4.2 – The Mean Value Theorem

Theorem: (Rolle's Theorem)

Suppose, (i) $f(x)$ is continuous on $[a, b]$

(ii) $f(x)$ is differentiable on (a, b)

(iii) $f(a) = f(b)$

Then, there is a number c in (a, b) such that $f'(c) = 0$.

Proof: There are three cases:

Case I: $f(x) = k$, a constant
Then $f'(x) = 0$, so the number c can be *any* number in (a, b).

Case II: $f(x) > f(a)$ for some x in (a, b)
By the Extreme Value Theorem (can be used due to (i)), f has a maximum value somewhere in [a, b]. Since $f(a) = f(b)$, it must attain this maximum value at a number c in an open interval (a, b). Then f has a *local* maximum at c and, by (ii), f is differentiable at c. Thus, $f'(c) = 0$.

Case III: $f(x) < f(a)$ for some x in (a, b)
By the Extreme Value Theorem, f has a minimum value somewhere in [a, b] and, since $f(a) = f(b)$, it attains this minimum value at a number c in an open interval (a, b). Then f has a *local* minimum at c and f is differentiable at c. Thus, $f'(c) = 0$.

Ex4.4) Show that the equation $1+2x+x^3+4x^5=0$ has exactly one real root.

Theorem: (Mean Value Theorem)

Suppose, (i) $f(x)$ is continuous on $[a, b]$

(ii) $f(x)$ is differentiable on (a, b)

Then, there is a number c in (a, b) such that

$$f'(c)=\frac{f(b)-f(a)}{b-a} \qquad \text{OR} \qquad f(b)-f(a)=f'(c)(b-a)$$

Consequences of the Mean Value Theorem

Let $f(x)$ and $g(x)$ be differentiable functions. Then,

(i) If $f'(x) = 0$, then $f(x)$ is a constant function.

(ii) If $f'(x) = g'(x)$, then $f(x) = g(x) + k$ for some constant k.

Proof:

Ex4.5) Find the function with a derivative of $f'(x) = \cos(2x)$ that goes through the point $\left(\frac{\pi}{2}, 3\right)$.