# GEOMETRY OF LOOP EISENSTEIN SERIES

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CONTENTS

## Part 1. **Introduction and Generalities**

### 1. Introduction

**(1.1)** The aim of this paper is to make precise the relation between two proposals for Eisenstein series on loop groups, one group theoretic in nature and the other geometric. The group theoretic Eisenstein series were defined by the first named author [Gar04] who has used the arithmetic theory of loop groups and algebras [Gar78, Gar80] to study the analytic properties of these series. On the other hand, motivated by a desire to mathematically explain the modularity phenomenon underlying the $S$-duality principle from string theory, Kapranov [Kap00] has introduced a formal generating function counting bundles (equipped with certain data) on an algebraic surface. These generating functions satisfy the same functional equations as the loop Eisenstein series and their very definition parallels the construction of [Gar04] in many ways. Our purpose here is to interpret the elements of the group theoretical construction geometrically and thereby obtain exactly Kapranov's generating series.

**(1.2)** Let us briefly review the classical case of which this paper is an infinite-dimensional analogue. Let $G$ be an algebraic group (simply connected, split, semi-simple for convenience) defined over a function field $F/\mathbb{F}_q$ of finite characteristic in one variable. Assume also that $\mathbb{F}_q$ is the exact field of constants of $F$, i.e., that $F \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ is a field, where $\overline{\mathbb{F}}_q$ is the algebraic closure of $\mathbb{F}_q$. Denote by $\mathbb{A}_F$ the ring of adeles over $F$ and for any place $v$ of $F$ we let $\mathscr{O}_v \subset F_v$ be the ring of integers. Let

$$K_{\mathbb{A}} := \prod_v G(\mathscr{O}_v) \subset G_{\mathbb{A}_F}$$

be the maximal compact subgroup and denote by $B$ a Borel subgroup and $H$ a split torus contained in $B$. The Eisenstein series we are interested in are constructed from quasi-characters $\lambda : H \to \mathbb{C}^*$ on the split torus. Using the Iwasawa decomposition, we can extend $\lambda$ to a map $\Phi_\lambda : G_{\mathbb{A}} \to \mathbb{C}^*$. We then form the Eisenstein series,

$$E_\lambda(g) := \sum_{\gamma \in G_F/B_F} \Phi_\lambda(g\gamma)$$

which is defined on the double coset space

$$K_{\mathbb{A}} \setminus G_{\mathbb{A}}/G_F. \tag{1.2.1}$$

The construction above has a analogue in the case when $F$ is a number field, and in this context, Langlands [Lan76] has established the basic analytic properties of $E_\lambda(g)$ : he has shown that for $\lambda$ in a suitable half-space specified by conditions known as Godement's criterion, $E_\lambda(g)$ converges uniformly in a suitable sense. Moreover, it is a result of Langlands that $E_\lambda(g)$ admits a meromorphic continuation in the parameter $\lambda$ and satisfies a functional equation. By now, Eisenstein series have achieved a near ubiquitous role in the modern theory of automorphic forms. Let us only mention here two of Langlands original motivations for studying them: (a) the functional equations which govern their constant terms may be unravelled to provide strong analytic information about classical $L$-functions in number theory [Lan71]; and (b) the continuous part of the spectral decomposition of $L^2(G_{\mathbb{A}}/G_F)$ is governed by the Eisenstein series coming from cusp forms [Lan76] on lower rank groups. In the function field case, the above results also remain true [Har74] and in general are much simpler.

In the case when $F$ is a function field, the basis for the geometric interpretation of Eisenstein series (and in general for automorphic forms) is the classical observation due to A. Weil that the adelic double quotient space on which the Eisenstein series is defined (1.2.1) parametrizes $G$-bundles on a non-singular projective model of $F$. One might then ask whether the Eisenstein series itself has a geometric meaning. In [Har74], Harder has in fact interpreted the series $E_\lambda(g)$ as a generating function counting rational sections of a flag bundle associated to the $G$-bundle corresponding to the element $g \in K_{\mathbb{A}} \backslash G_{\mathbb{A}}/G_F$. This idea proved instrumental in Harder's computation of certain Tamagawa numbers as well as his proof of the Hasse principle for Chevalley groups [Har74]. Several years later, Laumon [Lau88] used a sheaf-theoretic version of Harder's construction to obtain some of earliest examples of automorphic sheaves. This idea was then vastly generalized by Braverman and Gaitsgory [BG02] in their work on the geometric Langlands conjecture.

(**1.3**) Parallel to classical theory of automorphic forms on finite dimensional groups, there is an arithmetic theory for loop groups [Gar78, Gar80] which allows one to introduce the central object of this paper, the *loop Eisenstein series.* The analytic properties of this series are also now well understood: in [Gar04] , the convergence and meromorphic continutation and functional equations of the constant term were studied; in [Gar06], the absolute convergene was proven; and in [Gar07a, Gar07b, Gar07c, Gar05], the Maass-Selberg relations have been established from which the meromorphic continuation of the loop Eisenstein series, as a locally integrable function in the number field case and as an actual function in the function field case, follows.

The starting point for this paper is the work of Kapranov [Kap00], who defines, in the spirit of [Har74, Lau88] a geometric generating function involving now the moduli space of $G$-bundles on a *surface S*. We note in passing here that through the work of Nakajima and others, moduli spaces of bundles on a surface have been intimately connected with the representation theory of loop algebras. In analogy with the theory of Eisenstein series, Kapranov establishes certain functional equations for his generating functions, and in so doing offers a mathematical explanation for the modularity phenomenon underlying *S*-duality. From this geometric point of view, however, analytic properties remain somewhat obscure and one consequence of this paper is that Kapranov's geometric Eisenstein series enjoys the same analytic properties as the loop Eisenstein series.

The idea underlying the comparison between the group theoretical and geometric series is quite simple and we proceed to sketch it here in the complex analytic case (though in the remainder of this paper, we shall work over finite fields): let $\mathsf{G}$ be a group and $\mathsf{LG}$ the corresponding loop group which can be viewed as the set of maps from $\mathbb{C}^* \to \mathsf{G}$. Let $X$ be a complex curve, and suppose we would like to study $\mathsf{LG}$-bundles on $X$. The usual Čech formalism suggests that we cover choose an open cover of $\{U_i \to X\}$ and then consider trandition functions

$$g_{ij}(x) : U_i \cap U_j \to \mathsf{LG}.$$

But by the "law of exponents," this is the same thing as considering the maps

$$g_{ij}(z,x) : (U_i \cap U_j) \times \mathbb{C}^* \to \mathsf{G}.$$

In other words, $\mathsf{LG}$-bundles on the *curve* $X$ may be regarded as $\mathsf{G}$-bundles on the *surface* $X \times \mathbb{C}^*$. Using this idea with the arithmetic theory of [Gar80], we can almost completely give the identification between the two definitions of loop Eisenstein series. The essential difficulty is that that we are not just dealing with loop groups $\mathsf{LG}$, but their central extensions! Kapranov has elucidated the meaning of the central extensions through the (relative) second Chern class of a bundle on a surface. Thus, one of our main points will be to identify this (relative) second Chern class in group theoretical terms. To do so, we need a local Riemann-Roch type on the level of Čech cocycles. This is similar in spirit to a local Deligne-Riemann-Roch theorem [Del87] and one way to understand our result is that it provides a group theoretical construction of this Riemann-Roch isomorphism.

(**1.4**) This paper is organized as follows: in the remainder of this chapter, we discuss some preliminary material on sheaves, torsors, gerbes, and groups. In Chaper 2, we discuss some of the arithmetic theory of loop groups and introduce the loop Eisenstein series. In Chapter 3, we interpret the elements of the group theoretical definition of loop Eisenstein series in geometric terms. Finally in the last chapter, we make precise the connection between central extensions of loop groups and local Riemann-Roch theorems for bundles on a surface.

## 2. SITES AND SHEAVES

**(2.1)** Let $\mathscr{C}$ be a category. Recall that a *Grothendieck topology* on $\mathscr{C}$ consists of a set $\mathrm{Cov}(\mathscr{C})$ of families of maps $\{U_i \to U\}_{i\in I}$ (where $I$ here is some index set) in $\mathscr{C}$ called *coverings* satisfying the following conditions,

(1) If $V \to U$ is an isomorphism in $\mathscr{C}$, then the set $\{V \to U\}$ is a covering.
(2) If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathscr{C})$ is a covering and $V \to U$ is any map in $\mathscr{C}$, then the fibered product $U_i \times_U V$ exists for $i \in I$, and the

$$\{U_i \times_U V \to V\}_{i\in I} \in \mathrm{Cov}(\mathscr{C}).$$

(3) If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathscr{C})$, and for each $i$ we are given a covering $\{V_{ij} \to U_i\}_{j\in J}$, then the composition

$$\{V_{ij} \to U_i \to U\}_{i\in I, j\in J} \in \mathrm{Cov}(\mathscr{C}).$$

A category with a Grothendieck topology is called a *site*.

Let $\mathsf{Sch}$ ($\mathsf{Sch}_k$) denote cateory of schemes (resp. schemes over $k$, for some base $k$). If $S \in \mathsf{Sch}$, we denote by $S_{Zar}$ the big Zariski site consisting of some full subcategory of schemes over $S$ equipped with the Zariski topology. The exact choice of subcategory will be fairly unimportant for us. We can usually choose schemes of finite type, but sometimes may need to work with more general cases when dealing with infinite dimensional groups. We shall be more precise when we need to be.

**(2.2)** Generalizing the classical notion of a sheaf on a topological space, we have the following. Let $S$ be a site, $C$ a category with products. Then a *presheaf* on $S$ with values in $C$ is a functor $\mathsf{F} : S \to C$. A *sheaf* if a presheaf satisfying the following condition: For every $\{U_i \to U\} \in \mathrm{Cov}(T)$, the diagram,

$$\mathsf{F}(U) \to \Pi_i \mathsf{F}(U_i) \rightrightarrows \Pi_{i,j} \mathsf{F}(U_i \times_U U_j)$$

is exact. Often we will be interested in sheaves of groups on a site $S$ which are defined in an analogous manner.

Let $T$ be some category of sheaves on a site. For example, $T$ may be the category of locally free sheaves of $G$-bundles for $G$ some group. Often we would like to view an element $X$ of $T$ as a space in its own right. This is achieved by considering the topos $T/X$ consisting of pairs $(F, f)$ where $\mathsf{F}$ is a sheaf in $T$ and $f : \mathsf{F} \to X$ is a map of sheaves. One can then consider some category of sheaves over this topos, etc.

## 3. TORSORS AND GERBES

**(3.1)** Let $S$ be some topological space, and $G$ a sheaf of groups on $S$. We call a sheaf $\mathsf{F}$ on $S$ a $G$-torsor (or sometimes we shall just write (principal) $G$-bundle) if $\mathsf{F}$ is equipped with a right action $\rho : \mathsf{F} \times G \to \mathsf{F}$ such that locally on $S$, the pair $(\mathsf{F}, \rho)$ is equivalent to the pair $(G, m)$ where $m : G \times G \to G$ is the multiplication map. Morphisms of $G$-torsors are defined in the natural way, and we denote the category of $G$-torsors on $S$ by $\mathsf{TORS}(S, G)$. Actually, for what follows $S$ only needs to be a topos and Giraud has developed the general theory of torsors in this context [Gir71]. Let $H^1(S, G) := \pi_0(\mathsf{TORS}(S, G))$ denote the set of isomorphism classes of the category $\mathsf{TORS}(S, G)$. In case $G$ is abelian, this is the usual sheaf cohomology group, but in general $H^1(S, G)$ only has the structure of a pointed set, the distinguished point corresponding to the trivial torsor.

There are several operations of torsors which provide an interpretation for the functorial properties one would expect the (non-abelian) cohomology set $H^1(S, G)$ to have. More precisely, we have the following operations,

(1) *Pushforward:* Let $\mathsf{F}_1$ be a $G_1$ torsor, and let $f : G_1 \to G_2$ be a morphism of groups. Then there exists a $G_2$ torsor $\mathsf{F}_2 := f_*(\mathsf{F}_1)$ defined uniquely by the condition that there should exists a map of sheaves also denoted $f : \mathsf{F}_1 \to \mathsf{F}_2$ satisfying $f(tg) = f(t)f(g)$ locally. This pushforward operation descends to a map
$$H^1(S, G_1) \xrightarrow{f_*} H^1(X, G_2).$$

(2) *Pullback:* Let $\phi : S_1 \to S_2$ be a map of spaces. Let $G$ a sheaf of groups on $S_2$ and $\mathsf{F}$ a $G$-torsor. Then the pullback of sheaves equips $\phi^* F$ with the structure of a $\phi^* G$-torsor on $S_1$. This operation descends to a map
$$\phi^* : H^1(S_2, G) \to H^1(S_1, \phi^* G).$$

(3) *Sum:* Let $\mathsf{F}_i$ be a $G_i$ torsor for $i = 1, 2$. Then $\mathsf{F}_1 \times \mathsf{F}_2$ is a $G_1 \times G_2$-torsor. This operation descends to a map,
$$H^1(S, G_1) \times H^1(S, G_2) \to H^1(S, G_1 \times G_2).$$
In the case when $H$ is abelian, we may push by the map $G \times G \to G$ to obtain a sum on torsors which recovers the usual group structure on $H^1(S, G)$.

Let $S$ be a fixed space, and $u : H \to G$ a morphism of sheaves of groups on $S$. Let $\mathsf{P}$ be a $G$-torsor. Then by a *reduction of* $\mathsf{P}$ *to* $H$, we shall mean a pair $(\mathsf{Q}, q)$ where $\mathsf{Q}$ is a $H$-torsor and $q : \mathsf{Q} \to \mathsf{P}$ is a morphism covering $u$. If $(\mathsf{Q}, q)$ and $(\mathsf{Q}', q')$ are two such reductions, then a morphism $(\mathsf{Q}, q) \to (\mathsf{Q}', q')$ is a $\mathsf{H}$-morphism $f : \mathsf{Q} \to \mathsf{Q}'$ such that $q' f = f$. Let us note that all morphisms of reductions are actually isomorphisms, and if $u$ is a monomorphism, two isomorphic reductions are canonically so.

**Proposition 3.1.1.** [Gir71, Ch. III §3.2.1] *Let $u : H \to G$ be a monomorphism of sheaves of groups, and let $\mathsf{P}$ be a $G$-torsor. Then the set of reductions of $\mathsf{P}$ to $H$ is isomorphic to the set $H^0(S, \mathsf{P}/H)$, where $\mathsf{P}/H$ is the quotient sheaf on $S$.*

(**3.2**) For us, it will be useful to view torsors from a Čechy perspective. Let $S$ be a space and $\mathscr{U} = \{U_i \to S\}$ a covering. We adopt the following notation: Let $\pi_k : X_i \times X_j \times X_k \to X_i \times X_j$ be the natural projection. If $\mathscr{F}$ is any sheaf, and $a_{ij} \in \mathscr{F}(X_i \times X_j)$, we shall denote by $\pi_k^*(a_{ij})$ by ${a_{ij}}^k$. Similarly, we define ${a_i}^j{}_k$ and $a^i{}_{jk}$, where raised index corresponds to which factor is being projected out.

A Čech 1-*cocycle* of a sheaf of groups $G$ with respect to the cover $\mathscr{U}$ is a family

$$g_{ij} \in G(U_{ij}) \quad \text{where} \quad U_{ij} = U_i \times_U U_j,$$

such that one has

$$g_i{}^j{}_k = {g_{ij}}^k \cdot g^i{}_{jk} \in G(U_{ijk}) \quad \text{where} \quad U_{ijk} = U_i \times_U U_j \times_U U_k$$

Two cocycles $\{g_{ij}\}$ and $\{h_{ij}\}$ are said to be cohomologous if there exist a family of elements $a_i \in G(U_i)$ such that

$$h_{ij} = a_i^j \cdot g_{ij} \cdot (a_j^i)^{-1} \in G(U_{ij}).$$

The notion of Čech 1-cocycles is essentially equivalent to the notion of a torsor. Given a cocycle $\{g_{ij}\}$ we can construct a torsor $\mathsf{F}$ on $S$ by gluing together the trivial torsors on $U_i$ by the maps $g_{ij} : G(U_i \cap U_j) \to G(U_i \cap U_j)$ which sends $g \mapsto gg_{ij}$. Similarly, given a torsor $\mathsf{F}$ on $S$ which trivializes with respect to the cover $\mathscr{U}$, we obtain a Čech 1-cocyle. The natural operations of pushforward, pullback, and sum have easy Čech descriptions.

(**3.3**) In the same way as torsors correspond to elements of $H^1$, *gerbes* are the geometric objects corresponding to $H^2$. Let $S$ be a space. Then recall that a stack $\mathsf{C}$ over $S$ is a sheaf of groupoids: this consists of the data of a groupoid $\mathsf{C}(U)$ for every open set as well as a notion of comparison of objects (restriction) for $\mathsf{C}(U)$ and $\mathsf{C}(V)$ for $V \subset U$ which satisfies certain descent conditions. See [Gir71] for the precise definitions.

A *gerbe* on $X$ will be a stack $\mathscr{G}$ on $X$ which locally has an object and such that locally any two objects are isomorphic. Suppose $\mathscr{G}$ is a gerbe on $X$, and suppose that for all objects $P \in \mathscr{G}(U)$ for $U \subset X$ open, the sheaf $\mathsf{Aut}(P)$ over $U$ of automorphisms of $P$ is commutative. Then the sheaf $\mathsf{Aut}(P)$ does not depend on the choice of $P$ and the sheaves $\mathsf{Aut}(P)$ may be glued to form a sheaf of abelian groups on $X$ called the band of $\mathscr{G}$. For a sheaf $\mathsf{A}$ of abelian groups on $X$, we say that $\mathscr{G}$ is a *gerbe with band* $\mathsf{A}$ if $\mathscr{G}$ is a gerbe with band isomorphic to $\mathsf{A}$.

Analogously to torsors, gerbes with band also admit a Čechy description. A trivialization of a gerbe $\mathscr{G}$ over $X$ will be an object of $\mathscr{G}(X)$. The set of isomorphism classes of gerbes on $X$ with band shall be denoted by $H^2(X,\mathsf{A})$. We shall give a Čech interpretation of gerbes now: let $c_{ijk}$ be a 2-cocycle with respect to a cover $\{U_i\}_{i\in I}$. Then we can define a gerbe $\mathscr{G}((c_{ijk}))$ as follows: over an open set $U$, the objects shall be $\mathsf{F}$-torsors $P_i$ on $U\cap U_i$ and isomorphism of torsors $p_{ij}:P_i \xrightarrow{\sim} P_j$ on $U\cap U_i\cap U_j$ verifying the condition $p_{jk}p_{ij}=p_{ik}c_{ijk}$. Conversely, given a gerbe $\mathscr{G}$ with band $\mathsf{A}$, we can construct a 2-cocyle as follows: choose a cover $\{U_i\}$ and objects $Q_i\in G(U_i)$ equipped with isomorphisms $e_{ij}:Q_i|_{U_i\cap U_j}\xrightarrow{\sim} Q_j|_{U_i\cap U_j}$ which satisfy the condition $e_{jk}e_{ij}=e_{ik}c_{ijk}$. The gerbe $\mathscr{G}$ can then be identified with $\mathscr{G}((c_{ijk}))$ by associating to each object $P\in\mathscr{G}(U)$ the $\mathsf{A}$- torsors $\mathrm{Hom}(Q_i,P)$ on $U\cap U_i$.

**(3.4)** Let $f:X'\to X$ be a morphism of schemes. Suppose that $\mathscr{G}$ is a gerbe with band $\mathsf{A}$ on $X'$. Then we can pushforward the underlying stack of $\mathscr{G}$ to get a stack on $E$. This will however not be a gerbe, and we denote by $\mathsf{Ger}(\mathscr{G})$ the sheaf on $X$ of maximal sub-gerbes of the pushforward stack $f_*\mathscr{G}$. Suppose the class of $\mathscr{G}\in H^2(X',\mathsf{A})$ lies in the kernel

$$H^2(X',\mathsf{A})^{tr} := \mathsf{kernel}(H^2(X',\mathsf{A})\to H^0(X,\mathrm{R}^2f_*\mathsf{A})).$$

Then it is known [Gir71, p.327] that the sheaf $\mathsf{Ger}(\mathscr{G})$ is a $\mathrm{R}^1f_*\mathsf{A}$-torsor on $E$ whose isomorphism class is given by the image under

$$H^2(E',\mathsf{A})^{tr}\to H^1(E,\mathrm{R}^1f_*\mathsf{A}).$$

*Notation:* Abusing notation, we shall often denote this torsor by $f_*\mathscr{G}$ instead of $\mathsf{Ger}(\mathscr{G})$.

On the other hand, suppose we are working over a fixed $X$, and let Let $f:\mathsf{A}\to\mathsf{B}$ be a morphism of sheaves of abelian groups on $X$. Then we have a pushforward operation which takes gerbes with band $\mathsf{A}$ to gerbes with band $f_*A$. For a given gerbe $\mathscr{G}$ with band $\mathsf{A}$, the objects in the pushforward gerbe $f(\mathscr{G})$ are the same as those of $\mathscr{G}$ but morphisms are obtained by pushing the $\mathsf{A}$-torsor of morphisms in $\mathscr{G}$ by $f$.

If $\mathscr{G}_i$ is a gerbe with band $\mathsf{A}_i$ for $i=1,2$, then the product stack $\mathscr{G}_1\times\mathscr{G}_2$ is a gerbe with band $\mathsf{A}_1\times\mathsf{A}_2$. This corresponds to the map

$$H^2(X,\mathsf{A}_1)\times H^2(X,\mathsf{A}_2)\to H^2(X,\mathsf{A}_1\times\mathsf{A}_2).$$

In the case that $\mathsf{A}_1=\mathsf{A}_2=\mathsf{A}$ we can combine this product map with the pushforward $\mathsf{A}\times\mathsf{A}\to\mathsf{A}$ to obtain a sum operation on gerbes.

## 4. Central Extensions

**(4.1)** Let $S$ be a site, and let $A$ be a sheaf of abelian groups and $G$ a sheaf of groups on $S$. An *extension of $G$ by $A$* will be an exact sequence of sheaves of groups $(E,a,b)$

$$A \xrightarrow{a} E \xrightarrow{b} G.$$

This is to be interpreted in the sheaf theoretic sense, so for each test object $T \in S$, we have an exact sequence of groups,

$$0 \to H^0(T, A_T) \to E(T) \to G(T) \to H^1(T, A_T).$$

An extension $E$ is said to be *central* if $H^0(T, A_T) \subset E(T)$ is a central subgroup for any $T \in S$. We shall need the following simple,

**Lemma 4.1.1.** [Blo74, p. 353] *Let $S$ be a regular, separated, quasi-compact scheme, and let*

$$A \to E \to G$$

*be a central extension of sheaves of groups on $S$. Suppose that for every $x \in S$, and every $f \in R = \mathcal{O}_{S,x}$, the map $E(\mathrm{Spec}(R_f)) \to G(\mathrm{Spec}(R_f))$ is surjective. Then there exists an open affine cover $\{U_i\}$ of $S$ such that on each $U_{ij} = U_i \cap U_j$, the map $E(U_{ij}) \to G(U_{ij})$ is surjective.*

**(4.2)** Grothendieck has reformulated the notion of central extension as follows. We follow the exposition in [BD01]: Let $A \xrightarrow{a} E \xrightarrow{b} G$ is a central extension of groups. The set $E$ becomes an $A$-torsor under the action

$$e \mapsto ea \;\text{ for } e \in E, a \in A.$$

But $E$ is equipped with more than just the structure of an $A$-torsor: we also have a product map

$$E \times E \to E$$

which fits into the following commutative diagram,

$$\begin{array}{ccc} E \times E & \longrightarrow & E \\ \downarrow & & \downarrow \\ G \times G & \xrightarrow{\mu} & G \end{array},$$

where $\mu$ is the multiplication map on $G$. The full structure on the set $E$ which allows us to recover the fact that $E$ is a central extension is given by the following,

**Definition 4.2.1.** A *multiplicative* $\mathsf{A}$*-torsor on* $\mathsf{G}$ as an $\mathsf{A}$-torsor $\mathsf{E}$ provided with a morphism of $\mathsf{A}$-torsors on $\mathsf{G} \times \mathsf{G}$,

$$m : pr_1^* \mathsf{E} + pr_2^* \mathsf{E} \to \mu^* \mathsf{E}$$

where $\mu : \mathsf{G}\times\mathsf{G}\to\mathsf{G}$ is the multiplication map and $pr_i : \mathsf{G}\times\mathsf{G}\to\mathsf{G}$ is the projection onto the $i$-th factor. Furthermore, the map $m$ must make the following diagram of $\mathsf{A}$-torsors on $\mathsf{G}\times\mathsf{G}\times\mathsf{G}$ commute,

$$\begin{array}{ccc} pr_1^*\mathsf{E}+pr_2^*\mathsf{E}+pr_3^*\mathsf{E} & \longrightarrow & pr_{12}^*\mu^*\mathsf{E}+pr_3^*\mathsf{E} \\ \downarrow & & \downarrow \\ pr_1^*\mathsf{E}+pr_{23}^*\mu^*\mathsf{E} & \longrightarrow & \mu_{123}^*\mathsf{E} \end{array} .$$

We then have the following useful reformulation of the notion of central extension,

**Theorem 4.2.2.** *There is an equivalence of categories between the category of central extensions of* $\mathsf{G}$ *by* $\mathsf{A}$ *and the category of multiplicative* $\mathsf{A}$*-torsor on* $\mathsf{G}$.

