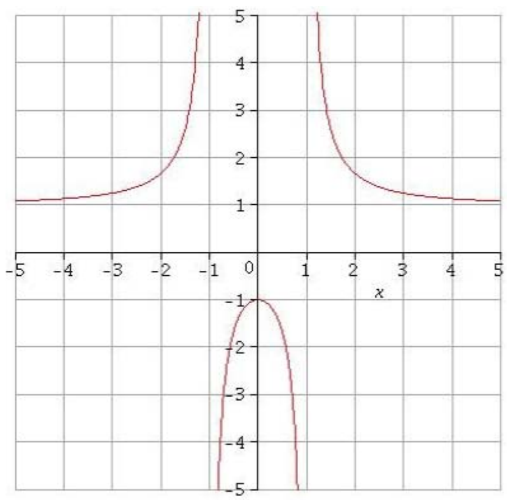
1.1 Functions and Their Graphs

Def'n: A <u>function</u> *f* from a set *X* to a set *Y* is a rule that assigns a single value $f(x) \in Y$ to each $x \in X$.

From this, *X* is called the <u>domain</u> of *f* and the set of all values *f*(*x*) is called the <u>range</u> of *f*. Notice that the range of *f* is a subset of *Y*.

Ex1.1)

Function	**Domain**	**Range**
$y = x^2$		
$y = \frac{1}{x}$		
$y = \sqrt{x}$		
$y = \sqrt{4-x}$		
$y = \sqrt{1-x^2}$		
$y = \frac{x^2+1}{x^2-1}$ 		

Remark: A curve in the *xy*-plane is the graph of a function if and only if no vertical line intersects the graph more than once. This is called the <u>vertical line test</u>.

Def'n: A piecewise function uses different formulas on different parts of its domain.

Ex1.2)

$$f(x)=\begin{cases}1-x^2 & , \quad x<0\\ x-1 & , \quad x=0\\ \sqrt{x} & , \quad x>0\end{cases}$$

Def'n: A function f that satisfies $f(-x)=f(x)$ is called an **even** function. Even functions have reflective symmetry about the y-axis.

A function f that satisfies $f(-x)=-f(x)$ is called an **odd** function. Odd functions have 180º rotational symmetry about the origin.

Ex1.3)

Def'n: A function f is called <u>increasing</u> on an interval I if

$$f(x_1) < f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

A function f is called <u>decreasing</u> on an interval I if

$$f(x_1) > f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

Ex1.4)

1.2 Important Graphs (to memorize)

1. Constant function

2. Linear function

3. $f(x) = x^2$

4. $f(x) = \sqrt{x}$

5. $f(x) = x^3$

6. $f(x) = \frac{1}{x}$

7. $f(x) = a^x$, $a > 0$ (exponential functions)

8. $f(x) = \log_a x$, $a > 0$ (logarithmic functions)

1.3 New Functions from Old

Ex1.5) If $f(x)=\sqrt{x^2-x+1}$ and $g(x)=\sin x$, then

$f(x)+g(x)=$

$f(x)-g(x)=$

$f(x)g(x)=$

$\frac{f(x)}{g(x)}=$

Def'n: If *f* and *g* are functions, the composite function $f\circ g(x)$ is defined by

$$f\circ g(x)=f(g(x))$$

Note: $f\circ g(x)$ is defined whenever both *g*(*x*) and *f*(*g*(*x*)) are defined.

Ex1.6) If $f(x)=\sqrt{x^2-x+1}$ and $g(x)=\sin x$, then

$f\circ g(x)=$

$g\circ f(x)=$

$f\circ f(x)=$

$g\circ g(x)=$

Shifting and Scaling Graphs of Functions

- Vertical shifts: $y = f(x) + k$ shifts the graph of $f(x)$ up k units if $k > 0$ and down $|k|$ units if $k < 0$.

- Horizontal shifts: $y = f(x - k)$ shifts the graph of $f(x)$ right k units if $k > 0$ and left $|k|$ units if $k < 0$.

- Scaling: $y = cf(x)$, $c > 1$ stretches the graph of $f(x)$ vertically by a factor of c.

