Effective Theory of Classical and Quantum Particle Dynamics in Rapidly Oscillating Fields

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We study the large-scale dynamics of charged particles in a rapidly oscillating field and formulate its classical and quantum effective theory description. The high-order perturbative results for the effective action are presented. Remarkably, the action models the effects of post-Newtonian general relativity on the motion of nonrelativistic particles, with the values of the emergent curvature and speed of light determined by the field spatial distribution and frequency. Our results can be applied to a wide range of physical problems including the high-precision analysis and design of the charged particle traps and Floquet quantum materials.

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Since the classical work [1] the dynamics of particles in a rapidly oscillating field has been studied in a wide range of problems from dynamical chaos [2] to quantum computing [3,4] and Floquet engineering of quantum materials [5] with the renowned application in the design of the Paul traps [6]. Theoretical description of this class of systems is based on the concept of averaging, when the effect of the oscillating field is smeared out and the long-time evolution is governed by the resulting effective interaction naturally obtained within the high-frequency expansion as a series in the ratio of the oscillation period to a characteristic time-scale of the averaged system. The method is well known in classical mechanics [7] and has been extended to quantum systems [8–11]. Many subsequent works were dedicated to the quantum physics applications and the method has been refined and generalized to include many-body systems, spin, adiabatic variation of the oscillating field, etc. [12–19]. However, given the importance of the problem, surprisingly little is know about the high-order perturbative behavior of the generic three-dimensional systems even at the classical level. The existing analysis of the quantum systems based on Floquet theory quickly becomes tedious in high orders too, and often lacks the proper power counting. Hence, it is no surprise that the theory of the charged particles confined in the Paul traps [20] is far less accurate than the one for the Penning traps [21]. The goal of this work is to introduce a new foundation for a systematic analysis of the periodically driven systems in the high-frequency limit. Its core is the effective field theory approach ideal for the perturbative treatment of the multiscale problems. We start with the discussion of a classical system to identify the relevant scales, expansion parameters, and power counting rules. Then we elaborate an asymptotic method to compute the classical effective action to high orders in high-frequency expansion. Remarkably, the resulting effective interaction models the dynamics of the nonrelativistic particle in the pseudo-Riemann space, which gives a new nontrivial example of "analog gravity" [22]. To quantize the effective action we develop the high-frequency effective theory (HFET), being guided by an analogy between the high-frequency expansion and the nonrelativistic expansion of quantum electrodynamics (QED).

Our starting point is the classical equation of motion for a particle of mass m subjected to a static force $-\boldsymbol{G}$ and a periodic force $-\boldsymbol{F}\cos\omega t$

$$m\ddot{\boldsymbol{R}} + \boldsymbol{G}(\boldsymbol{R}) + \boldsymbol{F}(\boldsymbol{R})\cos\omega t = 0, \qquad (1)$$

where the dot stands for the time derivative d/dt and the bold fonts indicate three-dimensional vectors. The periodic drive is limited to a single harmonic for the clarity of the presentation but the inclusion of higher harmonics is rather straightforward. We do not specify the nature of the external fields to keep the discussion general and consider the limit of fast oscillation. Let us quantify this condition as it plays a crucial role for the determination of the expansion parameter and the power counting rules. For a system of a characteristic size L the typical velocity acquired by the particle under the action of the time-independent force is $v \sim (GL/m)^{1/2}$. One can define a "reference" velocity $c = L\omega$ and the oscillations are considered fast when $v/c \ll 1$. The main idea of the effective theory approach is to separate the "slow" large-scale dynamics characterized by the velocity v from the "fast" small-scale dynamics characterized by the velocity c and manifested through the power corrections in the scale ratio to the effective action. As we will see, the expansion in v/c shares many features with the nonrelativistic expansion of the relativistic field theories, with c playing a role of the speed of light. It is convenient to introduce the dimensionless variables $\omega t \to t$, $R/L \to R$ so that the equation of motion becomes

