Journal of Algebra **246,** 641–646 (2001)
doi:10.1006/jabr.2001.8973, available online at http://www.idealibrary.com on IDEAL®

Isometry Dimension of Finite Groups

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Communicated by Walter Feit

Received July 31, 2000

We show that the isometry dimension of a finite group G is equal to the dimension of a minimal-dimensional faithful real representation of G. Using this result, we answer several questions of Albertson and Boutin [*J. Algebra* **225** (2000), 947–955].

1. INTRODUCTION

A very general problem is to realize a given group as the group of automorphisms of some object. This problem can be studied in many different contexts by varying the type of object realizing the group. Once we have chosen a class of objects, a natural question always arises, "Which groups can be realized?"

A class of objects is termed (*finitely*) *universal* if every finite group can be realized by an object in the class. In [3], Cayley identified the first universal class, which are now called "Cayley color graphs" in honor of his work. The Cayley color graph associated to a finite group of order n is a directed graph whose edges can have one of n colors. Generalizing Cayley's construction, Frucht [5, pp. 241–244] showed how to represent colored directed edges by graph gadgets, thereby establishing the universality of graphs. Since Frucht's work, several other classes of objects have been identified as universal, including 3-colorable graphs, topological spaces, and subsets of Euclidean space; see [8, p. 523; 6, p. 96; 1, p. 949]. On the other hand, trees, groups, and planar graphs are not universal, and it remains an important open question whether Galois extensions of $\mathbf{Q}$ are universal.

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0021-8693/01 $35.00

In this paper, we study the universal class of subsets W of Euclidean n-space, where an automorphism of W is an isometry or distance preserving bijection of W onto itself. Albertson and Boutin [1, p. 949] show that any group of order n can be realized by a finite subset of Euclidean n-space containing $n + \binom{n}{2}$ points. Their proof sets up a geometric analog to the Cayley color graph. It is not constructive since it uses the Implicit Function Theorem in a crucial way.

Albertson and Boutin [1, p. 953] also introduce the notion of *isometry dimension* of a finite group: it is the least n such that the group can be realized by a subset of Euclidean n-space. The main result of this paper is that the isometry dimension of a group is equal to the dimension of a minimal-dimensional faithful real representation.

In Section 2, we prove our main result. In Section 3, we answer a number of questions posed in [1, p. 955], and we also restate some unresolved questions found in that same article.

2. MAIN RESULT

Let G be a finite group. Recall from Section 1 that the *isometry dimension* $\delta(G)$ is defined as the smallest n such that G can be realized as the group of isometries of a subset of $\mathbf{R}^n$. In this section, we prove the following theorem, which is our main result.

THEOREM 1. *Let G be a finite group. Then the isometry dimension $\delta(G)$ is equal to the dimension of a minimal-dimensional faithful real representation of G.*

Before discussing the proof, let us establish three useful lemmas. Say that the points $A_0, \ldots, A_k$ in $\mathbf{R}^n$ are *affinely independent* if there is no $(k-1)$-dimensional hyperplane containing them.

LEMMA 2. *Let $A_0, \ldots, A_n$ be affinely independent points in $\mathbf{R}^n$. Let α be an isometry of a subset $W \subset \mathbf{R}^n$ containing these points. Then, for any point $P \in \mathbf{R}^n$, there is at most one way to extend α to an isometry of $W \cup \{P\}$.*

Proof. Choose points $B_0, \ldots, B_n$ in W such that $\alpha(B_i) = A_i$. Set $r_i :=$ dist(P, B_i), and denote by $S(A_i, r_i)$ the $(n-1)$-dimensional sphere in $\mathbf{R}^n$ centered at A_i with radius r_i. Since the point $\alpha(P)$ must lie in the set $\bigcap_{i=0}^n S(A_i, r_i)$, it suffices to show that this intersection consists of a single point.

By translation, we can assume that $A_n = 0$. Let $A_i = (a_{i1}, \ldots, a_{in})$. Then, for $i = 0, \ldots, n-1$, there exist hyperplanes,

$$H_i := \{(x_1, \ldots, x_n) \mid a_{i1}x_1 + \cdots + a_{in}x_n = k_i\},$$

for certain $k_i \in \mathbf{R}$ such that $S(A_i, r_i) \cap S(A_n, r_n) \subset H_i$. Indeed, we may choose $k_i = (r_n^2 - r_i^2 + a_{i1}^2 + \cdots + a_{in}^2)/2$. Since $A_0, \ldots, A_{n-1}$ are linearly independent, the intersection of the n hyperplanes $H_0, \ldots, H_{n-1}$ consists of a single point. ∎

LEMMA 3. *Let W be a subset of $\mathbf{R}^n$, and let $\alpha \in \mathrm{Aut}(W)$ be an isometry. Then there exists an isometry $\tilde{\alpha} \in \mathrm{Aut}(\mathbf{R}^n)$ with $\tilde{\alpha}|_W = \alpha$.*

