1.1 Probabilities

Def'n: A random experiment is a process that, when performed, results in one and only one of many observations (or outcomes).

The sample space S is the set of all *elementary* outcomes of an experiment.

Experiment	**Outcomes**	**Sample Space**
Toss a coin	Head, Tail	
Toss 2-headed coin	Head	
Toss a $5 bill	Get it back, Lose money	
Pick a suit	Spades, Clubs, Diamonds, Hearts	

Def'n: A Venn diagram is a picture that depicts S.

A probability tree shows outcomes represented by tree branches.

Ex1.1) Two successive draws from a deck, considering only the suit

(example diagrams drawn in class)

If $S = \{O_1, O_2, \ldots, O_n\}$, where O_i is the i^{th} elementary outcome, and p_i is the probability of the i^{th} elementary outcome, then

$$P(O_i) = p_i \qquad \text{and} \qquad 0 \le P(O_i) \le 1$$

If each of n outcomes are equally likely, then $P(O_i) = p_i = 1/n$. Also, in some experiments, "*with*" or "*without replacement*" affects probabilities. Lastly, note that

$$p_1 + p_2 + p_3 + \ldots + p_n = P(O_1) + P(O_2) + P(O_3) + \ldots + P(O_n) = 1 = P(S)$$

1.2/1.3 Events and Combinations of Events

Def'n: An event A (or B) is a subset of the sample space; $A \subset S$.

- A complement of an event (event does **not** happen) is denoted by A'.
  - $P(S) = P(A) + P(A') = 1 \rightarrow P(A') = 1 - P(A)$
- An intersection of 2 events (A **and** B happen together) is denoted by $A \cap B$.
  - $P(A \cap B) + P(A \cap B') = P(A) \rightarrow$ (a.k.a. "total probability rule")
  - If A and B are mutually exclusive (or disjoint), then $P(A \cap B) = 0$.
- A union of 2 events (A, B, **or** both happen) is denoted by $A \cup B$.
  - $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$

1.4 Conditional Probability

Table 1X0 – 2-way table of responses

	Like Hockey (A)	Indifferent (B)	Dislike Hockey (C)	**Total**
Male (M)				
Female (F)				
Total				

Def'n: Marginal probability is the probability of a single event without consideration of any other event.

Ex1.2) $P(M) =$ $P(F) =$

$P(A) =$ $P(B) =$ $P(C) =$

Conditional probability is the probability that an event will occur given that another event has already occurred. If A and B are 2 events, then the conditional probability of A given B is written as $P(A \mid B)$. Keywords: **given**, **if**, **of**

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \quad \text{and} \quad P(B \mid A) = \frac{P(B \cap A)}{P(A)}$$

such that $P(A) \neq 0$ and $P(B) \neq 0$.

Ex1.3) a) If you are male in this class, what is the probability that you like hockey?

$$P(\text{like hockey} \mid \text{male}) = P(A \mid M) = \frac{P(A \cap M)}{P(M)} =$$

b) What is the probability of being female in this class, given that you are indifferent to hockey?

$$P(\text{female} \mid \text{indifferent}) = P(F \mid B) = \frac{P(F \cap B)}{P(B)} =$$

c) Complements: What is the probability of NOT being female in this class?

$$P(\text{female}') = P(F') = 1 - P(F) =$$

Note: $P(A' \mid B) = 1 - P(A \mid B)$ Does $P(A \mid B') = 1 - P(A \mid B)$? Not necessarily.

Ex1.4) deck of cards: $P(Face') = 1 - \frac{12}{52} =$

$P(Face \mid Black) =$ $P(Face' \mid Black) =$

$P(Face \mid Black') =$

Ex1.5) deck of cards: $P(Heart \mid Red) = \frac{P(H \cap R)}{P(R)} = \frac{P(H)}{P(R)} =$

1.5/1.3(cont'd)

Def'n: Two events are independent if the occurrence of one does not affect the probability of the occurrence of the other. In other words,

$P(A \mid B) = P(A)$ OR $P(B \mid A) = P(B)$

Ex1.6) From Table 1X0, $P(F) =$ $P(F \mid B) =$

Since probabilities are not equal, the 2 events are dependent.

Ex1.7) deck of cards: $P(B) =$ $P(B \mid F) =$

Since the probabilities ARE equal, the 2 events are independent.

Mutually exclusive (or disjoint) events are events that cannot occur together.

