VERTEX ALGEBRAS AS TWISTED BIALGEBRAS: ON A THEOREM OF BORCHERDS

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To Joe Gallian on his 65th birthday

ABSTRACT. Following Borcherds, we show a certain class of vertex algebras can be uniquely constructed from a bialgebra together with a twisted multiplication by a bicharacter. We illustrate this construction in the case of Heisenberg and lattice vertex algebras. As a consequence, we see that these vertex algebras can be recovered from their 2-point correlation functions and their underlying bialgebra structure.

CONTENTS

1. INTRODUCTION

(1.1) Today, vertex algebras have become rather ubiquitous in mathematics. Among the subjects on which they bear direct influence, we can list representation theory, finite group theory, number theory (both through the classical theory of elliptic functions and the modern geometric Langlands theory), combinatorics, and algebraic geometry. Despite their firm entrenchment within the world of pure mathematics, vertex algerbas (or rather their consituent elements *vertex operators*) actually first arose in the 1970s within the early string theory literature concerning dual resonance models [S]. Independently of this, [LW] Lepowsky and Wilson were interested in constructing representations of a twisted form of the affine Lie algebra $\widehat{sl}_2$ using differential operators acting on the ring of infinite polynomials $\mathbb{C}[x_1, x_2, \ldots]$. To do so, the main difficulty they faced was in representing a certain infinite dimensional Heinsenberg algebra. This they achieved using rather complicated looking formulas. Howard Garland then observed that essentially the same formulas occurred within the physics literature, and he thus imported the "vertex operator" into mathematics. Shortly thereafter, Igor Frenkel and Victor Kac [FK] (and G. Segal [Se] independetly) constructed the *basic representation* of untwisted affine Lie algebras, again using the newly christened vertex operators. Within the world of representation theory of affine Lie algebras, vertex operators were thus seen to play a central role.

Another significant impetus to the development of vertex operators came from the theory of finite groups. In the classification of finite groups, the Fischer-Greiss Monster (aka, "friendly giant") is the largest of the 26 sporadic simple groups. A number of very mysterious empirical phenomena concerning this group began to surface [see Bo3 for a review]. These results collectively went under the title *Monstrous*

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moonshine, and one particular facet of moonshine was the prediction was that most natural representation of this finite simple group was actually of infinite dimension! The question then arose as to explicitly construct such a representation. Again, vertex operators proved to be essential here, and in [FLM1], Frenkel-Lepowsky-Meurman construted the sought-after Moonshine module with the aid of vertex operators. Richard Borcherds [Bo1] then proposed an axiomatic framework to deal systematically with all these occurrences of vertex operators, and define what we today refer to as vertex operator *algebras.* Then in [FLM2], the authors showed that their Moonshine module had a natural vertex algebra structure and moreover that the Monster group could be realized as the group of automorphisms of this vertex operator algebra. Motivated by these results, it was widely believed by the mid 1980s that other vertex operator algebras should similarly encapsulate other sporadic simple groups. However, it was not until the work of John Duncan [Du1, 2], nearly two decades after the pioneering work of [FLM], that this was to come to fruition.

Though there is no geometry in this paper, it was motivated by a connection between vertex algebras and algebraic geometry. H. Nakajima [Na] and I. Grojnowski [Gr] have constructed the *Fock space* representation of an infinite dimensional Heisenberg algebras on the cohomology of the Hilbert scheme of points on an algebraic surface. This Fock space is none other than the ring of symmetric functions in infinitely many variables, the working ground for many combinatorialists. The fact that this same Fock space also possesses a vertex operator algebra structure has facilitated the migration of results back and forth between combinatorics and the theory of vertex algebras. For example, the Boson-Fermion correspondence from vertex algebras has been shown by I. Frenkel to encode (or be encoded by) the combinatorics of Littlewood-Richardson coefficients [FJ]. Returning to our geometric setup, certain "halves" of vertex operators have been shown to act on the cohomology of Hilbert schemes by means of cup-product with the Chern classes of certain tautological bundles [Na]. Essentially this observation was enough to unravel the cup-product structure on Hilbert schemes, a problem that had otherwise seemed intractable [LS]. I. Frenkel has then posed the question of understanding the full vertex algebra structure geometrically. Despite interesting progress [Le], the answer to this question still remains quite mysterious. Our approach was a synthetic one which attempted to first pare down the axioms of a vertex algebra so that each part may have a clear geometric meaning. This we do by means of Borcherds' theorem.

(1.2) *What exactly is a vertex algebra?* Unfortunately, this question does not have a very short answer. For example, the statement that a vertex algerba is a particular type of algera is false! Rather one may think of a vertex algebra as a vector space V (usually infinite dimensional) endowed with an infinite family of multiplications $\circ_n$,

$$\begin{aligned} \circ_n : V \times V &\rightarrow V \\ a \times b &\mapsto a \circ_n b \end{aligned}$$

These products are neither associative nor commutative, but satisfy conditions somewhere in between them, the so called Borcherds identities [Bo1]. An alternative starting point for vertex algerbas is the axiomatic treatment of Frenkel-Huang-Lepowsky [FHL]. Here one begins with a vector space V equipped with a "state-field correspondence," i.e. a linear map,

$$\begin{aligned} Y(\cdot, z) : V &\rightarrow \mathrm{End}(V)[[z, z^{-1}]] \\ a &\mapsto Y(a, z) \end{aligned}$$

The $Y(\cdot, z)$ may be viewed as a generating function encoding all the infinite products $\circ_n$ into one object: indeed, applying $Y(a, z)$ to b, we obtain

$$Y(a,z)b = \sum_{n \in \mathbb{Z}} (a \circ_n b) z^n.$$

One then develops a formal calculus to deal with these series and phrases the axioms of a vertex algebra in terms of properties of $Y(\cdot, z)$. This viewpoint is reviewed in section 3.