$$\ddot{\boldsymbol{R}} + \boldsymbol{g}(\boldsymbol{R}) + \boldsymbol{f}(\boldsymbol{R})\cos t = 0, \tag{2}$$

with $\boldsymbol{g} = \boldsymbol{G}/(Lm\omega^2)$ and $\boldsymbol{f} = \boldsymbol{F}/(Lm\omega^2)$. Note that in the rescaled variables $c = 1$ and the expansion parameter is v. While $\boldsymbol{g} = O(v^2)$ by definition, the scaling of the oscillating term needs to be determined. The leading contribution of the oscillating field to the effective action is quadratic in its amplitude and we are interested in the physical systems where the large-scale dynamics is essentially determined by the effect of the periodic drive, which should be comparable to the one of the static field. This requires $\boldsymbol{f} = O(v)$, i.e., with the rest of the parameters fixed, the amplitude of the oscillating field should scale linearly with its frequency. This does not necessarily mean the actual dependence of the amplitude on the frequency but rather determines the relevant range for the ratio of the static and oscillating field magnitude at a given ω. The above problem appears in a variety of physical systems and a number of methods have been developed to disentangle the slow and fast dynamics in perturbation theory. They share the principal idea of introducing independent variables for the fast and slow evolution with subsequent averaging over the fast one. Its particular realization, however, is crucial to get an efficient tool for the high-order analysis. We follow the general idea of the asymptotic method [7] and look for the solution in the form

$$\boldsymbol{R} = \boldsymbol{r} + \sum_{n=1}^{\infty} [\boldsymbol{c}_n(\boldsymbol{r})\cos(nt) + \boldsymbol{s}_n(\boldsymbol{r})\sin(nt)], \tag{3}$$

where the vector $\boldsymbol{r}$ describes the large-scale slow evolution, $\dot{\boldsymbol{r}} \equiv \boldsymbol{v} = \sum_{m=1}^{\infty} \boldsymbol{v}^{(m)}(\boldsymbol{r})$ with $\boldsymbol{v}^{(m)} = O(v^m)$. The method [7] has been originally developed for the nonlinear oscillation theory and its characteristic feature is that the oscillation amplitude itself is taken as a slow variable. In the case of nonquasiperiodic motion at hand a natural choice of the slow variable is the path along the smeared trajectory $\boldsymbol{r}(t)$. Then the total time derivative splits into the slow and fast components as follows $d/dt = \boldsymbol{v}\cdot\boldsymbol{\partial}_r + \partial_t$. Substituting Eq. (3) into Eq. (2) and reexpanding in the Fourier harmonics one can find the coefficients $\boldsymbol{c}_n(\boldsymbol{r})$ and $\boldsymbol{s}_n(\boldsymbol{r})$ order by order in v^2. The zero harmonic then defines the equation of motion for the slow evolution of the form $\ddot{\boldsymbol{r}} + \boldsymbol{\mathcal{F}}_{\text{eff}}(\boldsymbol{r},\boldsymbol{v}) = 0$. At $\mathcal{O}(v^2)$ we get the well known leading order expression

$$\boldsymbol{\mathcal{F}}_{\text{eff}} = \boldsymbol{g} + \frac{1}{2} f_i \partial_i \boldsymbol{f}. \tag{4}$$

