Emergent Riemann Space for Rapidly Oscillating Fields

Natalia Litvak
Superviser: Alexander Penin

I. INTRODUCTION

In 1989, the Nobel Prize in Physics was awarded in part to Hans G. Dehmelt and Wolfgang Paul for development of the ion trap, a method of using electromagnetic fields to hold a charged particle in place. Many areas of physics use the ideas of trapping atoms, particles, or other tiny objects. Optical micromanipulation of nanoparticles using radiation pressure from a laser invented by Arthur Ashkin [1] proved to be a useful tool in biology, and is applied for holding single cells or bacteria for study. Acoustic tweezers are used for particle and fluid micromanipulation [2].

If we consider a static electric field, we know that the electric potential is harmonic, that is,

$$\Delta\Phi = 0 \tag{1}$$

Due to the maximum principle, the local maxima and minima of the potential can be only on the boundary of the region. So, any charged particle will eventually move to the boundary for any configuration of static electric fields. This shows that static electric fields cannot be used to trap charged particles.

However, we can look at making traps with other means, such as by using oscillating fields, or combining electric and magnetic fields. Penning traps [3], developed by Hans G. Dehmelt, use the combination of homogeneous magnetic and quadrupole electric fields to increase the time that a cloud electron is kept in a compact region. An axial magnetic field causes the electrons to move in orbits around the magnetic field lines and slows the diffusion of electrons to the walls [3].

Wolfgang Paul studied multi-pole electric and magnetic fields as tool to focus beams of neutral particles that have an electric or magnetic dipole moment [4]. He proved that in a static electric quadrupole field, ions traveling along the z-axis could be focused in one transverse direction, such as along the x-axis, but not along the y-axis at the same time [4]. This problem is important in the context of high energy particle beams in accelerators, which need to be focused over long distances as they accelerate. It was then found that focusing is possible in the transverse plane of charged particles in a beam passing through a regular sequence of alternately converging and diverging electric or magnetic lenses [4]. The spatially periodic focusing and defocusing along one axis and defocusing and focusing along the other produce a beam that overall stays focused across the distance it travels.

The idea of a particle under the effect of oscillating fields has many interesting applications. The paper "Oscillating fields, Emergent Gravity and Particle Traps" [5] looks at how high frequency oscillations affects the motions of a charged particle, and how it can be described in terms of effective field theory. This work studies the motion of a massive particle of mass m subjected to a static force $\mathbf{G}$ and a high frequency periodic force $\mathbf{F}\cos\omega t$. The nature of the force is not assumed, and the analysis can be done without specific assumptions. The paper proved that the systems are well described by the motion in an effective gravitational field [5].

One assumption on the external force used in the derivation in the paper came from considering of electromagnetic particle traps that typically have vanishing charge density. This assumption holds for magnetic or electric fields. In terms of the vector function $\mathbf{f}$, this assumption reads $\partial_n f^n = 0$. In the case of a charge in external oscillating electric field this force $\mathbf{f} = q\nabla\Phi$, and when Φ satisfies equation (1) the condition $\partial_n f^n = 0$ is satisfied.

My goal was to find the emergent Riemann space for a non divergence free field. After the Riemann space and scalar curvate are known, my next goal was to find the effective quantum Hamiltonian in terms of the scalar curvature. This new setup of the problem will be applicable to more general set of systems when the external fields are not divergence free, for example acoustic waves. My analysis found that the effective Hamiltonian was

$$H_{eff} = \frac{1}{2}\hat{p}_i\gamma^{ij}\hat{p}_i - \frac{\hbar^2}{12}R^{(3)} - \frac{3}{8}\hbar^2(\partial_j\partial_i\partial_i f^n)(\partial_j f^n) - \frac{5}{16}\hbar^2(\partial_i\partial_i f^n)(\partial_j\partial_j f^n) \tag{2}$$

II. KAPITZA PENDULUM

One model demonstrating how high frequency oscillations of the external field may stabilize the system is the Kapitza pendulum [6]. This is a rigid pendulum with the pivot point oscillating in a vertical direction. Without oscillations of the pivot, there are two equilibrium position of the pendulum: the down-position $\theta = 0$ and the up-position $\theta = \pi$.