There is yet another related viewpoint put forth by Borcherds which purports to "make the theory of vertex algebras trivial" [Bo4]. In other words, a fairly elaborate categorical framework is constructed in which one can view vertex algebras as commutative ring objects. Borcherds' motivation for such a description seems to have come from the important question of quantizing vertex algebras, but in our exposition we will not be concered with any such quantum deformation. Neither will we deal with the categorial framework which Borcherds has constructed. Rather, our goal will be to present in as simple-minded a way as possible the new construction of vertex algebras contained in [Bo4]. The reader is warned here of the obvious danger that such a simplification necessarily introduces and is referred to the original works [Bo3, 4] for further motivations.

In our formulation, the essence of Borcherds' construction is that starting from some rather innocuous algebraic elements– a commutative, cocommutative bialgebra V equipped with a derivation $T : V \to V$ and a symmetric bicharacter $r : V \otimes V \to \mathbb{C}[(z-w)^{-1}]$– one can show how to construct the seemingly richer structure of a vertex algebra on V. In this light, a natural predecessor to Borcherds' result is the simple fact (due to Borcherds too?) that a holomorphic vertex algebra structure on V, i.e., one in which the fields $Y(a,z)v \in End(V)[[z]]$ are just power series, is equivalent to the structure of a commutative algebra with derivation on V. Briefly, this equivalence is constructed as follows: given a holonomic vertex algebra, we may define a product on V by

$$a \cdot b := [Y(a,z)b]_{z^0}$$

where by $[A(z)]_{z^0}$ denotes the coefficient of the constant term of $A(z) \in V((z))$. The axioms of a vertex algebra then provide V with a natural derivation compatible with this algebra structure. Conversely, given a commutative unital algebra with derivation T, we can construct a state-field correspondence by the formula

$$Y(a,z)b = (e^{zT}a) \cdot b.$$

The axioms for a vertex algebra then follow readily from the ring properties V.

Since the most interesting vertex algebras are not holomorphic, we might wonder how to modify the above construction by somehow introducing singularities (i.e., negative powers of z) into the picture. Borcherds answer to this question is as follows: given a *bi*algebra V, a derivation T compatible with the bialgebra structure, and bicharacter $r : V \otimes V \to \mathbb{C}[(z-w)^{-1}]$ he defines a state-field correspondence through the the following elegant formula which combines all the data which we are given:

$$Y(a,z)b = \sum_{(a)} T^{(k)}(a') \cdot b' \cdot r(a'', b''),$$

where the coproduct on V is written as $\Delta(a) = \sum_{(a)} a' \otimes a''$. Our first goal will be to directly prove that this formula does in fact define a vertex algebra on V.

The next question we address is what types vertex algebras can be constructed using the new approach. Along these lines, we define a certain class, called r-vertex algebras, which contain the vertex algebras constructed as above from commutative, cocommutative bialgebras V. For example, Heisenberg and Lattice vertex algebras are contained in this class. The second main result in this note is a uniqueness theorem which shows that if the underlying bialgebra of an r-vertex algebra V is *primitively generated*, then it is *the* vertex algebra as constructed above from a bialgebra V, derivation T, and bicharacter r determined by the 2-point functions. There are many other natural examples of vertex algebra in representation theory, among them the vacuum modules for affine Kac-Moody and Virasoro algebras. These vertex algebras do not have a commutative algebra structure and thus do not fall under the twisted bicharacter construction presented in this paper. It is an interesting question as to how this construction can be modified to account for such examples.

We would also like to add that since this paper was written, Anguelova and Bergvelt have extended and clarified many aspects of Borcherds proposal for quantizing vertex algebras in their interesting paper [AB].

(1.3) This paper is organized as follows: In section 2, we set up some preliminary algebraic machinery. The reader is advised to atleast skim over this part as many terms like bialgebra, derivation, etc. are given slightly nonstandard meanings. In section 3, we review some basics from the theory of vertex algebras, and in section 4, we state and prove Borcherds's theorem. In section 5, we state a converse, and we conclude with some examples in section 6.

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Finally, I would like to dedicate this paper to Joe Gallian on the occassion of his 65^{th} birthday. During the summer of 2000, I had the wonderful opportunity to participate in his REU program and to be so pleasantly initiated into the world of mathematical research. His unrivaled enthusiasm and dedication have been an inspiration ever since.

2. Algebraic Preliminaries

We fix an algebraically closed field k of characteristic zero.

(2.1) For us, an **algebra** A will be a commutative $\mathbb{Z}_+$-graded k-algebra with unit, $A = \oplus_{n=0}^{\infty} A_n$, where each A_n is a finite dimensional k-vector space.

Dually, a **coalgebra** will be a cocommutative $\mathbb{Z}_+$-graded k-coalgebra (C, Δ, ε) where $\Delta : C \to C \otimes C$ is the comultiplication. We adopt Sweedler notation to write the coproduct, $\Delta(x) = \sum_{(x)} x' \otimes x''$. Generalizing, we will also write for example

$$\Delta_2(x) = (\Delta \otimes 1)\Delta(x) = (1 \otimes \Delta)\Delta(x) = \sum_{\Delta_2(x)} x^{(1)} \otimes x^{(2)} \otimes x^{(3)}.$$

Writing $C = \oplus_{n=0}^{\infty} C_n$, then we assume C_n is a finite k-vector space, $\varepsilon(C_n) = 0$ for $n \neq 0$, and $\Delta C_n \subset \sum_{i=0}^{n} C_i \otimes C_{n-i}$. We say that a coalgebra is **connected** if $C_0 = k$.

We define a **bialgebra** to be an algebra A with a coalgebra structure (A, Δ, ε) such that the gradings on A as an algebra and coalgebra agree and Δ and ε are algebra morphisms. We say that an element of a bialgerba is **primitive** if $\Delta(x) = x \otimes 1 + 1 \otimes x$ and **group-like** if $\Delta(x) = x \otimes x$.

