Chapter 5 – Integrals

5.1 – Calculating Areas using Riemann Sums

We have "nice" formulas for calculating the area of a circle, rectangle, parallelogram, triangle, etc. What about the area under the curve $f(x) = x^2$, above the x-axis between $x = 0$ and $x = 5$?

We can approximate this area by the sum of the areas of the five rectangles with width 1 and height $f(x)$ evaluated at $x = 1, 2, 3, 4, 5$.

This is not a great approximation, however, if we use the ten rectangles of width $\frac{1}{2}$, we get a better approximation.

The more rectangles we take, the better the approximation is.

Notice that our estimates in the last example were too large. These are called upper estimates; we could have estimated the area by using lower estimates:

We also could have used rectangles whose heights are $f(x)$ evaluated at the midpoint of each interval.

Sigma Notation and Limits of Finite Sums

Def'n: Sigma notation enables us to write a sum with many terms in a compact form.

$$\sum_{k=1}^{n} a_k$$

$\sum$ stands for "sum".

Ex5.1) 1. $\sum_{k=1}^{5} k$

2. $\sum_{k=2}^{6} k^2$

3. $\sum_{k=1}^{5} (-1)^k$

4. $\sum_{k=10}^{12} (-1)^k \frac{k}{k+1}$

Properties of Sigma Notation

1. $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

2. $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

3. $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$

4. $\sum_{k=1}^{n} c = nc$

5. $\sum_{k=1}^{n} 1 = n$

Some Helpful Formulas:

1. $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

2. $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

3. $\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$

We can use all of this information (and limits) to help calculate the area under the graph of a function $f(x)$, above the x-axis and between two values $x = a$ and $x = b$.

We estimate this area by the sum of the areas of the rectangles of equal width Δx.
If there are n rectangles, then $\Delta x = \frac{b-a}{n}$. The height of the k^{th} rectangle is $f(x_k)$, where $x_k = a + k(\Delta x)$. That is, x_k is the right endpoint of the k^{th} interval.

So, $A \approx f(x_1)\Delta x + f(x_2)\Delta x + ... + f(x_n)\Delta x = \sum_{k=1}^{n} f(x_k)\Delta x$

But this approximation becomes better as the number of rectangles increases. We get the exact area when the number of rectangles goes to infinity. That is, when $n \to \infty$.

If $f(x) > 0$ on all $a \le x \le b$, then the area of the region under the graph of $f(x)$, above the x-axis and between $x = a$ and $x = b$ is given by:

$$A = \lim_{n\to\infty}[f(x_1)\Delta x + f(x_2)\Delta x + ... + f(x_n)\Delta x] = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x$$

Ex5.2) Calculate the area of the region under the graph of $f(x) = x^2 + 1$, above the x-axis and between $x = 1$ and $x = 5$.