Ex1.8) deck of cards
R = get red suit → diamond or heart
B = get black suit → spade or club
F = get face card → jack, queen, or king
Which pairs are disjoint?

Ex1.9) a single die
E = even = {2, 4, 6}
O = odd = {1, 3, 5}
Pr = prime = {2, 3, 5}

Note:
1. Two events are either disjoint or independent, not both (unless one has zero prob.).
2. Disjoint events are always dependent.
3. Dependent events may or may not be disjoint.

Def'n: The multiplication law to find joint probability (a.k.a. intersection of 2 events) is

$$P(A \cap B) = P(A) \times P(B \mid A)$$

- If A and B are two *independent* events, $P(A \cap B) = P(A) \times P(B)$.
- If A and B are two *disjoint* events, $P(A \cap B) = 0$.

Ex1.10) From Table 1X0,
$P(M \cap A) = P(M) \times P(A \mid M) =$
$P(B \cap F) = P(B) \times P(F \mid B) =$

Ex1.11) deck of cards: $P(B \cap F) = P(B) \times P(F) =$

$P(B \cap R) = ?$
- B and R are disjoint (you can't have a card that is black AND red);
thus, $P(B \cap R) = 0$.

With 3 or more independent events, the multiplication law becomes

$$P(A_1 \cap A_2 \cap \ldots \cap A_k) = P(A_1) \times P(A_2) \times \ldots \times P(A_k)$$

Def'n: The addition law to find the probability of a union of A and B is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- If A and B are two *mutually exclusive* events, $P(A \cup B) = P(A) + P(B)$.

Ex1.12) $P(M \cup A) = P(M) + P(A) - P(M \cap A) =$
$P(B \cup F) = P(B) + P(F) - P(B \cap F) =$

Ex1.13) deck of cards: $P(B \cup R) = P(B) + P(R) =$

$$P(B \cup F) = P(B) + P(F) - P(B \cap F) =$$

$$P(Face \cup Ace) = P(Face) + P(Ace) =$$

With 3 or more events, the addition law becomes quite complicated. If the events are all mutually exclusive, however, then

$$P(A_1 \cup A_2 \cup \ldots \cup A_k) = P(A_1) + P(A_2) + \ldots + P(A_k)$$

<u>Overall examples:</u>
Ex1.14) Suppose the probability of liking Gretzky is 0.86, the probability of liking Crosby is 0.79, and the probability of liking both is 0.71.

a) What is the probability of liking neither Gretzky nor Crosby? The probability of liking Gretzky but not Crosby?

b) What is the probability of liking Gretzky or Crosby?

c) What is the probability of liking Gretzky or not liking Crosby?

d) What is the probability of liking Crosby, given you like Gretzky?

Ex1.15) Suppose 30% of calls to an Oilers ticket phone line result in a sale being made. Assume all calls are independent. Suppose an operator handles 10 calls.

a) What is the probability that none of the 10 calls results in a sale?

b) What is the probability that at least one call results in a sale being made?

Ex1.16) Three friends play tennis (call them A, B, and C). The probability that A beats B is 0.7, the probability that A beats C is 0.8 and the probability that B beats C is 0.6. Assume all events are independent and that each player plays another at most once.

a) What is the probability that A wins both of its games?

P(A wins both games) =

b) What is the probability that A loses both of its games?

P(A loses both games) =

c) What is the probability that everyone wins a game?

Ex1.17) Assume that 70% of engineers who take the midterm next month have studied for the test. Of those who study for the midterm, 95% pass; of those who do not study for the test, 60% pass. What is the probability that an engineer did not study for the midterm, given that he passes the midterm?

1.7 Counting Techniques

Def'n: If an experiment consists of k steps, with step i resulting in n_i outcomes, then

Total number of outcomes in experiment = $n_1 \times \ldots \times n_k = \prod_{i=1}^{k} n_i$

Ex1.18) If picking 3 cards based on suit, how many ordered sequences are there?

A permutation of a set is an ordered sequence of the elements in the set.

$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$

Ex1.19) How many ways can the word LIQUOR be arranged?

If r ordered elements of a set of size n are desired, then

$$P_r^n = \frac{n!}{(n-r)!}$$

Ex1.20) How many ways can 3 of the letters in LIQUOR be arranged?

If order is NOT important when choosing r elements from a set of size n, then

$$C_r^n = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Ex1.21) How many ways can 3 letters be chosen from the word LIQUOR?