(2.2) By a **bialgebra with derivation**, we shall mean bialgebra V with a linear map $T : V \to V$ be a degree 1 satisfying

(1) $T(1) = 0$
(2) $T(a \cdot b) = T(a) \cdot b + a \cdot T(b)$
(3) $T(\Delta(a)) = \sum_{(a)} T(a') \otimes a'' + \sum_{(a)} a' \otimes T(a'')$

We say that a set $\{a_\alpha | a_\alpha \in V\}_{\alpha \in S}$ is a ***T*-generating** set of an algebra V if $\{T^{(k)} a_\alpha\}_{\alpha \in S}$ generates V as an algebra. Here, as usual, $T^{(k)} := \frac{T^k}{k!}$

(2.3) We next describe the additional piece of information on a bialgebra with derivation which allows us to construct interesting vertex algebras.

Definition 1. Let V be a bialgebra with derivation T. Then a **bicharacter** is a bilinear map $r_{z,w} : V \times V \to k[(z-w)^{\pm 1}]$ satisfying

(1) $r(a,b) = r(b,a)$;
(2) $r(a,1) = \varepsilon(a)$;
(3) $r(a \cdot b, c) = \sum_c r(a,c') r(b,c'')$
(4) $r(Ta, b) = -\frac{d}{dz} r(a,b)$
(5) $r(a, Tb) = \frac{d}{dw} r(a,b)$.
(6) For $a \in V_m$ and $b \in V_n$, $r(a,b) = \frac{c(a,b)}{(z-w)^{-m-n}}$, where $c(c,b)$ is some scalar.

Suppose now (V, T, r) is a bialgebra with derivation and bicharacter. In what follows, we will also make use of the following multivariable generalization of r.

Definition 2. Let $a_1, \ldots, a_n \in V$. Then we recursively define

$$r_{z_1,\ldots,z_n} : V \times V \cdots V \to k[(z_i - z_j)^{\pm 1}]_{1 \leq i \neq j \leq n},$$

to be

$$r_{z_1,\ldots,z_n}(a_1,\ldots,a_n) = \sum_{\Delta_{n-1}(a_1),(a_2),\ldots,(a_n)} r_{z_2,\ldots,z_n}(a'_2,\ldots,a'_n)\Pi_{j=2}^{n} r_{z_1,\ldots,\hat{z_j},\ldots z_n}(a_1^{(j-1)}, a''_2, \ldots, \widehat{a''_j}, \ldots a''_n)$$

Example 1. To make things more transparent, we unravel the above formula to get,

$$r_{x,y,z}(a,b,c) = \sum_{(a),(b),(c)} r_{y,z}(b',c')r_{x,y}(a',b'')r_{x,z}(a'',c'')$$

and similarly

$$r_{x,y,z,w}(a,b,c,d) = \sum_{\Delta_2(a),\Delta_2(b),\Delta_2(c),\Delta_2(d)} r_{x,y}(a^{(1)},b^{(1)})r_{x,z}(a^{(2)},c^{(1)})r_{x,w}(a^{(3)},d^{(1)})r_{y,z}(b^{(2)},c^{(2)})r_{y,w}(b^{(3)},d^{(2)})r_{z,w}(c^{(3)},d^{(3)})$$

Using the properties of r and the (co)commutativity of V, we then have,

Lemma— *Let $a_1, \ldots, a_n \in V$ and $\sigma \in S_n$ a permutation. Then*

$$r_{z_1,\ldots,z_n}(a_1,\ldots,a_n) = r_{z_{\sigma(1)},\ldots,z_{\sigma(n)}}(a_{\sigma(1)},\ldots,a_{\sigma(n)}).$$

(2.4) Given a bialgebra V, we can form its graded dual $V^\vee = \oplus_{n=0}^{\infty} V_n^*$. By a **bilinear form** on V we will mean the following a non-degenerate, symmetric, bilinear form $(\cdot,\cdot)$ on the vector space V satisfying

(1) $(V_m, V_n) = 0$ unless $m = n$ (identifying V with $V^\vee$ naturally)
(2) $(1, v) = \varepsilon(v)$

3. Vertex Algebras

(3.1) We must first introduce the concept of a *field*.

Definition 3. Let V be a vector space over k. A formal power series

$$A(z) = \sum_{j \in \mathbb{Z}} A_{(j)} z^{-j} \in \operatorname{End} V[[z^{\pm 1}]]$$

is called a **field** on V if for any $v \in V$ we have $A_{(j)} \cdot v = 0$ for large enough j.

In case V is $\mathbb{Z}_+$ graded, we can define a field of **conformal dimension** $d \in \mathbb{Z}$ to be a field where each $A_{(j)}$ is a homogeneous linear operator of degree $-j + d$

A useful procedure for dealing with fields will be normal ordering, defined as follows.

Definition 4. Let Let $A(z), B(w)$ be fields, and denote by $A(z)_+$ and $A(z)_-$ the non-negative and negative parts (in powers of z) of $A(z)$. Then define $: A(z)B(w) := A(z)_+ B(w) + B(w)A(z)_-$ to be **normally ordered product** of $A(z)$ and $B(w)$.

(3.2) We are now in a position to define the main object of study,

Definition 5. A **vertex algebra** $(V, |0\rangle, T, Y)$ is a collection of data:

- (state space) a $\mathbb{Z}_+$ graded k-vector space $V = \oplus_{m=0}^{\infty} V_m$, with each $\dim V_m < \infty$
- (vacuum vector) $|0\rangle \in V_0$
- (translation operator) $T : V \to V$ of degree 1.
- (vertex operators) $Y(\cdot, z) : V \to \operatorname{End}(V)[[z^{\pm 1}]]$ taking each $A \in V_m$ to a field

$$Y(A,z) = \sum_{n \in \mathbb{Z}} A_{(n)} z^{-n-1}$$

of conformal dimension m (i.e., $degA_{(n)} = -n + m - 1$).

satisfying the following axioms:

- (vacuum axiom) $Y(|0\rangle, z) = Id_V$. Furthermore, for any $A \in V$, $Y(A,z)|0\rangle \in V[[z]]$ and $Y(A,z)|0\rangle|_{z=0} = A$.
- (translation axiom) For every $A \in V$, $[T, Y(A,z)] = \frac{d}{dz}Y(A,z)$ and $T|0\rangle = 0$
- (locality 1) For every $A, B \in V$, there exists a positive integer N such that $(z-w)^N[Y(A,z), Y(B,w)] = 0$ as formal power series in $\mathrm{End}(V)[[z^{\pm 1}, w^{\pm 1}]]$.