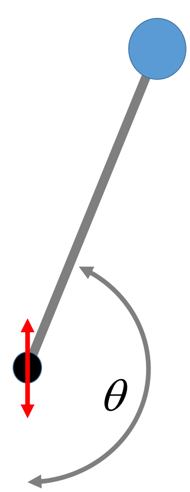

FIG. 1. The Kapitza pendulum. The pivot is oscillating vertically with some high frequency ω.

The equation of motion for such a pendulum (setting the frequency of oscillation for $\omega = 1$) takes the form

$$\ddot{\theta} + [\delta + \epsilon\cos t]\sin\theta = 0, \tag{3}$$

where δ and ϵ are some constants satisfying the condition $\delta << \epsilon << 1$.

Then, we split the dynamics into two parts: the slow motion and the fast oscillating motion driven by the external force. This can be achieved by the assumption

$$\theta = \theta_s + a(\theta_s)\cos\theta \tag{4}$$

Which holds if the slow evolution of the angle satisfies the low frequency condition $\dot{\theta}_s << 1$ and the amplitude of the fast vibrations is also small, $a(\theta_s) << 1$.

The equation for the fast oscillations is

$$(-a + \epsilon \sin\theta_s)\cos t = 0. \tag{5}$$

And for the slow evolution it is

$$\ddot{\theta}_s + \delta\theta_s + \epsilon a \cos^2 t \cos\theta_s = 0. \tag{6}$$

Then, we time-average the equation for the slow evolution, $\cos^2 t \to 1/2$, and obtain the effective equation for the slow evolution

$$\ddot{\theta}_s + \delta\theta_s + \frac{1}{4}\epsilon a \sin 2\theta_s = 0. \tag{7}$$

The Lagrangian describing this system is

$$L = \frac{\dot{\theta}_s}{2} - V \tag{8}$$

where the potential $V(\theta_s)$ is given by:

$$V = -\delta\cos\theta_s - \frac{\epsilon^2}{8}\cos(2\theta_s) \tag{9}$$

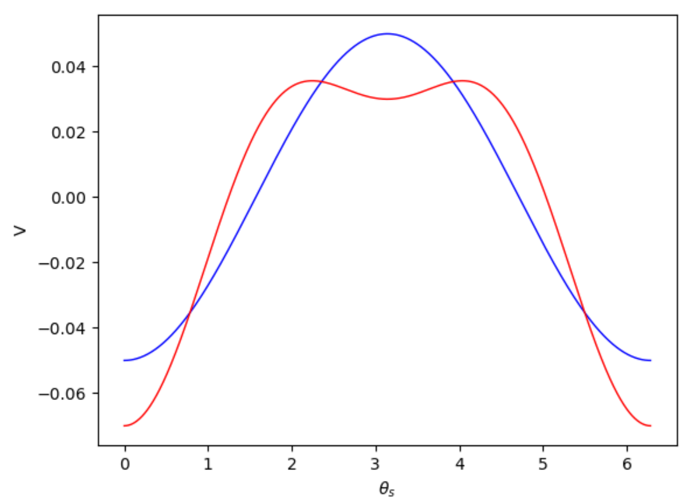

FIG. 2. The potential $V(\theta_s)$ for $\delta = 0.05$.

From the shape of the potential term, we can see that when the amplitude $\epsilon = 0$ the $\theta_s = \pi$ equilibrium point is unstable, so small deviations from it will amplify with time and move the pendulum further away from this equilibrium point. However, for a sufficiently large amplitude there appears a minimum in the upper part of the potential. The critical value of the parameters when the new local minimum of the potential appears is

$$\epsilon^2 = 2\delta. \tag{10}$$

So, due to the high frequency oscillations, a new stable equilibrium appears, and the overall motion of the pendulum at that equilibrium point is changed. The pendulum will oscillate with a low frequency defined by the parameters δ and ϵ around the point $\theta_s = \pi$. This example demonstrates induced stability of a system in an oscillating field.

