Effective Field Theory
of Unstable Top

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Topics discussed

- Top-antitop threshold production
  - brief introduction and review
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- Top width effect:
  - beyond the complex energy shift
  - effective theory of unstable particles “ρNRQCD”
  - unstable top production in NLO and NNLO
  - spurious divergences in pNRQCD
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  - effective theory of unstable particles “ρNRQCD”
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Top-antitop threshold production at the ILC

Would be an ideal place to study the top
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Theory:
- Allows for the first principle QCD predictions
- High order results are available (NNNLO is coming!)
Top-antitop threshold production at the ILC

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Phenomenology:

- $m_t$, $\alpha_s$, $\Gamma_t$, $y_t$, $M_H$
Top-antitop threshold production at the ILC

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Phenomenology:
- $m_t$, $\alpha_s$, $\Gamma_t$, $y_t$, $M_H$

Higgs, EWSB, GUT, SUSE, ...
Born cross section

\[ R(e^+e^- \rightarrow t\bar{t}) \]
Coulomb and finite width effects

\[ R(e^+e^- \to t\bar{t}) \]

\[ E \text{(GeV)} \]

\[ R_{\text{res}} \sim \frac{\alpha_s^3}{m_t \Gamma_t}, \quad E_{\text{res}} \sim \alpha_s^2 m_t \]
Perturbation theory for heavy quarkonium

Pinnacle of modern effective field theory!
Perturbation theory for heavy quarkonium

Pinnacle of modern effective field theory!

- Apparent slow convergence

- Possible reasons:
  - Renormalons: $n!(\beta_0 \alpha_s)^n$
  - Threshold logs: $\alpha_s^n \ln^m \alpha_s$
Perturbation theory for heavy quarkonium

Pinnacle of modern effective field theory!

Apparent slow convergence

Possible reasons:

- Renormalons: \( n!(\beta_0\alpha_s)^n \)
- Threshold logs: \( \alpha_s^n \ln^m \alpha_s \)

Full N\(^3\)LO analysis is mandatory
The N$^3$LO ground state energy can be expressed as:

$$\frac{\delta E_1^{N^3LO}}{E_1^{LO}} = \alpha_s^3 \left( 58.205 + 15.297 \ln(\alpha_s) + 26.654 \right)$$

This contribution is referred to as the renormalon contribution.
$N^3LO$ ground state energy

![Graph showing NLO, NNLO, and $N^3LO$ ground state energy as functions of $\mu$ (GeV).](image)
Finite top lifetime

- **Resonant approximation**
  - **complex energy shift** \[ E \rightarrow E + i\Gamma_t \]

(V. Fadin, V. Khoze, JETP Lett. 46 (1987) 525)

- *not consistent in pNRQCD beyond LO!*
Finite top lifetime

- **Resonant approximation**
  - *complex energy shift* $E \rightarrow E + i\Gamma_t$
  
  \[(V.\text{Fadin}, V.\text{Khoze}, JETP Lett. 46 (1987) 525)\]

- *not consistent in pNRQCD beyond LO!*

- **Nonresonant contribution**
  - *Suppression of nonfactorizable contribution*
    

  - *Phase space matching (tight cuts on top invariant mass)*
    
    \[(A. \text{Hoang}, C. \text{Rei\ss}er, P. \text{Ruiz-Femenía}, Phys. Rev. D82 (2010) 014005)\]

  - *QCD effective theory of unstable particles to NLO*
    

  - *NRQCD effective theory of unstable particles to NNLO*
    
    \[(A. \text{Penin}, J. \text{Piclum}, JHEP 1201 (2012) 034)\]
Optical theorem:

\[ R_{\text{Born}}^{\text{res}} \sim \Im \int \frac{d^3 p}{(2\pi)^3} \frac{1}{p^2 - m_t E - i\epsilon} \sim \Im \sqrt{-E - i\epsilon}, \]

On-shell top:

\[ \Im \left[ \frac{1}{p^2 - m_t E - i\epsilon} \right] \sim \delta(p^2 - m_t E), \]
Unstable top

\[ e^- \rightarrow \gamma, Z \rightarrow e^- + e^- + \bar{e} + \bar{e} + e^+ + e^+ \]