(3.3) *Example: Heisenberg Vertex Algebra* Let $\mathfrak{h}$ be an l-dimensional complex vector space equipped with a non-denegerate, symmetric, bilinear form $(\cdot,\cdot)$. The *Heisenberg Lie algebra* is defined to be $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ with relations $[K, \mathfrak{h}] = 0$ and $[a \otimes t^m, b \otimes t^n] = (a,b)m\delta_{m,-n}K$. We will denote $a \otimes t^n, a \in \mathfrak{h}$ by $a(n)$. For each λ in the dual space $\mathfrak{h}^*$, we may construct the Fock space by the following induction:

$$\Pi^\lambda = U(\hat{\mathfrak{h}}) \otimes_{U(\mathfrak{h}\otimes\mathbb{C}[t]\oplus\mathbb{C}K)} \mathbb{C}$$

where $\mathbb{C}$ is the one dimensional space on which $\mathfrak{h} \otimes t\mathbb{C}[t]$ acts as zero, K acts as the identity, and $\mathfrak{h} \otimes t^0$ acts via the λ. As vector spaces,

$$\Pi^\lambda = \mathrm{Sym}(\mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}]).$$

The space $\Pi := \Pi^0$ has a vertex algebra with vertex operators are described as follows: for $h \in \mathfrak{h}$, set $h(z) = \sum_{n\in\mathbb{Z}} h(n)z^{-n-1}$ where $h(n)$ acts as the operator $h \otimes t^n$ on the $\hat{\mathfrak{h}}$-module Π. Explicitly, K acts as the identity, $h(n)$ acts by multiplication for $n < 0$, derivation defined by $h(n)a(-s) = n\delta_{n,s}(h,a)$ for $n > 0$, and 0 for $n = 0$. Then set

$$Y(a_1(j_1)\cdots a_k(j_k), z) = \frac{1}{(-j_1-1)!\cdots(-j_k-1)!} : \partial_z^{j_1-1}a_1(z)\cdots\partial_z^{-j_k-1}a_k(z).$$

The verification of the vertex algebra axioms is left as an exercise, and will also follow from our theorem below.

(3.4) Let $(V, |0\rangle, T, Y)$ be a vertex algebra, and assume that it is equipped with a non-degenerate, symmetric, bilinear form $(\cdot,\cdot)$ which identifies V with its restricted dual $V^\vee$. With this assumption, denote $\langle 0| \in V$ the dual to the element $|0\rangle \in V$. Then, we introduce the

Definition 6. Given $a_1, \ldots, a_n \in V$. We define the ***n*-point functions** to be

$$(\langle 0|, Y(a_1, z_1)\cdots Y(a_n, z_n)|0\rangle) \in k[[z_1, z_2, \ldots, z_n]][(z_i - z_j)]_{1\le i<j\le n}$$

4. BORCHERDS'S THEOREM

(4.1) In this section, we state and prove Borcherds's theorem [?]. Let V be a bialgebra with derivation T and bicharacter r. We aim to construct a vertex algebra on V with translation operator given by V and 2-point functions specified by r. We first define the following maps which will allow us to construct vertex operators,

Definition 7. Let

$$\Phi^n(z_1, \ldots, z_n) : V \otimes V \otimes \cdots \otimes V \to V[[z_1, z_2, \cdots, z_n]][(z_i - z_j)^{-1}]$$

send

$$a_1 \otimes a_2 \cdots a_n \mapsto \sum_{(a_1)\cdots(a_n)} \sum_{j_1,\ldots j_k \ge 0} T^{(j_1)}(a_1') \cdot T^{(j_2)}(a_2') \cdots T^{(j_n)}(a_n') r_{z_1,\ldots,z_n}(a_1'', a_2'', \ldots, a_n'').$$

So, for example, we have that

$$\Phi^2(z,w)(a,b) = \sum D^{(j)}(a')D^{(k)}(b')r_{z-w}(a'', b'').$$

The following is easy to see from (co)commutativity of V and the corresponding facts for r,

Lemma— *Let $a_1, \ldots, a_n \in V$ and $\sigma \in S_n$ a permutation. Then*

$$\Phi^n(z_1, \ldots, z_n)(a_1, \ldots, a_n) = \Phi^n(z_{\sigma(1)}, \ldots, z_{\sigma(n)})(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$$

as elements of $V[[z_1, \ldots, z_n]][(z_i - z_j)^{-1}]_{1\le i<j\le n}$.

(4.2) Following Borcherds, we make the following fundamental definition:

Definition 8. Let $a,b \in V$. Then define the field $Y(a,z)$ by

$$Y(a,z_1)Y(b,z_2)|0\rangle = \Phi(z_1,z_2)(a,b).$$

In particular, we have $Y(a,z_1)b = \Phi(z_1,0)(a,b)$.

With the above definition, we can verify the following formula for multiplication.

Lemma— *Let $a_1,\dots,a_n \in V$. Then $Y(a_1,z_1)\cdots Y(a_n,z_n)|0\rangle = \Phi(a_1,\dots,a_n)$*

Proof. The proof is a straightforward computation. For ease of notation, we just sketch the case $n=3$ which already illustrates all the main ideas. Let $a,b,c \in V$. Then from the above definition, we see that

$$Y(a,z_1)Y(b,z_2)Y(c,z_3)|0\rangle = Y(a,z_1)\sum_{(b),(c)}\sum_{k,l\geq 0} z_2^k z_3^l T^{(k)}(b')T^{(l)}(c')r_{z_2,z_3}(b'',c'').$$