**Remark 4.2.3.** Using Theorem 4.2.2 as a starting point, we can reinterpret several notions regarding central extensions in terms of multiplicative torsors. For example, we have,

(1) If $H\subset G$ is a subgroup. Then there is an equivalence of categories between multiplicative $A$-torsor on $G/H$ and central extensions of $G$ by $A$ split over $H$.
(2) Let $G$ and $H$ be sheaves of groups such that $H$ acts on $G$. Then there is an equivalence of categories from the category of multiplicative $A$-torsors $E_0$ on the semi-direct product $H\rtimes G$ to the category of triples (a) (b) (c) as follows: (a) a multiplicative $A$-torsor $E$ on $G$; (b) a multiplicative $A$-torsor $F$ on $H$; and (c) an action of $H$ on $(G,E)$ "lifting" (see [BD01, p. 16]) the action of $H$ on $G$.

# **Part** 2. **Arithmetic of Loop Groups**

## 5. Central Extensions by $\mathsf{K}_2$

**(5.1)** Let $X$ be a scheme. Then we denote by $\mathsf{K}_2(X)$ Quillen's higher $\mathsf{K}$-group of the category of locally free sheaves on $X$. The assignment to every open subset $U\subset X$ of the group $\mathsf{K}_2(U)$ defines a (Zariski) presheaf on $X$, whose associated sheaf we denote by $\mathsf{K}_{2,X}$. Usually we suppress the $X$ from the notation and just write $\mathsf{K}_2$. For $X=\mathrm{Spec}(A)$ affine, we simply write $\mathsf{K}_2(A)$ for $\mathsf{K}_2(\mathrm{Spec}(A))$.

**(5.2)** Let $A$ be any domain, and $\mathfrak{p}\subset A$ be a prime of height 1. Then $A_{\mathfrak{p}}$ is a discrete valuation ring with fraction field $K$ coinciding with the fraction field of $A$. We shall denote by $v_{\mathfrak{p}}$ (or just $v$ for short) the valuation $K^*\to\mathbb{Z}$.

The residue field of $A_\mathfrak{p}$ will be denoted by $k(\mathfrak{p})$, and for an element of $a \in A_\mathfrak{p}$ we denote by $\overline{a}$ its image in the residue field $k(\mathfrak{p})$. The tame symbol

$$\begin{aligned} T_\mathfrak{p} : \mathsf{K}_2(K) &\to k(\mathfrak{p})^* \\ x, y &\mapsto (-1)^{\nu(x)\nu(y)} \overline{x^{\nu(y)} y^{-\nu(x)}} \end{aligned}$$

fits into the following exact sequence of groups,

$$0 \to \mathrm{K}_2(A_\mathfrak{p}) \to \mathrm{K}_2(K) \xrightarrow{T_\mathfrak{p}} k(\mathfrak{p})^* \to 0.$$

See for example, [DS72]. When $A$ is local, we omit the $\mathfrak{p}$ from the notation.

**(5.3)** Let $G$ be (sheaf of) reductive groups on $\mathsf{Sch}_k$ Then in [BD01], central extension of $G$ by $\mathsf{K}_2$ have been classified. More precicely, they have constructed a functorial equivalence between the categories of multiplicative $\mathsf{K}_2$-torsors on $G$ and some category $\mathscr{C}$. In the case when $\mathsf{G}$ is split, then the set of equivalence classes of this category $\pi_0(\mathscr{C})$ is equal to $\mathbb{Z}$ and hence corresponding to $1 \in \mathbb{Z}$, we have a unique central extension (defined up to unique isomorphism) which we shall henceforth denote by $\widetilde{G}$. So, we have an extension of sheaves of groups on $\mathsf{Sch}_k$,

$$\mathsf{K}_2 \to \widetilde{G} \to G.$$

Recall that for any $T \in \mathsf{Sch}_k$, we have an exact sequence of groups,

$$1 \to H^0(T, \mathsf{K}_2) \to \widetilde{\mathsf{G}}(T) \to \mathsf{G}(T) \to H^1(T, \mathsf{K}_2).$$

**Example 5.3.1.** When $T = \mathrm{Spec}(k)$ for $k$ a field, then we have get an exact sequence of groups,

$$1 \to \mathsf{K}(k) \to \widetilde{\mathsf{G}}(k) \to \mathsf{G}(k) \to 1$$

which agrees with the central extension constructed by Matsumoto.

**Example 5.3.2.** Let $R$ be an essentially smooth local $k$-algebra and $f \in R$ a local parameter. Then if $T = \mathrm{Spec}(R_f)$, we have by [Qui73] that $H^1(T, \mathsf{K}_2) = 0$. Thus in this case we get an exact sequence,

$$1 \to \mathsf{K}_2(R_f) \to \widetilde{\mathsf{G}}(R_f) \to \mathsf{G}(R_f) \to 1$$

**(5.4)** For any ring $\mathrm{Spec}(R) \in \mathsf{Sch}_k$, we may consider the rings $R[t, t^{-1}]$ and its formal version $R((t))$. The results of [BD01] then give central extensions (not necessarily exact on the right),

$$\mathsf{K}_2(R[t,t^{-1}]) \to \widetilde{G}(R[t,t^{-1}]) \to G(R[t,t^{-1}]) \tag{5.4.1}$$

$$\mathsf{K}_2(R((t))) \to \widetilde{G}(R((t))) \to G(R((t))) \tag{5.4.2}$$

Actually, to obtain the second, one needs to use the generalization of Gersten's conjecture by Panin [Pan00] to case of equicharacteristic local rings (see also the recent work of [Moc07] and the the remark in [BD01, p.10]).

Now, let us define sheaves locally by the following,

$$\begin{aligned} G[t,t^{-1}] : R &\mapsto G(R[t,t^{-1}]), \\ G((t)) : R &\mapsto G(R((t))) \end{aligned}$$

Similarly, we define sheaves $\mathsf{K}_2[t,t^{-1}], \mathsf{K}_2((t))$ and $\widetilde{G}[t,t^{-1}]$ and $\widetilde{G}((t))$. *Notation:* We shall often drop the $R$ from our notation and just write for example $\mathsf{K}_2[t,t^{-1}]$ in place of $\mathsf{K}_2 R[t,t^{-1}]$. The sequences 5.4.1 and 5.4.2 may then be written as central extension of sheaves on $\mathsf{Sch}_k$

$$\mathsf{K}_2[t,t^{-1}] \to \widetilde{G}[t,t^{-1}] \to G[t,t^{-1}] \tag{5.4.3}$$

$$\mathsf{K}_2((t)) \to \widetilde{G}((t)) \to G((t)) \tag{5.4.4}$$

**(5.5)** We shall really be interested in "twisted" versions of the groups $\widetilde{G}((t))$ where the twist comes from rotation (and in general automorphism) of the loop parameter $t$. So, let us first define the sheaf $\mathrm{Aut}[[t]]$ on $\mathsf{Sch}_k$ locally via,

$$\mathrm{Aut}[[t]] : R \mapsto \{\sum_{i\geq 0} a_i t^i | a_1 \in R^*, a_i \in R\}.$$

We also have a subsheaf $\mathbb{G}_m \subset \mathrm{Aut}[[t]]$ where $\mathbb{G}_m(R) = R^*$. There is a natural action of $\mathrm{Aut}[[t]]$ on $G((t))$ which is localy given by the following: let $a(t) \in \mathrm{Aut}(R[[t]])$ and $g(t)$ on $G(R((t)))$. Then

$$a(t).g(t) = g(a(t)).$$

This action lifts to the central extension $\widetilde{G}((t))$: given an element of $\mathrm{Aut}(R[[t]])$, we obtain an isomorphism

$$K_2(R((t))) \to K_2(R((t))).$$

Pushing 5.4.2 by this map, we obtain a new central extension which is canonically isomorphic to $\widetilde{G}((t))$ and hence we have a map from $\widetilde{G}((t)) \to \widetilde{G}((t))$. It is clear that this action covers the action of $\mathrm{Aut}[[t]]$ on $G((t))$, and thus we obtain a diagram,

$$\mathsf{K}_2((t)) \to \widetilde{G}((t)) \ltimes \mathrm{Aut}[[t]] \to G((t)) \ltimes \mathrm{Aut}[[t]].$$

We obtain similar diagrams in the case of Laurent polynomials and also when $\mathrm{Aut}[[t]]$ is replaced by $\mathbb{G}_m$.

## 6. Loop Algebra Basics

**(6.1)** Let $k$ be any field and $\mathfrak{g}$ be a simple Lie algebra over $k$ with invariant bilinear form $(\cdot,\cdot)$. The *affine Lie algebra* $\widehat{\mathfrak{g}}$ is the vector space

$$\widehat{\mathfrak{g}} := \mathfrak{g} \otimes_k k((t)) \oplus k\mathbf{c},$$

equipped with the unique Lie algebra structure satisfying the conditions,

(1) $\mathbf{c}$ is central;

(2) if $x \otimes \sigma, y \otimes \tau \in \mathfrak{g} \otimes k((t))$. Then,

$$[x \otimes \sigma, y \otimes \tau] = [x,y] \otimes \sigma\tau - (x,y)\,\mathrm{Res}(\sigma d\tau)\mathbf{c},$$

where Res is the residue at 0 :

$$\mathrm{Res} : k((t)) \to k \, , \sum a_j t^j \mapsto a_{-1}$$

*Remark*: We may replace $k((t))$ by the Laurent polynomials $k[t,t^{-1}]$ above in which case we obtain a Lie algebra which we denote by $\widehat{\mathfrak{g}}_{\mathrm{pol}}$.

We consider the *degree operator* which is the $k$-linear map $\mathbf{D} : \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}$ defined by the rules,

$$\mathbf{D}(t^n \xi) = nt^n \xi, \quad \text{for} \quad \xi \in \mathfrak{g}$$
$$\mathbf{D}(\mathbf{c}) = 0.$$

With these rules, it is easy to see that $\mathbf{D}$ is a derivation of the Lie algebra $\widehat{\mathfrak{g}}$ and thus we may form the semi-direct product Lie algebra of $\widehat{\mathfrak{g}}$ and the the one dimensional Lie algebra $k\mathbf{D}$, which we denote as the *extended affine algebra*

$$\widehat{\mathfrak{g}}^e = \widehat{\mathfrak{g}} \oplus k\mathbf{c} \oplus k\mathbf{D}.$$

**(6.2)** Let $\mathfrak{h} \subset \mathfrak{g}$ denote a Cartan subalgebra and denote by $\Delta \subset \mathfrak{h}^\vee$ the set of roots with respect to $\mathfrak{h}$. Let $\alpha_0 \in \Delta$ be the highest root. Define the extended affine Cartan subalgebra by

$$\widehat{\mathfrak{h}}^e = \widehat{\mathfrak{h}} \oplus k\mathbf{D}.$$

We shall denote by $(\widehat{\mathfrak{h}}^e)^\vee$ the $k$-dual of $\widehat{\mathfrak{h}}^e$. Then we define the *affine roots* $\widehat{\Delta} \subset (\widehat{\mathfrak{h}}^e)^\vee$ as follows. Each classical root $\alpha \in \Delta$ extends to an element $\alpha \in (\widehat{\mathfrak{h}}^e)^\vee$ by requiring that

$$\alpha(\mathbf{c}) = \alpha(\mathbf{D}) = 0.$$

Furthermore, we construct an *imaginary root* $\iota \in (\widehat{\mathfrak{h}}^e)^\vee$ by requiring

$$\iota|_{\mathfrak{h} \oplus k\mathbf{c}} = 0 \quad \text{and} \quad \iota(\mathbf{D}) = 1 \ .$$

Then we may define,

$$\widehat{\Delta} := \{\alpha + n\iota\}_{\alpha\in\Delta, n\in\mathbb{Z}} \bigcup \{n\iota\}_{n\in\mathbb{Z}\setminus 0}.$$

We shall call

$$\widehat{\Delta}_W = \{\alpha + n\iota\}_{\alpha\in\Delta, n\in\mathbb{Z}}$$

the set of *Weyl or real roots* and

$$\widehat{\Delta}_I = \{n\iota\}_{n\in\mathbb{Z}\setminus 0}$$

the set of *imaginary roots.* Setting $\alpha_{l+1} = -\alpha_0 + \iota$, we shall write $\{\alpha_1, \dots, \alpha_{l+1}\}$ for the set of *affine simple roots.* Let $h_{l+1} = -h_{\alpha_0} + \mathbf{c}$, and denote by $\{h_1, \dots, h_{l+1}\}$ the set of *affine simple coroots.*

For each $\alpha \in \widehat{\Delta}$, we define as usual,

$$\mathfrak{g}^{\alpha} = \{x \in \mathfrak{g} | [x,h] = \alpha(h)x, \text{ for all } h \in \widehat{\mathfrak{h}}^e\}.$$

Then we obtain the root space decomposition for the extended affine Lie algebra,

$$\widehat{\mathfrak{g}}^e = \widehat{\mathfrak{h}}^e \bigoplus_{\alpha\in\widehat{\Delta}} \widehat{\mathfrak{g}}^{\alpha}$$

where it is easy to verify that

$$\begin{aligned} \widehat{\mathfrak{g}}^{\alpha} &= t^n\mathfrak{g}^{\beta}, \alpha = \beta + n\iota \in \widehat{\Delta}_W \\ \widehat{\mathfrak{g}}^{\alpha} &= t^n\mathfrak{h}, \alpha = n\iota \in \widehat{\Delta}_I \end{aligned}$$

**(6.3)** We call that $\lambda \in (\widehat{\mathfrak{h}}^e)^{\vee}$ a *dominant integral weight* if

$$\lambda(h_i) \in \mathbb{Z}_{\geq 0} \quad \text{for} \quad i = 1, \dots, l+1$$

and if in addition

$$\lambda(h_i) \in \mathbb{Z}_{>0} \quad \text{for some} \quad i = 1, \dots, l+1.$$

Associated to such a dominant integral weight, there exists a $\widehat{\mathfrak{g}}^e$-module $V^{\lambda}$ over $k$, which admits a decomposition,

$$V^{\lambda} = \bigoplus_{\mu\in(\widehat{\mathfrak{h}}^e)^{\vee}} V^{\lambda}_{\mu}$$

where

$$V^{\lambda}_{\mu} := \{v \in V^{\lambda} | h.v = \mu(h)v \; h \in \widehat{\mathfrak{h}}^e\}.$$

If $V^{\lambda}_{\mu} \neq 0$, then we call $\mu$ a weight and we must have

$$\mu = \lambda - \sum_{i=1}^{l+1} k_i\alpha_i, \; k_i \in \mathbb{Z}_{\geq 0}.$$

We then define the depth of such a weight weight to be

$$\mathsf{dp}(\mu) := \sum_{i=1}^{l+1} k_i.$$

Finally, we define a *coherently ordered basis* of $V^\lambda$ as a basis $\mathscr{B} = \{v_0, v_1, \dots\}$ such that every $v_i$ lies in $V^\lambda_\mu$ for some weight $\mu$ and so that if $v_i \in V^\lambda_\mu$ and $v_j \in V^\lambda_{\mu'}$ and $i < j$, then we must have $\mathsf{dp}(\mu) \leq \mathsf{dp}(\mu')$. Moreover, we assume that for each weight $\mu$ the $v_i$ which occur in the basis $\mathscr{B}$ which belong to $V^\lambda_\mu$ occur consecutively.

**(6.4)** In [Gar78], Chevalley forms $\mathfrak{g}_\mathbb{Z} \subset \mathfrak{g}$ and $\widehat{\mathfrak{g}}_\mathbb{Z} \subset \widehat{\mathfrak{g}}$ have been constructed. Let

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{h}} \bigoplus_{\alpha \in \widehat{\Delta}} \widehat{g}^\alpha$$

be a root space decomposition as above. Then set

$$\widehat{\mathfrak{h}}_\mathbb{Z} := \widehat{\mathfrak{h}} \cap \widehat{\mathfrak{g}}_\mathbb{Z}$$

and for any $\alpha \in \widehat{\Delta}$,

$$\widehat{\mathfrak{g}}^\alpha_\mathbb{Z} := \widehat{\mathfrak{g}}^\alpha \cap \widehat{\mathfrak{g}}_\mathbb{Z}.$$

Then it is shown in [Gar78] that the coroots $h_1, \dots, h_{l+1}$ defined above can be taken to form a $\mathbb{Z}$-basis for $\widehat{\mathfrak{h}}_\mathbb{Z}$. Furthermore, for each $\beta \in \Delta$, there exist elements $X_\beta \in \mathfrak{g}^\beta$ such that if $\alpha = \beta + n\delta \in \widehat{\Delta_W}$, then

$$\widehat{\mathfrak{g}}^\alpha_\mathbb{Z} = \mathbb{Z}\xi_\alpha \quad \text{where} \quad \xi_\alpha = t^n \otimes X_\beta,\ i = 1, \dots, l\ .$$

On the other hand, if $\alpha = n\delta \in \widehat{\Delta}_I$, then we have that

$$\widehat{\mathfrak{g}}^{n\iota}_\mathbb{Z} := \widehat{\mathfrak{g}}^{n\iota} \cap \widehat{\mathfrak{g}}_\mathbb{Z} = \mathbb{Z}\xi_1(n) \oplus \cdots \oplus \mathbb{Z}\xi_l(n) \quad \text{where} \quad \xi_i(n) = t^n \otimes h_i$$

**(6.5)** The representations $V^\lambda$ also admit *Chevalley forms* $V^\lambda_\mathbb{Z} \subset V^\lambda$. Recall the weight space decomposition

$$V^\lambda = \oplus_\mu V^\lambda_\mu.$$

Letting $V^\lambda_{\mu,\mathbb{Z}} = V^\lambda_\mu \cap V^\lambda_\mu$, we have that

$$V^\lambda_\mathbb{Z} = \oplus_\mu V^\lambda_{\mu,\mathbb{Z}}.$$

There is a notion of coherently ordered basis similar to that descriped above. The Chevalley forms of these reprsentations also satisfy the following important properties with respect to the Chevalley forms $\widehat{\mathfrak{g}}_\mathbb{Z}$ described above.

**Proposition 6.5.1.** *Let $\alpha = \beta + n\delta \in \widehat{\Delta}_W$ and let $\xi_\alpha$ as above.*

(1) *For each non-negative integer m, the divided powers* $\xi_\alpha^{(m)} := \frac{\xi_\alpha^m}{m!}$ *stabilize* $V_{\mathbb{Z}}^\lambda$
(2) *Given* $v \in V^\lambda$, *there exists* $m_0 \in \mathbb{Z}_{\geq 0}$ *such that for all* $m \geq m_0$, *we have*
$$\xi_\alpha^{(m)} v = 0$$
(3) *Given* $v \in V^\lambda$ *and* $\beta \in \Delta$, *there exists* $n_0 \in \mathbb{Z}_{\geq 0}$ *such that for all* $n \geq n_0$, *we have*
$$\xi_{\beta+n\delta} \cdot v = 0.$$

**Remark 6.5.2.** Using the Chevalley forms above, we may define, for any commutative ring with unit $R$,

$$\begin{aligned} \widehat{\mathfrak{g}}_R &:= R \otimes_{\mathbb{Z}} \widehat{\mathfrak{g}}_{\mathbb{Z}}, \\ \widehat{\mathfrak{g}}_R &:= R \otimes_{\mathbb{Z}} \widehat{\mathfrak{h}}_{\mathbb{Z}}, \\ V_R^\lambda &:= R \otimes_{\mathbb{Z}} V_{\mathbb{Z}}^\lambda. \end{aligned}$$

The constructions above extend to this case and in particular the analogue of Proposition 6.5.1 holds.

## 7. Central Extensions from Representation Theory

**(7.1)** Let $R$ be a commutative domain. We shall fix $\lambda$ a dominant integral weight throughout, and write $V_R^\lambda$ for the corresponding highest weight module. For $\alpha \in \widehat{\Delta}_W$, and $s \in R$, the formal sum

$$\chi_\alpha(s) = \sum_{m \in \mathbb{Z}_{\geq 0}} \xi_\alpha^{(m)} s^m$$

defines an element of $\mathrm{Aut}(V_R^\lambda)$ by Proposition 6.5.1 (2). Furthermore, if $\sigma = \sum_{i \geq i_0} a_i t^i \in R((t))$, then we using Proposition 6.5.1 (3) above, we can also make sense, for $\alpha \in \Delta$ of the infinite product

$$\chi_\alpha(\sigma) := \Pi_{i \geq i_0} \chi_{\alpha+i\delta}(a_i)$$

as an element of $\mathrm{Aut}(V_R^\lambda)$.

**Definition 7.1.1.** For a given $\lambda$, we define a sheaf of groups, $\widehat{G}^\lambda$ locally by the assignment,

$$R \mapsto \langle \chi_\alpha(\sigma) | \alpha \in \Delta, \sigma \in R((t)) \rangle \subset \mathrm{Aut}(V_R^\lambda),$$

where for any subset $X \subset \mathrm{Aut}(V_R^\lambda$, we denote the group generated by $X$ by $\langle X \rangle$.

**Remark 7.1.2.** We shall often drop the superscript $\lambda$ and just denote this sheaf by $\widehat{G}$ when no confusion will arise.

**(7.2)** We would now like to define certain useful subgroups of $\widehat{G}$. To do this, we need the following notations: let

$$\begin{aligned} w_\alpha(\sigma) &= \chi_\alpha(\sigma)\chi_{-\alpha}(-\sigma^{-1})\chi_\alpha(\sigma) \\ h_\alpha(\sigma) &= \widehat{w}_\alpha(\sigma)\widehat{w}_\alpha(1)^{-1} \end{aligned}$$

For a ring $R$, we then define the following subgroups,

(1) $\widehat{B}(R)$ is the subgroup generated by elements of the form $\chi_\alpha(\sigma)$, where,
  (a) $\alpha \in \Delta_+(A)$ and $\sigma(t) \in R[[t]]$
  (b) $\alpha \in \Delta_-$ and $\sigma \in tR[[t]]$;
  (c) $h_\alpha(\sigma)$ for $\alpha \in \Delta_+$ , $\sigma \in R[[t]]^*$;
  (d) $h_{a_{l+1}}(s)$ for $s \in R^*$.

(2) $\widehat{U}(R)$ is the *pro-unipotent radical* of $\widehat{B}$ and is generated by elements of the form (a) and (b) as above. Fix an order on the positive, classical roots $\Delta_+(A)$ so that $\alpha < \beta$ when the height of $\alpha$ is less than the height of $\beta$. Similarly, fix an order on $\Delta_-$ which is consistent with respect to heights. Then, it can be shown [Gar98, 5.2] that each element of $u \in \widehat{U}(R)$ has an expression,

$$u = \prod_{\alpha \in \Delta_+(A)} \chi_\alpha(\sigma_\alpha(t)) h \prod_{\alpha \in \Delta_-(A)} \chi'_\alpha(\sigma_\alpha(t)) \tag{7.2.1}$$

where

$$\begin{aligned} &\sigma_\alpha(t) \in R[[t]],\ \alpha \in \Delta_+(A), \\ &\sigma'_\alpha(t) \in tR[[t]],\ \alpha \in \Delta_-(A) \end{aligned}$$

where the products are taken with respect to the fixed orders on $\Delta_\pm(A)$ and where

$$h = \prod_{i=1}^{l} h_{\alpha_i}(\sigma_i(t))$$

with $\sigma_i(t) \in R[[t]]^*$ satisfying

$$\sigma_i(t) \equiv 1 (\mathrm{mod}\, t).$$

We also define the congruence subgroups of level $j$, say $\widehat{U}^{(j)}(R)$, to consist of elements $u \in \widehat{U}(R)$ such that in an expression of the form 7.2.1, we have

$$\sigma_\alpha(t) \equiv 0 \ (\mathrm{mod}\ t^j) \text{ for } \alpha \in \Delta_+(A),$$

$$\sigma'_\alpha(t) \equiv 0 \ (\mathrm{mod}\ t^j) \text{ for } \alpha \in \Delta_-(A),$$

and

$$\sigma_i(t) \equiv 1 (\mathrm{mod}\, t^j) \text{ for } i = 1, \ldots, l.$$

(3) $\widehat{H}(R)$ is the *split torus* and is generated by elements of the form $h_\alpha(s)$ for $s \in R^*$ and $\alpha \in \widehat{\Delta}_W$.

**Remark 7.2.1.** With respect to a coherently ordered basis $\mathscr{B}$ of $V_R^\lambda$ the groups above admit the following descriptions,

(1) $\widehat{B}$ is the subgroup of all $b \in \widehat{G}$ which are upper triangular
(2) $\widehat{U}$ is the subgroup $u \in \widehat{G}$ represented by upper triangular, unipotent matrices.
(3) $\widehat{H}$ is the subgroup of all $a \in \widehat{G}$ which are diagonal.