Imaginary part of mass operator:

Here \( \rho = 1 - M_W/m_t, \ z = (p^2 - m_t E)/m_t^2 \ll 1 \)

\[ \Im[\Sigma(z)] = \frac{\Gamma_t}{2} - \frac{\Gamma_t}{2} \left[ \theta(z - \rho) + \left( \frac{2z}{\rho} - \frac{z^2}{\rho^2} \right) \theta(\rho - z) + O(\rho, z) \right] \]
Unstable top

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\]

Resonant contribution  Nonresonant contribution
Resonant contribution

Complex energy shift:

- **Dyson resummation**

\[
\frac{1}{p^2 - m_t E - i\epsilon} \to \frac{1}{p^2 - m_t E - i m_t \Gamma_t}
\]

- **Breit-Wigner resonance**

\[
\delta(p^2 - m_t E) \to \frac{1}{\pi} \frac{\Gamma_t}{(p^2/m_t - E)^2 + \Gamma_t^2},
\]

- **Born cross section**

\[
R_{res}^{Born} \sim \Im \left[ \sqrt{-E - i\Gamma_t} \right]
\]

Invariant mass distribution:

\[
2p^2 \approx m_t^2 - (p_W + p_b)^2
\]
Nonresonant contribution

On-shell $t \leftrightarrow$ on-shell $W$ and $b$

- Kinematical constraint $M_W^2 < (p_W + p_b)^2 < m_t^2$
- Natural cutoff on spatial momentum $0 < p^2 < \rho m_t^2$
- $\Im[\Sigma] - \Gamma_t/2 \neq 0$ for $p^2 \neq 0 \Rightarrow \text{"nonresonant"}$
Nonresonant contribution

- **On-shell** $t \rightarrow$ on-shell $W$ and $b$

- **Kinematical constraint** \[ M_W^2 < (p_W + p_b)^2 < m_t^2 \]

- **Natural cutoff on spatial momentum** \[ 0 < p^2 < \rho m_t^2 \]

- $\Im [\Sigma] - \Gamma_t/2 \neq 0$ for $p^2 \neq 0 \Rightarrow "\text{nonresonant}"$

- **Approximation** $\rho \ll 1$

- **Nonrelativistic** $t$ and $W$, **ultrarelativistic** $b$

- **Expansion in** $\rho$ similar to pNRQCD expansion in $v^2 \sim E/m_t$

- **Actual value** $\rho = 0.53 \ldots$
Nonrelativistic effective theory of unstable top

Scales

$pNRAQCD$:
- **hard** $m_t$
- **soft** $v m_t$
- **ultrasoft** $v^2 m_t$

$\rho NRAQCD$:
- **hard** $m_t$
- **$\rho$-soft** $\rho^{1/2} m_t$
- **$\rho$-ultrasoft** $\rho m_t$
Nonrelativistic effective theory of unstable top

Scales

\( pNRQCD: \)

- hard \( m_t \)
- soft \( v m_t \)
- ultrasoft \( v^2 m_t \)

\( \rho NRQCD: \)

- hard \( m_t \)
- \( \rho \)-soft \( \rho^{1/2} m_t \)
- \( \rho \)-ultrasoft \( \rho m_t \)

Scale hierarchy and power counting

\( pNRQCD \) scaling: \( \alpha_{ew}^{1/2} \sim \alpha_s \sim v \ll 1, \quad \Gamma_t / m_t \sim \alpha_{ew} \)

complimentary expansion in \( \rho \) with \( v \ll \rho^{1/2} \ll 1 \)

\( \rho \)-Coulomb terms \( \alpha_s / \rho^{1/2} \ll 1 \)
Nonrelativistic effective theory of unstable top

How to expand?

\( \rho \)-pNRQCD Feynman rules

expansion by regions
Nonrelativistic effective theory of unstable top

How to expand?