Expanding

$$\sum_{(b),(c)} r_{z_2,z_3}(b'',c'')\sum_{k,l\geq 0} Y(a,z_1)[T^{(k)}(b')T^{(l)}(c')]z_2^k z_3^l,$$

we get

$$\sum_{(b),(c)} r_{z_2,z_3}(b'',c'')\sum_{k,l\geq 0} z_2^k z_3^l \sum_{(a)} r_{z_1}(a'',[T^{(k)}(b')T^{(l)}(c')]'')\sum_{j\geq 0} z_1^j T^{(j)}(a')[T^{(j)}(b')T^{(k)}(c')]'',$$

which using the fact that $\Delta(a\cdot b) = \Delta(a)\cdot\Delta(b)$ and property (3) of r becomes,

$$\sum_{(b),(c)} r_{z_2,z_3}(b'',c'')\sum_{k,l\geq 0} z_2^k z_3^l \sum_{(a)} r_{z_1}(a^{(2)},[T^{(j)}(b')]'')r_{z_1}(a^{(3)},[T^{(k)}(c')]'')\sum_{j\geq 0} z_1^j T^{(j)}(a')[T^{(j)}(b')]'[T^{(k)}(c')]'.$$

Observing that,

$$\Delta T^{(k)}(a) = \sum_{j=0}^{k}\sum_{(a)} D^{(k-j)}(a')\otimes D^{(j)}a'',$$

we collect terms in the above coefficients $T^{(j)}(a^{(1)})T^{(k)}(b^{(1)})T^{(l)}(c^{(1)})$ to get

$$\sum_{j,k,l\geq 0}\sum_{(a),(b),(c)} T^{(j)}(a^{(1)})T^{(k)}(b^{(1)})T^{(l)}(c^{(1)})\sum_{p,q\geq 0} z_1^j z_2^{k+p} z_3^{l+q} r_{z_1}(a^{(2)},T^{(p)}(b^{(2)})r_{z_1}(a^{(2)},T^{(q)}(c^{(2)}).$$

But Taylor's formula and the property (5) of r give that

$$r_{z,w}(a,b) = \sum_{n\geq 0} r_z(a,T^{(n)}(b))w^n.$$

Applying this fact to the above sum, we conclude that the terms with coefficient $T^{(j)}(a^{(1)})T^{(k)}(b^{(1)})T^{(l)}(c^{(1)})$ is just $r_{z_1,z_2,z_3}(a'',b'',c'')$ which concludes the proof.

□

(4.3) We are now ready to state the first theorem of this note,

Theorem 1— *Let V be a bialgebra with derivation T and bicharacter r. Define fields $Y(a,z)$ as in Definition 8. Then $(V,|0\rangle,T,Y)$ is a vertex algebra.*

Proof. It is obvious that the map $Y(\cdot,z): V \to V[[z]][z^{-1}]$ takes $a \in V_m$ to a field of conformal dimension m. So, let us verify the vacuum, translation, and locality axioms.

Vacuum: For $a \in V$, the above definition gives that $Y(a,z)|0\rangle = \sum_{k=0}^{\infty} T^{(k)}(a)z^k$, from which the vacuum axiom follows easily.

Translation: The translation axiom requires that

$$[T,Y(a,z)]b = \frac{d}{dz}Y(a,z)b.$$

Expanding the left hand side, we get

$$TY(a,z)b = T\left[\sum_{(a),(b),k\geq 0} T^{(k)}(a')b'r_z(a'',b'')z^k\right]$$

and

$$Y(a,z)Tb = \sum_{(a),(b),k\geq 0} T^{(k)}(a')T(b')r_z(a'',b'')z^k + \sum_{(a),(b),k\geq 0} T^{(k)}(a')b'r_z(a'',Tb'')z^k$$

Since

$$T\left[T^{(k)}(a')b'\right] = (k+1)T^{(k+1)}(a')b' + T^{(k)}(a')T(b'),$$

this difference equals

$$\sum_{(a),(b),k\geq 0} (k+1)T^{(k+1)}(a')b'r_z(a'',b'') - T^{(k)}(a')b'r_z(a'',Tb'')z^k$$

which is easily seen to be equal to

$$\frac{d}{dz}Y(a,z)b = \sum_{(a),(b),k\geq 0} T^{(k)}(a')\cdot b'\frac{d}{dz}[r_z(a'',b'')]z^k + T^{(k)}(a')\cdot b'r_z(a'',b'')\frac{d}{dz}[z^k]\ .$$

Locality: Locality essentially follows from the Lemma above. Indeed, let $a,b\in V$. Then

$$\Phi^3_{(z_1,z_2,0)}(a,b,c) = Y(a,z_1)Y(b,z_2)c$$

and similarly

$$\Phi^3_{(z_2,z_1,0)}(b,a,c) = Y(b,z_2)Y(a,z_1)c.$$

So, by Lemma Φ, we see that both

$$\Phi^3_{(z_1,z_2,0)}(a,b,c) = \Phi^3_{(z_2,z_1,0)}(b,a,c) \in End(V)[[z_1^{\pm 1}, z_2^{\pm 1}]][(z_1-z_2)^{-1}]$$

which means that there is some positive integer N such that

$$(z-w)^N[Y(a,z_1),Y(b,z_2)]c = 0.$$

Furthermore, this number N does not depend on c as is clear from looking at the explicit formula for Φ.

□

In case our bialgebra V also has a bilinear form, we have the following formula for the n-point functions.

Corollary— *Let $(V,|0\rangle,T,Y)$ be the vertex algebra constructed above and let $(\cdot,\cdot)$ be a bilinear form on V. Then*

$$(\langle 0|, Y(a_1,z_1)\cdots Y(a_n,z_n)|0\rangle) = r_{z_1,z_2,\ldots,z_n}(a_1,a_2,\ldots,a_n)$$

5. Converse to Borcherds's theorem

In this previous section, we showed how starting from a bialgebra V together with a derivation T and a bicharacter r we can construct a vertex algebra. Now, given a vertex algebra $(V,|0\rangle,T,Y)$ whose underlying space V also has a bialgebra structure with derivation T, we might wonder to what extent does this completely determine the vertex algebra structure. By itself, V and T cannot know about the singularities (hence, 2-point functions) of $Y(\cdot,z)$, so in order to get a meaningful answer to our question, we have to also feed in the information of r. If the bialgebra structure on V is not reflected in the vertex algebra structure, we will have no hope of recovering the vertex algebra structure. So, in addition to r, we need to assume some compatability conditions between the vertex and bialgebra structures. These considerations are formalized in the following definition.