  $y = cf(x)$, $0 < c < 1$ compresses the graph of $f(x)$ vertically by a factor of $\frac{1}{c}$.

  $y = f(cx)$, $c > 1$ compresses the graph of $f(x)$ horizontally by a factor of c.

  $y = f(cx)$, $0 < c < 1$ stretches the graph of $f(x)$ horizontally by a factor of $\frac{1}{c}$.

- Reflecting: $y = -f(x)$ reflects the graph of $f(x)$ about the x-axis.

  $y = f(-x)$ reflects the graph of $f(x)$ about the y-axis.

Ex1.7) Sketch the graph of the function $y = 2x^2 - 12x + 23$.

Continue with 1.3 ODDS.

1.5 Exponential Functions

Def'n: An exponential function is a function of the form $y = a^x$, where $a > 0$ and x is a variable.

Remarks: The domain of $y = a^x$ is $\mathbb{R}$.

The range of $y = a^x$ is $(0, \infty)$. (unless $a = 1$)

Sketch:

Remember, $a^0 = 1$ for any $a \neq 0$.

Ex1.8) 1. Sketch $f(x) = 3^x - 1$.

2. Sketch $y = 2^{x-4}$.

Fact: If $a^m = a^n$, then $m = n$.

Ex1.9) 1. Solve $4^x = 16^{2x-2}$.

2. Solve $3^x(3^x - 3) = 0$.

3. Solve $2^{2x} - 5 \cdot 2^x + 4 = 0$.

Def'n: In the case where $a = e \approx 2.71828\ldots$, one can obtain the natural exponential function, $y = e^x$. This value is special because the tangent line at (0, 1) has $m = 1$.

1.6 Logarithms

Def'n: The logarithmic function, $y = \log_a x$, is defined as the inverse of the exponential function, $y = a^x$. As with exponential functions, $a > 0$. Now also, $a \neq 1$.

Sketch:

Remarks: The domain of $y = \log_a x$ is $(0, \infty)$.

The range of $y = \log_a x$ is $\mathbb{R}$.

$\log_a x$ is the power a must be raised to in order to get x.

Ex1.10) 1. $\log_2 16 =$

2. $\log_{10} 1000 =$

3. $\log_{17} \sqrt{17} =$

4. $\log_5 \frac{1}{25} =$

Fact: $\log_a B = C$ is equivalent to $a^C = B$.

Note that $y = \log_a x$ is the reflection of $y = a^x$ about the line $y = x$. A one-to-one function is a function that never takes the same value twice: $f(x_1) \neq f(x_2)$ when $x_1 \neq x_2$.

Remark: A function is one-to-one if and only if no horizontal line intersects its graph more than once. This is called the horizontal line test.

Ex1.11) Using the function from Ex0.12), $y = x^2 + x + 1$, find its inverse.

Laws of Logarithms: If x and y are positive numbers, then

1. $\log_a(xy) = \log_a x + \log_a y$
2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
3. $\log_a(x^r) = r\log_a x$ (where r is any real number)
4. $\log_a(1) = 0$
5. $\log_a(a) = 1$

Def'n: The inverse function of the natural exponential function, $y = e^x$, is called the natural logarithm and is denoted by $y = \ln x$ instead of $y = \log_e x$.

Remark: Since $y = e^x$ and $y = \ln x$ are inverse functions, then

$$y = \ln(e^x) = x \quad \text{and} \quad y = e^{\ln x} = x$$

Fact: To convert other logs to natural logs, use

$$\log_a x = \frac{\ln x}{\ln a} \qquad (a \neq 1)$$

Ex1.12) 1. $e^{\ln 2} =$

2. $2 \ln 4 - \ln 8 - \ln 5 =$

3. Solve $\ln(4x - 3) = 7$.

4. Solve $\ln(\ln x) = 1$.

5. Solve $\ln x + \ln(x + 7) = \ln 4 + \ln 2$.

6. Find the domain of $y = \ln(2x - 7)$.

7. Find the domain of $y = \dfrac{6x^2 - 4}{\ln(x+3)}$.