**(7.3)** Let now $F$ be a function field of finite characteristic and for each place $\nu$ we denote by $F_\nu$ and $\mathscr{O}_\nu$ the completion and maximal compact subring respectively. In this case, we can define a maximal compact subgroup $\mathsf{K}_\nu \subset \widehat{G}(F_\nu)$ as the subgroup which preserves the lattice $V_{\mathscr{O}_\nu} \subset V_{F_\nu}$. We shall also adopt the shorthand notation $\widehat{G}_\nu := \widehat{G}(F_\nu), \widehat{U}_\nu := \widehat{U}(F_\nu)$, and $\widehat{H}_\nu := \widehat{H}(F_\nu)$. With these notations and methods similar to that of [Gar80, §16], we have

**Theorem 7.3.1** (Iwasawa Decomposition)**.**

$$\widehat{G}_\nu = \widehat{K}_\nu \widehat{H}_\nu \widehat{U}_\nu \tag{7.3.1}$$

**(7.4)** Keep the same notations as in the previous paragraph, and let $\widehat{R}_\nu := \widehat{G}(\mathscr{O}_\nu)$. Then clearly $\widehat{R}_\nu \subset \widehat{K}_\nu$, and the aim of this paragraph is to show that

**Proposition 7.4.1.** *If the characteristic of $F$ is sufficiently large, then*

$$\widehat{R}_\nu = \widehat{K}_\nu.$$

*Proof.* We need to show that if $x \in \widehat{K}_\nu$, then $x \in \widehat{R}_\nu$. The arguments in [Gar80, §16] applied to $\widehat{R}_\nu$ instead of $\widehat{K}_\nu$ allows us conclude that

$$\widehat{G}_\nu = \widehat{R}_\nu \widehat{H}_\nu \widehat{U}_\nu. \tag{7.4.1}$$

Then the proposition will follow from

**Claim 7.4.2.** *If the characteristic of $F$ is sufficiently large, then*

$$\widehat{H}_\nu \widehat{U}_\nu \cap \widehat{K}_\nu \subset \widehat{R}_\nu$$

Indeed, suppose that Claim 7.4.2 has been verified. Then decomposing $x$ according to 7.4.1 as $x = r_\nu a_\nu u_\nu$, we have that

$$r_\nu^{-1} x \in \widehat{H}_\nu \widehat{U}_\nu \cap \widehat{K}_\nu \subset \widehat{R}_\nu$$

and hence

$$x = r_\nu r_\nu^{-1} x \in \widehat{R}_\nu.$$

Now we proceed to the verification of Claim 7.4.2. This will in turn be reduced to the verification of

**Claim 7.4.3.** *If the characteristic of $F$ is sufficiently large, then*

$$\widehat{U}_\nu \cap \widehat{K}_\nu \subset \widehat{R}_\nu.$$

Indeed, suppose that Claim 7.4.3 has been verified. Then we may factor

$$(\widehat{H}_\nu \widehat{U}_\nu) \cap \widehat{K}_\nu = (\widehat{H}_\nu \cap \widehat{K}_\nu)(\widehat{U}_\nu \cap \widehat{K}_\nu).$$

With respect to this decompostion, let us write $x \in (\widehat{H}_\nu \widehat{U}_\nu) \cap \widehat{K}_\nu$ as $x = a'_\nu u'_\nu$ for $a'_\nu \in \widehat{H}_\nu \cap \widehat{K}_\nu$ and $u'_\nu \in \widehat{U}_\nu \cap \widehat{K}_\nu$. But clearly $\widehat{H}_\nu \cap \widehat{K}_\nu \subset \widehat{R}_\nu$ since an element $h = \prod_{i=1}^{l} h_{\alpha_i}(s_\nu)$ with $s_\nu \in F_\nu^*$ will preserve $V_{\mathscr{O}_\nu}$ if and only if $s_\nu \in \mathscr{O}_\nu^*$. So, if Claim 7.4.3 is verified, then $x = a'_\nu u'_\nu \in \widehat{R}_\nu$.

Finally, we turn to the verification of Claim 7.4.3. By 7.2.1, we may write

$$u_\nu = \prod_{\alpha \in \Delta_+(A)} \chi_\alpha(\sigma_\alpha(t)) h \prod_{\alpha \in \Delta_-(A)} \chi'_\alpha(\sigma_\alpha(t)) \tag{7.4.2}$$

where

$$\sigma_\alpha(t) \in F_\nu[[t]],\ \alpha \in \Delta_+(A),$$

$$\sigma'_\alpha(t) \in tF_\nu[[t]],\ \alpha \in \Delta_-(A)$$

where the products are taken with respect to the fixed orders on $\Delta_\pm(A)$ and where

$$h = \prod_{i=1}^{l} h_{\alpha_i}(\sigma_i(t))$$

with $\sigma_i(t) \in F_\nu[[t]]^*$ satisfying

$$\sigma_i(t) \equiv 1 (\operatorname{mod} t).$$

An argument similar to [Gar98, Lemma 5.6] shows that if $u_\nu \in \widehat{K}_\nu$, then we must have $\sigma_\alpha \in \mathscr{O}_\nu[[t]]$ and $\sigma'_\alpha \in t\mathscr{O}_\nu[[t]]$. Hence it suffices to show that if $h = \prod_{i=1}^{l} h_{\alpha_i}(\sigma_i(t))$ as above is in $\widehat{K}_\nu$, then $\sigma_i(t) \in \mathscr{O}_\nu[[t]]^*$, for $i = 1, \ldots, l$.

For each $j = 1, \ldots, l$, we have

$$h \chi_{\alpha_j}(1) h^{-1} = \chi_{\alpha_j}(\tau_j) \in \widehat{K}_\nu$$

where

$$\tau_j = \prod_{i=1}^{l} \sigma_i^{\frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}},\ j = 1, \ldots, l.$$

But again using the methods of [Gar98, Lemma 5.6], we may conclude that for each $j = 1, \dots, l$

$$\tau_j = 1 + \sum_{k=1}^{\infty} c_k t^k$$

where $c_k \in \mathscr{O}_\nu$. Now, set

$$c_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}\ i, j = 1, \dots, l$$

and denote the matrix $C = (c_{ij})$. The elements of $C$ have integral entries, and hence so does the determinant. Now for any field $F$ in which the primes dividing this determinant are invertible, the matrix $C$ has an inverse $D = (d_{ij})$ with integral coefficients. Then we have,

$$\tau_j = \prod_{i=1}^{l} \sigma_i^{c_{ij}}$$

and so

$$\begin{aligned}\prod_{j=1}^{l} \tau_j^{d_{jk}} &= \prod_j \prod_i \sigma_i^{c_{ij} d_{jk}} \\ &= \prod_i \sigma_i^{\sum_j c_{ij} d_{jk}} = \sigma_k.\end{aligned}$$

It follows from a that the coefficients of $\sigma_k$ are then in $\mathscr{O}_\nu$. This concludes the proof of Claim 7.4.3 and hence of the proposition. □

**Definition 7.4.4.** We say that the characteristic of $F$ is *good* if we can identify $\widehat{K}_\nu$ and $\widehat{R}_\nu$.

**(7.5)** The groups $\widehat{G}$ introduced above are essentially central extensions of the loop group $G((t))$. We make this relation explicit in this section. We now have to be careful which of the groups $\widehat{G}^\lambda$ we are dealing with, so we reintroduce the $\lambda$ into our notation.

We begin by noticing that the same arguments in [Gar80] applied to case of $\widehat{G}^\lambda(R)$, where $R$ is an integral domain (instead of a field) show that the following relations hold in $\widehat{G}(R)$ :

(1)

$$\chi_\alpha(\sigma)\chi_\alpha(\tau) = \chi_\alpha(\sigma + \tau)$$

(2)

$$(\chi_\alpha(\sigma), \chi_\beta(\tau)) = \Pi_{i,j>0,\, i\alpha + j\beta \in \Delta(A)} \chi_{i\alpha + j\beta}(c_{ij}\sigma^i \tau^j)$$

where the $c_{ij}$ are constants which depend only on $\alpha, \beta, i, j$ (so not on $R$).

If we further impose the relation

$$h_\alpha(\sigma\tau) = h_\alpha(\sigma)h_\alpha(\tau)$$

upon the group $\widehat{G}^\lambda(R)$ we obtain a quotient group which we denote by $\overline{G}^\lambda(R)$.

**Theorem 7.5.1.** [Gar80, §12] *For each field $F$ there is a central extension of groups*

$$F^* \to \widehat{G}^\lambda(F) \to \overline{G}^\lambda(F).$$

*Moreover, this extension splits over the image of the subgroup $\widehat{U}(F)$ in $\overline{G}^\lambda(F)$.*

**(7.6)** The groups $\overline{G}^\lambda$ are related to the loop groups $G((t))$ when $\lambda$ is suitable chosen. Let $G$ be the simply connected Chevalley group associated to $\mathfrak{g}$. Then let $\widehat{L}_\lambda$ be the lattice of weights on $V^\lambda$ and denote by $L_\lambda := \widehat{L}_\lambda \cap \mathfrak{h}^\vee$ the associated finite dimensional lattice. Then we say that a dominant integral weight $\lambda$ is *clasically maximal* if $L_\lambda$ is the lattice spanned by the fundamental weights of $\mathfrak{h}$. Then, as follows from the arguments in [Ste67], the map

$$G((t)) \to \overline{G}^\lambda$$

which in general is only surjective with finite central kernel, is an isomorphism for $\lambda$ clasically maximal. From now on, we shall assume that $\lambda$ satisfies this hypothesis in addition to being dominant and integral.

*Notation:* Henceforth, we drop the $\lambda$ from our notation for groups. Let $B((t))$, $U((t))$ and $H((t))$ be the images of the corresponding groups in $\widehat{G}$. We sometimes refer to these groups as $\overline{B}, \overline{U}$, and $\overline{H}$.

**(7.7)** We would now like to introduce a twisted version of the groups above as in §1.4. To do so, we first construct an action of the subgroup $\mathbb{G}_m$ on $V_R^\lambda$ for any $k$-algebra $R$. Recall that $\lambda$ was chosen to be dominant integral and satisfy $\lambda(\mathbf{D}) \in \mathbb{Z}$. For $r \in R^*$ We then define the maps,

$$\eta(r) : V_R^\lambda \to V_R^\lambda$$

on the weight space $V_{\mu,R}^\lambda$ where $\mu = \lambda - \sum_{i=1}^{l+1} k_i\alpha_i$ as the multiplication operator by $r^{\lambda(\mathbf{D})-k_{l+1}}$. Then it is an easy to see that $\eta(r)$ normalizes $\widehat{G}(R)$, and in fact,

$$\eta(r)\chi_\alpha(\sigma)\eta(r)^{-1} = \chi_\alpha(\sigma')$$

where

$$\sigma = \sum a_i t^i \text{ and } \sigma' = \sum a_i (rt)^i.$$

On the quotient $G((t))$ the action of $\eta(r)$ is thus rotation by the loop parameter, which agrees with the action introduced in §1.4.

Now we may form the semi-direct product group, $\widehat{G} \ltimes \mathbb{G}_m$. Analogously to Theorem 7.5.1 we have an sequence

$$\mathscr{O}^* \to \widehat{G} \ltimes \mathbb{G}_m \to G((t)) \ltimes \mathbb{G}_m.$$

**(7.8)** Finally, we can relate the central extension 7.7 and 5.5 in the following way,

**Theorem 7.8.1** ([Gar80, BD01])**.** *Let $F$ be a field and assume that $\lambda$ is dominant integral and classically maximal. Then there exists an map of exact sequences of groups,*

$$\begin{array}{ccccc} \mathsf{K}_2(F((t))) & \longrightarrow & \widetilde{\mathsf{G}}(F((t))) \ltimes \mathbb{G}_m & \longrightarrow & \mathsf{G}(F((t))) \ltimes \mathbb{G}_m \\ \downarrow \partial & & \downarrow & & \downarrow \\ F^* & \longrightarrow & \widehat{G}(F) \ltimes \mathbb{G}_m & \longrightarrow & G(F((t))) \ltimes \mathbb{G}_m \end{array},$$

*where $\partial : \mathsf{K}_2(F((t))) \to F^*$ is a power of the tame symbol depending only on $\lambda$.*

## 8. Adelic Groups

**(8.1)** Let $F$ be a global field of finite characteristic over a finite field $k = \mathbb{F}_q$. Suppose $F = k(X)$, the function field of an algebraic curve $X$. We denote the set of places of $F$ by $\mathscr{V}$ and identify this set with the closed points of $X$. We shall denote an element of $\mathscr{V}$ by $\nu \in \mathscr{V}$ if we want to regard it algebraically and $P \in X$ if we wish to think of it geometrically. For each $\nu \in \mathscr{V}$, we denote the completion of $F$ with respect to this place any $F_\nu$ and denote

$$\nu : F_\nu^* \to \mathbb{Z}$$

the corresponding discrete valuation. Let $\mathscr{O}_\nu$ the ring of integers of this valuation, $m_\nu$ the maximal ideal, and $\pi_\nu$ a uniformizing element. Furthermore, we consider the norm,

$$|\cdot|_\nu : F_\nu^* \to q^{\mathbb{Z}} \subset \mathbb{R} \tag{8.1.1}$$

$$a \mapsto |a|_\nu := q^{-\nu(a)} \tag{8.1.2}$$

The adeles over $F$ by are given by

$$\mathbb{A}_F := \prod_{\nu \in \mathscr{V}}{}' F_\nu$$

where the restricted direct product is considered with respect to $\prod_{v\in\mathscr{V}}\mathscr{O}_v$. Similarly, we define the ideles over $F$ by

$$\mathbb{I}_F := \prod_{v\in\mathscr{V}}{}' F_v^*$$

where the restricted direct product is with respect to $\prod_{v\in\mathscr{V}}\mathscr{O}_v^*$. If $\Xi\subset\mathscr{V}$ is a finite set of valuations and $W=\mathscr{V}\setminus\Xi$, then we shall denote by

$$\mathbb{A}_W := \prod_{v\in W}{}' F_v$$

where the restricted product is with respect to $\prod_{v\in W}\mathscr{O}_v$. Similarly, we construct $\mathbb{I}_W$. Note that $\mathbb{A}_X=\mathbb{A}_F$ and $\mathbb{I}_X=\mathbb{I}_F$. When there is no danger of confusion, we write shall omit the subscript $W$ from our notation. The local norms 8.1.1 above induces one on the ideles which shall be denoted by $|\cdot|_{\mathbb{I}_F}$ or just $|\cdot|_{\mathbb{I}}$ for short.

**(8.2)** We fix a dominant integral weight $\lambda$ which is classically maximal. Recall that for each place $v$ of $F$ we have constructed groups $\widehat{G}_v, \widehat{\mathsf{H}}_v, \widehat{\mathsf{U}}_v, \widehat{\mathsf{K}}_v$, and $\widehat{\mathsf{B}}_v$ in section §3.3. Let us define,

$$\widehat{\mathsf{K}}_{\mathbb{A}} := \prod_{v\in\mathscr{V}}\widehat{\mathsf{K}}_v.$$

Then we can then form the adelic groups $\widehat{G}_{\mathbb{A}}, \widehat{\mathsf{H}}_{\mathbb{A}}, \widehat{\mathsf{U}}_{\mathbb{A}}$, and $\widehat{\mathsf{B}}_{\mathbb{A}}$ by the usual restricted direct product construction with respect to the subgroups $\prod_{v\in\mathscr{V}}\widehat{\mathsf{K}}_v$ , $\prod_{v\in\mathscr{V}}\widehat{\mathsf{K}}_v\cap\widehat{\mathsf{H}}_v$ , $\prod_{v\in\mathscr{V}}\widehat{\mathsf{K}}_v\cap\widehat{\mathsf{U}}_v$, and $\prod_{v\in\mathscr{V}}\widehat{\mathsf{K}}_v\cap\widehat{\mathsf{B}}_v$.

From the local Iwasawa decompositions 7.3.1 we have

$$\widehat{G}_{\mathbb{A}} = \widehat{\mathsf{K}}_{\mathbb{A}}\widehat{\mathsf{H}}_{\mathbb{A}}\widehat{\mathsf{U}}_{\mathbb{A}}.$$

If $g\in\widehat{G}_{\mathbb{A}}$ and we may write (nonuniquely),

$$g = k_g h_g u_g \text{ , for } k_g\in\widehat{\mathsf{K}}_{\mathbb{A}},\ h_g\in\widehat{\mathsf{H}}_{\mathbb{A}}\ u_h\in\widehat{\mathsf{U}}_{\mathbb{A}},$$

then the element $h_g$ has an expression

$$h_g = \prod_{i=1}^{l+1} h_{\alpha_i}(\sigma_i)\ ,\sigma_i\in\mathbb{I}$$

with

$$|h_g| := \prod_{i=1}^{l+1} h_{\alpha_i}(|\sigma_i|)\in\widehat{\mathsf{H}}(\mathbb{R})$$

uniquely determined by $g$.

Finally, we shall need a twisted version of the Iwasawa decomposition. Let $\tau\in\mathbb{I}$ consider the automorphism of $\widehat{G}_{\mathbb{A}}$ defined by $\eta(\tau)$. Denote the subgroup of the semi-direct product $\widehat{G}_{\mathbb{A}}\ltimes\mathbb{I}$ (where an element of $\mathbb{I}$ acts by rotation as usual) generated by $\widehat{G}_{\mathbb{A}}$ and $\eta(\tau)$ by $\widehat{G}_{\mathbb{A}}\eta(\tau)$ for short. Then as $\eta(\tau)$ normalizes $\widehat{\mathsf{U}}_{\mathbb{A}}$, may write,

$$\widehat{G}_{\mathbb{A}}\eta(\tau) = \widehat{K}_{\mathbb{A}}\widehat{\mathsf{H}}_{\mathbb{A}}\eta(\tau)\widehat{\mathsf{U}}_{\mathbb{A}}.$$

(**8.3**) We would now like to introduce an arithmetic subgroup of $\widehat{G}_{\mathbb{A}}$. For each $\nu$, we have an embdedding

$$\widehat{G}_F \hookrightarrow \widehat{G}_{F_\nu}$$

which induces a map

$$i : \widehat{G}_F \hookrightarrow \prod_{\nu \in \mathscr{V}} \widehat{G}_{F_\nu}.$$

However, as discussed in [Gar04, §2] the image of this map is not contained in $\widehat{G}_{\mathbb{A}}$. To remedy this situation, we define,

$$\widehat{\Gamma}_F := \{g \in \widehat{G}_F | i(g) \in \widehat{G}(\mathbb{A})\}.$$

We then have the following,

$$(8.3.1) \quad \widehat{\Gamma}_F/(\widehat{\Gamma}_F \cap \widehat{\mathsf{B}}_F) \cong \widehat{G}_F/\widehat{\mathsf{B}}_F \cong \overline{G}_F/\overline{B}_F = G(F[t,t^{-1}]/B(F[t,t^{-1}])$$

## 9. On Arithmetic Quotients

The loop Eisenstein series introduced below in §6 is a function defined on the double quotient space,

$$\widehat{\mathsf{K}}_{\mathbb{A}} \backslash \widehat{G}_{\mathbb{A}}\eta(\tau)/\widehat{\Gamma}_F$$

for $\tau \in \mathbb{I}_F$. Later in Chapter 3 we shall see that this doube coset space parametrizes *formal* $G$-bundles on some surface. On the other hand, we would like to interpret the space as a moduli space for *actual* $G$-bundles on a surface. To do so, we need to be able to choose for each $g\eta(\tau) \in \widehat{G}_{\mathbb{A}}\eta(\tau)$ a representative modulo $\widehat{\mathsf{K}}_{\mathbb{A}}$ and $\widehat{\Gamma}_F$ which is a Laurent polynomial valued loop rather than a Laurent power series valued loop. The aim of this section is to use a part of the reduction theory developed in [Gar80] to show how such a representative can be chosen. Note that the full strength of reduction theory is not used here, just the fact that one can find a fundamental domain for the pro-unipotent radical $\widehat{\mathsf{U}}_{\mathbb{A}}$ under $\widehat{\Gamma}_F \cap \widehat{\mathsf{U}}_{\mathbb{A}}$.

(**9.1**) For our given function field $F$, let $k$ be the field of constants. Our first goal will be to state an approximation theorem for $\mathbb{A}_F$. To state it, let us introduce some notation. By a *divisor,* we shall mean a formal sum

$$\mathfrak{a} = \sum_{\nu \in \mathscr{V}} a(\nu)\nu$$

with $a(\nu) \in \mathbb{Z}$ and $a(\nu) = 0$ for almost all $\nu$. For each place $\nu$, recall that $\pi_\nu \in \mathscr{O}_\nu$ was a uniformizing element. Define the degree of $\nu$ by

$$\deg(\nu) = [\mathscr{O}_\nu / \pi_\nu \mathscr{O}_\nu : k],$$

and the degree of the divisor $\mathfrak{a}$ by the expression

$$\deg(\mathfrak{a}) := \sum_{\nu \in \mathscr{V}} a_\nu \deg(\nu).$$

Given any divisor $\mathfrak{a}$, we define the subgroup a $\mathbb{A}$,

$$\Omega(\mathfrak{a}) = \prod_{\nu \in \mathscr{V}} \pi_\nu^{-a(\nu)} \mathscr{O}_\nu.$$

Then it is essentially a consequence of the Riemann-Roch theorem for curves (over finite fields) that,

**Proposition 9.1.1** ([Wei74]). *There exists a positive integer $g$ called the genus of $F$ such that if $\mathfrak{a}$ is a divisor whose degree is $> 2g-2$, then*

$$\mathbb{A}_F = F + \Omega(\mathfrak{a})$$

The form in which we shall use this result is as follows: Let $\{P_1, \dots, P_n\}$ be a set of points of $X(k)$ and let $Q$ be a point distinct from the $P_i$. Then, given non-negative integers $a_i$ for $i = 1, \dots, n$, we have,

**Corollary 9.1.2.** *Let $\mathfrak{a}$ be the divisor*

$$\mathfrak{a} = \sum_i -a_i P_i + mQ.$$

*Let $U = X(k) \setminus \{P_1, \dots, P_n, Q\}$. If*

$$m \deg(Q) \geq m \geq 2g - 2 + \sum_i a_i \deg(P_i),$$

*then we have*

$$\mathbb{A}_F = F + (\prod_{i=1}^{n} p_{\nu_i}^{a_i} \mathscr{O}_{\nu_i} \times p_{\nu_Q}^{m} \mathscr{O}_{\nu_Q}) \times \prod_{\nu \in U} \mathscr{O}_\nu$$

**(9.2)** We begin with the following,

**Definition 9.2.1.** We say that an element $\xi \in \widehat{G}(\mathbb{A})$ has finite $t$-expansion if it may be written as a product of $\chi_\alpha(\sigma)$ where $\sigma = (\sigma_\nu)$ and $\sigma_\nu \in F_\nu[t, t^{-1}]$. Also, we say that an element $\xi \eta(\tau) \in \widehat{G}(\mathbb{A}) \eta(\tau)$ has finite $t$-expansion if $\xi$ does.

The main result of this section is then,

**Theorem 9.2.2.** *Let $\tau = (\tau_P)_{P\in X(k)} \in \mathbb{I}_F$ be such that $0 < |\tau|_{\mathbb{I}} < 1$. Then for any element in the double coset space*

$$\widehat{\mathsf{K}}_{\mathbb{A}} \setminus \widehat{G}_{\mathbb{A}}\eta(\tau)/\widehat{\Gamma}$$

*we can find a represenative $\xi\eta(\tau) \in \widehat{G}_{\mathbb{A}}\eta(\tau)$ which has finite $t$-expansion.*

**Remark 9.2.3.** It is crucial here that $0 < |\tau|_{\mathbb{I}} < 1$. See Chapter 3, §2.3 for a geometric interpretation.

*Proof. Step 1:* The Iwasawa decomposition for $\widehat{G}_{\mathbb{A}}\eta(\tau)$ allows us to write $\widehat{G}_{\mathbb{A}}\eta(\tau) = \widehat{\mathsf{K}}_{\mathbb{A}}\widehat{\mathsf{H}}_{\mathbb{A}}\eta(\tau)\widehat{\mathsf{U}}_{\mathbb{A}}$. Given $g\eta(\tau)$, let us write this factorization as

$$g\eta(\tau) = (k_P)_P(h_P)_P\eta((\tau_P)_P)(u_P)_P$$

where $P$ denotes a place of $F$. Now, we can choose a set $\Xi$ such that for $P \in X\setminus\Xi$, we have $h_P \in \widehat{\mathsf{H}}_{\mathscr{O}_P}$, $u_P \in \widehat{U}_{\mathscr{O}_P}$, and $\tau_P \in \mathscr{O}_P^*$. Then set $k_\Xi = (k_P)_{P\in\Xi}$ and similarly define $h_\Xi, u_\Xi, \tau_\Xi$. It is then enough to show that

$$g_\Xi\eta(\tau_\Xi) := k_\Xi h_\Xi\eta(\tau_\Xi)u_\Xi \in \prod_{P\in\Xi}\widehat{G}_{F_P}\eta(\tau_P)$$

has a $t$-finite expansion since for all other points $P \in X\setminus\Xi$, $g_P\eta(\tau_P)$ is integral and hence contained in $\widehat{\mathsf{K}}_P\eta(\tau_P)$.

