

Ch. 23 – CI for a Population Mean

- common situation is that σ is unknown, so the sample data must estimate it.

Recall $Z = \frac{\bar{Y} - \mu}{\sigma / \sqrt{n}}$. We now have $Z \neq \frac{\bar{Y} - \mu}{s / \sqrt{n}} = t$, where t is a diff. standardized variable.

The value of s may not be all that close to σ , especially when n is small. Consequently, there is extra variability and the distribution of t is more spread out than the z curve.

t-distributions:

As with normal curves, there exists a family of t -curves. The normal distribution has 2 parameters: μ and σ ; the t -distribution has a single parameter: *degrees of freedom* (df).

Range of t : similar to range of z ; range of df : 1, 2, 3, ..., ∞

Properties of the t-distribution:

1. The t -curve, with any fixed df , is bell-shaped and centered at 0 (just like z -curve).
2. Each t -curve is more spread out than the z -curve. $t_{\alpha/2} > z_{\alpha/2}$
3. As df increases, the spread of corresponding t -curve decreases.
4. As df increases, the corresponding sequence of t -curves approaches the z -curve.

Let $y_1, y_2, \dots, y_n$ constitute a random sample from a normal population distribution. Then the probability distribution of the standardized variable

$$t = \frac{\bar{Y} - \mu}{s / \sqrt{n}} \sim t_{df} \quad (df = n - 1)$$

One-Sample t Confidence Interval:

Assumptions:

1. Sample is random.
2. The population distribution is normal *OR* the sample size is large ($n \geq 30$).
3. σ , the population standard deviation, is unknown.

$$\bar{y} \pm (t_{\alpha/2, n-1}) \times \left(\frac{s}{\sqrt{n}} \right)$$

Table T gives critical values appropriate for the most common confidence levels.

		Sample size (n)	
		$n \geq 30$	$n < 30$
Normality	(approx.) normal	OK	OK
	NOT normal	OK	DON'T use t !

Ex23.1) A naïve astrophysicist wants to determine the mean radius of all of Jupiter's numerous moons. Only taking the 4 largest moons, he discovers a sample mean of 2103.75 km with a standard deviation of 495 km. Construct and interpret a 90% confidence interval for the population mean radius of Jupiter's moons. (Note: changes were made in class to this example to make the assumptions hold.)

- Parameter = μ
- Sample size: $n = 4$
- Estimate: $\bar{y} = 2103.75$
- Sample standard deviation: $s = 495$
- Standard Error = $s/\sqrt{n} = 495/\sqrt{4} = 247.50$
- Confidence Level is 90%, so critical value $t_3 = 2.353$
- Interval is $2103.75 \pm (2.353)(247.50) = 2103.75 \pm 582.37 \rightarrow (1521.38, 2686.12)$

With 90% confidence, the population mean radius of Jupiter's moons is between 1521.38 km and 2686.12 km.

Choosing n for a population mean:

Consider the CI as $\bar{y} \pm ME$, where $ME = t_{\alpha/2} \frac{s}{\sqrt{n}} \approx z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$

Solving for n here gives

$$n \approx \left(\frac{z_{\alpha/2} \sigma}{ME} \right)^2$$

Round up n to next integer. Replace σ by a prior estimate or, for a population that is not too skewed, use $\sigma \approx (\text{range})/6$ since 99.7% of the data is approximately within $3s$ of the sample mean. (This situation is rare.)

Ex23.2) You wish to estimate the mean cost of textbooks so that the margin of error is within \$7 of the true population mean. Most textbook prices are between \$80 and \$200. What is the minimum sample size required to be 99% confident in achieving the mentioned level of accuracy? Assume textbook cost is normally distributed.

$$\sigma \approx \frac{\text{range}}{6} = \frac{200 - 80}{6} = \frac{120}{6} = 20$$

$$n \approx \left(\frac{z_{\alpha/2} \sigma}{ME} \right)^2 = \left(\frac{(2.576)(20)}{7} \right)^2 = 54.170 \approx 55$$

Significance Tests about Means

As with intervals, it is rare to know σ , but for the test statistic t , the appropriate P -value can be found under the t -curve with $df = n - 1$.

Assumptions:

1. Sample is random.
2. The population distribution is normal *OR* the sample size is large ($n \geq 30$).
3. σ , the population standard deviation, is unknown.

Ex23.3) A website claims the average length of an episode is 21.104 min. A bored student takes a random sample of 7 episodes and found their mean length to be 21.16 min with a standard deviation of 0.20 min. It is known that the length of all episodes has an approximate normal distribution. Test the above claim using both approaches and $\alpha = 0.01$.

The sample is random. n is small, yet the population is normally distributed. Consequently, we may use a t -distribution; specifically, t_6 .

$$H_0: \mu = 21.104 \quad H_A: \mu \neq 21.104$$

$$t = \frac{\bar{Y} - \mu_0}{s / \sqrt{n}} = \frac{21.16 - 21.104}{0.20 / \sqrt{7}} = 0.741$$

Using the row of $df = 6$ in the t -table,

$$0 < t = 0.741 < 1.440$$

$$0.5 > \quad > 0.10$$

$$1 > P\text{-value} > 0.20 \quad \rightarrow P\text{-value range is } (0.20, 1)$$

In fact, software produces an exact P -value of 0.487. Using the “judgment approach”, we have weak evidence against H_0 . Using the “significance level approach” and $\alpha = 0.01$, we do not reject H_0 . Either way, we conclude that the website’s claim may indeed be valid.

(Try one-tailed tests with this example as well. Another example using both two-tailed and one-tailed tests was discussed in class. More quick examples to understand the t -distribution were also done in class.)