III. PAUL TRAPS

Paul ion traps use fluctuating electric fields to trap charged particles [4]. The traps create potential wells that are up to several electron volts deep, and don't depend on the internal state of the ion [4].

To describe the physics of this trap, we consider an external field satisfying Laplace's equation (1) and consisting of static and oscillating parts. The review by Blatt et.al. [7] starts with the electric potential,

$$\Phi(x, y, z, t) = \frac{1}{2}U(\alpha x^2 + \beta y^2 + \gamma z^2) + \frac{1}{2}\tilde{U}\cos(\omega t)(\alpha' x^2 + \beta' y^2 + \gamma' z^2). \tag{11}$$

Due to Laplace's equation, we have the following conditions on the coefficients:

$$\begin{aligned} \alpha + \beta + \gamma &= 0, \\ \alpha' + \beta' + \gamma' &= 0. \end{aligned} \tag{12}$$

Different choices of these constants gives different configurations of trapping regions. For example if

$$\begin{aligned} \alpha = \beta = \gamma &= 0, \\ -(\alpha' + \beta') = \gamma &\neq 0, \end{aligned} \tag{13}$$

in which case the oscillating field leads to a three-dimensional confinement [7]. The other choice

$$\begin{aligned} -(\alpha + \beta) = \gamma &> 0, \\ \alpha' = -\beta' &\neq 0, \end{aligned} \tag{14}$$

describes the case when oscillatory part confines the ion in the $x - y$ plane, while the static potential confines the charged ions in the z direction [7].

If we consider the motion to be only in the x-direction, the equation of motion of the charge q in the x direction becomes

$$\ddot{x} = -\frac{q}{m}\partial_x \Phi = -\frac{q}{m}[U\alpha + \tilde{U}\cos(\omega t)\alpha']x \tag{15}$$

Then, they introduce the new rescaled constants [7]

$$\xi = \frac{\omega t}{2}, \qquad a_x = \frac{4qU\alpha}{m\omega^2}, \qquad b_x = \frac{4q\tilde{U}\alpha'}{m\omega^2} \tag{16}$$

which make equation (15) take the form of the Mathieu equation

$$\ddot{x} + [a_x - 2b_x \cos(2\xi)]x = 0 \tag{17}$$

They looked for the solution to this equation perturbatively in the form

$$x(\xi) = Ae^{i\beta_x\xi} \sum_{n=-\infty}^{\infty} C_{2n}e^{i2n\xi} + Be^{-i\beta_x\xi} \sum_{n=-\infty}^{\infty} C_{2n}e^{-i2n\xi} \tag{18}$$

where β_x and C_{2n} are functions of a_x and b_x only and do not depend on initial conditions. The constants A and B are defined to satisfy the initial conditions [7].

After the substitution (18) to (15) we can find the recursion relation for C_{2n}

$$\begin{aligned} &C_{2n+2} - D_{2n}C_{2n} + C_{2n-2} = 0, \\ &D_{2n} = [a_x - (2n + \beta_x)^2]/b_x. \end{aligned} \tag{19}$$

In the leading order approximation, when $a_x \ll 1$ and $b_x^2 \ll 1$ the coefficient $C_4 \approx 0$. Then from the initial condition $A = B$ it follows that

$$\beta_x \approx \sqrt{a_x + \frac{1}{2}b_x^2} \tag{20}$$

and the solution for $x(t)$ reads

$$x(t) \approx 2AC_0 \cos\left(\frac{1}{2}\beta_x\omega t\right)\left[1 - \frac{1}{2}b_x\cos(\omega t)\right] \tag{21}$$

In this form, it's seen that in the leading approximation, this equation describes harmonic oscillations with the frequency

$$\Omega = \frac{1}{2}\beta_x\omega \tag{22}$$

and there are small high frequency ω oscillations around this trajectory. So, after after averaging over time scale greater than ω^{-1} the system behaves as a harmonic oscillator with frequency Ω [7].