*Step 2:* By 7.2.1 every $u_\Xi \in U_\Xi$ has an expression of the form

$$u_\Xi = \prod_{\alpha\in\Delta_+}\chi_\alpha(\sigma_\alpha)h\prod_{\alpha\in\Delta_-}\chi_\alpha(\sigma_\alpha)$$

where

$$h = \prod_{i=1}^{l}h_{\alpha_i}(\sigma_i)$$

and $\sigma_\alpha, \sigma_i$ are $|\Xi|$-tuples of formal power series of the form, $\sum_{j=0}^{\infty}c_jt^j$ where $c_0 = 0$ if $\alpha\in\Delta_-$, and $c_0 = 1$ for each component of $\sigma_i$. Furthermore, using Proposition 9.1.1 and an argument similar to [Gar80, Thm. 18.16] we may conclude that the magnitudes of $c_j$ in the above expression are uniformly bounded.

*Step 3:* For $u_\Xi$ as above and $q\in\mathbb{N}$, we would like to introduce the notion of a $q$-truncation of $u_\Xi$ which we shall denote by $u_\Xi[q]$ as follows: for $\sigma = \sum_{j\geq 0}c_jt^j$, let $\sigma(q) = \sum_{j=0}^{q}c_jt^j$, and extend this definition to $|\Xi|$-tuples of power series in the obvious manner. Then we define,

$$u_\Xi[q] = \prod_{\alpha\in\Delta_+}\chi_\alpha(\sigma_\alpha(q))h[q]\prod_{\alpha\in\Delta_-}\chi'_\alpha(\sigma_\alpha(q))$$

where it remains to define $h(q)$. This we do as follows: recall that

$$h_{\alpha_i}(\sigma_i) = \chi_{\alpha_i}(\sigma_i)\chi_{-\alpha_i}(-\sigma_i^{-1})\chi_{\alpha_i}(\sigma_i)w_{\alpha_i}(1)^{-1}.$$

We then define

$$h[q] = \prod_{i=1}^{l} \chi_{\alpha_i}(\sigma_i(q))\chi_{-\alpha_i}(-\sigma_i^{-1}(q))\chi_{\alpha_i}(\sigma_i(q))w_{\alpha_i}(1)^{-1}.$$

With this notation, we can factor $u_\Xi \in \widehat{\mathsf{U}}_\Xi$ as

$$u_\Xi = u'_\Xi[q]u_\Xi[q]$$

where $u'_\Xi[q] \in \widehat{\mathsf{U}}_\Xi^{(q)}$ is the product over $\Xi$ of the congruence subgroups of level $q$ introduced in §3.2. For $\gamma \in \widehat{\Gamma}_F \cap \widehat{\mathsf{U}}_\Xi^{(q)}$ we have

$$u_\Xi\gamma = u'_\Xi[q]u_\Xi[q]\gamma = u'_\Xi(\gamma)[q]u_\Xi[q]$$

where $u'_\Xi(\gamma)[q] \in \widehat{\mathsf{U}}_\Xi^{(q)}$.

The proof of the theorem is now reduced to,

**Claim 9.2.4.** *There exists* $q \geq 0$ *and* $\gamma \in \Gamma \cap \widehat{U}_\Xi^{(q)}$ *such that*

$$h_\Xi\eta(\tau_\Xi)\, u'_\Xi(\gamma)[q]\, \eta(\tau_\Xi)^{-1}h_\Xi^{-1} \in \widehat{K}_\Xi.$$

Indeed, suppose we have verified Claim 9.2.4 , then

$$g_\Xi\eta(\tau_\Xi)\gamma = k_\Xi h_\Xi\eta(\tau_\Xi)\, u'_\Xi(\gamma)[q]\, \eta(\tau_\Xi)^{-1}h_\Xi^{-1}h_\Xi\eta(\tau_\Xi)u_\Xi[q].$$

So, modulo $\widehat{K}_\Xi$, we can write $g_\Xi\eta(\tau_\Xi)\gamma$ as a product

$$h_\Xi\eta(\tau_\Xi)u_\Xi[q]$$

which clearly has a finite $t$-expansion.

*Step 4:* For any given $0 < \varepsilon < 1$, define the subset

$$\widehat{K}_\nu^\varepsilon := \langle \chi_\alpha(\sigma) \in \widehat{K}_\nu | \sigma = \sum_{i \geq i_0} c_i t^i, |c_i|_\nu \leq \varepsilon \rangle \subset \widehat{\mathsf{K}}_\nu.$$

Then form

$$\widehat{\mathsf{K}}_\Xi^\varepsilon = \prod_{\nu \in \Xi} \widehat{K}_\nu^\varepsilon.$$

The proof of Claim 9.2.4 can then be reduced to the following,

**Claim 9.2.5.** *Given any* $0 < \varepsilon < 1$, *there exists* $q \in \mathbb{N}$ *and* $\gamma \in \widehat{\Gamma} \cap \widehat{U}_\Xi^{(q)}$ *such that*

$$\eta(\tau_\Xi)\, u'_\Xi(\gamma)[q]\, \eta(\tau_\Xi)^{-1} \in \widehat{K}_\Xi^\varepsilon.$$

Indeed, suppose we have verified Claim 9.2.5. Then we may choose $\varepsilon$ such that

$$h_\Xi\widehat{K}_\Xi^\varepsilon h_\Xi^{-1} \subset \widehat{K}_\Xi,$$

as follows from knowledge of the the explicit action of $\widehat{A}$ on $\widehat{U}$ (see, [Gar80]13.10, 14.11).

*Step 5:* We would now like to prove Claim 9.2.5. As $\prod_{P\in\Xi}|\tau_P| < 1$, we may break up $\Xi$ into a union of disjoint sets $\Xi_j$ where in each $\Xi_j$ there exists exactly one point $Q^j \in \Xi_j$ such that $|\tau_{Q^j}| < 1$. We enumerate the elements of the set $\Xi_j$ by

$$\{Q^j, P_1^j, \ldots, P_{r_j}^j\}.$$

Let us first verfiy the following,

**Lemma 9.2.6.** *Let $\Xi = \{Q, P_1, \ldots, P_r\}$ where $|\tau_Q| < 1$, $|\tau_{P_i}| > 1$, and $\prod_{P\in\Xi}|\tau_{P_i}| < 1$. For each pair of positive integers $j$ and $D$ consider the following truncated version of the sets $\widehat{\mathsf{K}}_\nu^\varepsilon$ and $\widehat{\mathsf{K}}_\Xi^\varepsilon$ introduced in Step 4,*

$$\widehat{\mathsf{K}}_\nu^\varepsilon[D, D+j] := \{\prod \chi_\alpha(\sigma) \in \widehat{\mathsf{K}}_\nu^\varepsilon | \sigma = \sum_{i\geq i_0} c_i t^i \ , |c_i| < \varepsilon \ \text{for } D \leq i \leq D+j\}$$

$$\widehat{\mathsf{K}}_\Xi^\varepsilon[D, D+j] := \prod_{\nu\in\Xi} \widehat{\mathsf{K}}_\nu^\nu[D, D+j].$$

*Then consider the following statement which we call $P(D, j)$:*

(1) *For each $D \leq k \leq D+j$ there exists $\gamma_k \in \widehat{\Gamma}_F$ such that $\gamma_k = \gamma_{k+1}$ modulo $\widehat{\Gamma}_F \cap U_\Xi^{(j+1)}$.*
(2) *Letting $u'_j = u\gamma_j$, the coefficients of $t^k$ for $D \leq k \leq j$ in the expression $\eta(\tau_\Xi)u'\eta(\tau_\Xi)^{-1}$ lie in $\widehat{K}_\Xi^\varepsilon[D, D+j]$.*

*Then there exists $D >> 0$ such that $P(D, j)$ holds for all $j \in \mathbb{N}$.*

*Proof of Lemma.* Let us first show that there exists a $D >> 0$ such that $P(D, 0)$ is true. From the Corollary 9.1.2 above: given $(a_P)_{P\in\Xi}$, $a_P \in F_P$ with each $|a_P|$ bounded by $2^a$, we can find for each $n \in \mathbb{N}$ an element $\phi_n \in F$ such that

$$(a'_P)_{P\in\Xi} = (a_P)_{P\in\Xi} + \phi_n$$

satisfies the conditions,

- if $P \neq Q$, then $|a'_{P_i}| < 2^{a-n\deg(P)}$, and
- $|a'_Q| < 2^{a+n\deg(P)+c}$ where $c$ is some fixed constant (depending only on the genus of the field).

Let $\chi_\alpha(\sigma) \in \prod_{P\in\Xi}\widehat{G}_P$, where $\sigma = (\sigma_P)$, and $\sigma_P = \sum_{n\geq 0} c_{n,P}t^n$. Then translating an element $\chi_\alpha(\sigma_\Xi)$ by $\chi_\alpha(\phi_n t^D)$, we obtain an element $\chi_\alpha(\sigma')$ where the coefficient of $t^D$ in $\sigma'_P$ satisfies the condition:

- if for $P_i$, $i = 1, \ldots r$, then $|c'_{P_i,D}| < 2^{a-n_i\deg(P_i)}$,
- $|c'_{Q,D}| < 2^{a+n+c}$ where $c$ is as above and $n = \sum_i^r n_i \deg(P_i)$.

Hence the coefficient of $t^D$ in $\eta(\tau_\Xi)\chi_\alpha(\sigma')\eta(\tau_\Xi)^{-1}$ satisfies the condition,

- for $P_i$, $i = 1, \ldots r$, then $|c'_{P_i,D}| < 2^{Dp_i + a - n_i \deg(P_i)}$, and
- $|c'_{Q,D}| < 2^{Dq_i + a + n + c}$ where $c$ is as above and $n = \sum_i^r n_i \deg(P_i)$.

Finally, we conclude the proof of the lemma from the following,

**Claim 9.2.7.** *Let $p_1, \ldots, p_r, q$ be integers such that*

$$p_1 + \cdots p_r + q < 0.$$

*For fixed positive numbers $a, c, M$, there exist numbers $D, n_i \in \mathbb{N}$ such that*

$$Dq + a + \sum_i n_i \deg(P_i) + c < -M$$

*and*

$$Dp_i + a - n_i \deg(P_i) < -M.$$

*Proof of claim.* Suppose that $|\tau_{P_i}| = 2^{p_i}$ and $|s_Q| = 2^q$ for some $p_i > 0$ and $q < 0$ satisfying

$$p_1 + \cdots + p_n + q < 0.$$

Thus, there exists $D >> 0$ such that $D(p_1 \cdots + p_n + q)$ can be made as small as possible. Choose $D$ such that this value is smaller that $-(r+1)M - r - ra - c$.

For this fixed $D$ choose $n_i$ such that

$$Dp_i + a - n_i \deg(P_i) = -M - 1$$

for all $i$. We then have that

$$\sum_i^r (Dp_i + a - n_i \deg(P_i)) + (Dq + a + \sum_i n_i \deg(P_i) + c) = \sum_i^r Dp_i + Dq + ra + c < -(r+1)M - r.$$

So then

$$Dq + a + \sum_i n_i \deg(P_i) + c < -(r+1)M - r - r(-M-1) = -M,$$

and the claim is proven. □

From this, it follows readily that there exists $D$ such that $P(D, 0)$ is true. An induction similar to that of Theorem 18.16 from [Gar80] then allows one to conclude $P(j+1, D)$ from $P(j, D)$. This finishes the proof of the lemma. □

The validity of $P(j, D)$ for all $j \in \mathbb{N}$ implies statement Claim 9.2.5 for the special case of $\Xi$ as in the lemma. The case when $\Xi$ is more complicated and is the union of sets $\Xi = \cup \Xi_j$ also follows from a similar argument.

□

**Remark 9.2.8.** Let $\widehat{G}_{\mathbb{A}}^{\mathrm{pol}}\eta(\tau)$ be the polynomial version of the group $\widehat{G}_{\mathbb{A}}\eta(\tau)$, i.e., the group generated by $\chi_\alpha(\sigma)$ for $\sigma$ a Laurent *polynomial* rather than a Laurent series. Similarly, we define $\widehat{\Gamma}^{\mathrm{pol}}$ and $\widehat{\mathsf{K}}_{\mathbb{A}}^{\mathrm{pol}}$. The theorem above gives a surjective map,

$$\widehat{\mathsf{K}}_{\mathbb{A}}^{\mathrm{pol}} \setminus \widehat{G}_{\mathbb{A}}^{\mathrm{pol}}\eta(\tau)/\widehat{\Gamma}_F^{\mathrm{pol}} \to \widehat{\mathsf{K}}_{\mathbb{A}} \setminus \widehat{G}\eta(\tau)/\widehat{\Gamma}_F.$$

We do not know whether the above map is bijective.

## 10. Loop Eisenstein Series

**(10.1)** Let $\mu : \widehat{\mathsf{H}}(\mathbb{R}) \to \mathbb{C}^*$ be a quasi-character. Our notation shall be

$$a^\mu := \mu(a)$$

for $a \in \widehat{\mathsf{H}}(\mathbb{R})$. Such a $\mu$ corresponds uniquely to an $\mathbb{R}$-linear map

$$\widehat{\mathfrak{h}} \to \mathbb{C}^*.$$

We say that a quasi-character $\mu$ satisfies *Godement's criterion* if

$$\mathsf{Re}\lambda(h_i) \leq -2 \quad \text{for} \quad i = 1, \ldots, l+1 \ .$$

Using the Iwasawa decomposition, we can also extend our quasi-characters to the whole group $\widehat{G}_{\mathbb{A}}\eta(\tau)$ as follows: writing $g\eta(\tau) = k_g a_g u_g \eta(\tau)$, we know that $|a_g| \in \widehat{\mathsf{H}}(\mathbb{R})$ is well-defined by §4.2. We then define,

$$\begin{array}{rcl} \Phi_\mu : \widehat{G}_{\mathbb{A}}\eta(\tau) & \to & \mathbb{C}^* \\ g\eta(\tau) & \mapsto & |a_g|^\mu \end{array}$$

**(10.2)** Let $\mu$ be a fixed quasi-character. Then we can form the formal sum

$$E_\mu(g\eta(\tau)) = \sum_{\widehat{\Gamma}_F/\widehat{\Gamma}_F \cap \widehat{\mathsf{B}}_F} \Phi_\mu(g\eta(\tau)\gamma),$$

for $g \in \widehat{G}_{\mathbb{A}}$, $\tau \in \mathbb{I}$. The above series is called the *loop Eisenstein series*, and its basic analytic property is given by the following

**Theorem 10.2.1.** *The series $E_\mu(g\eta(\tau))$ is defined on*

$$\widehat{K}_{\mathbb{A}} \setminus \widehat{G}_{\mathbb{A}}\eta(\tau)/\widehat{\Gamma}_F$$

*and it converges absolutely on the domain in which $\mu$ satisfies Godement's criterion. Furthermore, it has a meromorphic continutation to the region*

$$\mathsf{Re}(\mu+\rho)(\mathbf{c}) < 0$$

*where $\rho(h_i) = 1$ for $i = 1, \ldots, l+1$.*

**(10.3)** In [Gar04], the quotient space $\widehat{\mathsf{U}}_{\mathbb{A}}/(\widehat{\mathsf{U}}_{\mathbb{A}} \cap \widehat{\Gamma}_F)$ is realized as a projective limit of compact spaces, and is thus equipped with a measure $du$ normalized so that the volume of $\widehat{\mathsf{U}}_F/\widehat{\mathsf{U}}_F \cap \widehat{\Gamma}_F$ is equal to 1. Using this measure, we may define the *constant term of the Eisenstein series* as the integral,

$$E^{\sharp}_{\mu}(g\eta(\tau)) := \int_{\widehat{\mathsf{U}}_{\mathbb{A}}/(\widehat{\mathsf{U}}_{\mathbb{A}} \cap \widehat{\Gamma}_F}) E_{\mu}(g\eta(\tau)u)du.$$

which convrges on the same half-plane as $E(\mu, g\eta(\tau))$. By a loop analogue of the Gindikin-Karpelevich formula, we have,

**Theorem 10.3.1** ([Gar04])**.** *The constant term $E^{\sharp}_{\mu}(g\eta(\tau))$ admits an expression,*

$$E^{\sharp}_{\mu}(g\eta(\tau)) = \sum_{w \in \widehat{W}} (a_g\eta(\tau))^{w(\mu+\rho)-\rho}\widetilde{c}(\mu, w)$$

*where we have the following notation: $a_g$ is the $\widehat{A}$ component of $g$ in the Iwasawa decomposition;*

$$\widetilde{c}(\mu, w) = \prod_{\alpha \in \widehat{\Delta}_+ \cap w^{-1}(\widehat{\Delta}_-)} \frac{\zeta_F(-(\mu+\rho)(h_\alpha))}{\zeta_F(-(\mu+\rho)(h_\alpha)+1)},$$

*with*

$$\zeta_F(s) = \prod_{v}(1 - q_v^{-s})^{-1}$$

*the zeta function of the curve corresponding to $F$.*

## **Part** 3. **Geometric Construction of Loop Eisenstein Series**

### 11. BLOCH'S MAP

**(11.1)** Globalizing the situation of Chapter 2 §1.2, we have the have the following: Let $S \in \mathsf{Sch}_k$ be integral with function field $F_S$, and let $D \subset S$ be an irreducible divisor with function field $F_D$. The local ring $A = \mathscr{O}_{S,D}$ (which is $\mathscr{O}_{S,x}$, where $x \in D$ is the generic point), is a discrete valuation ring. We denote by $v_D$ the corresponding valuation on $F_S$. For each $x \in F_S$, there are only finitely many irreducible divisors $D$ such that $v_D(x)$ is non-zero. Thus, we may form the map,

$$T = \oplus T_{v_D} : \mathsf{K}_2(F_S) \to \oplus_D F_D^*$$

where the sum is over all irreducible divisors $D$ on $S$. Viewing $\mathsf{K}_2(F_S)$ as a constant sheaf on $S$ and $F_D^*$ as a constant sheaf on $S$ supported on $D$, we may view the map $T$ above as a map of sheaves on $S$.

**Proposition 11.1.1.** *Let $\mathscr{C}_2$ be the cokernel of the map of sheaves $T : \mathsf{K}_2(F_S) \to \oplus_D F_D^*$. Then there exists an exact sequence of sheaves on $S$,*

$$0 \to \mathsf{K}_{2,S} \to \mathsf{K}_2(F_S) \xrightarrow{T} \oplus_D F_D^* \to \mathscr{C}_2 \to 0$$

*where $T$ is the sheafified tame symbol introduced above.*

**Remark 11.1.2.** Let $S \in \mathsf{Sch}_k$ be a regular scheme of finite type. Denote by $S^{(j)}$ the set of codimension $j$ points of $S$. For $x \in S^{(j)}$, we denote by $i_x : \{x\}^- \hookrightarrow S$ the corresponding closed embedding. Denote the sheaves

$$G_n^j := \oplus_{x \in S^{(j)}} (i_x)_* \mathsf{K}_{n-i}(k(x)),$$

where $k(x)$ is the function field of $\{x\}^-$, we know that

$$0 \to \mathsf{K}_{n,S} \to G_n^*$$

is a flasque resolution called the *Gersten resolution*. In the case when $n = 2$, and $S$ is integral, we are reduced to situation above.

**(11.2)** Let $S \in \mathsf{Sch}_k$ be smooth, and let $X \subset S$ be an irreducible divisor. Then following Bloch [**?**], we may define a map

$$\mathscr{B} : \mathrm{Pic}(X) \to H_X^2(S, \mathsf{K}_{2,S}).$$

In the notation of [**?**, Theorem 5.11], this map is called $a$ and it is also shown that in [**?**, Theorem 5.11] that this map is injective. We shall need this injectivity in the proof of Theorem 14.5.2.

Let $\mathscr{C}_{1,X}$ be the sheaf of Cartier divisors on $X$. It fits into the exact sequence of sheaves on $X$,

$$0 \to \mathscr{O}_X^* \to F_X^* \to \mathscr{C}_{1,X} \to 0,$$

where $F_X^*$ is the constant sheaf on $X$. Taking global sections we have that

$$F_X^* \to \Gamma(\mathscr{C}_{1,X}) \to Pic(X) \to 0$$

since $H^1(X, F_X^*) = 0$.

Let $j : X \hookrightarrow S$ be the closed embedding. Then we may define a map of sheaves on $S$

$$\Psi : j_* \mathscr{C}_{1,X} \to \underline{H}_X^0(S, \mathscr{C}_2)$$

on the level of presheaves as follows: let $V \subset S$, and write $U = V \cap X$ and choose $(U, f) \in \mathscr{C}_{1,X}(U)$, where $f \in F_X^* (\bmod\ \mathscr{O}_X^*)$. Denote by $f|_X \in \oplus_D F_D^*$ the element whose component in $F_D^*$ is 0 unless $D = X$, in which case the component is $f$. The presheaf map $\Psi$ defined by

$$(V, f) \mapsto (V, f|_X (\bmod\ \mathsf{im}(T))$$

is well-defined. Hence it defines a map of sheaves, which is again denoted by $\Psi$. Taking global sections, we obtain a map, also denoted $\Psi$,

$$\Psi : \Gamma(X, \mathscr{C}_{1,X}) \to H^0_X(S, \mathscr{C}_2).$$

The Gersten complex may be split into two short exact sequences as follows,

$$\begin{aligned} 0 &\to \mathsf{K}_{2,S} \to \mathsf{K}_2(F_S) \to \mathsf{im}(T) \to 0 \\ 0 &\to \mathsf{im}(T) \to \oplus_D F_D^* \to \mathscr{C}_2 \to 0 \end{aligned}$$

Having done so, we may apply the functor $\Gamma_X$, and taking the associated long exact sequence on cohomology, we obtain a map $\delta$ as the composition,

$$\delta : H^0_X(S, \mathscr{C}_2) \xrightarrow{\delta_0} H^1_X(S, \mathsf{im}(T)) \xrightarrow{\delta_1} H^2_X(S, \mathsf{K}_2).$$

Bloch's map $\mathscr{B}$ is then defined through the following diagram,

$$\begin{array}{ccccccc} F_X^* & \longrightarrow & H^0_X(S, \mathscr{C}_2) & \xrightarrow{\delta} & H^2_X(S, \mathsf{K}_2) & \longrightarrow & 0 \\ \Big\downarrow{=} & & \Psi\Big\uparrow & & \mathscr{B}\Big\uparrow & & \\ F_X^* & \longrightarrow & H^0(S, \mathscr{C}_{1,X}) & \longrightarrow & Pic(X) & \longrightarrow & 0 \end{array} .$$

**(11.3)** In what follows, we need the following explict form of the map $\mathscr{B}$. Let $\mathscr{L} \in \mathrm{Pic}(X)$, and choose an open cover $\{V_i \to S\}$ such that $U_i = V_i \cap X$ is a cover of $X$ such that we may choose $\{(U_i, f_i)\} \in \Gamma(X, \mathscr{C}_{1,X})$ representing $\mathscr{L}$. Then we may explicitly describe a Čech representative of $\mathscr{B}(\mathscr{L})$ as follows: Choose lifts $\widetilde{f_{ij}}$ of the elements $f_i/f_j \in \mathscr{O}_X^* \subset F_X^*$ under the map $T_{v_X} : \mathsf{K}_2(F_S) \to F_X^*$. Then the desired cocycle representative of $\mathscr{B}(\mathscr{L})$ is given on the open set $V_{ijk} = V_i \cap V_j \cap V_k$ by

$$\widetilde{f_{ij}}\widetilde{f_{jk}}\widetilde{f_{ik}}^{-1}.$$

To see this, we need only trace through the sequence of maps defining $\mathscr{B}$,

$$\begin{aligned} \Gamma(X, \mathscr{C}_{1,X}) \to \Gamma_X(S, \mathscr{C}_2) &\to H^1_X(S, \mathsf{im}(T)) \to H^2_X(S, \mathsf{K}_{2,S}) \\ f_i \mapsto f_i \,(\mathrm{mod}\ \mathsf{im}(T)) &\mapsto f_i f_j^{-1} = T(\widetilde{f_{ij}}) \mapsto \widetilde{f_{ij}}\widetilde{f_{jk}}\widetilde{f_{ik}}^{-1} \end{aligned}$$

## 12. ADELEIC LOOP GROUPS AND $G$-BUNDLES ON A PUNCTURED SURFACE

**(12.1)** Let $X \in \mathsf{Sch}_k$ be a smooth, irreducible, projective curve with function field $F = k(X)$. Then let

$$X[t, t^{-1}] = X \times_k (\mathbb{A}^1_k \setminus \{0\})$$

be the punctured affine line over $X$. Let $G$ be a sheaf of groups, assumed to be a simply-connected Chevalley group. Let $P$ be a $G$-torsor on $X[t,t^{-1}]$. We say that $P$ is *rationally trivial* if there exists an open subset $U \subset X$ and an isomorphism between

$$\sigma : P|_{U[t,t^{-1}]} \cong U[t,t^{-1}] \times G.$$

This is equivalent to the existence of a section $s_\sigma \in P(F[t,t^{-1}])$. Similarly, we say that $P$ is *locally trivial at $x \in X$* if there exists an isomorphism

$$\tau_x : P|_{\mathscr{O}_x[t,t^{-1}]} \cong \mathscr{O}_x[t,t^{-1}] \times G.$$

This is equivalent to the existence of a section $s_{\tau_x} \in P(\mathscr{O}_x[t,t^{-1}])$.

