3.10 Linear Approximation

The tangent line of a differentiable function $f(x)$ at point $(a, f(a))$ is

$$y = f'(a)(x-a) + f(a)$$

For x values very close to a, the graph of $f(x)$ is very close to the graph of the tangent line. For x values close to a, we can use the tangent line to obtain a linear approximation (or tangent line approximation) of $f(x)$. The function

$$L(x) = f'(a)(x-a) + f(a)$$

is called the linearization of f at a.

Ex3.7) Find the linearization of $f(x) = \sqrt{x}$ at $a = 4$. Use this to approximate $\sqrt{3.9}$ and $\sqrt{4.01}$.

Ex3.8) Find the linearization of $f(x) = \sqrt{x+4}$ at $a = 5$. Use this to approximate $\sqrt{9.1}$ and $\sqrt{8.99}$. What do you get when you use this linearization to approximate $\sqrt{4}$?

If we use this linearization at $a = 5$ to approximate $\sqrt{0+4}$, we get a very bad approximation. This is because linear approximations are only accurate for x-values very close to the a value, and in this example, $x = 0$ is not close to $a = 5$.

Differentials

If $y = f(x)$ is a differentiable function, we get differentials dx and dy so that

$$dy = f'(x)dx$$

On a small interval (a, b), if we let $x = a$ and $dx = \Delta x = b - a$, then Δy is the change in the actual function value from $f(a)$ to $f(b)$. The differential dy, however, is the linear approximation of the change from $f(a)$ to $f(b)$. That is, $dy = f'(a)(b - a)$.

To approximate $f(b)$ using differentials, we use

$$f(b) \approx f(a) + dy$$

This approximation is only accurate for small intervals (a, b).

Ex3.9) For $y = f(x) = \cos(3x^2)$, what is dy?

Ex3.10) Use differentials to approximate 2.01^3.

Ex3.11) Use differentials to determine how much paint it will take to coat a ball with a 5-cm radius so that the paint is 1 mm thick.