IV. EMERGENT GRAVITY AND PARTICLE TRAPS

Next, we turn to the case of a charged particle in a rapidly oscillating field, and aim to find the emergent Riemann space for the particle. We start with the equation of motion for the particle,

$$\ddot{\boldsymbol{R}} + \boldsymbol{g}(\boldsymbol{R}) + \boldsymbol{f}(\boldsymbol{R})\cos t = 0. \tag{23}$$

which describes a system affected by a high frequency external force with a small vector amplitude $\boldsymbol{f} = f^n$, as well as a static force $\boldsymbol{g}$. The perturbation theory leads to the dynamics of the system described by the effective Lagrangian in the form

$$\mathcal{L}_{eff} = \frac{v_i v_j}{2}\left[\delta_{ij} - \frac{3}{2}\partial_i f^n \partial_j f^n\right] - V_{eff} \tag{24}$$

Where in the quadratic order

$$V_{eff} = V_g + \frac{1}{4} f^n f^n + \frac{1}{4} f^n f^i \partial_i g^n \tag{25}$$

The term in square brackets in Eq.(24) can be seen as the effective metric

$$\gamma_{ij} = \delta_{ij} - \frac{3}{2} (\partial_i f^n)(\partial_j f^n) \tag{26}$$

where the Einstein notations were used for summation over the repetitive indices. In the quadratic approximation, we can combine the terms in the effective Lagrangian depending on the oscillating force f^n to be written in terms of the Ricci scalar for this metric. Doing this means that after averaging over the high frequency oscillations any system obeying equation (23) behaves as if the particle is moving in a curved spacetime [5].

My goal was to check this property for a more general setup, when $\partial_n f^n \neq 0$. I rederived the expressions for the three and four dimensional curvatures for this more general case.

To do this, we consider an arbitrary metric $\bar{g}_{\alpha\beta}$ and perturb it $g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \Delta g_{\alpha\beta}$. The inverse metrics are defined by the requirements: $g^{\alpha\epsilon} g_{\beta\epsilon} = \delta^\alpha_\beta$, $\bar{g}^{\alpha\epsilon} \bar{g}_{\beta\epsilon} = \delta^\alpha_\beta$. In the linear approximation, when $\Delta g_{\alpha\beta} << \bar{g}_{\alpha\beta}$ we have $g^{\alpha\beta} = \bar{g}^{\alpha\beta} + \Delta g^{\alpha\beta}$ and according to the variational rule $\delta g^{\alpha\beta} = -g^{\alpha\mu} g^{\beta\nu} \delta g_{\mu\nu}$ we can write

$$\Delta g^{\alpha\beta} = -\bar{g}^{\alpha\mu} \bar{g}^{\beta\nu} \delta g^{\mu\nu} + O(\Delta g^2_{\alpha\beta}) \tag{27}$$

V. THREE-DIMENSIONAL RICCI SCALAR

In order to derive the properties of the effective metric that appears in the Lagrangian (24) we need to compute the Riemann curvature tensor to the quadratic order in the small parameter v.

To compute the Ricci scalar, we first find the Riemann tensor and then contract it with the metric to get the Ricci tensor and the Ricci scalar. In three dimensions $g_{\alpha\beta} \to \gamma_{ij}$, $\bar{g}_{\alpha\beta} \to \delta_{ij}$ and $\Delta g_{\alpha\beta} \to -\frac{3}{2} \partial_i f^n \partial_j f^n$. We get:

$$\gamma^{ij} = \delta^{ij} - \delta^{ik} \delta^{jl} \Delta g_{kl} = \delta^{ij} + \frac{3}{2} (\partial_k f^n)(\partial_l f^n) \delta^{ki} \delta^{lj}$$

$$\gamma^{ij} = \delta^{ij} + \frac{3}{2} (\partial^i f^n)(\partial^j f^n) + O(\Delta g^2_{ij})$$