**Definition 12.1.1.** We call $P$ *admissible* if it admits both a rational trivialization and a local trivialization for each $x \in X$, and denote the set of isomorphism classes of admissible $G$-bundles on $X[t,t^{-1}]$ by $\mathsf{Bun}_G^{adm}(X[t,t^{-1}])$.

A natural question now is to determine how large is the class of admissible $G$-bundles. Along these lines, we have the following,

**Proposition 12.1.2.** *Let $P \in \mathsf{Bun}_G(X[t,t^{-1}])$ be a $G$-bundle which can be extended to a $G$-bundle on $X[t]$. Then the $P$ is admissible.*

*Proof.* The statement is local, and so it suffices to prove the following local statement: Let $R$ be a domain, and $M$ a projective module over $R[t]$. Then there exists $f \in R$ such that $M_f = M \otimes_{R[t]} R_f[t]$ is free.

Let $R[t] \to R$ be the map $t \mapsto 0$ which corresponds to the closed embedding, $i : \mathrm{Spec}(R) \to \mathrm{Spec}(R[t])$. Since $M$ is projective, so is $i^*M$, and so there exists $f \in R$ such that $(M \otimes_{R[t]} R) \otimes_R R_f \cong M \otimes_{R[t]} R_f$ is free. We would like to show that $M \otimes_{R[t]} R_f[t]$ is then free. In other words, we are reduced to the following claim: *Let $A$ be a domain, and $N$ a projective $A[t]$-module such that $N \otimes_{A[t]} A$ is free. Then $N$ is a free $A[t]$-module.*

To prove this last statement, fix an embedding $N \subset A[t]^n$ for some $n$. Then, choose non-zero elements $n_1, \cdots, n_r \in N$, and suppose there is some relation

$$a_1(t)n_1 + \cdots + a_r(t)n_r = 0$$

in $N$, where each $a_i(t)$ is of minimal degree. Then the same relation persists in $N \otimes_{A[t]} A$, and so we must have

$$a_1(0)n_1 + \cdots + a_r(0)n_r = 0.$$

Since $N \otimes_{A[t]} A$ was assumed to be free, we can assume that $t$ divides $a_1(t), \ldots, a_r(t)$, and so we may write our original relation as,

$$t(a_1'(t)n_1 + \cdots + a_r'(t)n_r) = 0.$$

Since $N \subset A[t]^n$, and $A$ was assumed to be a domain, we must have that

$$a'_1(t)n_1 + \cdots + a'_r(t)n_r = 0,$$

contradicting the assumption that $a_i(t)$ were of minimal degree.

□

**(12.2)** Recall that we have defined a non-centrally extended loop group $\overline{G}$ in section Chapter 2 §3.6. In this section, we shall consider a polynomial version of the same group and retain the same notation. So for example, the group $\overline{G}$ shall now refer to the functor which locally is defined on a domain $R$ by,

$$R \mapsto \overline{G}(R) := G(R[t,t^{-1}]).$$

Similarly, we define $\overline{K}, \overline{B}, \overline{H}$, and $\overline{U}$. We denote by $\overline{G}_{\mathbb{A}}$ the adelization of this group as in Chapter 2, §4.2 with respect to the group $\overline{K}_{\mathbb{A}}$. Note that in this polynomial setting, $\overline{\Gamma}_F = \overline{G}_F$.

Generalizing A. Weil's observation (see Chapter 1, §1.2), we have

**Proposition 12.2.1.** *There is a bijective correspondence between the set of isomorphism classes of admissible G-bundles* $\mathsf{Bun}_G^{adm}(X[t,t^{-1}])$ *and the double coset space*

$$\overline{K}_{\mathbb{A}} \setminus \overline{G}_{\mathbb{A}} / \overline{G}_F.$$

*Proof.* The proof is very similar to the corresponding result in the finite dimensional case, so we just sketch the details here.

*Step 1:* Given an admissible bundle $(P, \sigma, \{\tau_x\})$, we obtain an element of $\overline{G}_{\mathbb{A}}$ as follows. From $\sigma$ we obtain an element $s \in P(F[t,t^{-1}])$ and hence an element $s_x \in P(F_x[t,t^{-1}])$ for each point $x \in X$. From $\tau$ we obtain, for each $x \in X$ an element of $t_x \in P(\mathscr{O}_x[t,t^{-1}])$ and hence also an element $t_x \in P(F_x[t,t^{-1}])$. We can then find an element $g_x \in \overline{G}(F_x)$ such that

$$s_x = t_x g_x.$$

The collection $(g_x) \in \overline{G}_{\mathbb{A}}$ as is easily verified.

*Step 2:* Conversely, given an element $g \in \overline{G}_{\mathbb{A}}$, we can obtain a Čech representative for a $G$-bundle on $X[t,t^{-1}]$. To do so, we shall need the following approximation theorem: for any proper open set $U \subset X$, we denote by

$$\overline{K}_U := \prod_{x \in U} \overline{K}_v.$$

Then using Chapter 2 Proposition 9.1.1 and an easy induction, we can show that for any such $U$,

$$\overline{G}_{\mathbb{A}_U} = \overline{K}_U \overline{G}_F.$$

Hence, if we are given an open cover cover $\{U_i \to X\}$, we obtain factorizations

$$\overline{G}_{\mathbb{A}_{U_i}} = \overline{K}_{U_i}\overline{G}_F.$$

So, for each $g \in \overline{G}_{\mathbb{A}}$, we may write

$$g = k_i\beta_i = k_j\beta_j$$

where $\beta_i, \beta_j \in \overline{G}_F$ and $k_i \in \overline{K}_{U_i}$ and $k_j \in \overline{K}_{U_j}$. Then the elements,

$$g_{ij} := \beta_j\beta_i^{-1} = k_j^{-1}k_i \in G(U_i \cap U_j)[t, t^{-1}]$$

as they are both in $\overline{G}_F$ and $\overline{K}_{U_i \cap U_j}$. The transition functions for the bundle constructed from $g$ are then $g_{ij}$.

*Step 3:* The two constructions above are clearly inverses to one another. For example: start with an $g \in \overline{G}_{\mathbb{A}}$ and construct a $G$-bundle by Step 2 called $P_g$. Then $P_g$ is equipped with rational trivializations $\beta_i$ (with notation as in Step 2) and local trivializations obtained from $k_i$. But at any point $x \in X$, we have

$$g_x = (\beta_i)_x(k_i)_x$$

where $(\beta_i)_x$ is the image of $\beta_i$ in $\overline{G}(F_x)$ and similarly for $(k_i)_x$. But this just means that the element of $\overline{G}_{\mathbb{A}}$ obtained from Step 1 can be chosen to be $(g_x)$. □

**(12.3)** Let $L$ be a line bundle on $X$. We may then form the surface which is the total space of $L$,

$$p : S_L := \underline{\mathrm{Spec}}(L) \to X.$$

We have a closed embedding corresponding to the zero section $z : X \to S_L$. We denote the punctured surface

$$S_L^o := S_L \setminus z(X).$$

Recalling the isomorphism

$$\mathrm{Pic}(X) = \mathbb{I}/(F^* + \prod_{v \in X} \mathscr{O}_v^*) \tag{12.3.1}$$

we now have the following easy consequence of [Ful98, p. 67, 3.3.2]

**Proposition 12.3.1.** *Let $s_L \in \mathbb{I}$ denote an idele representing $L \in \mathrm{Pic}(X)$. Then the self-intersection of L in $S_L$ is given by*

$$(L.L) = |s_L|_{\mathbb{I}}$$

We also have the notion of rationally trivial, locally trivial, and admissible $G$-bundles on $S_L^o$. We denote by $\mathsf{Bun}_G(S_L^o)$ the set of isomorphism classes of admissible $G$-bundles on $S_L^o$. The same arguments as in Prop 12.1.2 give,

**Proposition 12.3.2.** *Let $P \in \mathsf{Bun}_G(S_L^o)$ be a G-bundle which can be extended to a G-bundle on $S_L$. Then the P is admissible.*

**(12.4)** Our next task will be to describe admissible $G$-bundles on $S_L^o$ in group theoretical terms. Usually, we think of $G$-bundles on $S_L^o$ as equivalent to Čech 1-cocyles in $H^1(S_L^o, G)$. However, the admissibility condition allows us to view $G$-bundles on the surface $S_L^o$ as (twisted) loop group bundles on the curve $X$. This is of course implicit in the Proposition 12.2.1, and we formalize it and extend it to the case of the surface $S_L^o$ as follows: As in Chapter 2 §1.4 , we have defined the action of the group $\mathbb{G}_m$ on $\overline{G}$ by "rotation of the loop." Given $r \in R^*$ and $g(t) \in \overline{G}(R)$, we have an action $\rho : \mathbb{G}_m \to \mathrm{Aut}(\overline{G})$ defined by

$$\rho(r).g(t) = g(rt).$$

Thus we form the semi-direct product group $\overline{G} \ltimes \mathbb{G}_m$. The following proposition is then essentially a twisted restatement of the simple "law of exponents,"

$$A^{B\times C} = (A^B)^C$$

where $A, B, C$ are spaces and $X^Y$ denotes the space of maps $Y \to X$.

**Proposition 12.4.1.** *There exists a morphism of stacks*

$$\mathsf{TORS}(S_L^o, G) \to \mathsf{TORS}(X, \overline{G} \ltimes \mathbb{G}_m)$$

*Proof.* Fix an open cover $\{U_i \hookrightarrow X\}$. This gives a covering $\{V_i := U_i[t, t^{-1}] \hookrightarrow S_L^o\}$. By descent, the surface $S_L^o$ is equivalent to the affine pieces $V_i$ equipped with the gluing map,

$$\begin{array}{rcl} \lambda_{ij} : V_i \cap V_j & \to & V_j \cap V_i \\ (U_i \cap U_j)[t, t^{-1}] & \to & (U_i \cap U_j)[t, t^{-1}] \\ f(t) & \mapsto & f(\lambda_{ij} t) \end{array}$$

where $\{\lambda_{ij}\}$ is a Čech cocycle for $L$. Note that the map $\lambda_{ij}$ also induces a map which we denote by the same symbol,

$$\lambda_{ij} : G(V_i \cap V_j) \to G(V_i \cap V_j).$$

For a given $P \in \mathsf{TORS}(S_L^o, G)$, assume that $P$ trivializes over such an open set. Then $P$ is equivalent to the its descent data with respect to the cover $V_i$, i.e., trivializations $P|_{V_i} \xrightarrow{\phi_i} G(V_i)$, which induce maps $\phi_{ij}$ fitting into the following commutative diagram,

$$\begin{array}{ccc} P|_{V_i \cap V_j} & \xrightarrow{\phi_{ij}} & P|_{V_i \cap V_j} \\ \downarrow \phi_i & & \downarrow \phi_j \\ G(V_i \cap V_j) & \xrightarrow{\lambda_{ij}} & G(V_i \cap V_j) \\ \downarrow = & & \downarrow = \\ G(U_{ij}[t,t^{-1}]) & \xrightarrow{\lambda_{ij}} & G(U_{ij}[t,t^{-1}]) \end{array}$$

and satisfying the condition,

$$\phi_{ij} \circ \phi_{jk} = \phi_{ik}.$$

Unraveling this last condition, we find that the $\phi_{ij}$ may be viewed as a Čech cocyle with values in the twisted loop group,

$$\phi_{ij} \in H^1(\{U_i\}, \overline{G} \ltimes \mathbb{G}_m).$$

This construction works not just on the level of isomorphism classes, but on actual torsors so we obtain an element of $\mathsf{TORS}(X, \overline{G} \ltimes \mathbb{G}_m)$. □

**(12.5)** We use the identification of line bundles with ideles 12.3.1 and the the same arguments as in Prop 12.2.1 to show,

**Proposition 12.5.1.** *Let $L$ be a line bundle on $X$ which is represented by an element $\tau \in \mathbb{I}_F$. Then there is a bijective correspondence between the set of isomorphism classes $\mathsf{Bun}_G^{adm}(S_L^o)$ and the double coset space,*

$$\overline{K}_{\mathbb{A}} \setminus \overline{G}_{\mathbb{A}} \eta(\tau) / \overline{G}_F.$$

## 13. Affine flag varieties and extensions of $G$-bundles

**(13.1)** Let us keep the notations of the previous section. Let $P^o$ be an admissible $G$-bundle on $S_L^o$. Let $B \subset G$ be a Borel subgroup.

**Definition 13.1.1.** Let $(P, \phi, r)$ be a triple where $P$ is a $G$-bundle on $S_L$ equipped with an isomorphism $P|_{S_L^o} \xrightarrow{\phi} P^o$ and a reduction $r$ of the structure group of $P$ to $B$ along $X$. There is an obvious notion of isomorphism of such triples $(P, \phi, r)$ and we denote by

$$\mathsf{Bun}_{G,B}^{P^o}(S_L)$$

the set of isomorphism classes of such triples.

The goal of this section is to establish a bijective correspondence between $\mathsf{Bun}_{G,B}^{P^o}(S_L)$ and a certain affine flag variety.

**(13.2)** In §2.4 we have seen how to view our bundle $P^o$ as an $\overline{G} \ltimes \mathbb{G}_m$ bundle on $X$. The elements of the set $\mathsf{Bun}_{G,B}^{P^o}(S_L)$ have the following description,

**Lemma 13.2.1.** *The set $\mathsf{Bun}_{G,B}^{P^o}(S_L)$ is in bijective correspondence with reductions of $\overline{G} \ltimes \mathbb{G}_m$-torsor $P^o$ to a $\overline{B} \ltimes \mathbb{G}_m$-torsor.*

Using this lemma, we can then prove the following

**Proposition 13.2.2.** *The set of extensions of $\mathsf{Bun}_{G,B}^{P^o}(S_L)$ is in bijective correspondence with the set $\overline{\Gamma}_F/(\overline{B}_F \cap \overline{\Gamma}_F)$.*

*Proof.* By Chapter 1, Proposition 3.1.1, we know that reductions of $P^o$ to an $\overline{B} \ltimes \mathbb{G}_m$-torsor are in bijective correspondence with sections

$$H^0(X, P^o/\overline{B} \ltimes \mathbb{G}_m).$$

But, locally over $X$ we know that $P^o/\overline{B} \ltimes \mathbb{G}_m$ is just the affine flag variety, which we know has an ind-proper scheme structure. As $P^o$ is assumed to be rationally trivial, we have

$$\mathrm{Hom}(\mathrm{Spec}(F), P^o/\overline{B} \ltimes \mathbb{G}_m) = \overline{\Gamma}_F/(\overline{B}_F \cap \overline{\Gamma}_F).$$

This concludes the proof. ☐

**(13.3)** *Explicit reduction:* We can make the reduction procedure described in the last paragraph explicit, on the level of cocycles, as follows. Let $\gamma \in \overline{\Gamma}_F$ be a representative for an element of $\overline{\Gamma}_F/(\overline{B}_F \cap \overline{\Gamma}_F)$. Analogously to Step 2 of Proposition 2.2.1, we can associate to each

$$x = g\eta(\tau) \in \overline{G}_{\mathbb{A}}\eta(\tau) \;\text{ for } \tau \in \mathbb{I}_F$$

a cocycle representative $\{x_{ji}\}$ for a bundle $P^o_x$ on the surface associated to the line bundle corresponding to $\tau$. This is achieved by using the factorization over an open set $U \subset X$ of the element $x$ as

$$x = k_U \eta(\kappa_U) \beta_U \eta(\iota_U)$$

where $\beta_U \in \overline{G}_F, \iota_U \in F, k_U \in \overline{K}_U$, and $\kappa_U \in \prod_{v \in U} \mathscr{O}_v^*$. Now, given a covering $\{U_i \to X\}$, the bundle $P^o_x$ is represented by the cocyle,

$$x_{ji} = \eta(\kappa_{U_j})^{-1} k_{U_j}^{-1} k_{U_i} \eta(\kappa_{U_i}) \tag{13.3.1}$$

$$= \beta_{U_j} \eta(\iota_{U_j}) \eta(\iota_{U_i})^{-1} \beta_{U_i}^{-1} \in H^1(\{U_i\}, \overline{G} \ltimes \mathbb{G}_m). \tag{13.3.2}$$

To obtain a reduction of the $\{x_{ji}\}$, we first begin with the following simple,

**Claim 13.3.1.** *Let $\gamma, x$ as above. Then for any open set $U \subset X$, we can write,*

$$x\gamma = k_U r_U \eta(\kappa_U) b_U \eta(\iota_U)$$

*where $b_U \in \overline{B}_{\mathbb{A}_U}$, $r_U \in \overline{K}_U$, and $\iota_U, \kappa_U$ are as above.*

*Proof.* Indeed, we have,

$$\begin{aligned} x\gamma &= k_U \eta(\kappa_U) \beta_U \eta(\iota_U)\gamma \\ &= k_U \eta(\kappa_U) \beta_U (\eta(\iota_U)\gamma' \eta(\iota_U)^{-1}) \eta(\iota_U) \\ &= k_U \eta(\kappa_U) r'_U b_U \eta(\iota_U) \\ &= k_U r_U \eta(\kappa_U) b_U \eta(\iota_U) \end{aligned}$$

where $r'_U \in \overline{K}_U$. In the third line we have use the Iwasawa decomposition $\overline{G}_{\mathbb{A}} = \overline{K}_{\mathbb{A}} \overline{B}_{\mathbb{A}}$, and we have also used that $\eta(\iota_U)$ normalizes $\overline{G}_F$ and $\eta(\kappa_U)$ normalizes $\overline{K}_U$. □

Now, suppose we are given open sets $U_i, U_j \subset X$. Abbreviate $b_{U_i}$ by $b_i$ etc. Then we have,

$$k_i r_i \eta(\kappa_i) b_i \eta(\iota_i) = k_j r_j \eta(\kappa_j) b_j \eta(\iota_j)$$

and so,

$$\begin{aligned} b_j \eta(\iota_j) b_i^{-1} \eta(\iota_i)^{-1} &= \eta(\kappa_j)^{-1} r_j^{-1} k_j^{-1} k_i r_i \eta(\kappa_i) \\ &= \eta(\kappa_j)^{-1} r_j^{-1} \eta(\kappa_j) \eta(\kappa_j)^{-1} k_j^{-1} k_i \eta(\kappa_i) \eta(\kappa_i)^{-1} r_i \eta(\kappa_i) \\ &= R_j^{-1} \eta(\kappa_j)^{-1} k_j^{-1} k_i \eta(\kappa_i) R_i \\ &= R_j^{-1} x_{ji} R_i \end{aligned}$$

where $R_i = \eta(\kappa_i)^{-1} r_i \eta(\kappa_i) \in \overline{K}_{U_i}$, $R_j^{-1} = \eta(\kappa_j)^{-1} r_j^{-1} \eta(\kappa_j) \in \overline{K}_{U_j}$, and where $x_{ji}$ are the transition functions representing $P^o$ as above. A simple computation shows further that

$$b_{ji} := b_j \eta(\iota_j) b_i^{-1} \eta(\iota_i)^{-1} \in \overline{B}(U_i \cap U_j) \ltimes \mathbb{G}_m(U_i \cap U_j).$$

In summary, given $x \in \overline{G}_{\mathbb{A}} \eta(\tau)$ and $\gamma \in \overline{\Gamma}_F$, we have constructed elements $\{x_{ji}\} \in H^1(X, \overline{G} \ltimes \mathbb{G}_m)$ and $\{R_i\} \in \overline{G}(U_i)$ such that

$$R_j^{-1} x_{ji} R_i \in H^1(X, \overline{B} \ltimes \mathbb{G}_m) \tag{13.3.3}$$

represents a reduction of $\{x_{ji}\}$.

**Remark 13.3.2.** Starting with $x \in \overline{G}_{\mathbb{A}} \eta(\tau)$ and $\gamma \in \overline{G}_F$, we have constructed an explicit cocycle $\{x_{ji}\}$ for a $G$-bundle on $S^o_\tau$ as well as a cocycle $\{b_{ji}\}$ for extension of this bundle to $S_\tau$. The cocycle $\{x_{ji}\}$ constructed above can also be interpreted (using the law of exponents, see Proposition 2.4.1) as encoding a $\overline{G} \ltimes \mathbb{G}_m$-torsor on $X$ and the cocycle $\{b_{ji}\}$ then represents a reduction of this torsor to a $\overline{B} \ltimes \mathbb{G}_m$-torsor on $X$. On the other hand, exactly the same group-theoretical procedure can be carried out in the centrally extended groups, i.e. to elements $\widehat{x} \in \widehat{G}_{\mathbb{A}}\eta(\tau)$ and $\widehat{\gamma} \in \widehat{\Gamma}_F$, we can construct cocycles $\widehat{x}_{ji}$ and $\widehat{b}_{ji}$. These cocycles then represent $\widehat{G} \ltimes \mathbb{G}_m$-bundles on $X$ and a reduction of this bundle to a $\widehat{B} \ltimes \mathbb{G}_m$-bundle, i.e., $\widehat{x}_{ji} \in H^1(X, \widehat{G} \ltimes \mathbb{G}_m)$ and $\widehat{b}_{ji} \in H^1(X, \widehat{B} \ltimes \mathbb{G}_m)$. Furthermore, there exist elements $\widehat{R}_j$ as in 13.3.3 which satisfy,

$$\widehat{R}_j^{-1} \widehat{x}_{ji} \widehat{R}_i = \widehat{b}_{ji}.$$

The reduction of $\widehat{x}_{ij}$ and $\widehat{b}_{ji}$ coincide with $x_{ji}$ and $b_{ji}$, the cocycles corresponding to the reductions of $\widehat{x}$ and $\widehat{\gamma}$ modulo the center. To simplify notation, we shall often drop the hat from the notation of an element in the central extension, and continue to denote an element of $\widehat{G}\eta(\tau)$ by $x$ when no confusion shall arise. The cocyles corresponding to $x$ will still have a hat over them though.

## 14. Relative Chern classes and Central extensions

**(14.1)** Let $X$ be a projective curve contained in a regular surface $S$ (we do not assume that $S$ is projective). Let $P^o$ be a $G$-bundle on $S^o$ and $P_1$ and $P_2$ be two extensions of $P^o$ to a $G$-bundle on $S$. Recall that we have a central extension of sheaves of groups on $S$,

$$\mathsf{K}_2 \to \widetilde{G} \to G.$$

Thus, for each $G$-bundle $P_i$, we obtain a gerbe of liftings, $\mathsf{Lift}_{P_i}^{\mathsf{K}_2}$. Since both $P_i$ are isomorphic on $S^o$, we know that the difference gerbe

$$\mathsf{Lift}_{P_1,P_2}^{\mathsf{K}_2} := \mathsf{Lift}_{P_1}^{\mathsf{K}_2} - \mathsf{Lift}_{P_2}^{\mathsf{K}_2}$$

has a section on $S^o$. Hence it defines a class,

$$\mathsf{Lift}_{P_1,P_2}^{\mathsf{K}_2} \in H^2_X(S, \mathsf{K}_2).$$

**Remark 14.1.1.** In the case when $P_i$ are vector bundles, then the image of the class $\mathsf{Lift}_{P_1,P_2}^{\mathsf{K}_2}$ under the map $H^2_X(S,\mathsf{K}_2) \to CH^2(S)$ is equal to $c_2(P_1) - c_2(P_2)$, the difference of the second Chern classes (see [**?**]).

**(14.2)** The Leray map

$$H^2_X(S, \mathsf{K}_2) \xrightarrow{f_*} H^1(X, \underline{H}^1(S, \mathsf{K}_2))$$

can be interpreted in terms of a pushforward of gerbes (see Chapter 1, §3.4: since the local cohomology group $H^2_X(S, \mathsf{K}_2) = 0$, this pushforward of gerbes returns a torsor). By the Gersten resolution, we also have an identification $\underline{H}^1_X(S,\mathsf{K}_2) \cong \mathscr{O}^*_X$. Hence, we obtain an $\mathscr{O}^*_X$-torsor which we denote by $\mathscr{C}_2(P_1,P_2)$ on $X$ by pushing forward $\mathsf{Lift}^{\mathsf{K}_2}_{P_1,P_2}$ :

$$\mathscr{C}_2(P_1,P_2) := f_*(\mathsf{Lift}^{\mathsf{K}_2}_{P_1,P_2}).$$

Recall here the conventions of Chapter 1, §3.4: $f_*(\mathsf{Lift}^{\mathsf{K}_2}_{P_1,P_2})$ is not the underlying pushforward stack, but rather the torsor of maximal subgerbes of this pushforward stack.