The new next-to-leading $\mathcal{O}(v^4)$ result reads

$$\begin{aligned}\boldsymbol{\mathcal{F}}_{\text{eff}} &= \boldsymbol{g} + \frac{1}{2} f_i\partial_i\boldsymbol{f} - \frac{3}{2} v_i v_j(\partial_i\partial_j f_k)\partial_k\boldsymbol{f} + \frac{1}{4} f_i f_j \partial_i\partial_j\boldsymbol{g} \\ &+ \left[\frac{3}{2} g_i(\partial_i f_k) + \frac{1}{2} f_i(\partial_i g_k) + \frac{25}{32} f_i(\partial_i f_j)(\partial_j f_k)\right. \\ &+ \left.\frac{3}{16} f_i f_j(\partial_i\partial_j f_k)\right]\partial_k\boldsymbol{f} + \frac{1}{32} f_i f_j(\partial_i f_k)\partial_j\partial_k\boldsymbol{f} \\ &+ \frac{1}{16} f_i f_j f_k \partial_i\partial_j\partial_k\boldsymbol{f},\end{aligned} \tag{5}$$

where the summation over repeating vector indices is implied. So far we did not make any assumption about the properties of the fields. If we assume the existence of the corresponding potentials $\boldsymbol{g} = \boldsymbol{\partial} V_g$ and $\boldsymbol{f} = \boldsymbol{\partial} V_f$, Eqs. (4) and (5) follow from the effective Lagrangian

$$\mathcal{L}_{\text{eff}} = \frac{v_i v_j}{2}\left[\delta_{ij} - \frac{3}{2}\partial_i\boldsymbol{f}\partial_j\boldsymbol{f}\right] - V_{\text{eff}}, \tag{6}$$

where the effective potential reads

$$V_{\text{eff}} = V_g + \frac{\boldsymbol{f}^2}{4} + \frac{1}{4} f_i \boldsymbol{f}\partial_i\boldsymbol{g} + \frac{1}{64} f_i f_j(\partial_i\boldsymbol{f})\partial_j\boldsymbol{f} + \frac{1}{16} f_i f_j \boldsymbol{f}\partial_i\partial_j\boldsymbol{f}. \tag{7}$$

In the quadratic approximation in the oscillating field the effective interaction has a distinctive form. The velocity dependent term in Eq. (5) can be associated with the geodesic equation for the affine connection $\Gamma^k_{ij} = -\frac{3}{2}(\partial_i\partial_j\boldsymbol{f})\partial^k\boldsymbol{f}$ corresponding to the three-dimensional metric $\gamma_{ij} = \delta_{ij} - \frac{3}{2}\partial_i\boldsymbol{f}\partial_j\boldsymbol{f}$. For an arbitrary field $\boldsymbol{f}$ with nonvanishing second derivative this metric describes a non-Euclidean space. In the region of vanishing charge density $\boldsymbol{\partial f} = 0$, which corresponds to a harmonic potential $\boldsymbol{\partial}^2 V_f = 0$ relevant for most physical applications, the expression for the corresponding Riemann curvature scalar takes a particulary simple form $R^{(3)} = \frac{3}{2}(\partial_i\partial_j\boldsymbol{f})^2$ and is non-negative. Moreover, in the quadratic approximation Eq. (6) coincides with the post-Newtonian expansion of the relativistic Lagrangian for a particle moving in a gravitational field $\mathcal{L} = -(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)^{1/2}$, where $x^\mu = (t, \boldsymbol{r})$ and the metric of the $3+1$ dimensional pseudo-Riemann space is [23]

$$g_{00} = 1 + \boldsymbol{f}^2/2, \qquad g_{0i} = 0, \qquad g_{ij} = -\gamma_{ij}. \tag{8}$$

The corresponding scalar curvature reads

$$R^{(4)} = (\partial_i\boldsymbol{f})^2 - \frac{3}{2}(\partial_i\partial_j\boldsymbol{f})^2. \tag{9}$$