Then, we see that

$$\partial_i f^n \partial_j f^n = (\partial_i f^x)(\partial_j f^x) + (\partial_i f^y)(\partial_j f^y) + (\partial_i f^z)(\partial_j f^z) \tag{28}$$

Because f^n is assumed to be small, $f^n \approx v$, and $\Delta g_{ij} \approx v^2$ we get:

$$\gamma^{ij} = \delta^{ij} + \frac{3}{2} (\partial^i f^n)(\partial^j f^n) + O(v^4) \tag{29}$$

We see that the Christoffel symbols are linear in derivatives of the metric:

$$\Gamma_{\delta\alpha\beta} = \frac{1}{2} (\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\alpha\delta} - \partial_\delta g_{\alpha\beta}) \tag{30}$$

And so in the case of three-dimensional metric γ_{ij}, we substitute $g_{\alpha\beta} \to \gamma_{ij} = \delta_{ij} + O(v^2)$. Then $\partial_\alpha g_{\beta\delta} \approx O(v^2)$, so Γ_{kij} is on the order of v^2. From this, we can compute the Riemann tensor in the leading approximation in the small parameter v. Then, the leading term is of order v^2.

$$\begin{aligned} \Gamma_{kij} =& \frac{1}{2}(\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij}) \\ =& \frac{1}{2}(\partial_i(-\frac{3}{2}\partial_k f^n \partial_j f^n) + \partial_j(-\frac{3}{2}\partial_k f^n \partial_i f^n) - (-\frac{3}{2}\partial_k(\partial_i f^n \partial_j f^n)) \end{aligned} \tag{31}$$

After expansion of partial derivatives we obtain

$$\begin{aligned} \Gamma_{kij} =& -\frac{3}{4}(\partial_i\partial_k f^n(\partial_j f^n) + (\partial_k f^n)\partial_i\partial_j f^n \\ &+ \partial_j\partial_k f^n(\partial_i f^n) + (\partial_k f^n)\partial_i\partial_j f^n - \partial_k\partial_i f^n(\partial_j f^n) - (\partial_i f^n)\partial_k\partial_j f^n) \end{aligned} \tag{32}$$

And then we find

$$\Gamma_{kij} = -\frac{3}{2}(\partial_k f^n)(\partial_i\partial_j f^n) \tag{33}$$

When we raise the first index with the inverse metric γ^{ij} we obtain

$$\begin{aligned} \Gamma^l_{ij} = \gamma^{lk}\Gamma_{kij} &= [\delta^{lk} + \frac{3}{2}(\partial^l f^n)(\partial^k f^n) + O(v^2)][-\frac{3}{2}(\partial_k f^n)(\partial_i\partial_j f^n)] \\ &= \delta^{lk}[-\frac{3}{2}(\partial_k f^n)(\partial_i\partial_j f^n)] - \frac{9}{4}(\partial^l f^n)(\partial^k f^n)[-\frac{3}{2}\partial_k f^n \partial_i\partial_j f^n] \end{aligned}$$

$$\Gamma^l_{ij} = -\frac{3}{2}(\partial^l f^n)(\partial_i\partial_j f^n) + O(v^4) \tag{34}$$

Then we can see that $\Gamma^l_{ij} \approx v^2$.

Similarly, we evaluate the components of the three-dimensional Riemann tensor. Using the observation that $\Gamma^l_{ij} \approx v^2$ in application to the Riemann tensor we see

$$R_{kilj} = \frac{1}{2}[g_{kj,il} + g_{il,kj} - g_{kl,ij} - g_{ij,kl}] + O(v^4) \tag{35}$$