Definition 9. An ***r*-vertex algebra** consists of the following data:

- a vertex algebra $(V,|0\rangle,T,Y)$
- a bialgebra structure with derivation (V,T) with compatible gradings.
- a bilinear form $(\cdot,\cdot)$ on V

satisfying the following additional axioms:

- for $a \in V$ primitive, $a_{(-1)}b = a \cdot b$
- for $a, b \in V$ primitive, $a_{(0)}b = 0$

With the above definition, we can state a partial converse to Borcherds's theorem. This is not the strongest statement that can be made, but the following theorem (or a slight variation of it) suffices to deal with all of the r-vertex algebras which we know.

Theorem 2— *Let V be a connected bialgebra with derivation T, bicharacter r and bilinear form $(\cdot,\cdot)$. Assume that V is T-generated by a set of primitive elements $\{a_\alpha\}_{\alpha\in S}$. Then there exists a unique r-vertex algebra structure on V such that the 2-point functions of V are given by r.*

Proof. In the previous section, we showed how to construct a vertex algebra on V with two-point functions specified by r. It is clear that this vertex algebra is actually a r-vertex algebra.

To prove the uniqueness, we shall make us of the following.

Reconstruction Theorem [FB], — *Let $V = \oplus_{n=0}^{\infty} V_n$ be a $\mathbb{Z}_+$-graded vector space, $|0\rangle \in V_0$ a non-zero vector, and T a degree 1 endomorphism of V. Let S be a (countable) set and $\{a^\alpha\}_{\alpha\in S}$ a collection of homogenous vectors in V. Suppose we are given fields*

$$a^\alpha(z) = \sum_{n\in\mathbb{Z}} a^\alpha_{(n)} z^{-n-1}$$

such that the following conditions hold:

(1) *For all α, $a^\alpha(z)|0\rangle = a^\alpha + z(\cdots)$*
(2) *$T|0\rangle = 0$ and $[T, a^\alpha(z)] = \partial_z a^\alpha(z)$ for all $\alpha \in S$.*
(3) *The fields $a^\alpha(z)$ are mutually local*
(4) *V is spanned by the vectors*

$$a^{\alpha_1}_{(j_1)} \cdots a^{\alpha_m}_{(j_m)}|0\rangle, j_i < 0$$

Then these structures together with the vertex operation

$$Y(a^{\alpha_1}_{(j_1)} \cdots a^{\alpha_m}_{(j_m)}|0\rangle, z) = \frac{1}{(-j_1-1)!} \cdots \frac{1}{(-j_m-1)!} : \partial_z^{-j_1-1} a^{\alpha_1}(z) \cdots \partial_z^{-j_m-1} a^{\alpha_m}(z) :$$

give rise to a unique vertex algebra structure on V satisfying (1)-(4) above and such that $Y(a^\alpha, z) = a^\alpha(z)$.

Assume that we have a r-vertex algebra satisfying the hypothesis of Theorem 2. First we contend that the vectors $a^{\alpha_1}_{(j_1)} \cdots a^{\alpha_m}_{(j_m)}|0\rangle$, $j_i < 0$ generate V as a vector space. Indeed,

Lemma— *Let $a \in V$ be primitive and $b \in V$. For $k \geq 0$, we have that*

$$T^{(k)}(a) \cdot b = a_{(-k-1)}b.$$

Proof. For $k = 0$, we have that $a \cdot b = a_{(-1)}b$ since a is primtive and V is a r-vertex algebra. The general case follows by induction using the following two facts: first, if a is primtive, then so is $T^{(k)}(a)$; and second, for any vertex algebra $Y(Ta, z) = \frac{d}{dz}Y(a, z)$. □

This shows that any r-vertex algebra satisfying the hypothesis of Theorem 2, it will also satisfy the hypothesis of the Reconstruction theorem. Therefore, in order to show uniqueness, we just need to verify that the fields $Y(a_\alpha, z)$ are uniquely specified by the given information. So, let us suppose that $(V, |0\rangle, T, Y)$ and $(V, |0\rangle, T, \widetilde{Y})$ are two b-vertex algebra structures satisfying the hypotheses of the theorem. We may as well assume that $(V, |0\rangle, T, Y)$ is the vertex algebra constructed in Theorem 2.

Write $\widetilde{Y}(a, z) = \sum_{n\in\mathbb{Z}} \widetilde{a}_{(n)} z^{-n-1}$ and $Y(a, z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$. Then the lemma above allows us to conclude that $\widetilde{Y}(a, z)_+ = Y(a, z)_+$.

To show that $\widetilde{Y}(a, z)_- v = Y(a, z)_- v$, we first check the result for v a primitive element of minimal positive degree d. Note than since V is primitively generated, the minimal degree of a primitive element is also the minimal positive degree of any element of V. By assumption, $a_{(0)}v = 0$, so let us focus on $a_{(n)}v$ for $n > 0$. Each $a_{(n)}$ has strictly negative degree as an operator on V, so by our assumption on v we conclude that

$a_{(n)}v = 0$ unless $n = d$, in which case $a_{(n)}v = \lambda|0\rangle$. Furthermore, we can determine this value of λ from the 2-point functions:

$$r(a,b) = (\langle 0|, Y(a,z)b) = (\varepsilon, \lambda|0\rangle)z^{-d-1} = \lambda z^{-d-1}.$$

Hence for a primitive and v primitive of minimal degree $Y(a,z)v = \widetilde{Y}(a,z)v$.