**Definition 14.2.1.** Let $P_1,P_2$ as above. Then we shall define the *$P_2$-relative K-theoretic Chern class of $P_1$*, or just relative second Chern class for short, to be the degree of the line bundle $\mathscr{C}_2(P_1,P_2)$. We write this as $c_2^{P_1}(P_2)$.

By its very construction, the Bloch map can be seen to be the inverse of the Leray map $f_*$ (see [Blo74, 5.11]). In other words, we have,

**Proposition 14.2.2.** *Under the Bloch map $\mathscr{B}:\mathrm{Pic}(X) \to H^2_X(S,\mathsf{K}_2)$ we have an equality in $H^2_X(S,\mathsf{K}_2)$,*

$$\mathscr{B}(\mathscr{C}_2(P_1,P_2)) = \mathsf{Lift}^{\mathsf{K}_2}_{P_1,P_2}$$

**(14.3)** Our next goal will be to identify the isomorphism class of the line bundle $\mathscr{C}_2(P_1,P_2)$ on $X$ in group theoretical terms. We begin with a general construction $\mathbf{c}:\widehat{G}_{\mathbb{A}}\eta(\tau) \to \mathrm{Pic}(X)$ which associates to any element

$$x \in \widehat{G}_{\mathbb{A}_F}\eta(\tau) \rightsquigarrow \mathbf{c}(x) \in \mathbb{I}_F$$

whose image in $\mathrm{Pic}(X)$ is well-defined. The procedure is as follows: given such an $x$, we may write it using the Iwasawa decomposition as

$$x = ka\eta(\tau)u$$

where

$$a = \prod_{i=1}^{l+1} h_i(\sigma_i) \in \widehat{H}_{\mathbb{A}} \text{ with } \sigma_i \in \mathbb{I}_F.$$

As mentioned before, the $\sigma_i$ are not unique, but their image in the idele class group is unique.

**Lemma 14.3.1.** *Let $\alpha_0 = \sum_{i=1}^{l} n_i h_i$ be the longest root. For every $x \in \mathbb{I}_F$, denote by*

$$h_c(x) = h_{l+1}(x)\prod_{i=1}^{l} h_i(x^{n_i}).$$

*Then every element $h \in \widehat{H}_{\mathbb{A}}$ has an expression*

$$h = h_c(\sigma)\prod_{i=1}^{l} h_i(\sigma_i) \text{ for } \sigma, \sigma_i \in \mathbb{I}_F.$$

*Moreover, the idele class of $\sigma$ is uniquely defined.*

*Proof.* Every such element $h$ has an expression

$$h = \prod_{i=1}^{l+1} h_i(\tau_i).$$

We choose $\sigma = \tau_{l+1}$ and $\tau_i = \tau_i\sigma^{-n_i}$. Uniqueness of the idele class follows from the uniqueness of the idele classes of each $\tau_i$. □

*Construction:* To each $x \in \widehat{G}_{\mathbb{A}_F}\eta(\tau)$, write $x = ua\eta(\tau)k$ as above. Then using Lemma 14.3.1 we can associate to $a \in \widehat{H}_{\mathbb{A}}$ an idele $\sigma \in \mathbb{I}_F$ such that

$$a = h_c(\sigma)\prod_{i=1}^{l} h_i(\sigma_i).$$

The image of $\sigma$ in $\mathrm{Pic}(X)$ only depends on $x$ and will be denoted $\mathbf{c}(x)$.

**(14.4)** *Alternative construction of* $\mathbf{c}(x)$: The idele class $\mathbf{c}(x)$ above defines a line bundle on $X$, whose Čech representative can be obtained as follows. The central extension $\widehat{G} \ltimes \mathbb{G}_m$ splits over the subgroup $\overline{B} \ltimes \mathbb{G}_m \subset \overline{G} \ltimes \mathbb{G}_m$. Let us denote the section by $\Phi : \overline{B} \ltimes \mathbb{G}_m \to \widehat{G} \ltimes \mathbb{G}_m$. Fix a cover $\{U_i = \mathrm{Spec}(R_i) \hookrightarrow X\}$ (We may have to refine it below, but this can be done by Chapter 1 Lemma 4.1.1) Then given an element $x \in \widehat{G}_{\mathbb{A}}\eta(\tau)$, we can construct a cocycle $\widehat{b}_{ij} \in \widehat{\mathsf{B}}(R_{ij}) \ltimes R_{ij}^*$ as in 13.3.3. Denote by $b_{ji}$ the image of $\widehat{b_{ji}}$ under the map,

$$\widehat{G} \ltimes \mathbb{G}_m \to \overline{G} \ltimes \mathbb{G}_m.$$

**Lemma 14.4.1.** *The line bundle $\mathbf{c}(x)$ is isomorphic to the line bundle with transition functions*

$$\lambda_{ij} := \widehat{b}_{ij}^{-1}\Phi(b_{ij}).$$

**Remark 14.4.2.** If $R \hookrightarrow S$ is a map of domains then we have the following commutative diagram,

$$\begin{array}{ccc} G(R[t]) \ltimes R^* & \xrightarrow{\Phi_R} & \widehat{G}(R) \ltimes R^* \\ \downarrow & & \downarrow \\ G(S[t]) \ltimes S^* & \xrightarrow{\Phi_S} & \widehat{G}(S) \ltimes S^* \end{array},$$

where both vertical arrows are injective. Hence, viewing $\widehat{b}_{ji} \in \widehat{G}(F) \ltimes F^*$, we may equivalently write

$$\lambda_{ji} = \widehat{b}_{ji}^{-1}\Phi_F(b_{ji}).$$

**Remark 14.4.3.** The section $\Phi_F$ above is actually uniquely defined. This follows from the following simple observation: let $A \to E \xrightarrow{\pi} G$ is a central extension of groups such that $G = [G,G]$. If $s,t$ are two sections of $\pi$, then $s = t$. Indeed, let $x \in G$ be written as $x = [a,b]$. Then $s(a) = t(a)\alpha$ and $s(b) = t(b)\beta$ where $\alpha, \beta \in A$. But then,

$$s(x) = [s(a), s(b)] = [t(a)\alpha, t(b)\beta] = [t(a), t(b)]$$

since $\alpha, \beta$ are central. The fact that $\overline{B}_F \ltimes F^*$ is equal to its commutator group follows from an argument similar to [Ste67, Lemma 32'].

**(14.5)** Let $\tau \in \mathbb{I}$ be such that $0 < |\tau|_{\mathbb{I}} < 1$. Let $L_\tau$ be the corresponding line bundle on $X$ and denote by $S_\tau$ and $S_\tau^o$ the total surface and punctured total surface attached to $L$. Pick $x \in \widehat{G}_{\mathbb{A}}\eta(\tau)$. By Chapter 2 Theorem 9.2.2, we can choose a polynomial representative for the coset of $x$ modulo left translation by $\widehat{\mathsf{K}}_{\mathbb{A}}$ and right translation by $\widehat{\Gamma}_F$. Fix one such polynomial representative and continue to denote it by $x$. Then, the reduction of $x$ modulo the center gives an element of $\overline{K}_{\mathbb{A}} \setminus \overline{G}_{\mathbb{A}}\eta(\tau)/\overline{\Gamma}_F$. By Proposition 2.5.1, this element corresponds to a $G$-torsor on $S_\tau^o$ which we shall denote by $P_x^o$.

**Remark 14.5.1.** We do not know whether this bundle depends on the polynomial representative of $x$ which was chosen. However, we shall not need this fact in what follows.

We have seen that to each $\gamma \in \widehat{\Gamma}_F$, or actually to its coset in $\widehat{\Gamma}_F/\widehat{\Gamma}_F \cap \widehat{\mathsf{B}}_F$, we can construct a well-defined reduction of $P_x^o$ to an element in $\mathsf{Bun}_{G,B}^{P_x^o}(S_\tau)$ which we shall denote by $P_{x\gamma}$. The main result of this section is then the following,

**Theorem 14.5.2.** *Let $x \in \widehat{G}_{\mathbb{A}}\eta(\tau)$ correspond to the G-bundle $P_x^o$ on $S_\tau^o$. Let $P_{x\gamma_1}$ and $P_{x\gamma_2}$ be extensions to G-bundles on $S_\tau$ corresponding to elements $\gamma_1, \gamma_2$ in $\widehat{\Gamma}_F$. Then, we have an equality in* $\mathrm{Pic}(X)$

$$(\mathbf{c}(x\gamma_1) \otimes \mathbf{c}(x\gamma_2)^{-1})^N = \mathscr{C}_2(P_{x\gamma_1}, P_{x\gamma_2})$$

*where N is the power of the tame symbol in Chapter 2, Theorem 3.7.1 (depending only on the group $\widehat{G}$).*

*Proof.* By Proposition 14.2.2 above and the injectivity of the Bloch map, it suffices to verify the equality in $H_X^2(S_\tau, \mathsf{K}_2)$,

$$\mathscr{B}((\mathbf{c}(x\gamma_1) \otimes \mathbf{c}(x\gamma_2)^{-1})^N) = \mathsf{Lift}_{P_{x\gamma_1}, P_{x\gamma_2}}^{\mathsf{K}_2}$$

*Step 1*: Let us understand the right hand side of the above equality. In other words, given a bundle $P_{x\gamma}$, let us compute its gerbe class $\mathsf{Lift}_{P_{x\gamma}}^{\mathsf{K}_2} \in H^2(S_\tau, \mathsf{K}_2)$ group theoretically. Let $\{U_i = \mathrm{Spec}(A_i)\}$ be any covering of $X$, which we may need to refine later. From section §3.3, Remark 3.3.2, we can associate to $x \in \widehat{G}_{\mathbb{A}}\eta(\tau)$ a cocycle $\widehat{x}_{ji} \in H^1(X, \widehat{G} \ltimes \mathbb{G}_m)$. The reduction of this cocycle modulo the center will be denoted by $\{x_{ji}\}$ and it represents the bundle $P_x^o$. Moreover, given $\gamma \in \widehat{\Gamma}_F$, we have also constructed elements $\widehat{R}_j$ and a cocycle $\widehat{b}_{ji} \in H^1(X, \widehat{B} \ltimes \mathbb{G}_m)$ such that

$$\widehat{b}_{ji} = \widehat{R}_j^{-1} \widehat{x}_{ji} \widehat{R}_i^{-1}.$$

The reduction of $\widehat{b}_{ji}$ modulo the center will be denoted by $\{b_{ji}\}$ and it represents the bundle $P_{x\gamma}$. The reduction of the $\widehat{R}_j$ modulo the center will be called $R_j$ and it will provide the explicit reduction of the cocycle $x_{ji}$ to $b_{ji}$.

Then (refining the cover if necessary), we may choose lifts $\widetilde{b}_{ij} \in \widetilde{G}(A_{ij}[t]) \ltimes A_{ij}^*$ of $b_{ji}$ under the map

$$\mathsf{K}_2(A_{ij}[t]) \to \widetilde{G}(A_{ij}[t]) \ltimes A_{ij}^* \to G(A[t]) \ltimes A_{ij}^*.$$

A representative for the gerbe $\mathsf{Lift}_{P_{x\gamma}}^{\mathsf{K}_2}$ is then given by

$$\widetilde{b}_{ji} \widetilde{b}_{ik} \widetilde{b}'^{-1}_{jk} \in \mathsf{K}_2(A_{jik}[t]).$$

*Step 2:* Let $F_X$ denote the function field of $X$. Then we have the following diagram of groups,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathsf{K}_2(F_X(t)) & \longrightarrow & \widetilde{G}(F_X(t)) \ltimes F_X^* & \xrightarrow{\widetilde{\pi}} & G(F_X(t)) \ltimes F_X^* & \longrightarrow & 1 \\ & & \downarrow \partial & & \downarrow \omega & & \downarrow = & & \\ 1 & \longrightarrow & F_X^* & \longrightarrow & \widehat{G}(F_X) \ltimes F_X^* & \longrightarrow & G(F_X(t)) \ltimes F_X^* & \longrightarrow & 1 \end{array}$$

where the map $\partial$ is some power of the tame symbol (see Chapter 2, Proposition 3.7.1) and the map $\omega$ is surjective. We may regard $\widehat{G}(A) \subset \widehat{G}(F_X)$ for any ring $A \subset F_X$. Thus we may choose lifts $\widetilde{R}_j \in \widetilde{G}(F_X(t))$ and $\widetilde{x}_{ji} \in \widetilde{G}(F_X(t)) \ltimes F_X^*$ of $\widehat{R}_j$ and $\widehat{x}_{ji}$ under $\omega$. Denoting by

$$\widetilde{h}_{ji} = \widetilde{R}_j \widetilde{x}_{ji} \widetilde{R}_i^{-1} \in \widetilde{G}(F_X(t)) \ltimes F_X^*,$$

we have that $\widetilde{h}_{ji}$ is a lift of $b_{ji}$ under $\widetilde{\pi}$ by commutativity of the above diagram.

*Step 3:* For every ring $A \subset F_X$, we also have a commutative diagram by base change,

$$\begin{array}{ccc} \widetilde{G}(A[t]) \ltimes A^* & \longrightarrow & G(A[t]) \ltimes A^* \\ \downarrow & & \downarrow \\ \widetilde{G}(F_X(t)) \ltimes F^* & \longrightarrow & G(F_X(t)) \ltimes F^* \end{array}$$

Let us continue to denote by $\widetilde{b}_{ji}$ the image of $\widetilde{b}_{ji}$ under the left vertical map and $b_{ji}$ for the image of $b_{ji}$ under the right vertical map. Then as $\widetilde{b}_{ji}$ is a lift of $b_{ji} \in G(F_X(t))$ under $\widetilde{\pi}$, there exist elements $\alpha_{ji} \in \mathsf{K}_2(F_X(t))$ such that

$$\widetilde{b}_{ji} = \widetilde{h}_{ji}\alpha_{ji} \in \widetilde{G}(F_X(t)) \ltimes F^*.$$

*Step 4:* For any $A \subset F_X$, the map $\mathsf{K}_2(A[t]) \to \mathsf{K}_2(F_X(t))$ is injective by the Gersten resolution. Thus, we regard computations in $\mathsf{K}_2(A[t])$ as actually occurring in $\mathsf{K}_2(F_X(t))$. So, the gerbe class of $\mathsf{Lift}^{\mathsf{K}_2}_{P_{x\gamma}}$ then has cocycle representative

$$\begin{aligned} \widetilde{b}_{ji}\widetilde{b}_{ik}\widetilde{b}_{jk}^{-1} &= (\widetilde{R}_j\widetilde{x_{ji}}\widetilde{R}_i^{-1})(\widetilde{R}_i\widetilde{x_{ik}}\widetilde{R}_k^{-1})(\widetilde{R}_k^{-1}\widetilde{x_{kj}}^{-1}\widetilde{R}_j)(\alpha_{ji}\alpha_{ik}\alpha_{jk}^{-1}) \\ &= (\widetilde{x_{ji}}\widetilde{x_{ik}}\widetilde{x_{jk}}^{-1})(\alpha_{ji}\alpha_{ik}\alpha_{jk}^{-1}) \end{aligned}$$

where in the first line we have used that $\alpha_{ji}$ is central and in the second that the product $\widetilde{x_{ji}}\widetilde{x_{ik}}\widetilde{x_{jk}}^{-1}$ is central.

*Step 5:* Our next goal will be to compute the image of $\alpha_{ji}$ under the map $\partial$. In order to do this, consider the following diagram of groups,

$$\begin{array}{ccccc} \mathsf{K}_2(F_X[t]) & \longrightarrow & \widetilde{G}(F_X[t]) \ltimes F^* & \xrightarrow{\pi'} & G(F_X[t]) \ltimes F^* \\ \downarrow & & \downarrow & & \downarrow \\ \mathsf{K}_2(F_X(t)) & \longrightarrow & \widetilde{G}(F_X(t)) \ltimes F^* & \xrightarrow{\widetilde{\pi}} & G(F_X(t)) \ltimes F^* \\ \downarrow \partial & & \downarrow \omega & & \downarrow \\ F_X^* & \longrightarrow & \widehat{G}(F_X) \ltimes F^* & \xrightarrow{\widehat{\pi}} & G(F_X(t)) \ltimes F^* \end{array},$$

where all rows are exact (where the top row is exact by [BD01, 3.1.1, p.26], and the composition of the maps $\mathsf{K}_2(F_X[t]) \to \mathsf{K}_2(F_X(t)) \to F_X^*$ is zero, and $\omega$ is surjective. Using the above diagram, we can define the following map from

$$\Psi : G(F_X[t]) \to \widehat{G}(F_X)$$

as follows: let $b \in G(F_X[t]) \ltimes F^*$, and choose a lift $\widetilde{b} \in \widetilde{G}(F_X[t]) \ltimes F^*$. Note that the element $\omega(\widetilde{b}) \in \widehat{G}(F_X) \ltimes F^*$ is then independent of the choice of $\widetilde{b}$, and so we have a well-defined map $\Psi : G(F_X[t]) \ltimes F^* \to \widehat{G}(F_X) \ltimes F^*$, which is easily seen to be a section of the map $\widehat{\pi}$. But such a section is unique by Remark 4.4.3, so we must have $\Psi = \Phi_F$ where $\Phi_F$ was the section constructed previously in §4.4.

Recall that we have defined Čech cocycles for $\mathbf{c}(x\gamma)$ in (5.2) as

$$\mathbf{c}(x\gamma)_{ji} := \widehat{b}_{ji}^{-1}\Psi_F(b_{ji}).$$

We then compute,

$$\begin{aligned}
\mathbf{c}(x\gamma)_{ji} &= \widehat{b}_{ji}^{-1}\Phi_F(b_{ji}) \\
&= \widehat{b}_{ji}^{-1}\Psi(b_{ji}) \\
&= \widehat{b}_{ji}^{-1}\omega(\widetilde{b}_{ji}) \\
&= \widehat{b}_{ji}^{-1}\omega(\widetilde{h}_{ji}\alpha_{ji}) \\
&= \widehat{b}_{ji}^{-1}\omega(\widetilde{h}_{ji})\omega(\alpha_{ji}) \\
&= \widehat{b}_{ji}^{-1}\widehat{b}_{ji}\partial(\alpha_{ji})
\end{aligned}$$

In summary, $\alpha_{ji}$ is a lift of $\mathbf{c}(x\gamma)_{ji}$ under a power of the tame symbol.

*Step 6:* Suppose we are given $P_{x\gamma_1}$ and $P_{x\gamma_2}$. Then there exist $\alpha_{ji}^1, \alpha_{ji}^2 \in \mathsf{K}_2(F_X(t))$ satisfying

$$\mathbf{c}(x\gamma_1)_{ji} = \partial(\alpha_{ji}^1) \text{ and } \mathbf{c}(x\gamma_2)_{ji} = \partial(\alpha_{ji}^2)$$

and such that the gerbe classes of $\mathsf{Lift}_{P_{x\gamma_1}}^{\mathsf{K}_2}$ and $\mathsf{Lift}_{P_{x\gamma_2}}^{\mathsf{K}_2}$ are given by,

$$(\widetilde{x_{ji}}\widetilde{x_{ik}}\widetilde{x_{jk}}^{-1})(\alpha_{ji}^1\alpha_{ik}^1(\alpha_{jk}^1)^{-1})$$

and

$$(\widetilde{x_{ji}}\widetilde{x}_{ik}\widetilde{x}_{jk}^{-1})(\alpha_{ji}^2\alpha_{ik}^2(\alpha^2)_{jk}^{-1})$$

respectively. Taking the difference, we find that the class of $\mathsf{Lift}_{P_{x\gamma_1},P_{x\gamma_2}}^{\mathsf{K}_2}$ is given by

$$(\alpha_{ji}^1\alpha_{ik}^1(\alpha_{jk}^1)^{-1})(\alpha_{ji}^2\alpha_{ik}^2(\alpha_{jk}^2)^{-1})^{-1}.$$

But recalling how the map $\mathscr{B}$ was constructed explicitly on the level of cocycles in §1.4, we conclude the proof of the theorem.

□

## 15. Loop Eisenstein Series and Geometric Generating Functions

**(15.1)** In this section, we summarize our work so far on geometrizing the loop Eisenstein series of Chapter 2 §6. Keep the conventions of §4.5. Also, we introduce the following notation: for $P_x^o \in \mathsf{Bun}_G(S_\tau^o)$, and $Q \in$

$\mathsf{Bun}_{G,B}^{P_x^o}(S_\tau)$, we define a relative multi-degree which takes values in $\widehat{\mathfrak{h}} = k\mathbf{c} \oplus \mathfrak{h}$,

$$\begin{aligned} (15.1.1)\quad \widetilde{\deg}_Q : \mathsf{Bun}_{G,B}^{P_x^o}(S_\tau) &\to \widehat{\mathfrak{h}} \\ (15.1.2)\quad P &\mapsto \widetilde{\deg}_Q(P) = \langle c_2^Q(P)^{1/N}, \deg(P|_X)\rangle, \end{aligned}$$

where $N$ is as in Theorem 4.5.1 and $c_2^Q(P)$ was the relative $K$-theoretic second Chern class of $P$ defined as the degree of $\mathscr{C}_2(P,Q)$.

**Theorem 15.1.1.** *Up to a constant, the loop Eisenstein series $E_\mu(x)$ is equal to the generating function,*

$$\sum_{P \in \mathsf{Bun}_{G,B}^{P_x^o}(S_\tau)} \mu(\widetilde{\deg}_Q(P))$$

*Proof.* Given our previous results, this is essentially obvious. Let us just note that we have identified,

$$\mathsf{Bun}_{G,B}^{P_x^o}(S_\tau) \leftrightarrow \widehat{\Gamma}_F / \widehat{\Gamma}_F \cap \widehat{B}_F.$$

So it remains to show the relation between $\mu(\widetilde{\deg}_Q)$ and $\Phi_\mu(x\gamma)$ (notation as in Chapter 2, §6). But, recall that

$$\Phi_\mu(x\gamma) = \mu(|h_{x\gamma}|),$$

where

$$h_{x\gamma} = h_c(\sigma) \prod_{i=1}^{l} h_i(\sigma_i).$$

By Proposition 4.4.1, we understand that $|\sigma|_{\mathbb{I}}^N$ and $c_2^Q(P_{x\gamma})$ are related by a constant (depending on our choice of $Q$). Furthermore, the degrees of $\sigma_i$ are related to the $\deg P|_X$ by Harder's work in the finite dimensional case [Har74, Lemma 2.1.1]. The theorem follows from these obsevations.

## 16. Ribbons and a Formal Analogue

As we have seen in §5, we can interpret the loop Eisenstein series as a generating function for bundles on the affine surface $S_\tau$. Suppose we are in the following more general situation: let $X \hookrightarrow S$ be a curve contained in any arbitrary *projective* surface (all schemes are over $k$ a finite field) with self-intersection $(X.X) < 0$. Let $S^o = S \setminus X$, and fix $P^o \in \mathsf{Bun}_G(S^o)$. Then Kapranov has defined the generating series,

$$E_{G,P^o}(q,z) = \sum_{n \in \mathbb{Z}, a \in L} \chi(\mathscr{M}_{G,P^o}(n,a)) q^n z^a$$

where $\chi(\mathscr{M}^{P^o}_{G,B}(S;n,a))$ is the number of $k$-rational points in the moduli scheme $\mathscr{M}^{P^o}_{G,B}(S;n,a)$ parametrizing $G$-bundles on $S$ extending $P^o$, whose second Chern number is $n$ (*note:* $S$ is now a projective surface so we have a well-defined notion of second Chern number), and whose restriction $P|_X$ to the curve $X$ is equipped with a reduction to $B$ which has degree $a \in L \subset \mathfrak{h}_{\mathbb{Z}}$. The above series also admits an adelic interpretation in terms of Einsenstein series on loop groups twisted by the higher idele group $\mathbb{I}[[t]]$ described below.

**(16.1)** Let $S$ be a noetherian scheme and $X$ a closed subscheme defined by a sheaf of ideals $\mathscr{I}$. We denote the formal completion of $S$ along $X$ by the pair $(\widehat{S}_X, \mathscr{O}_{\widehat{S}_X})$. Recall that this is the ringed space whose underlying topological space is $X$ and whose sheaf of rings is

$$\mathscr{O}_{\widehat{S}_X} := \lim \mathscr{O}_S/\mathscr{I}_X^n,$$

where $\mathscr{I}_X$ is the ideal sheaf of $X$. Similarly, if $\mathscr{F}$ is a coherent sheaf on $S$, we denote by $\widehat{\mathscr{F}} := \lim \mathscr{F}/\mathscr{I}^n\mathscr{F}$ the completion of $\mathscr{F}$ along $X$.

By a noetherian formal scheme (or just formal scheme for short), we shall mean a locally ringed space $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ with an open cover $\{\mathfrak{U}_i\}$ such that for each $i$, the pair $(\mathfrak{U}_i, \mathscr{O}_{\mathfrak{X}}|_{\mathfrak{U}_i})$ is isomorphic, as a locally ringed space, to the completion of some noetherian scheme $X_i$ along a closed subscheme $Y_i$. Morphisms of formal schems are morphisms as *locally* ringed spaces.