We see that only the first term in square brackets is important in the leading v^2 approximation because $\Gamma \sim v^2$. In our case $g_{ij} \equiv \gamma_{ij}$

$$\gamma_{ij} = \delta_{ij} + \Delta g_{ij}$$

$$\Delta g_{ij} = -\frac{3}{2}(\partial_i f^n)(\partial_j f^n)$$

and

$$R_{kilj} = \frac{1}{2}(-\frac{3}{2})[\partial_i\partial_l(\partial_k f^n \partial_j f^n) + \partial_k\partial_j(\partial_i f^n \partial_l f^n) - \partial_i\partial_j(\partial_k f^n \partial_l f^n) - \partial_k\partial_l(\partial_i f^n \partial_j f^n)]$$

Then, we expand all partial derivatives

$$\begin{aligned} R_{kilj} = -\frac{3}{4}[&(\partial_i\partial_l\partial_k f^n)(\partial_j f^n) + (\partial_i\partial_k f^n)(\partial_l\partial_j f^n) + (\partial_l\partial_k f^n)(\partial_i\partial_j f^n) + (\partial_k f^n)(\partial_i\partial_l\partial_j f^n) \\ &+ (\partial_k\partial_j\partial_j f^n)(\partial_l f^n) + (\partial_k\partial_i f^n)(\partial_j\partial_l f^n) + (\partial_j\partial_i f^n)(\partial_k\partial_l f^n) + (\partial_i f^n)(\partial_k\partial_j\partial_l f^n) \\ &- (\partial_i\partial_j\partial_k f^n)(\partial_l f^n) - (\partial_i\partial_k f^n)(\partial_j\partial_l f^n) - (\partial_j\partial_k f^n)(\partial_i\partial_l f^n) - (\partial_k f^n)(\partial_i\partial_j\partial_l f^n) \\ &- (\partial_k\partial_l\partial_i f^n)(\partial_j f^n) - (\partial_k\partial_i f^n)(\partial_l\partial_j f^n) - (\partial_l\partial_i f^n)(\partial_k\partial_j f^n) - (\partial_i f^n)(\partial_k\partial_l\partial_j f^n)] \end{aligned}$$

And this expression reduces to

$$R_{kilj} = -\frac{3}{2}[(\partial_i\partial_j f^n)(\partial_k\partial_l f^n) - (\partial)_i\partial_l f^n)(\partial_j\partial_k f^n)] + O(v^4) \tag{36}$$

$$R_{kilj} \approx v^2$$

The Ricci tensor is obtained by contraction of this expression with the inverse metric γ^{ij}

$$R_{ij} = \gamma^{kl} R_{kilj} = [\delta^{kl} + \frac{3}{2}(\partial_k f^n)(\partial_l f^n)]R_{kilj} \tag{37}$$

So, with an accuracy to order v^2 we get

$$R_{ij} = \delta^{kl} R_{kjilj} + O(v^4) \tag{38}$$

and

$$R_{ij} = -\frac{3}{2}[(\partial_i\partial_j f^n)(\partial^k\partial_k f^n) - (\partial_i\partial_k f^n)(\partial_j\partial^k f^n)] \tag{39}$$

The Ricci scalar in the same order in v^2 is computed similarly

$$R = \gamma^{ij} R_{ij} = (\delta^{ij} + \Delta\gamma^{ij})R_{ij} = \delta^{ij} R_{ij} + O(v^4) \tag{40}$$

So, we obtain the required general expression for the three-dimensional Ricci scalar

$$R = \frac{3}{2}[(\partial_i\partial_j f^n)(\partial^i\partial^j f^n) - (\partial_i\partial^i f^n)(\partial_j\partial^j f^n)] \tag{41}$$

In (41) there appear the new terms of the form

$$(\partial_i\partial^i f^n)(\partial_j\partial^j f^n) = (\partial_i\partial^i f^x)^2 + (\partial_i\partial^i f^y)^2 + (\partial_i\partial^i f^z)^2 \neq 0 \tag{42}$$

that are not present in the original paper [5].

VI. FOUR-DIMENSIONAL RICCI SCALAR

In four dimensions the Riemann tensor can be computed similarly.