The above method readily generates the higher order terms of the effective Lagrangian and is limited mainly by the size of the resulting expressions. We present a relatively compact $\mathcal{O}(v^6)$ Lagrangian in one dimension since many physical systems can be reduced or decomposed into the one-dimensional problems. For a single generalized coordinate q we get the next-to-next-to-leading result

$$\begin{aligned}\mathcal{L}_{\rm eff} = {}& \frac{\dot{q}^2}{2}\left[1-\frac{3}{2}f'^2-10f'f''g-10f'^2g'-3ff'g''-\frac{379}{128}f'^4-\frac{691}{64}ff'^2f''-\frac{3}{128}f^2f''^2-\frac{9}{8}f^2f'f'''\right]\\ &+\frac{\dot{q}^4}{12}\left[-5f''^2+10f'f'''\right]+\left[g+\frac{f^2}{4}+\frac{f^2g'}{4}+\frac{f^2f'^2}{64}+\frac{f^3f''}{16}-\frac{5}{4}f'^2g^2-\frac{5}{4}ff'^3g+\frac{f^2g'^2}{4}+\frac{9}{256}f^2f'^2g'\right.\\ &\left.+\frac{3}{16}f^3f''g'+\frac{f^3f'g''}{64}+\frac{f^4g'''}{64}-\frac{1435}{4608}f^2f'^4+\frac{65}{4608}f^3f'^2f''+\frac{41}{1152}f^4f''^2+\frac{f^4f'f'''}{192}+\frac{f^5f''''}{384}\right],\end{aligned} \tag{10}$$

where dash stands for the derivative d/dq. Note that the emergent Lorentz invariance of the effective action is broken by the $\dot{q}^4$ term of Eq. (10) in agreement with the general argument [24].

The convergence of the high frequency expansion Eq. (10) depends on the system and for a given system on the particular values of the adjustable parameters. It can be roughly estimated by evaluating the expression for v^2/c^2 with the characteristic values of the system parameters. At the same time the new high-order results make the assessment of the convergence for a given system much more reliable.

The result Eq. (10) has an interesting connection to the theory of parametric resonance and stability of dynamical systems, which is crucial for the further discussion of the effective theory power counting. Namely, for $g(q)=\delta q+\mathcal{O}(q^2)$ and $f(q)=\epsilon q+\mathcal{O}(q^2)$ with some parameters δ and ϵ the system has an equilibrium point $\mathcal{F}=0$ at $q=\dot{q}=0$. Then the equation $\partial\mathcal{F}(q,\dot{q})/\partial q|_{q=\dot{q}=0}=0$ controls the change of its stability. This equation defines δ as a function of ϵ, i.e., one of the stability curves in the parameter space which play a crucial role in the analysis of chaotic and regular behavior of dynamical systems. For $\epsilon\ll 1$ through the next-to-next-to-leading approximation we get

$$\delta=-\frac{\epsilon^2}{2}+\frac{7}{32}\epsilon^4-\frac{29}{144}\epsilon^6+\mathcal{O}(\epsilon^8), \tag{11}$$

which is consistent with the scaling $\boldsymbol{g}\sim\boldsymbol{f}^2$. Equation (11) agrees with the result obtained within Floquet theory analysis of the Mathieu equation [25], being a nontrivial test of our analysis. This equation, in particular, defines the corrections to the classical result on the stability of inverted pendulum with the natural frequency $\sqrt{-\delta}$ and the forced oscillation amplitude ϵ [1]. Recently, the analysis of the periodically driven pendulum with $f,g\propto\sin(q)$ has been performed to very high orders of perturbation theory [26]. Equation (10) agrees with the next-to-leading effective Lagrangian presented there. For higher orders the comparison of the results is not straightforward since in [26] the velocity dependent terms are eliminated from the equation of motion by using the energy conservation. Hence, the resulting effective potential depends on the total energy of the system, while we use the standard definition of the Lagrangian independent of the initial conditions.

