

2.5 Continuity

Def'n: A function $f(x)$ is continuous at $x = a$ if

$$(i) \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) \quad (\text{the limit exists})$$

$$(ii) \lim_{x \rightarrow a} f(x) = L = f(a)$$

(iii) $f(a)$ is defined.

Def'n: $f(x)$ is continuous if it is continuous at each point in its domain.

Remark: Another way of saying $f(x)$ is continuous is to say it can be drawn without lifting your pencil.

If a is an endpoint of an interval, then $f(x)$ is continuous at a if either $\lim_{x \rightarrow a^-} f(x) = f(a)$ or $\lim_{x \rightarrow a^+} f(x) = f(a)$.

Ex2.14) Where is $f(x)$ continuous?

$$f(x) = \begin{cases} x^2 - 1 & , \quad -1 \leq x < 0 \\ 2x & , \quad 0 < x < 1 \\ 1 & , \quad x = 1 \\ -2x + 4 & , \quad 1 < x < 2 \\ 0 & , \quad 2 \leq x < 3 \end{cases}$$

There are four types of discontinuities:

1. Removable discontinuity: A discontinuity that can be “removed” by redefining f .
2. Infinite discontinuity: A discontinuity where one or both of the one-sided limits does not exist or is infinite.
3. Jump discontinuity: A discontinuity where the function jumps from one value to another.
4. Oscillating discontinuity: A discontinuity where the function does not approach a fixed number as x approaches a .

Theorem: If f and g are both continuous at a , then so are:

- $f + g$
- $f - g$
- $f g$
- $\frac{f}{g}$ (as long as $g \neq 0$)
- kf (where k is a constant)

If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a .

Ex2.15) 1. Let $h(t) = \frac{t^2 + 3t - 10}{t - 2}$. What value should we assign to $h(2)$ to make $h(t)$ continuous there?

2. For what value of b is $g(x)$ continuous?

$$g(x) = \begin{cases} x^2 - b^2 & , \quad x < 4 \\ bx + 20 & , \quad x \geq 4 \end{cases}$$

Intermediate Value Theorem:

Let $f(x)$ be a continuous function on the closed interval $[a, b]$. Let $f(a) \leq y_0 \leq f(b)$.

Then,

Ex2.16) Prove that $\cos x = x$ has at least one solution.

Ex2.17) 1. Is there a number that is exactly one more than its cube? If so, which two consecutive integers does this number fall between?

$$x = x^3 + 1$$

2. Is there a number where adding one equals its cube? If so, which two consecutive integers does this number fall between?

$$x + 1 = x^3$$