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}[g_{\alpha\delta,\beta\gamma} + g_{\beta\gamma,\alpha\delta} - g_{\alpha\gamma,\beta\delta} - g_{\beta\delta,\alpha\gamma}] + O(v^4) \tag{43}$$

The four-dimensional metric $g_{\alpha\beta}$ is given by

$$g_{00} = 1 + \frac{1}{2} f^n f^n, \qquad g_{ij} = -\gamma_{ij} \tag{44}$$

The Ricci tensor and the Ricci scalar are defined as follows

$$R_{\beta\delta} = g^{\alpha\gamma} R_{\alpha\beta\gamma\delta}$$

$$R = g^{\beta\gamma} R_{\beta\delta} = g^{00} R_{00} + g^{ij} R_{ij}$$

And, after writing separately the temporal and spatial components we get the expression

$$R_{ij} = g^{\alpha\gamma} R_{\alpha i \gamma j} = g^{00} R_{0i0j} + g^{kl} R_{kilj}$$

$$R = g^{00} R_{00} + g^{00} R_{0i0j} g^{ij} + g^{kl} R_{kilj} g^{ij}$$

Because $g_{ij} = -\gamma_{ij}$ we have $R^{[g_{ij}]}_{kilj} = -R^{(3)[\gamma_{ij}]}_{kilj}$, where $R^{(3)[\gamma_{ij}]}_{kilj}$ denotes the Riemann tensor of the three dimensional metric γ_{ij}. So

$$g^{00} R_{00} = g^{00} g^{ij} R_{0i0j}$$

$$R = 2g^{00} g^{ij} R_{0i0j} + R^{[g_{ij}]}$$

Because R_{0i0j} is of the order of v^2 we can use $g^{00} = 1 + O(v^2)$, $g^{ij} = \delta^{ij} + O(v^2)$ and $R = 2R_{0i0j}\delta^{ij} + R^{[g_{ij}]}$

$$R_{0i0j} = \frac{1}{2}[-\partial_0 \partial_0 g_{ij} - \partial_i \partial_j g_{00}]$$

$$R_{0i0j} = -\frac{1}{2}(\partial_i \partial_j g_{00}) = -\frac{1}{2}[\partial_i \partial_j (1 + \frac{1}{2} f^n f^n)]$$

$$R_{0i0j} = -\frac{1}{4}\partial_i [2(\partial_j f^n) f^n] = -\frac{1}{2}[(\partial_i \partial_j f^n) f^n + (\partial_i f^n)(\partial_j f^n)]$$

$$R^{[g_{ij}]} = -R^{[\gamma_{ij}]} = -\frac{3}{2}[(\partial_i \partial_j f^n)(\partial^i \partial^j f^n) - (\partial_i \partial^i f^n)(\partial_j \partial^j f^n)]$$

So we get the formula for the four-dimensional Ricci scalar in the leading v^2 order

$$R = [(\partial_i \partial^i f^n) f^n + (\partial_i f^n)(\partial^i f^n)] - \frac{3}{2}[(\partial_i \partial_j f^n)(\partial^i \partial^i f^n) - (\partial_i \partial^i f^n)(\partial_j \partial^j f^n)] \tag{45}$$

VII. EFFECTIVE HAMILTONIAN

To generalize the effective Hamiltonian Eq.(16) of the paper by Penin and Su I rederived it through the following steps. Consider the operator $\hat{p}_i = -i\hbar\partial_i$. We see that

$$[\hat{p}_i \hat{p}_i, \gamma^{ij}] = (\hat{p}_i \hat{p}_j \gamma^{ij}) + 2\hat{p}_i \gamma^{ij} \hat{p}_j \tag{46}$$

The effective hamiltonian then becomes

$$H_{eff} = \frac{1}{8}([\hat{p}_i \hat{p}_i, \gamma^{ij}] + 2\hat{p}_i \gamma^{ij} \hat{p}_j) + \frac{\hbar^2}{16}(\partial_i \partial_j f^n)(\partial_i \partial_j f^n) \tag{47}$$