Let us now consider the quantization of the effective action. The existing theory of quantum systems in a rapidly oscillating field is based on Floquet analysis of the Schrödinger equation with the time-periodic Hamiltonian $\mathcal{H}=\hat{\boldsymbol{p}}^2/2+V_g+V_f\cos t$, where $\hat{\boldsymbol{p}}=-i\hbar\boldsymbol{\partial}$ is the momentum operator and we keep the dependence on the Planck constant $\hbar\neq 1$ to separate the quantum corrections from the classical action. The general idea of the method is to construct within the high-frequency expansion a unitary operator $\hat{U}$ such that the effective Hamiltonian $\mathcal{H}_{\rm eff}=\hat{U}^\dagger\mathcal{H}\hat{U}-i\hbar\hat{U}^\dagger\partial_t\hat{U}$ is time independent and determines the quasienergy spectrum, i.e., the slow evolution of the quantum states. The particular realizations of this program may be different. However, in this framework the perturbative calculations quickly become tedious as the order of approximation increases. At the same time the multiscale problems are common to the quantum field theory, where very efficient methods based on the scale separation are elaborated and optimized for high-order calculations. As it was pointed out, the high-frequency expansion is similar to the nonrelativistic expansion and we suggest to realize it in the same way as the Dirac equation in an external field is expanded in inverse powers of the speed of light [27]. Let us consider a Green function of the original time-dependent Schrödinger equation $\mathcal{G}=(i\hbar\partial_t-\mathcal{H}+i\varepsilon)^{-1}$ and its Fourier transform $\tilde{\mathcal{G}}(\boldsymbol{p}_i,\boldsymbol{p}_f;E_i,E_f)$, which depends on the initial and final momentum and energy variables. In general the initial and final energy may differ due to the time dependence of the Hamiltonian. We, however, are interested in the low-energy behavior of the Green function with the kinematical constraints [28] $\boldsymbol{p}_{i,f}^2, E_{i,f}\ll\hbar\omega$. In this case the periodic character of the time dependence implies the energy conservation $E_i=E_f\equiv E$. Expanding the Green function in powers of the external fields we get a series

$$\tilde{\mathcal{G}}=\tilde{\mathcal{G}}_0+\tilde{\mathcal{G}}_0\tilde{V}_g\tilde{\mathcal{G}}_0+\tilde{\mathcal{G}}_0\tilde{V}_f\tilde{\mathcal{G}}_0\tilde{V}_f\tilde{\mathcal{G}}_0+\ldots, \tag{12}$$

where $\tilde{\mathcal{G}}_0(\boldsymbol{p},E)=(E-\boldsymbol{p}^2/2+i\varepsilon)^{-1}$ is the free particle propagator and $\tilde{V}_g$ $(\tilde{V}_f)$ is the Fourier transform of V_g

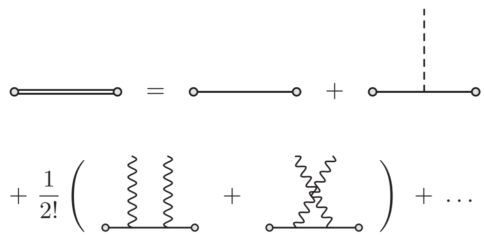

FIG. 1. The Feynman diagrams representing the expansion of the Green function Eq. (12). The double (single) line represents the exact (free) particle propagator while the dashed (wavy) line corresponds to the static (oscillating) external field.

($V_f \cos \omega t$). By using the standard diagrammatic rules for the particle propagators, external fields and the interaction vertices defined by the full theory Lagrangian, this expansion can be represented by the Feynman diagrams in Fig. 1. Note that the contribution with a single insertion of the oscillating field is forbidden by the "energy scale" conservation (for a single-harmonic oscillating field this is true for any odd power of $\boldsymbol{f}$). At the same time in the effective theory the Green function $\mathcal{G} = (i\hbar\partial_t - \mathcal{H}_{\rm eff} + i\varepsilon)^{-1}$ has the expansion

$$\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_0 + \tilde{\mathcal{G}}_0 \delta\mathcal{H}_{\rm eff} \tilde{\mathcal{G}}_0 + \ldots, \tag{13}$$