Now, we observe the following simple,

Lemma— *Suppose $Y(a,z)b = \widetilde{Y}(a,z)b$, then $Y(Ta,z)b = \widetilde{Y}(a,z)b$ and $Y(b,z)a = \widetilde{Y}(b,z)a$*

Since V is T-generated from a set of (minimal degree) primitives, we see that $Y(a,z)b$ is determined for all $a,b \in V$ primitive. So in particular, $a_{(n)}b$ is determined for all $a,b \in V$ primitive. It is easy to compute these to be:

- $a_{(0)}b = 0$.
- for $n > 0$, $a_{(n)}b = 0$ if $n \neq deg(b)$
- for $n > 0$, $a_{(n)}b = Coeff(r_{z,0}(a,b))$ otherwise.

To finish the proof of the theorem, we need to understand the action of $Y(a,z)_-$ on products of primitive elements. This is achieved through the following,

Proposition— *Let $\widetilde{Y}$ be a r-vertex algebra, and let a,v be a primitive. Then*

$$\widetilde{Y}(a,z)_-(v \cdot w) = \widetilde{Y}(a,z)_- v \cdot w + v \cdot \widetilde{Y}(a,z)_- w$$

Proof of Proposition. It is enough to show that for $n \geq 0$

$$\widetilde{a}_{(n)}[v \cdot w] = \widetilde{a}_{(n)}[v] \cdot w + w \cdot \widetilde{a}_{(n)}[w].$$

This will follow essentially from the well known Borcherds commutator formulas, a special case of which is,

$$[a_{(m)}, v_{(-1)}] = \sum_{j=0}^{n} \binom{m}{j} (a_{(j)}v)_{(m-1-j)}.$$

But since for a and v primitive, $a_{(j)}v$ is always a scalar, so the only way $(a_{(j)}v)_{(m-1-j)}$ can be a nonzero operator is when $m = j$. So, the formula reduces to

$$[a_{(m)}, v_{(-1)}]w = (a_{(m)}b)_{(-1)}w.$$

Expanding this formula out and recalling that for $x \in V$ primitive $x_{(-1)}y = x \cdot y$, we see that the claim is verified. ☐

From the Proposition and the fact that V is T-generated by the set $\{a_\alpha\}$, the theorem is proven.

☐

6. Examples

In this section, we show how the Heisenberg and Lattice Vertex algebras may be given the structure of a r-vertex algebra.

(6.1) *Heisenberg Vertex Algebras:* We can describe an r-vertex algebra structure on the space $\Pi := \Pi^0$ of example ?. First, we will need to define a grading on Π, which is achieved by setting $\deg 1 = 0$ and $\deg a_1(-j_1) \cdot a_2(-j_2) \cdots a_k(-j_k) \cdot 1 = j_1 + \cdots j_k$ and taking the grading by degree. Now Π has an algebra structure by construction, and it is also equipped with a derivation $T : \Pi \to \Pi$ defined by $T \cdot 1 = 0$ and $T(a(-j) \cdot 1) = ja(-j-1)$. Additionally, Π has the structure of a bialgebra with derivation if we take $h(-1), h \in \mathfrak{h}$ to be primitive elements and $\varepsilon(1) = 1$. Also, we note that Π has a bilinear form $(\cdot,\cdot)$ defined by $(1,1) = 1$ and such that $h(n)$ and $h(-n)$ are adjoint.

With these preliminaries, we can use Theorem 1 to construct an r-vertex algebra structure on Π with $r_{z,w}(h(-1), g(-1)) = \frac{(h,g)}{(z-w)^2}$. Using Theorem 2, we easily conclude that this vertex algebra structure coincides with the one introduced in Example 3.3.

(6.2) *Lattice Vertex Algebras:* Next, we consider the vertex algebra associated to a positive-definite, even lattice L. Suppose L is a free abelian group of rank l together with a positive, definite, integral bilinear form $(\cdot|\cdot) : L\otimes L \to \mathbb{Z}$ such that $(\alpha|\alpha)$ is even for $\alpha \in L$. Let $\mathfrak{h} = \mathbb{C}\otimes_{\mathbb{Z}} L$ equipped with the natural $\mathbb{C}$-linear extension of the form $(\cdot|\cdot)$ from L. Construct Π as above associated to $\mathfrak{h}$, and set $V_L = \Pi \otimes \mathbb{C}[L]$ where $\mathbb{C}[L]$ is the group algebra of L with basis $e^\alpha, \alpha \in L$. We will often write e^α for $1\otimes e^\alpha$ and $e^0 = 1$.

Since Π and $\mathbb{C}[L]$ are bialgebras, V_L also has a bialgebra structure with gradation given by $deg(e^\alpha) = \frac{1}{2}(\alpha|\alpha)$ and $deg(h\otimes t^n) = n$ as before for $h \in \mathfrak{h}$. There is also a derivation T acting on V_L by $T\cdot 1 = 0$, $T(h(-n)\otimes 1) = nh(-n-1)\otimes 1$, and $T(1\otimes e^\alpha) = \alpha(-1)\otimes e^\alpha$. With this data V_L is a bialgebra with derivation. V_L is also equipped with a bilinear form defined by $(1\otimes 1, 1\otimes 1) = 1$ and $(h\otimes e^\alpha, g\otimes e^\beta) = (h,h')(\alpha|\beta)$.

We proceed to give two descriptions of r-vertex algebra structures on V_L, and then show they are isomorphic. This vertex algebra is none other than the **Lattice vertex algebra** (of an even lattice). First, there is the usual vertex algebra constructed with vertex operators specified as follows: Let π_1 be the representation of $\hat{\mathfrak{h}}$ on Π defined above. Also define a representation π_2 of $\hat{\mathfrak{h}}$ on $\mathbb{C}[L]$ by setting $\pi_2(K) = 0$, $\pi_2(h(n))e^\alpha = \delta_{n,0}(\alpha,h)e^\alpha$ for $h\in H, \alpha \in L, n\in \mathbb{Z}$.