Proof. By Lemma 2, it is enough to prove the following claim: given affinely independent points $A_0, \ldots, A_m$ in $\mathbf{R}^n$ and an isometry α of any subset containing these points, there exists an isometry $\tilde{\alpha} \in \mathrm{Aut}(\mathbf{R}^n)$ such that $\tilde{\alpha}(A_i) = \alpha(A_i)$ for $i = 0, \ldots, m$. Indeed, suppose the above claim is true. Then, for any maximal set of affinely independent points $A_0, \ldots, A_k$ in W, there exists an isometry $\tilde{\alpha} \in \mathrm{Aut}(\mathbf{R}^n)$ agreeing with α on the points A_i for $i = 1, \ldots, k$. By Lemma 2, $\tilde{\alpha}$ must then agree with α on all of W.

We proceed to prove the above claim by induction. For $m = 1$, the result is clear. Suppose $A_0, \ldots, A_m$ are affinely independent points and α is an isometry of a subset containing these points. Since the points $A_0, \ldots, A_m$ are affinely independent, the points $A_1, \ldots, A_m$ must also be affinely independent. By induction, there is an isometry β of $\mathbf{R}^n$ such that $\beta(A_i) = \alpha(A_i)$ for $i = 1, \ldots, m$. An easy modification to the proof of Lemma 2 shows that once we know where an isometry sends A_i for $i = 1, \ldots, m$, there are at most two possibilities for the image of A_0, say E or F. Furthermore, the points $\alpha(A_i)$, for $i = 1, \ldots, m$, must lie on the perpendicular bisector of the segment from E to F. If $\beta(A_0) = \alpha(A_0)$, then set $\tilde{\alpha} = \beta$. Otherwise, let r denote reflection along the perpendicular bisector from E to F, and set $\tilde{\alpha} = r \circ \beta$. ∎

Remark 2.1. In Lemma 3, if W contains an affinely independent set of $n + 1$ points, Lemma 2 implies that $\tilde{\alpha}$ is unique.

If we are given a faithful real representation of G, the following technical result will help us construct a subset realizing G.

LEMMA 4. *Let H be a finite subgroup of the group of orthogonal matrices of dimension d, and let $S \subset \mathbf{R}^d$ be a finite set containing $d + 1$ affinely independent points. Suppose $H \subsetneq \mathrm{Aut}(S)$. Then there exists a finite subset $W \subset \mathbf{R}^d$ containing S such that $H \subset \mathrm{Aut}(W)$.*

Proof. Let $\sigma \in \mathrm{Aut}(S)$ be such that σ does not agree with any $h \in H$ on all of S. By Lemma 3, we may assume that σ is actually an isometry of $\mathbf{R}^d$. Since σ has finite order, it is an orthogonal operator on $\mathbf{R}^d$. Since σ and h are linear operators, the subspace $T_h \subset \mathbf{R}^d$ where σ and h coincide has dimension at most $d - 1$. Hence, $\bigcup_{h \in H} T_h$ is a set of measure zero, and its complement

$$V = \{x \in \mathbf{R}^d \mid \sigma(x) \neq h(x), \forall\, h \in H\}$$

is unbounded. So, we may choose $x \in V$ with $\operatorname{dist}(x, 0) > \max_{s\in S} \operatorname{dist}(s, 0)$.

Let $W = S \cup O_x$, where O_x is the orbit of x under H. By Lemma 3, any $\alpha \in \operatorname{Aut}(W)$ extends to an orthogonal operator which preserves distances from the origin. Since we chose x to satisfy $\operatorname{dist}(x, 0) > \operatorname{dist}(s, 0)$ for every $s \in S$, we must have $\alpha(W - S) \subset W - S$. Thus, α restricts to an isometry of S. On the other hand, $\sigma \in \operatorname{Aut}(S)$, but $\sigma \notin \operatorname{Aut}(W)$ since it does not send the point x to another point in W. ∎

We now proceed to prove our main result, Theorem 1.

Proof of Theorem 1. Let $n = \delta(G)$. By the definition of $\delta(G)$, there exists a subset W of $\mathbf{R}^n$ realizing G by isometries. Since n is the least dimension in which G may be realized, we may choose $n + 1$ affinely independent points $A_0, \dots, A_n \in W$. By Lemma 3, we get a map $G \to \operatorname{Aut}(\mathbf{R}^n)$, say $g \mapsto \tilde{g}$, which is certainly injective. It is also a homomorphism since $\widetilde{gh} = \tilde{g}\tilde{h}$ on W, and an isometry of $\mathbf{R}^n$ is uniquely specified by its values on W. Since G is finite, its elements are orthogonal operators on $\mathbf{R}^n$. Thus, we have an n-dimensional faithful real representation of G.