✘ $\rho$-pNRQCD Feynman rules

✓ expansion by regions
NLO nonresonant contribution

Power counting

- resonant contribution
- nonresonant contribution

\[ \Im \sqrt{-E - i\Gamma_t} \sim v \]
\[ \Gamma_t \sim v^2 \]
NLO nonresonant contribution

- **Power counting**
  - resonant contribution
    \[ \Im \sqrt{-E - i\Gamma_t} \sim v \]
  - nonresonant contribution
    \[ \Gamma_t \sim v^2 \]

- **Calculation steps**
  - treat \( \Im[\Sigma] - \Gamma_t/2 \) as a perturbation
  - add all the two-loop diagrams with \( t-W-b \) cut
  - expand in \( E/\rho m_t \), expand in \( \rho \) single region left:
    - \( t \) and \( W \) are \( \rho \)-potential, \( b \) is \( \rho \)-ultrasoft
  - recover nonrelativistic propagators and vertices
NLO nonresonant contribution

- **Power counting**
  - **resonant contribution** \[ \Im \sqrt{-E - i\Gamma_t} \sim v \]
  - **nonresonant contribution** \[ \Gamma_t \sim v^2 \]

- **Calculation steps**
  - **treat** \( \Im [\Sigma] - \Gamma_t/2 \) **as a perturbation**
  - **add all the two-loop diagrams with** \( t-W-b \) **cut**
  - **expand in** \( E/\rho m_t \), **expand in** \( \rho \rightarrow \) **single region left:**
    - \( t \) and \( W \) **are** \( \rho \)-**potential**, \( b \) **is** \( \rho \)-**ultrasoft**
  - **recover nonrelativistic propagators and vertices**

- **no expansion in** \( \rho \rightarrow \) **fully relativistic calculation** (M. Beneke et al.)
NLO diagrams
NLO diagrams
NLO result

leading term of $\rho$-expansion

\[ R_{nr}^{NLO} = -\frac{24}{\pi \rho^{1/2}} \frac{\Gamma_t}{m_t} \left[ \frac{4}{9} + "Z" - \frac{1}{\sin^4 \theta_W} \left( \frac{17}{48} - \frac{9\sqrt{2}}{32} \ln \left( 1 + \sqrt{2} \right) \right) \right] \]
NLO result

leading term of $\rho$-expansion

\[
R_{nr}^{NLO} = -\frac{24}{\pi \rho^{1/2}} \frac{\Gamma_t}{m_t} \left[ 4 \frac{9}{9} + "Z" - \frac{1}{\sin^4 \theta_W} \left( \frac{17}{48} - \frac{9\sqrt{2}}{32} \ln \left( 1 + \sqrt{2} \right) \right) \right]
\]

Convergence?

generally not bad

for some diagrams Padé is necessary
Convergence

*diagram "j"*
Convergence

Dash line - leading $\rho$-dependence
Solid line - exact $\rho$-dependence
NNLO nonresonant contribution

\( \rho \)-leading diagrams

Regions of gluon momentum

- (a) and (d) - hard, potential, \( \rho \)-potential
- (b) and (d) - hard, \( \rho \)-soft
NNLO nonresonant contribution

Diagram \((d)\):

- **box momentum is \(\rho\)-potential**
- **self-energy momentum is hard or \(\rho\)-soft**

\[
\text{Im}[\Pi^{(1)}(z)] = \frac{C_F \alpha_s}{\pi} \left[ \frac{9}{4} + \frac{1}{3} \pi^2 - \frac{3}{2} \ln(2\rho) - \frac{3}{2} \ln \left(1 - \frac{z}{\rho}\right) + \mathcal{O}(\rho, z) \right] \text{Im}[\Pi^{(0)}(z)]
\]

- **\(\ln(\rho)\)** term is absorbed into corrected \(\Gamma_t\)

\[
\Gamma_t = \left[ 1 + \frac{C_F \alpha_s}{\pi} \left( \frac{9}{4} - \frac{2}{3} \pi^2 - \frac{3}{2} \ln(2\rho) + \mathcal{O}(\rho) \right) \right] \Gamma_t^{LO}
\]
NNLO result

leading term of $\rho$-expansion

$$R_{nr}^{N^2 LO} = \frac{3C_F\alpha_s}{\pi^2 \rho^{1/2}/m_t} \left\{ \left[ \frac{4}{9} + \text{“Z”} \right] \frac{\pi^2}{\rho^{1/2}} \left( 3 \ln \left( \frac{E^2 + \Gamma_t^2}{\rho m_t} \right) + \frac{3}{2} + 6 \ln 2 \right) + (18 + 24 \ln 2) \right\}$$