**Definition 16.1.1.** Let $X$ be a scheme. Then by a *ribbon* over $X$ we shall mean a formal scheme $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ whose underlying topological space is $X$ and such that there exists an affine open cover $\{\mathfrak{U}_i = \mathrm{Spec}(R_i)\}$ of $X$ such that for each $i$, the pair $(\mathfrak{U}_i, \mathscr{O}_{\mathfrak{X}}|_{\mathfrak{U}_i})$ is isomorphic to $(\mathrm{Spec}(R_i), \mathscr{O}_{R_i[[t]]})$. By a morphism of ribbons, we shall mean a morphism of formal schemes which is the identity on $X$. Let $Y \to X$. We shall denote the *set* of isomorphism classes of ribbons over $X$ by $\mathsf{Rib}(X)$.

As in Chapter 2, §1.4, we define for any open affine $U = \mathrm{Spec}(R) \subset X$, the sheaf

$$\mathrm{Aut}(R[[t]]) = \{\sum_{i=1}^{\infty} r_i t^i | r_1 \in R^*\}.$$

It is clear that we have an isomorphism of (pointed) sets,

$$H^1(X, \mathrm{Aut}(\mathscr{O}_X[[t]])) = \mathsf{Rib}(X),$$

where the left hand side is interpreted as a non-abelian cohomology group. The map

$$\mathrm{Aut}(R[[t]]) \to R^*$$

obtained by projecting onto the coefficient of $t$ may be glued together to obtain a map

$$\mathrm{Aut}(\mathscr{O}_X[[t]]) \to \mathscr{O}_X^*$$

which induces a map,

$$\mathsf{Rib}(X) \to \mathsf{Pic}(X).$$

We denote the *degree* of a ribbon to the degree of its image under the above map.

**(16.2)** The rough structure of the group $\mathrm{Aut}_R(R[[t]])$ is given as follows. Let

$$R^* \subset \mathrm{Aut}(R[[t]])$$

consist of elements of the form $t \mapsto rt$ where $r \in R^*$. Denote by $U_R \subset \mathrm{Aut}(R[[t]])$ the subgroup of elements of the form

$$t \mapsto t + a_2t^2 + a_3t^3 + \dots\ , \text{ for } a_i \in R.$$

Then we have,

**Lemma 16.2.1.** $\mathrm{Aut}(R[[t]]) = U_R \ltimes R^*$.

*Proof.* First note $U_R.R^* = \mathrm{Aut}(R[[t]])$. Indeed, given $x = \sum_{i=1} b_i t^i$ then

$$x \circ b_1^{-1} t \in U_R.$$

It is easy to verify that $R^*$ normalizes $U_R$. □

**(16.3)** *Main Example:* The main example of a ribbon for us will be constructed by taking the formal neighborhood of a curve $X$ inside a surface $S$. We shall denote this ribbon by $\widehat{S}_X$ and the punctured ribbon $\widehat{S}_X \setminus X$ by $\widehat{S}_X^o$. If $L$ is a line bundle over $X$, then we may form the total space $S_L \to X$ as in the previous section. The zero section $z : X \hookrightarrow S_L$ defines ribbon which we denote by $\widehat{S}_L$. Thus we obtain a map,

$$\mathsf{Pic}(X) \to \mathsf{Rib}(X).$$

We say that a ribbon has *linear structure* if it lies in the image of this map. In the case when $X = \mathbb{P}^1$, all ribbons with negative degree have linear structure as we shall see below.

**(16.4)** We have already seen that line bundles on a curve $X$ are classified by elements of the idelic double quotient space $F^* \setminus \mathbb{I}_F / \prod_{v \in X} \mathscr{O}_v^*$. Yongchang Zhu has informed us of an analogous presentation for ribbons over $X$. The role of $\mathbb{I}$ is now played by the group,

$$\mathbb{I}[[t]] := {\prod_{v \in X}}' \mathrm{Aut}(F_v[[t]]),$$

where the restricted direct product is with respect to the subgroups $Aut(\mathscr{O}_v[[t]])$.

**Proposition 16.4.1.** *(Y. Zhu) Let X be a curve. Then we have a bijective correspondence between the sets,*

$$\mathsf{Rib}(X) \leftrightarrow \mathrm{Aut}(F[[t]]) \setminus \mathbb{I}[[t]] / \prod \mathrm{Aut}(\mathscr{O}_\nu[[t]])$$

*Proof.* For $V \subset X$ an open, we denote by

$$\Omega_V = \prod_{\nu \in X \setminus V} \mathrm{Aut}(F_\nu[[t]]) \times \prod_{\nu \in V} \mathrm{Aut}(\mathscr{O}_\nu[[t]]).$$

For $p \in X$ and $V_p = X \setminus \{p\}$, we shall write for short

$$\Omega(p) := \Omega_{V_p}.$$

Also, we write,

$$U_{\mathbb{A}} := \prod_{\nu \in X}{}' U_{F_\nu}$$

where the restricted direct product is with respect to $\{U_{\mathscr{O}_\nu}\}$.

*Step 1:* We have an approximation theorem,

$$\mathbb{I}[[t]] = \mathrm{Aut}_F(F[[t]])\widehat{\Omega}(p).$$

To prove this, let $x = (x_\nu) \in \mathbb{I}[[t]]$, and use the Lemma 6.2.1 above to write this as $x_\nu = r_\nu u_\nu$ for $(r_\nu) \in \mathbb{I} \subset \prod F_\nu^*$ and $(u_\nu) \in U_{\mathbb{A}}$. By the approximation theorem for $\mathbb{I}_F$, we may find $\phi \in F^*$ such that $r_\nu \phi \in \prod_{\nu \neq p} \mathscr{O}_\nu^*$. Hence,

$$x\phi = r_\nu \phi \phi^{-1} u_\nu \phi \in \prod_{\nu \neq p} \mathscr{O}_\nu^* \rtimes U_{\mathbb{A}}.$$

Thus it suffices to show that

$$U_{\mathbb{A}} = (U_{\mathbb{A}} \cap \Omega(p)) \times (U_F \cap U_{\mathbb{A}}).$$

But this follows from the additive approximation theorem for $\mathbb{A}$ and an easy induction.

*Step 2:* Let $\tau \in \mathbb{I}[[t]]$, and choose an open cover $\{U_\alpha = \mathrm{Spec}(R_\alpha)\}$ of $X$. Then over each open set, we may write,

$$\tau = x_\alpha r_\alpha \quad \text{for } r_\alpha \in \Omega_{U_\alpha} \quad \text{and } x_\alpha \in \mathrm{Aut}(F[[t]]) \ .$$

Then on overlaps $U_\alpha \cap U_\beta$, we find

$$\tau_{\alpha\beta} := x_\beta^{-1} x_\alpha = r_\beta r_\alpha^{-1} \in \mathrm{Aut}(R_{\alpha\beta}[[t]]).$$

The $\tau_{\alpha\beta}$ then form a cocycle for an $\mathrm{Aut}(\mathscr{O}_X[[t]])$-torsor (i.e., a ribbon on $X$).

*Step 3:* Conversely, suppose we are given a ribbon $Y$ over $X$. Then the ribbons $Y \otimes_X F$ and $Y \otimes \mathscr{O}_\nu$ for any $\nu \in X$ are both trivial. For each $\nu$, the difference between these two trivializations gives us an element in $\mathrm{Aut}(F_\nu[[t]])$. It is then easy to see that we actually obtain an element of $\mathbb{I}[[t]]$ in this way.

□

(**16.5**) Our next task will be to describe bundles over the ribbons. Let us regard a ribbon $\widehat{S}$ over $X$ as a site with covers of the form $\{U[[t]] \to \widehat{S}\}$, where $U \to X$ is a Zariski open set. Then $G$-torsors on $\widehat{S}$ are defined with respect to these covers. In this section, we shall revert back to our original convention from Chapter 2, §3.5 and a bar over a group will mean the formal non-centrally extended loop group. So for example, $\overline{G}(R) = G(R((t)))$. With this definition, the proof of the following statement follows as before,

**Proposition 16.5.1.** *Let $\tau \in \mathbb{I}[[t]]$ and $\widehat{S}_\tau$ $(\widehat{S}^o_\tau)$ the corresponding ribbon (resp. punctured ribbon) on X from the previous proposition.*

(1) *There is a bijective correspondence between: (a) G-torsors on $\widehat{S}^o_\tau$; and (b) elements of the double coset space,*
$$\overline{K}_{\mathbb{A}} \setminus \overline{G}_{\mathbb{A}} \rtimes \mathbb{I}[[t]] / \overline{\Gamma}_F.$$
(2) *Let $\widehat{P}^o$ be a given G-torsor on $\widehat{S}^o_\tau$. Then there is a bijective correspondence between: (a) the set of triples $(\widehat{P}, \tau, r)$ where hP is a G-torsor on $\widehat{S}_\tau$, $\tau : \widehat{P}|_{\widehat{S}^o_\tau} \to \widehat{P}^o$ is an isomorphism, and r is a reduction of $\widehat{P}|_X$ to an B-torsor; and (b) the elements of the set*
$$\overline{\Gamma}_F / (\overline{B}_F \cap \overline{\Gamma}_F).$$

(**16.6**) The loop Eisenstein series construction from of Chapter 2, §6 can be carried out when $\tau \in \mathbb{I}[[t]]$. In this context, and we define the *formal loop Eisenstein series* by the same formula as before,
$$\widehat{E}_\mu(x) := \sum_{\overline{\Gamma}_F/\overline{B}_F \cap \overline{\Gamma}_F} \Phi_\nu(x\gamma)$$
for $x \in \overline{K}_{\mathbb{A}} \setminus \widehat{G}_{\mathbb{A}} \eta(\tau) / \widehat{\Gamma}_F$, and $\tau \in \mathbb{I}[[t]]$.

**Remark 16.6.1.** The analytic properties of this series remain to be investigated.

The geometric description of this series is achieved as follows: Let $X \hookrightarrow S$ where $S$ is a projective surface. Let $P^o$ be a fixed $G$-bundle on $S^o$. To every extension $P \in \mathsf{Bun}^{P^o}_{G,B}(S)$, we denote its multi-degree by
$$\deg(P) = \langle c_2(P)^{1/N}, \deg P|_X \rangle \in \widehat{\mathfrak{h}}_{\mathbb{Z}},$$
where $N$ is as in Chapter 2, Theorem 4.5.1. Then we can form the formal generating series,
$$\mathscr{K}_\mu(P^o) := \sum_{P \in \mathsf{Bun}^{P^o}_{G,B}(S)} \mu(\deg(P).$$
This series is related to a series of the form $\widehat{E}_\mu$ as follows.

Let $\tau$ denote the element of $\mathbb{I}[[t]]$ which corresponds to the ribbon $\widehat{S}_X$, and denote the corresponding bundle on $\widehat{S}_X^o$ corresponding to $P^o$ by $\widehat{P}^o$. Then we may find $x \in \widehat{\mathsf{K}}_{\mathbb{A}} \setminus \widehat{G}_{\mathbb{A}} \eta(\tau) / \widehat{\Gamma}_F$ whose reduction modulo the center gives an element in $\overline{K}_{\mathbb{A}} \setminus \overline{G}_{\mathbb{A}} \rtimes \mathbb{I}[[t]] / \overline{\Gamma}_F$ which corresponds by Proposition 6.5.1 to a formal bundle $\widehat{P}_x^0$ which is isomorphic to $\widehat{P}^o$. The space of *formal* extensions of $\widehat{P}^o$ to a bundle on $\widehat{S}_X$ can be identified with the set

$$\overline{\Gamma}_F / \overline{B}_F \cap \overline{\Gamma}_F.$$

By the descent lemma of Beauville-Lazlo [BL94], we may identify the set of these formal extensions with the set of actual extensions $\mathsf{Bun}_{G,B}^{P^o}(S)$. Hence for this choice of $x$, the formal loop Eisenstein series $\widehat{E}_\mu(x)$ can be indentified (up to a constant) with $\mathscr{K}_\mu(P^o)$ as formal series. However, it remains to be seen whether this identification can be refined in any way. This again will involve investigating the analytic properties of the formal loop Eisenstein series.

## 17. Example: Loop Eisenstein Series on $\mathbb{P}^1$.

**(17.1)** In the case when $X = \mathbb{P}^1$, ribbons are particularly easy to describe.

**Lemma 17.1.1.** *Let $X = \mathbb{P}^1$. Let $Y$ be a ribbon whose degree is negative. Then $Y$ is linear (i.e., comes from a line bundle on $X$).*

*Proof:* Let us give a Čech computation of $H^1(\mathbb{P}^1, \mathrm{Aut}(\mathscr{O}_{\mathbb{P}^1}[[t]]))$. Denote by $U_0 = \mathrm{Spec}(k[s])$ and $U_1 = \mathrm{Spec}(k[s^{-1}])$ the two open sets covering $\mathbb{P}^1$. So, we are reduced to computing $H^1(\{U_i\}, \mathrm{Aut}(\mathscr{O}_X[[t]]))$.

Recall how the multiplication in $\mathrm{Aut}(R[[t]])$ is defined: let $f(t) = \sum_{i=1}^{\infty} a_i t^i$ and $g(t) = \sum_{i=1}^{\infty} b_i t^i$. Then multiplication is by composition,

$$f(t) \circ g(t) = a_1 b_1 t + (a_1 b_2 + a_2 b_1^2) t^2 + (a_1 b_3 + a_2 b_1 b_2 + a_2 b_2 b_1 + a_3 b_1^3) + \cdots$$

Let $f(t) = \sum_{i=1}^{\infty} a_i t^i \in \mathrm{Aut}(k[s, s^{-1}][[t]])$ represent the 1-cocyle in $H^1(\{U_i\}, \mathrm{Aut}(\mathscr{O}_X[[t]]))$. The negative degree hypothesis means that $a_1 = cs^{-n}$, for $n \in \mathbb{N}$. Hence

$$k[s] \subset a_1 k[s],$$

and it is clear form the above expression that we can choose $b_i \in k[s]$ such that

$$f(t) \circ g(t) = \sum_{i=1}^{\infty} d_i t^i$$

where $d_i \in k[s^{-1}]$. Now, precomposing $f(t) \circ g(t)$ with $h(t) \in \mathrm{Aut}(k[s^{-1}][[t]])$ we can see that

$$h(t) \circ f(t) \circ g(t) = ct$$

where $c \in k[s, s^{-1}]^*$. $\square$

**(17.2)** From Lemma 7.1.1, we know that all on $X = \mathbb{P}^1$, the set of ribbons with negative degree coincides with the space of line bundles with negative degree. Suppose we fix such a ribbon $\widehat{S}_\tau$ corresponding to an idele $\tau \in \mathbb{I}_F$. We now wish to clasify $G$-bundles on $\widehat{S}^o_\tau$. This is equivalent to looking at the double coset space,

$$\overline{K}_{\mathbb{A}} \setminus \overline{G}_{\mathbb{A}} \eta(\tau) / \overline{\Gamma}_F.$$

In the case $X = \mathbb{P}^1$, this set has a particularly nice description which can be seen as an analogue of Grothendieck's theorem for loop group bundles. We first begin with the case of centrally extended groups because it shall be useful for us in the next paragraph. But, of course the central extension does not play a significant role in the remainder of this section.

Recall that $\widehat{H}_{\mathbb{A}} \subset \widehat{G}_{\mathbb{A}}$ be the subgroup generated by elements of the form

$$\prod_{i=1}^{l+1} h_{\alpha_i}(\sigma_i) \ \text{ for } \sigma_i \in \mathbb{I}_F.$$

For $x \in \widehat{H}_{\mathbb{A}}$, we define

$$(x\eta(\tau))^{\alpha_j} = \prod_{i=1}^{l+1} \sigma_i^{\alpha_j(h_i)} \tau^{\delta_{j,l+1}},$$

where $\delta_{j,l+1}$ is the Kronecker delta function.

**Definition 17.2.1.** Let $\widehat{H}^1_{\mathbb{A}} \subset \widehat{H}_{\mathbb{A}}$ as the subset of elements with $|(x\eta(\tau))^{\alpha_j}| < 1$ for $j = 1, \ldots, l+1$, where the norm is the standard norm on $\mathbb{I}_F$.

**Proposition 17.2.2.** *Let $\tau \in \mathbb{I}_F$ such that $|\tau| < 1$. Let $\widehat{H}^1_{\mathbb{A}} \subset \widehat{H}_{\mathbb{A}}$ denote the subset defined above. Then we have,*

$$\widehat{G}_{\mathbb{A}} \eta(\tau) = \widehat{K}_{\mathbb{A}} \widehat{H}^1_{\mathbb{A}} \eta(\tau) \widehat{\Gamma}_F$$

*Proof.* Let us first construct a fundamental domain for $\widehat{U}_{\mathbb{A}}$ under $\widehat{\Gamma}_F \cap \widehat{U}_{\mathbb{A}}$. Let $p \in X$ be any point. Recall that we have

$$\mathbb{A}_F = F + \Omega,$$

where

$$\Omega = \pi_p^{-1} \mathscr{O}_p \times \prod_{v \neq p} \mathscr{O}_v.$$

From this, it is standard that a fundamental domain for $\widehat{U}_{\mathbb{A}}$ under $\widehat{\Gamma}_F \cap \widehat{U}_{\mathbb{A}}$ is given by the set $\widehat{U}_\Omega$ consisting of elements of the form,

$$x = \prod_{\alpha \in \Delta_+(A)} \chi_\alpha(\sigma_\alpha(t)) \prod_{i=1}^{l} h_{\alpha_i}(\sigma_i(t)) \prod_{\alpha \in \Delta_-(A)} \chi_\alpha(\sigma'_\alpha(t))$$

where

$$\sigma_i(t) = 1 + a_1 t + a_2 t^2 + a_3 t^3 + \dots \quad \text{with} \quad a_i \in \Omega, \ ,$$

and

$$\sigma_\alpha, \sigma'_\alpha(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots \quad \text{with} \quad a_i \in \Omega.$$

Let us now note that if $x \in \widehat{H}^1_{\mathbb{A}}$, then

(17.2.1) $$x \widehat{U}_\Omega x^{-1} \in \widehat{K}_{\mathbb{A}}.$$

Indeed this follows form the definition of $\widehat{H}^1_{\mathbb{A}}$, the formulas13.10 and 14.11 of [Gar80], and the fact that an element of $\Omega$ has a pole of order at most 1.

The proposition now follows from the Siegel set construction of [Gar80]. □

The above proposition also holds for the non centrally-extended loop group $\overline{G}$. As a consequence, we have a purely group-theoretical proof of the following theorem originally due to Kapranov.

**Corollary 17.2.3.** *Let $X = \mathbb{P}^1$ and $\tau \in \mathbb{I}_F$ an idele with negative degree $d$. There is a bijective correspondence between the set of $G$-bundles on $\widehat{S}^o_\tau$ and dominant affine coweights of the form $(0, a, d) \in \widehat{L}_{\mathbb{Z}} = \mathbb{Z} \oplus L \oplus \mathbb{Z}$, where the first $\mathbb{Z}$ corresponds to the center and the last one to the degree operator.*

*Proof.* This set in the corollary is in bijective correspondence to the space of double cosets,

$$\overline{K}_{\mathbb{A}} \setminus \overline{G}_{\mathbb{A}} \eta(\tau) / \overline{\Gamma}_F.$$

Let $\overline{H}^1_{\mathbb{A}}$ be the non-centrally extended analogue of $\widehat{H}^1_{\mathbb{A}}$ described above. Fix a prime $p \in X$. Then every element $x \in \overline{H}^1_{\mathbb{A}}$ can be written as a product,

$$x = \prod_{i=1}^{l} h_{\alpha_i}(\sigma_i) \ \text{ for } \sigma_i \in \mathbb{I}_F$$

can be factored as $x = x_{\mathscr{O}} x_p x_F$, where

$$x_{\mathscr{O}} = \prod_{i=1}^{l} h_{\alpha_i}(\sigma'_i) \ \text{ for } \sigma'_i \in \prod_{\nu \neq p} \mathscr{O}^*_\nu.$$

$$x_p = \prod_{i=1}^{l} h_{\alpha_i}(\sigma_i) \ \text{ for } \sigma''_i \in k^*_p$$

and

$$x_F = \prod_{i=1}^{l} h_{\alpha_i}(\sigma'''_i) \ \text{ for } \sigma'''_i \in F^*.$$

Picking a local parameter $z$ at $p$, we may write

$$x_p = \prod_{i=1}^{l} h_{\alpha_i}(z^{m_i}) \text{ for}$$

where

$$|z^{m_i}|_p = |\sigma_i|.$$

Since every element of the double coset space has a unique representative $x_p$ as above, the corollary follows. □

**(17.3)** Recall that we have defined the constant term as

$$E_\mu^\sharp(x) := \int_{\widehat{U}_{\mathbb{A}}/(\widehat{U}_{\mathbb{A}}\cap\widehat{\Gamma}_F)} E_\mu(xu)du.$$

Let us define for $x \in \widehat{G}_{\mathbb{A}}\eta(\tau)$ the function,

$$F_\mu(x) = E_\mu(x) - E_\mu^\sharp(x).$$

Our goal will be to show that

**Claim 17.3.1.** *For any $x \in \widehat{G}_{\mathbb{A}}\eta(\tau)$, we have $F_\mu(x) = 0$. Hence the Eisenstein series is equal to its constant term on $\mathbb{P}^1$.*

*Proof.* Let us first show that

$$E_\mu(x) = E_\mu(xu) \text{ for } u \in \widehat{U}_{\mathbb{A}}.$$

As

$$\widehat{U}_{\mathbb{A}} = \widehat{U}_\Omega(\widehat{\Gamma}_F \cap \widehat{U}_{\mathbb{A}}),$$

it suffices to show that

$$E_\mu(x) = E_\mu(xu) \text{ for } \omega \in \widehat{U}_\Omega$$

since the Eisenstein series is clearly $\widehat{\Gamma}_F$-invariant. Suppose that (modulo $\widehat{\Gamma}_F$) we have written

$$x = kh\eta(\tau)u$$

where $k \in \widehat{K}_{\mathbb{A}}, h \in \widehat{H}^1_{\mathbb{A}}, u \in \widehat{U}_\Omega$ Then again (modulo $\widehat{\Gamma}_F$) we may write

$$x\omega = kh\eta(\tau)u'$$

for $u' \in \widehat{U}_\Omega$. However, we may conjugate $h$ past both $u$ and absorb the difference into $\widehat{K}_{\mathbb{A}}$ by 17.2.1. Thus, we see that $E_\mu(x) = E_\mu(xu)$.

But, clearly $E_\mu^\sharp(x)$ is also right $\widehat{U}_{\mathbb{A}}$-invariant. So then $F_\mu(x)$ as well. So, we may compute, (by the way we have normalized the measure),

$$\int_{\widehat{U}_{\mathbb{A}}/(\widehat{U}_{\mathbb{A}}\cap\widehat{\Gamma}_F)} F_\mu(gu)du = F_\mu(g).$$

On the other hand, by definition, we have

$$\int_{\widehat{U}_{\mathbb{A}}/(\widehat{U}_{\mathbb{A}}\cap\widehat{\Gamma}_F)} F_\mu(gu)du = \int_{\widehat{U}_{\mathbb{A}}/(\widehat{U}_{\mathbb{A}}\cap\widehat{\Gamma}_F)} E_\mu(gu)du - \int_{\widehat{U}_{\mathbb{A}}/(\widehat{U}_{\mathbb{A}}\cap\widehat{\Gamma}_F)} E^\sharp_\mu(gu)du = 0.$$

□

**(17.4)** Combining the results of the previous paragraph with the computation of the constant term from Chapter 2 §6, we have the following,

**Theorem 17.4.1.** *When $F = k(\mathbb{P}^1)$, we can indentify the loop Eisenstein series*

$$E_\nu(g\eta(\tau)) = \sum_{w\in\widehat{W}} (a_g\eta(\tau))^{w(\nu+\rho)-\rho}\widetilde{c}(\nu,w) \tag{17.4.1}$$

*where we have the following notation: $a_g$ is the $\widehat{\mathsf{H}}$ component of $g$ in the Iwasawa decomposition;*

$$\widetilde{c}(\nu,w) = \prod_{\alpha\in\widehat{\Delta}_+\cap w^{-1}(\widehat{\Delta}_-)} \frac{\Phi_F(-(\nu+\rho)(h_\alpha))}{\Phi_F(-(\nu+\rho)(h_\alpha)+1)},$$

*with $\Phi_F(s) = \frac{1}{(1-s)(1-qs)}$ where $k = \mathbf{F}_q$.*

**Remark 17.4.2.** Both the left and right hand side of 17.4.1 have geometric definitions: the left hand side by Theorem 5.1.1; and the zeta functions on the right hand side in terms of symmetric products of a curve. It would be interesting to obtain a purely geometric proof do the 17.4.1.

