac{e^2}{2m^2\omega^4} (\partial E_i \partial E_j + \partial_i \mathbf{E} \partial E_j + \partial_j \mathbf{E} \partial E_i)$

● *effective potential* $V_{\text{eff}} = \frac{e^2}{4m\omega^2} \mathbf{E}^2$

Oscillating magnetic field

● Rotating magnetic field

$$\mathbf{B}(t, \mathbf{r}) = B (\cos(\omega t), -\sin(\omega t), 0)$$

- *already realized in TOP traps for magnetic moments*

➡ *Harmonic confining effective potential*

$$V_{\text{eff}} = \frac{m\omega_c^2}{16} (x^2 + y^2 + 2z^2)$$

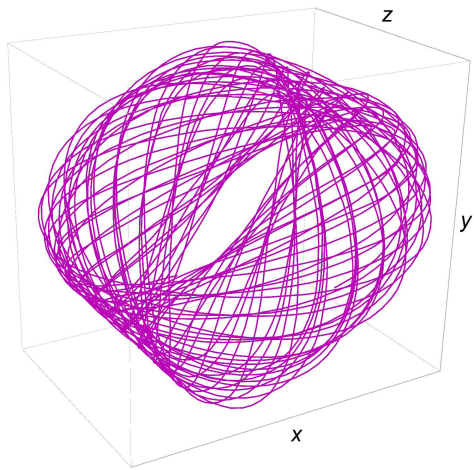
cyclotron frequency $\omega_c = eB/m$, *expansion parameter* ω_c^2/ω^2

● Crucial features

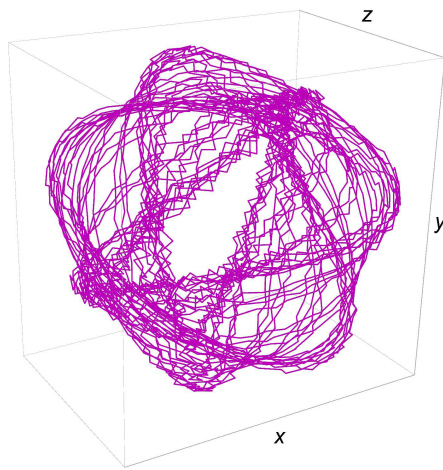
- *potential minimum is topologically protected (vortex structure)*
- *depth scales as $\omega_c \Phi$ with the total magnetic flux amplitude Φ*

Particle trajectories in the magnetic trap

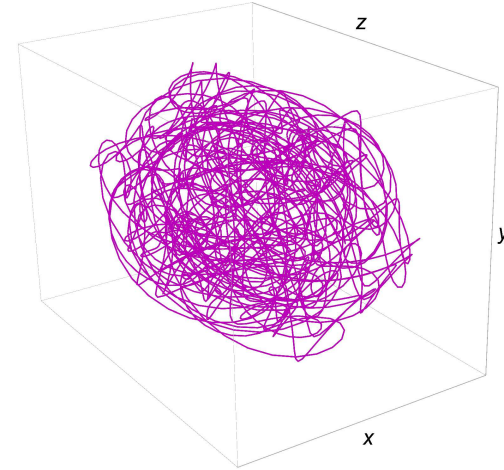
From quasiperiodic motion to chaos



$$\omega_c/\omega = 1/100$$



$$\omega_c/\omega = 1/10$$



$$\omega_c/\omega = 1$$

the confinement is lost only for $\omega_c \gtrsim \omega$

Summary

- Asymptotic method for rapidly oscillating field
 - *high order effective action*
- Emergent Riemann space
 - *field frequency* $\Leftrightarrow$ *speed of light*
 - *energy distribution* $\Leftrightarrow$ *curvature*
- Quantization
 - *NRQED* $\Leftrightarrow$ *HFET*
- Dynamical magnetic confinement
 - ➡ *new class of traps for electric charges*

Summary

a childhood dream came true

Feynman rules for Kapitsa pendulum at last!