$$+ \frac{1}{\sin^4 \theta_W} \left[ \frac{22}{3} + \frac{17\pi^2}{6} - \frac{17}{2} \ln 2 + (2 - 3\pi^2 + 9 \ln 2) \frac{3\sqrt{2}}{4} \ln \left( 1 + \sqrt{2} \right) - \frac{27\sqrt{2}}{8} \left( \ln^2 \left( 1 + \sqrt{2} \right) + \text{Li}_2 \left( 2\sqrt{2} - 2 \right) \right) \right] \}.$$
NNLO result

leading term of $\rho$-expansion

$$R_{nr}^{N^2 LO} = \frac{3C_F\alpha_s}{\pi^2\rho^{1/2}} \frac{\Gamma_t}{m_t} \left\{ \left[ \frac{4}{9} + "Z" \right] \left[ \frac{\pi^2}{\rho^{1/2}} \left( 3 \ln \left( \sqrt{E^2 + \Gamma_t^2/\rho m_t} \right) + \frac{3}{2} + 6 \ln 2 \right) + (18 + 24 \ln 2) \right] + \frac{1}{\sin^4 \theta_W} \left[ \frac{22}{3} + \frac{17\pi^2}{6} - \frac{17}{2} \ln 2 + (2 - 3\pi^2 + 9 \ln 2) \frac{3\sqrt{2}}{4} \ln \left(1 + \sqrt{2}\right) - \frac{27\sqrt{2}}{8} \left( \ln^2 \left(1 + \sqrt{2}\right) + \text{Li}_2 \left(2\sqrt{2} - 2\right) \right) \right] \right\}.$$

$\rho$-Coulomb term $\alpha_s/\rho^{1/2}$

new type of logs $\ln(E/\rho m_t) \sim \ln(v^2/\rho)$
Spurious divergences in pNRQCD

Problem with complex energy shift in NNLO

- integral over potential momentum $p^2/m_t \sim E$

is UV divergent $\delta R \sim \Im \left[ E \ln \left( E/\mu \right) \right] \to \Gamma_t \ln \left( E/\mu \right)$
Spurious divergences in pNRQCD

Problem with complex energy shift in NNLO

- integral over potential momentum \[ \frac{p^2}{m_t} \sim E \]

  is UV divergent \[ \delta R \sim \Im [E \ln(E/\mu)] \rightarrow \Gamma_t \ln(E/\mu) \]

Physical cutoff

- \[ \Im [\Sigma] = 0 \quad \text{for} \quad \frac{p^2}{m_t} > \rho m_t \]
Spurious divergences in pNRQCD

Problem with complex energy shift in NNLO

integral over potential momentum $p^2/m_t \sim E$

is UV divergent $\delta R \sim \Im [E \ln(E/\mu)] \rightarrow \Gamma_t \ln(E/\mu)$

Physical cutoff

$\Im [\Sigma] = 0 \text{ for } p^2/m_t > \rho m_t$

Expansion by regions

integral over $\rho$-potential region $p^2/m_t \sim \rho m_t$

is IR divergent $\delta R \sim \Gamma_t \ln(\mu/\rho m_t)$

$\Rightarrow$ finite sum $\delta R \sim \Gamma_t \ln(E/\rho m_t)$
Numerics

\[ R \]

- **dot line** - LO
- **dash line** - LO+NLO nonresonant
- **solid line** - LO+NNLO nonresonant

\[ E \text{ [GeV]} \]
Summary

Effective theory of $\rho$NRQCD
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- Effective theory of $\rho_{\text{NRQCD}}$
  - based on nonrelativistic expansion in $\rho = 1 - m_t/M_W$
  - systematically accounts for finite width effects in threshold top-antitop production
  - optimized for high-order calculations
  - solve the problem of the spurious divergences
Summary

Effective theory of $\rho$NRQCD

- based on nonrelativistic expansion in $\rho = 1 - m_t/M_W$
- systematically accounts for finite width effects in threshold top-antitop production
- optimized for high-order calculations
- solve the problem of the spurious divergences
- conceptually clear and aesthetically appealing