where $\mathcal{H}_{\rm eff} = \hat{\boldsymbol{p}}^2/2 + \delta\mathcal{H}_{\rm eff}$. By matching Eqs. (12) and (13) we get $\delta\mathcal{H}_{\rm eff}$ order-by-order in $1/\omega^2$. The second term in Eq. (12), i.e., the diagram with a single insertion of the static field in Fig. 1 defines the trivial V_g contribution to the effective Hamiltonian. Let us now consider the diagrams with the double insertion of the oscillating field. The intermediate state propagator carrying the momentum $\boldsymbol{p}$ and energy $E + \hbar\omega$ is far off-shell and can be expanded in a series

$$\tilde{\mathcal{G}}_0(\boldsymbol{p}, E + \hbar\omega) = \frac{1}{\hbar\omega} - \frac{E - \boldsymbol{p}^2/2}{(\hbar\omega)^2} + \cdots, \tag{14}$$

which gives rise to a local effective vertex, Fig. 2. This *seagull* vertex is well known in nonrelativistic QED where it is generated by a far off-shell positron in the intermediate state rather than the large timelike momentum transfer from the oscillating field. The odd powers in ω cancel between the planar and nonplanar diagrams and by the standard tools we readily get the leading $\mathcal{O}(1/\omega^2)$ contribution to the effective vertex in the coordinate space

$$\frac{\langle V_f(\hat{\boldsymbol{p}}^2/2 - E)V_f\rangle}{2(\hbar\omega)^2} = \frac{\boldsymbol{f}^2}{4\omega^2}, \tag{15}$$

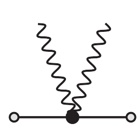

FIG. 2. The effective local vertex resulting from the expansion Eq. (14) of the off-shell propagator in Fig. 1.

where the matrix element is taken between on-shell states with $\boldsymbol{p}^2/2 = E$. For $\omega = 1$ we recover the leading contribution to the classical effective potential Eq. (7). At $\mathcal{O}(1/\omega^4)$ the contribution of the operator $V_f(\hat{\boldsymbol{p}}^2/2 - E)^3 V_f$ to the effective vertex can be computed in the same way with the result

$$\frac{1}{(2\omega)^4}[3(\{\hat{p}_i\hat{p}_j, \partial_i\boldsymbol{f}\partial_j\boldsymbol{f}\}_+ + 2\hat{p}_i\partial_i\boldsymbol{f}\partial_j\boldsymbol{f}\hat{p}_j) + \hbar^2(\partial_i\partial_j\boldsymbol{f})^2]. \tag{16}$$

The terms omitted in Eq. (12) give rise to the effective vertices with the higher powers of the external fields, which along with Eqs. (15) and (16) define the HFET Feynman rules. However, in the given order these vertices reduce to the classical effective potential as in Eq. (15), i.e., require no additional calculation. Setting $\omega = 1$ and switching back to the velocity power counting for the effective Hamiltonian trough $\mathcal{O}(v^4)$ we get

$$\mathcal{H}_{\rm eff} = \frac{1}{8}[\{\hat{p}_i\hat{p}_j, \gamma^{ij}\}_+ + 2\hat{p}_i\gamma^{ij}\hat{p}_j] + \frac{\hbar^2}{16}(\partial_i\partial_j\boldsymbol{f})^2 + V_{\rm eff}, \tag{17}$$

where $\gamma^{ij} = \delta_{ij} + \frac{3}{2}\partial_i\boldsymbol{f}\partial_j\boldsymbol{f} + \mathcal{O}(v^4)$ is the inverse of the metric tensor γ_{ij} and $V_{\rm eff}$ is given by Eq. (7). In one spatial dimension the first two terms of Eq. (17) agree with [10].