After expanding commutators of operators $\hat{p}_i$ we can write

$$H_{eff} = \frac{1}{8}((\hat{p}_i \hat{p}_i \gamma^{ij}) + 4\hat{p}_i \gamma^{ij} \hat{p}_j) + \frac{\hbar^2}{16}(\partial_i \partial_j f^n)(\partial^i \partial^j f^n) \tag{48}$$

Using

$$\gamma^{ij} = \delta_{ij} + \frac{3}{2}(\partial_i f^n)(\partial_j f^n)$$

we obtain

$$(\hat{p}_i\hat{p}_j\gamma^{ij}) = -\hbar^2(\partial_i\partial_j[\delta^{ij} + \frac{3}{2}(\partial_i f^n)(\partial_j f^n)]) = -\hbar^2\frac{3}{2}\partial_i\partial_j[(\partial_i f^n)(\partial_j f^n)] \tag{49}$$

and

$$(\hat{p}_i\hat{p}_i\gamma^{ij}) = -\hbar^2\frac{3}{2}(2(\partial_j\partial_i\partial_i f^n)(\partial_j f^n) + (\partial_i\partial_j f^n)(\partial_i\partial_j f^n) + (\partial_i\partial_i f^n)(\partial_j\partial_j f^n)) \tag{50}$$

The three-dimensional Ricci scalar has the form

$$R^{(3)} = \frac{3}{2}((\partial_i\partial_j f^n)(\partial_i\partial_j f^n) - (\partial_i\partial_i f^n)(\partial_j\partial_j f^n)) \tag{51}$$

From (50) and (51) we get that the the operators acting on the metric in terms of the Ricci scalar is

$$(\hat{p}_i\hat{p}_i\gamma^{ij}) = -\hbar^2\, R^{(3)} - 3\hbar^2(\partial_j\partial_i\partial_i f^n)(\partial_j f^n) - 3\hbar^2(\partial_i\partial_i f^n)(\partial_j\partial_j f^n) \tag{52}$$

and

$$H_{eff} = \frac{1}{2}\hat{p}_i\gamma^{ij}\hat{p}_i - \frac{\hbar^2}{12}R^{(3)} - \frac{3}{8}\hbar^2(\partial_j\partial_i\partial_i f^n)(\partial_j f^n) - \frac{5}{16}\hbar^2(\partial_i\partial_i f^n)(\partial_j\partial_j f^n) \tag{53}$$

Where no conditions on $\partial_i\partial_i f^n$ were assumed. The last two terms in (19) are new compared the result of the paper, and generalize their result for the case when there are no imposed conditions on the amplitude f^n except its smallness of the order of v^2 [5].

[1] A. Ashkin, "Acceleration and trapping of particles by radiation pressure," Phys. Rev. Lett. **24**, 156–159 (1970).

[2] A. Ozcelik, J. Rufo, and F. et al. Guo, "Acoustic tweezers for the life sciences," Nat Methods **15**, 1021–1028 (2018).

[3] Lowell S. Brown and Gerald Gabrielse, "Geonium Theory: Physics of a Single Electron or Ion in a Penning Trap," Rev. Mod. Phys. **58**, 233 (1986).

[4] Wolfgang Paul, "Electromagnetic traps for charged and neutral particles," Rev. Mod. Phys. **62**, 531–540 (1990).

[5] Alexander A. Penin and Aneca Su, "Effective Theory of Classical and Quantum Particle Dynamics in Rapidly Oscillating Fields," Phys. Rev. Lett. **132**, 051601 (2024), arXiv:2310.02311 [hep-ph].

[6] A. Penin, "Oscillating fields, emergent gravity and particle traps," in *Presentation at the TPI seminar, University of Alberta* (2023) pp. 1–19.

[7] D. Leibfried, R. Blatt, C. Monroe, and D. Wineland, "Quantum dynamics of single trapped ions," Rev. Mod. Phys. **75**, 281–324 (2003).