On the other hand, suppose we have a d-dimensional faithful real representation ρ: $G \to GL_d(\mathbf{R})$. Since G is finite, ρ is actually an orthogonal representation. We wish to construct a subset W of $\mathbf{R}^d$ such that $\operatorname{Aut}(W) \cong G$. We proceed by induction, using Lemma 4 repeatedly. Let $W_1 = \{0, O_{e_1}, \dots, O_{e_d}\}$, where the O_{e_i} is the orbit of the standard basis vector $e_i \in \mathbf{R}^d$ under the action of G. Then $\rho(G) \subset \operatorname{Aut}(W_1)$. Moreover, if $\rho(G) \neq \operatorname{Aut}(W_1)$, then we may use Lemma 4 to construct $W_2 \supset W_1$ such that $\rho(G) \subset \operatorname{Aut}(W_2) \subsetneq \operatorname{Aut}(W_1)$. Proceeding in a similar fashion, we eventually arrive at a set $W_k \subset \mathbf{R}^d$ such that $\rho(G) = \operatorname{Aut}(W_k)$. ∎

Remark 2.2. Since every group of order n has an n-dimensional faithful real representation (for example, the permutation representation), the proof above can be unwound to yield a finite subset $W \subset \mathbf{R}^n$ realizing G. However, such a construction might be quite difficult to carry out in practice.

3. QUESTIONS

We now answer several of the questions posed at the end of [1, p. 955].

QUESTION 1. *What finite groups have isometry dimension* 2 *or* 3?

Answer. Let $\mathbf{O}_d(\mathbf{R})$ be the group of orthogonal matrices of dimension d. A finite group G has isometry dimension d if and only if it is isomorphic to a subgroup of $\mathbf{O}_d(\mathbf{R})$, and it is not isomorphic to any subgroup of $\mathbf{O}_m(\mathbf{R})$ for $m < d$. Hence, the only finite groups with isometry dimension 2 are the cyclic groups Z_n and the dihedral groups D_n of order $2n$, for $n > 2$. All the finite subgroups of $\mathbf{O}_3(\mathbf{R})$ are listed in [4, p. 311].

QUESTION 2. *Let G, H be finite groups. Is $\delta(G \oplus H) = \delta(G) + \delta(H)$?*

Answer. Equality cannot always hold, as we can see by taking $G = Z_m$ and $H = Z_n$, cyclic groups of relatively prime orders m and n.

On the other hand, the following corollary is a simple consequence of our main result, Theorem 1.

COROLLARY 5. *Let G, H be finite groups. Then $\delta(G \oplus H) \leq \delta(G) + \delta(H)$.*

Proof. Let ρ_G be a faithful $\delta(G)$-dimensional real representation of G, and let ρ_H be a faithful $\delta(H)$-dimensional real representation of H. Set $d = \delta(G) + \delta(H)$. Then the direct sum of these two representations (ρ_G, ρ_H): $G \oplus H \to GL_d(\mathbf{R})$ is a d-dimensional faithful real representation of G. ∎

Corollary 5 gives an upper bound for $\delta(G \oplus H)$, which can be greatly refined for special classes of groups. For example, Karpilovskiĭ [7, Thm. 3] has computed the minimum dimension of a faithful real representation of finite abelian groups. Combining this result with Theorem 1, we have

THEOREM 6. *Let $G = G_1 \oplus G_2 \oplus \cdots \oplus G_s$, where G_i is a cyclic group of order m_i, and where m_i divides m_{i+1}. Then $\delta(G) = \sum_{i=1}^{s} \delta(G_i)$.*

Theorem 6 settles the conjecture in [1] that $\delta(\mathbf{Z}_2^n) = n$, which Boutin has proved by other means (private communication).

The proof of Theorem 1 gives an upper bound on the number of points needed to realize a group of order n in $\mathbf{R}^{\delta(G)}$. This bound is usually not sharp, and we have the natural question posed in [1, p. 955].

QUESTION 3. *Is there a relation between the order of a finite group G and the minimum number of points needed to realize it in $\mathbf{R}^{\delta(G)}$?*

The present paper studies only finite groups. The related problem for infinite groups is still open; see [1, p. 955].

QUESTION 4. *Which infinite groups may be realized by subsets of Euclidean space?*

ACKNOWLEDGMENTS

This work was done under the supervision of Dr. Joseph A. Gallian at the University of Minnesota, Duluth with financial support from the National Science Foundation (Grant DMS-92820179) and the National Science Agency (Grant 904-00-1-0026). The author wishes to thank Mike Develin, Stephen Hartke, Joseph Gallian, Kiran Kedlaya, Steven Kleiman, and David Moulton for their encouragement and excellent suggestions which greatly increased the quality of this paper.

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