## **Part** 4. **Appendix: Central Extensions and Riemann-Roch theorems**

### 18. Introduction

**(18.1)** In this chapter, we shall be interested in formulating a certain Riemann-Roch theorem for gerbes. Let us briefly recall, following [Del87], the general context in which this result fits. Let $f : S \to X$ be a proper, flat morphism of pure relative dimension $N$ (we shall be interested in the case when $N = 1$, i.e. a family of curves parametrized by a base $X$). Let $I$ be any finite set of indices and $\{C^i_p | p > 0, i \in I\}$ a set of indeterminates with $C^i_p$ having weight $p$. Let $P$ be any polynomial in the $C^i_p$, homogeneous of weight $N+1$. Given vector bundles $E_i$ on $S$ for $i \in I$, we may form the $p$-th Chern classes of the $i$-th bundle $c^i_p := c_p(E_i)$ and then define

$$c := \int_{S/X} P(c^i_p(E_i).$$

There are many contexts in which the above construction may be carried out, but we shall work with $S, X$ as quasi-projective algebraic varieties and take $c, c_p^i$ to be valued in the appropriate Chow group. In this case, we have

$$c \in CH^1(X) = Pic(X),$$

i.e., $c$ determines an *isomorphism class* of line bundles. Deligne then poses,

*Problem A:* Construct in a "functorial" (see [Del87, §2.1]) manner, a line bundle on $X$ called

$$I_{S/X}P(E_i | i \in I)$$

whose first Chern class is equal to $c$.

If $E$ is a vector bundle on $S$ and $T_{S/X}$ the relative tangent bundle of the map $f$, then the usual Riemann-Roch formula gives,

$$ch(Rf_*E) = \int_{S/X} ch(E).Td(T_{S/X}) \tag{18.1.1}$$

where $ch$ is the Chern character and $Td$ the Todd class. Focusing our attention on the component of degree 2, the class we obtain on the left hand side is

$$c_1(\det Rf_*E),$$

the first Chern class of the determinant of the cohomology line $\det Rf_*$ constructed in [KM76]. Let $RR_{N+1}$ be the component of degree $2(N+1)$ in $ch(E).Td(T_{S/X})$. It is a universal polynomial of weight $N+1$ in the Chern classes of $V$ and $T_{X/S}$ with *rational* coefficients. Assuming that the solution to Problem A posed above is known, then Deligne also poses,

*Problem B:* Find an integer $M$ such that $M.RR_{N+1}$ has integral coefficients and construct a canonical isomorphism of line bundles

$$(\mathrm{Det} Rf_*E)^{\otimes M} \cong I_{S/X}\, M.RR_{N+1}\, (E, T_{S/X})$$

**(18.2)** Let us now specialize to the case when $N = 1$. Then, Deligne has answered both of the above questions. Let us here sketch his solution. If $N = 1$, we are interested in homogeneous polynomials in the $C_p^i$ of weight 2, and so only in $C_1^1 C_1^1$ and $C_2^1$. The answer to Problem A then amounts to two functorial constructions:

(1) $I_{S/X}\, C_1^1.C_1^1\, (E_1, E_2)$ assigns to two vector bundles $E_1, E_2$ on $S$ a line bundle on $X$ whose first Chern class is the integral of the product of the first Chern classes of $E_1$ and $E_2$ on $S$. However, as the first Chern class of any vector bundle $E$ is only dependent on the line bundle $\wedge^{top} E$, we may restrict to the case when $E_i$ are actually line bundles.

In this case, Deligne has attached to any two line bundles $L,M$ on $S$ a line budle on $X$ which we denote by $\langle L,M\rangle$. It has functorial properties as described in [Del87, §6].

(2) $I_{S/X}\, C_2^1\,(E)$ assigns to each vector bundle $E$ on $S$ a line bundle on $X$ whose first Chern class is the integral of the second Chern class of $V$ on $S$. The construction and its functorial properties are described in [Del87, Prop 9.4].

Moreover, there is actually an extension of the construction of the above two constructions to virtual vector bundles (see [Del87] for a precise definition, but here we shall just consider the extension to differences $V_1 - V_2$ of vector bundles). Similarly, the determinant of the cohomology line bundle det $Rf_*$ can also be extended to virtual bundles. A relative form of the answer to Problem B is then the following,

**Theorem 18.2.1.** [Del87, §9.9] *Let $S \to X$ be as before with relative dimension* 1*. Let $E_1, E_2$ be vector bundles on $S$ with trivial determinant, i.e. $\wedge^{top} E_i = \mathscr{O}_S$ and such that there exists an isomorphism $E_1|_{S^o} \cong E_2|_{S^o}$. Then, there is a canonical isomorphism of line bundles on $X$,*

$$\mathsf{Det} f_*(E_1 - E_2) = I_{S/X}\, C_2\,(E_1 - E_2)$$

**(18.3)** The aim of this appendix is to provide a categorical interpretation of Theorem 18.3 above. In other words, we would like to find describe an equivalence of gerbes from which Theorem follows by looking at morphisms between objects in the corresponding gerbes. In our formulation, this statement is essentially a comparison between two central extensions and as such was inspired and is similar to the results of [KV07]. It would be interesting to establish the precise connection between these two results, though we note that [KV07] is for the reductive group $GL(n)$ whereas we are concerned with $SL(n)$ here.

## 19. Determinantal Central Extensions and Gerbes

**Let $G = SL(n)$ be fixed throughout this appendix.** Though this may be unnecessarily restrictive, we assume this for convenience.

**(19.1)** Given an irreducible $X \in \mathsf{Sch}_k$, we have constructed an extension of sheaves on $X$,

$$\mathsf{K}_2((t)) \to \widetilde{G} \to G((t)).$$

Pushing by the tame symbol $\partial : \mathsf{K}_2((t)) \to \mathscr{O}^*$ we obtain a central extension

$$\mathscr{O}_X^* \to \widehat{G} \to G((t))$$

which is usually called the *primitive* central extension.

We would like to identify this central extension in some more concrete terms. We do this using the notion of relative determinant lines on affine grassmannians. Recall from Chapter 1 §4, the dictionary between central extensions and multiplicative torors. Specializing to our case, we are interested in the category of multiplicative $\mathscr{O}^*$-torsors on the affine grassmannian $\widehat{Gr}_X := G((t))/G[[t]]$ over $X$.

**(19.2)** As a preliminary step in constructing such a multiplicative line bundle, let us recall the following notions from [KV07, §3]. Let $X$ be a scheme and $\mathscr{E}$ a locally free $\mathscr{O}_X((t))$-module of rank $N$. Then,

**Definition 19.2.1.** A *lattice* in $\mathscr{E}$ is a sheaf of $\mathscr{O}_X[[t]]$-submodules $\mathscr{F} \subset \mathscr{E}$ such that Zariski locally on $X$ the pair $(\mathscr{F}, \mathscr{E})$ is isomorphic to the pair $(\mathscr{O}_X[[t]]^n, \mathscr{O}_X((t))^N)$. A *special lattice* is a lattice equipped with a trivialization of $\wedge^{top}\mathscr{F}$ as an $\mathscr{O}_X[[t]]$-module.

The fundamental properties of such lattices are then given by,

**Proposition 19.2.2.** [KV07, Prop. 3.1] *Let $X$ be quasicompact and $\mathscr{F}_1, \mathscr{F}_2$ be two special lattices. Then,*
*(a) Zariski locally on $X$, there are integers $a, b$ such that*

$$t^a\mathscr{F}_1 \subset \mathscr{F}_2 \subset t^b\mathscr{F}_1$$

*(b) Suppose $\mathscr{F}_1 \subset \mathscr{F}_2$. Then $\mathscr{F}_2/\mathscr{F}_1$ is a locally free $\mathscr{O}_X$-module of of finite rank.*

**(19.3)** A line bundle on $\widehat{Gr}_X$ is specified by a family of line bundles $\mathscr{L}_U$ for every map $U \to X$. There is, however, no canonical choice of such a line bundle on $\widehat{Gr}_X$. Rather, there is a canonical line bundle on the product $\widehat{Gr}_X \times \widehat{Gr}_X$, described as follows: It is well known that each map $f : U \to Gr$ gives us a lattice $\mathscr{F}_U \subset \mathscr{O}_U((t))^n$. Given two such maps $f_1, f_2$, we obtain two special lattices $\mathscr{F}_{f_1}, \mathscr{F}_{f_2}$ in $\mathscr{O}_U((t))^n$. By Proposition 19.2.2 above, we then obtain a well-defined projective $\mathscr{O}_U^*$-bundle of rank 1 (i.e. line bundle on $U$) by the relative determinant,

$$(\mathscr{F}_{f_1}|\mathscr{F}_{f_2}) = \wedge^{top}\mathscr{F}_1/(\mathscr{F}_1 \cap \mathscr{F}_2) \otimes \wedge^{top}\mathscr{F}_2/(\mathscr{F}_1 \cap \mathscr{F}_2)^{-1}$$

The usual properties of the relative determinant [ACK87] give us canonical isomorphisms,

$$(\mathscr{F}_1|\mathscr{F}_2) \otimes (\mathscr{F}_2|\mathscr{F}_3) \to (\mathscr{F}_1|\mathscr{F}_3).$$

So we have constructed an $\mathscr{O}^*$-torsor on $\widehat{Gr}_X \times \widehat{Gr}_X$, i.e. for each $f, g : U \to \widehat{Gr}_X$ we have constructed a line bundle which we call $\Delta(f : g)$ on $U$. For a fixed $f_0 : X \to \widehat{Gr}_X$, we can then construct a line bundle on $\widehat{Gr}_X$ by defining for each $g : U \to \widehat{Gr}_X$ the line bundle

$$\Delta(g) := \Delta(f_0 : g).$$

The usual properties of determinant lines (see [ACK87]) then show that the $\Delta(g)$ form a multiplicative $\mathscr{O}^*$-torsor on $\widehat{Gr}_X$ and hence give rise to a central extension of $G((t))$ which splits over $G[[t]]$. One can also show that the central extension $\Delta$ obtained above is independent of the choice of $f_0$. It is known that this central extension agrees with the primitive central extension constructed from the tame symbol [Kum97].

**Remark 19.3.1.** In what follows, we shall actually need the following twisted version of the above constructions. Let Aut $[[t]]$ be the sheaf of groups constructed in Chapter 2 §1. Then, we have a central extension

$$\mathsf{K}_2((t)) \to \text{Aut } [[t]] \rtimes \widetilde{G}_X \to \text{Aut } [[t]] \rtimes G((t))$$

which we can again push by the tame symbol to obtain a central extension,

$$\mathscr{O}_X^* \to \text{Aut } [[t]] \rtimes \widehat{G}_X \to \text{Aut } [[t]] \rtimes G((t)).$$

This central extension will again be called the *twisted primitive central extension*.

**(19.4)** *Determintal Gerbes:* Let $\mathscr{E}$ be a locally free $\mathscr{O}_X((t))$-module. Then we can construct a sheaf of groupoids on $X$ as follows: the objects over an open $U \to X$ are given by special lattices in $\mathscr{F} \subset \mathscr{E}|_U((t))$ and morphisms between two such lattices $\mathscr{F}_1, \mathscr{F}_2$ are given by the relative determinant line $(\mathscr{F}_1|\mathscr{F}_2)$. Then by [KV07, Lemma 3.3.2], this sheaf of groupoids gives rise to a $\mathscr{O}_X^*$-gerbe on $X$ which we shall henceforth denote by $\mathsf{Det}_{\mathscr{E}}^{\mathscr{O}^*}$.

**Remark 19.4.1.** A twisted version of this construction also holds in which $\mathscr{E}$ is taken to be a Aut $[[t]] \rtimes G_X((t))$-torsor on $X$. We continue to denote the gerbe so obtained by $\mathsf{Det}_{\mathscr{E}}^{\mathscr{O}_X^*}$.

## 20. Central Extensions and Gerbes of Liftings

**(20.1)** Let $(A, E, G, a, b)$ be a central extension as in Chapter 1 §4. Let $P$ be a $G$-torsor. Then we obtain a gerbe of liftings which we denote by $\mathsf{Lift}_P^A$ whose equivalence class coincides with the image of $P$ under the boundary map

$$H^1(X, G) \to H^2(X, A).$$

Suppose we are given two central extensions $(A, E_1, G, a_1, b_1)$ and $(B, E_2, G, a_2, b_2)$, together with a map of central extensions,

$$\begin{array}{ccccc} A & \xrightarrow{a_1} & E_1 & \xrightarrow{b_1} & G \\ \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\ B & \xrightarrow{a_2} & E_2 & \xrightarrow{b_2} & G \end{array} .$$

Then, for each $G$-torsor $P$, we have the following equivalence of gerbes,

$$\partial_* \mathsf{Lift}_P^{E_1} \cong \mathsf{Lift}_P^{E_2}. \tag{20.1.1}$$

Recall that gerbe $\partial_* \mathsf{Lift}_P^{E_1}$ is constructed as follows: its objects are the same as those of $\mathsf{Lift}_P^{E_1}$, but the morphisms are obtained by pushing the $\mathsf{K}_2$ torsor by $\partial_*$. This equality of gerbe is essentially the compatibility of the gerbe of liftings construction with the commutativity of the square,

$$\begin{array}{ccc} H^1(X,G) & \longrightarrow & H^2(X,A) \\ \downarrow = & & \downarrow \partial_* \\ H^1(X,G) & \longrightarrow & H^2(X,B) \end{array} .$$

**(20.2)** Let $\mathscr{E}$ be a $G((t))$ torsor on $X$. Then associated to the central extension by $\mathsf{K}_2$ we obtain a gerbe of liftings with band $\mathsf{K}_2((t))$ on $X$ which we call $\mathsf{Lift}_{\mathscr{E}}^{\mathsf{K}_2}$. Associated to the primitive central extension we obtain a gerbe of liftings with band $\mathscr{O}^*$ on $X$ which we call $\mathsf{Lift}_{\mathscr{E}}^{\mathscr{O}^*}$.

We can also construct a determinantal gerbe $\mathsf{Det}_{\mathscr{E}}^{\mathscr{O}^*}$ on $X$. This gerbe is then equivalent to the gerbe of liftings $\mathsf{Lift}_{\mathscr{E}}^{\mathscr{O}^*}$ associated to the primitive central extension. The map is constructed as follows: given any $U \to X$, we can construct a functor

$$\Psi_U : \mathsf{Det}_{\mathscr{E}}(U) \to \mathsf{Lift}_{\mathscr{E}}^{\mathscr{O}^*}(U)$$

as follows. A special lattice $\mathscr{F} \subset \mathscr{E}$ gives a reduction of $\mathscr{E}$ to a $G[[t]]$-torsor. On the other hand, since $\widehat{G}$ splits over $G[[t]]$, we have a natural inclusion $G[[t]] \hookrightarrow \widehat{G}$. Pushing by this map, we obtain

$$H^1(X, G[[t]]) \to H^1(X, \widehat{G}).$$

The image of $\mathscr{F}$ under this map corresponds to a lifting of $\mathscr{E}$ to $\widehat{G}$, and so we define $\Psi_U(\mathscr{F})$ to be this image. The action of $\Psi_U$ on morphisms can be defined in a natural way so that $\Psi_U$ is an equivalence of categories which induces an equivalence of gerbes.

**(21.1)** Let $X$ be a curve embedded in a surface by $X \hookrightarrow S$, and suppose additionally that we are given a map $f : S \to X$ Let $S^o = S \setminus X$, $\widehat{S}_X$ be the formal completion of $S$ along $X$, and $\widehat{S}^o_X$ the complement of $X$ in $\mathscr{S}_X$. Let $\mathscr{E}$ be a projective $\mathscr{O}_{\widehat{S}^o_X}$-module with trivial determinant. Then we define may define a special lattice $\mathscr{F} \subset \mathscr{E}$ as in §2, as well as a determinantal gerbe $\mathsf{Det}_{\mathscr{E}}$.

**Lemma 21.1.1.** *Let $E$ be a bundle on $S$ with trivial determinant and let $\mathscr{E}$ the corresponding $\mathscr{O}_{\widehat{S}^o_X}$-bundle. Then, there is a bijective correspondence between the following sets,*
*(1) Extensions $F$ of $E$ to $S$ with trivial determinant*
*(2) Extensions $\mathscr{F}$ of $\mathscr{E}$ to $\widehat{S}_X$ with trivial determinant*
*(3) Special Lattices $\mathscr{F} \subset \mathscr{E}$.*

*Proof.* The equivalence of (1) and (2) is essentially the descent lemma of [BL94]. See also, [Kap00]. The equivalence of (2) and (3) is clear. □

**(21.2)** A bundle $\mathscr{E}$ on $\widehat{S}^o_X$ equipped with a trivialization of its determinant gives an $\mathrm{Aut}\,[[t]] \rtimes G((t))$-torsor by the procedure described in Chapter 2, §2 . Hence, we obtain a gerbe of liftings, $\mathsf{Lift}^{\mathscr{O}^*}_{\mathscr{E}}$ on $X$ which may be identified with $\mathsf{Det}_{\mathscr{E}}$.

**Proposition 21.2.1.** *Let $E$ be a bundle on $S^o$ with trivialization of its determinant and let $\mathscr{E}$ the corresponding* $\mathrm{Aut}\,[[t]] \rtimes G((t))$*-torsor. Then every extension of $F$ of $E$ to a bundle over $S$ with trivial determinant corresponds to a global section $\mathscr{F}$ of the gerbe* $\mathsf{Det}_{\mathscr{E}}$. *Given two global sections such bundles $F_1, F_2$ on $S$, we then have,*

$$\mathrm{Hom}_{\mathsf{Det}_{\mathscr{E}}}(\mathscr{F}_1, \mathscr{F}_2) = \mathsf{Det}\pi_*(F_1 - F_2)$$

*where the right hand side is the usual determinant of the cohomology of the map $\pi : S \to X$.*

*Proof.* The first part follows from the previous lemma, so let us focus on the second statement. First notice that the statement is local on $X$ so we may as well assume that $\mathscr{E}$ is trivial. To state the local assertion which the proposition follows from, we first need to introduce some terminology from [BL94]. Let $C$ be a smooth curve over a field $k$ and pick a point $p \in C$. Let $C^* = C \setminus \{p\}$. For a $k$-algebra $R$, we denote by $C_R := C \times_k \mathrm{Spec}(R)$ and $C^*_R = C^* \times_k \mathrm{Spec}(R)$. We shall also need the formal disc $D_R = \mathrm{Spec}(R[[t]])$ and its punctured analogue $D^*_R = \mathrm{Spec}(R((t)))$. We then have a cartesian square,

$$\begin{array}{ccc} D_R^* & \longrightarrow & D_R \\ \downarrow & & \downarrow g \\ C_R^* & \xrightarrow{j} & C_R^* \end{array} .$$

Let $F$ be a vector bundle on $C_R$ with trivial determinant. Then there exists an element $\gamma \in G(R((t)))$ such that $F$ is the kernel of the sequence,

$$0 \to F \to j_* \mathscr{O}^n_{C_R^*} \xrightarrow{\overline{\gamma}} g_*(K_{D_R}/\mathscr{O}_{D_R}) \to 0 \tag{21.2.1}$$

where $K_{D_R}$ is the sheaf on $D_R$ associated to the module $R((t))$, and $\overline{\gamma}$ is the inclusion of $j_*(\mathscr{O}_{C_R^*}) \to \mathscr{K}_{D_R}$ followed by the automorphism of $\mathscr{K}_{D_R}$ corresponding to the element $\gamma \in G((t))$. One can show that

$$\mathscr{F} = \Gamma(D_R, g^*F) \subset \mathscr{O}_R((t))^n$$

is a special lattice. The local statement from which the proposition follows is then

$$\wedge^{top} H^0(C_R, F) \otimes (\wedge^{top} H^1(C, F))^{-1} = (\mathscr{F} \,|\, t\mathscr{O}_R[[t]]), \tag{21.2.2}$$

where the right hand side is the relative determinant line from §2.3 and the left hand side is the determinant of the cohomology $\mathrm{Det}\pi_* F$ for $\pi : C_R \to R$.

Now we proceed to the proof of 21.2.2. From the sequence 21.2.1, we see that $H^0(C_R, F)$ and $H^1(C_R, F)$ arise as the kernel and cokernel of the map,

$$\Gamma(C_R, j_* \mathscr{O}^n_{C_R^*}) \to \Gamma(C_R, g_*(\mathscr{K}_{D_R}/\mathscr{O}_{D_R})).$$

On the other hand, consider the map,

$$\Gamma(D_R, F) \oplus \Gamma(C_R^*, E) \quad \to \quad \Gamma(D_R^*, E) \tag{21.2.3}$$

$$(f, g) \quad \mapsto \quad f - g. \tag{21.2.4}$$

Then it is easy to see that the kernel and cokernel of the sequences (1) and (2) are identical. Let $\mathscr{F}_- = t^{-1}R[t^{-1}]$ and $\mathscr{F}_+ = R[[t]]$.Then the kernel and cokernel of the above map is given by,

$$\mathrm{Ker} = \mathscr{F} \cap \mathscr{F}_- = \mathscr{F}/(\mathscr{F} \cap t\mathscr{F}_+)$$

and

$$\mathrm{coKer} = t\mathscr{F}_+/(\mathscr{F} \cap t\mathscr{F}_+).$$

Then 21.2.2 follows and the proposition is an easy consequence of this. □

(**21.3**) Finally, we obtain the main result of this appendix. Keep the notation of the previous proposition.

**Theorem 21.3.1.** *We have a natural equivalence of $\mathscr{O}^*$-gerbes on $X$,*

$$\partial_* \mathsf{Lift}_{\mathscr{E}}^{\mathsf{K}_2} \cong \mathsf{Det}_{\mathscr{E}}.$$

*Moreover, given any two extensions $F_1, F_2$ of $E$ to $S$, we let $\mathscr{V}_1$ and $\mathscr{V}_2$ be the corresponding objects in $\mathsf{Lift}_{\mathscr{E}}^{\mathsf{K}_2}$. Then we have,*

$$\mathsf{Det}\pi_*(F_1 - F_2) = \partial_* \operatorname{Hom}_{\mathsf{Lift}_{\mathscr{E}}^{\mathsf{K}_2}}(\mathscr{V}_1, \mathscr{V}_2)$$

**Remark 21.3.2.** We conclude with some assorted remarks.

(1) In this form, Theorem 21.3.1 looks very similar to Theorem 1.2.1 above. The analogy would be precise if we could identify $\partial_* \operatorname{Hom}_{\mathsf{Lift}_{\mathscr{E}}^{\mathsf{K}_2}}(\mathscr{V}_1, \mathscr{V}_2)$ with $I_{S/X}\, C_2\, (F_1, F_2)$. To prove such an assertion, one might try to show that our construction of $\partial_* \operatorname{Hom}(E_1, E_2)$ satisfies all the defining properties of $I_{S/X}\, C_2$ claimed in [Del87].

(2) It seems a bit awkward to have to resort to three gerbes above: two gerbes of liftings and the determinant gerbe. It would be preferable to eliminate all reference to the determinant gerbe.

(3) The above statement is closely related to in our Theorem 4.5.1 from Chapter 3. In each case the line bundle incarnating the second relative Chern class is obtained as follows: lift the cocycle representing the each bundle to the central extension by $\mathsf{K}_2$ and then take their difference. Applying the tame symbol to the difference gives a Čech cocyle to a line bundle on $X$ coming with the central extension by $\mathscr{O}^*$.

(4) Deligne's Riemann-Roch theorem can be stated more generally as follows: let $f : S \to X$ be as in the introduction with $N = 1$. Then,

$$\det f_* E^{\otimes 12} = I_{S/X}\, C_2(E)^{\otimes -12} \langle \omega_{S/X}, \omega_{S/X} \rangle^{\mathsf{rk}E} \langle \wedge^{top} E, \wedge^{top} E \otimes \omega_{S/X}^{-1} \rangle,$$

where $\omega_{S/X}$ is the relative dualizing sheaf of $S \to X$. Theorem 21.3.1 above though is only related to a relative version of Deligne's theorem. It would be interesting to obtain a purely categorical interpretation of the entire Riemann-Roch statement. Towards this end, we propose (see also [BS88, Appendix A.5]) the following gerby interpretation of $I_{S/X}\, C_2\, (E)$ and $\langle L, M \rangle$. Given a $f : S \to X$, consider the central extensions on $S$,

$$\begin{array}{ccc} \mathsf{K}_2 \to & \widetilde{G} & \to G \\ \mathsf{K}_2 \to & H & \to \mathscr{O}^* \times \mathscr{O}^* \end{array}$$

To a given $G$-bundle $E$, we may use the first to construct a gerbe $\mathsf{Lift}_E^{\mathsf{K}_2}$. Similarly to a pair of line bundles $L, M$, viewed as an $\mathscr{O}^* \times \mathscr{O}^*$-torsor, we can construct a gerbe $\mathsf{Lift}_{L,M}^{\mathsf{K}_2}$. Both of these gerbes have classes in $H^2(S, \mathsf{K}_2)$. Pushing these gerbes forward by $f_*$ we obtain a $R^1 f_* \mathsf{K}_2$-torsor on $X$ (using Chaper 1 §4 and the Gersten resolution) which we can in turn push to obtain an $\mathscr{O}^*$-torsor (again using the Gersten resolution). Applying this construction to the gerbes $\mathsf{Lift}_E^{\mathsf{K}_2}$ (when $G = GL(n)$ ) and $\mathsf{Lift}_{L,M}^{\mathsf{K}_2}$, we obtain line bundles $\mathscr{C}_2(E)$ and $(L, M)$ which we conjecture agree with Deligne's line bundles.

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