If we assume $\boldsymbol{\partial f} = 0$, Eq. (17) simplifies to $\mathcal{H}_{\rm eff} = \hat{p}_i\gamma^{ij}\hat{p}_j/2 - (\hbar^2/12)R^{(3)} + V_{\rm eff}$. It has an interesting property that the kinetic energy is not given by the covariant Laplace operator as required by the geometry of a genuine Riemann space. Thus, while classically the emergent nature of the metric is revealed by the $\mathcal{O}(v^4)$ Lorentz symmetry violating terms, at quantum level it is manifested already in the leading kinetic energy operator sensitive to the short-distance properties of the underlying fundamental theory.

The convergence of the quantum mechanical perturbation theory for the effective Hamiltonian Eq. (17) should be discussed. The high-frequency expansion is now the expansion in the expectation value of the velocity square operator. Thus, for the states of a high quantum number the quasiclassical approximation gives the same expansion parameter as for the classical system. However, for a low quantum number the quasiclassical approximation is not applicable and we need an alternative estimate. The low energy level splitting near a minimum of the unperturbed $\mathcal{O}(v^2)$ effective potential scales as $\sqrt{g'} \sim \epsilon$, while the expectation value of the $\mathcal{O}(v^4)$ perturbation scales as ϵ^2. Thus, the effective expansion parameter given by the corresponding ratio is $\epsilon \ll 1$. Note that the convergence may be affected if the leading order splitting accidently vanishes for some values of the parameters, e.g., near the bifurcation point.

The result Eq. (17) can be generalized to an arbitrary number of harmonics in the periodically oscillating field. The calculation of the classical action in this case is straightforward though the result is less elegant, and the quantum corrections are given by the sum of Eq. (16) over the harmonics weighted by the (square of) the corresponding Fourier coefficients. The spin structure and a different dispersion law of the quasiparticles in Floquet materials can be easily incorporated in the Feynman rules of HFET. The quantization of the theory through $\mathcal{O}(v^6)$ does not pose a technical challenge in the HFET framework as well.

Let us now compare our approach to the high-frequency expansion based on Floquet theory. For the problem discussed in this Letter the effective $\mathcal{O}(1/\omega^4)$ Hamiltonian in one spatial dimension has been derived for the first time in [10] (a formal general expression in a different representation can be found in [16]). This analysis relies on a formal power counting in $1/\omega$, with both $\boldsymbol{g}$ and $\boldsymbol{f}$ treated as $\mathcal{O}(1/\omega^2)$ quantities. Hence, the result does not account for the terms with the fourth power of $\boldsymbol{f}$ present in Eqs. (7) and (17). However, this power counting does not apply to the most interesting physical case of dynamical stabilization realized, e.g., in the Paul traps, where the oscillating field results in a qualitative change of the system behavior. The latter requires $\boldsymbol{g} \sim \boldsymbol{f}^2$ scaling, cf. Eq. (11). In general, the Floquet theory calculations in this order are already quite tedious even in one dimension and without the more challenging $\mathcal{O}(\boldsymbol{f}^4)$ terms, while the quantization of the Hamiltonian within HFET requires only a "one-line" derivation of Eq. (16).

To summarize, in this work we have presented a number of results connecting dynamical systems, general relativity, and quantum theory. We have elaborated an asymptotic method to systematically construct the effective action for particles moving in a rapidly oscillating field. The effect of the oscillating field on the large-scale dynamics models the pseudo-Riemann space of general relativity in the post-Newtonian limit, with the curvature determined by the field spatial distribution and the effective value of the speed of light determined by the oscillation frequency. While the appearance of emergent gravity in condensed matter systems has already been predicted [29,30] and observed experimentally (see, e.g., [31]) for quasiparticle propagation, the rapidly oscillating field creates the gravitylike effect for the classical charged particles. Guided by the analogy with the nonrelativistic expansion of QED, we have quantized the effective action and developed the high-frequency effective theory, apparently the most powerful analytic tool for the perturbative analysis of the periodically driven systems. It can be used in a wide range of physical applications from the high-precision analysis and design of the charged particle traps to the Floquet engineering of quantum materials.

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