Then V_L becomes a $\hat{\mathfrak{h}}$ module by $\pi = \pi_1\otimes 1 + 1\otimes \pi_2$. Then for $h\in\mathfrak{h}$, set $h(z) = \sum_{n\in\mathbb{Z}} h(n)z^{-n-1}$ where $h(n)$ acts as the operator $h\otimes t^n$ on the $\hat{\mathfrak{h}}$-module V_L. Then, set

$$Y(a_1(j_1)\cdots a_k(j_k)\otimes e^\alpha, z) = \frac{1}{(-j_1-1)!\cdots(-j_k-1)!} : \partial_z^{j_1-1}a_1(z)\cdots\partial_z^{-j_k-1}a_k(z)\Gamma_\alpha(z)$$

where

$$\Gamma_\alpha(z) = e^\alpha z^\alpha e^{-\sum_{j<0}\frac{z^{-j}}{j}\alpha(-j)}e^{-\sum_{j>0}\frac{z^{-j}}{j}\alpha(-j)}$$

where $z^\alpha(a\otimes e^\beta) = z^{(\alpha,\beta)}a\otimes e^\beta$ and $e^\alpha(a\otimes e^\beta) = \varepsilon(\alpha,\beta)a\otimes e^{\alpha+\beta}$, and $\varepsilon : L\times L \to \pm 1$ is a 2-cocycle satisfying $\varepsilon(\alpha,\beta)\varepsilon(\beta,\alpha) = (-1)^{(\alpha|\beta)+(\alpha|\alpha)(\beta|\beta)}$.

Secondly, we take take the r-vertex algebra defined by Theorem 1 using the bicharacter $r : V_L\otimes V_L \to \mathbb{C}[(z-w)^{-1}]$ uniquely specified by $r(e^\alpha, e^\beta) = \frac{\varepsilon(\alpha,\beta)}{(z-w)^{(\alpha,\beta)}}$.

In order to show that the two descriptions of V_L coincide, we would like to use Theorem 2, noting that they have the same two-point functions. Unfortunately, that theorem is not applicable since V_L is not primitively generated (e^α is group like). But, actually the proof of Theorem 2 can be modified in this case as follows: Denote by $(V_L, 1, T, \widetilde{Y})$ and $(V_L, 1, T, Y)$ the first and second vertex algebra descriptions above. V_L is T-generated from $\mathbb{C}[L]$, so by the Reconstruction Theorem, we just need to see that the fields $Y(e^\alpha, z) = \widetilde{Y}(e^\alpha, z)$.

Step 1: Given $h\in\mathfrak{h}$, let us show that the fields $Y(h(-1),z) = \widetilde{Y}(h(-1),z)$. The proof of Theorem 2 shows that $Y(h(-1),z)_+ = \widetilde{Y}(h(-1),z)_+$, so what remains is to show that $Y(h(-1),z)_- = \widetilde{Y}(h(-1),z)_-$. On Π these two agree by the same reasoning as in the proof of Theorem 2. Furthermore, on $\mathbb{C}[L]$, we contend that

$$Y(h(-1),z)_- e^\alpha = \frac{(h,\alpha)e^\alpha}{z}.$$

Indeed, since the 2-point functions are

$$(\varepsilon, \widetilde{Y}(h(-1),z)e^\alpha) = r(h(-1),e^\alpha) = \frac{(h,\alpha)e^\alpha}{z},$$

we know that

$$\widetilde{Y}(h(-1),z)e^\alpha = \frac{h(-1)_{(0)}e^\alpha}{z} = \frac{\lambda e^\alpha}{z},$$

since $\widetilde{a}_{(0)}$ is a scalar on group-like elements for $a\in V_L$ primitive. Clearly λ is specified by r and so we have shown that $\widetilde{Y} = Y$ on $\mathbb{C}[L]$ as well. Now, use Proposition [**?**] to conclude that $\widetilde{Y} = Y$ on all of V_L.

Step 2: $[h(-1)_{(n)}, Y(e^\alpha, z)] = (\alpha|h)z^n Y(e^\alpha, z)$ and $[h(-1)_{(n)}, \widetilde{Y}(e^\alpha, z)] = (\alpha|h)z^n \widetilde{Y}(e^\alpha, z)$. This follows easily from Borcherds's commutator formula and the fact that $Y(h(-1), z) = \widetilde{Y}(h(-1), z)$. Note that in particular $[h(0), Y(e^\alpha, z)] = (\alpha|h)Y(e^\alpha, z)$ and similarly $[h(0), Y(e^\alpha, z)] = (\alpha|h)\widetilde{Y}(e^\alpha, z)$. This means that

$$Y(e^\alpha, z) : \Pi \otimes \mathbb{C}[e^\beta] \to \Pi \otimes \mathbb{C}[e^{\beta+\alpha}],$$

and similarly for $\widetilde{Y}(e^\alpha, z)$.

Step 3: Since e^α has degree $\frac{1}{2}(\alpha, \alpha)$, we may write

$$Y(e^\alpha, z) = \sum_{n \in \mathbb{Z}} e^\alpha[n] z^{-n-\frac{(\alpha,\alpha)}{2}}$$

and

$$Y(e^\alpha, z) = \sum_{n \in \mathbb{Z}} \tilde{e}^\alpha[n] z^{-n-\frac{(\alpha,\alpha)}{2}}$$

where $e^\alpha[n]$ and $\tilde{e}^\alpha[n]$ have degree $-n$.

Step 4: We show that $Y(e^\alpha, z)e^\beta = \widetilde{Y}(e^\alpha, z)e^\beta$ for $\alpha, \beta \in L$. Indeed, by degree considerations, we have that

$$e^\alpha[s]e^\beta = 0 \text{ for } s > -\frac{(\alpha, \alpha)}{2} - (\beta, \alpha)$$

and

$$e^\alpha[s]e^\beta = c_{\alpha,\beta} e^{\alpha+\beta} \text{ for } s = -\frac{(\alpha, \alpha)}{2} - (\beta, \alpha).$$

Similarly for $\tilde{e}^\alpha[s]$, and examining 2-point functions we see that the $c_{\alpha,\beta}$ are completely determined. Now, using induction and the fact that

$$\frac{d}{dz} Y(e^\alpha, z) = Y(Te^\alpha, z) =: Y(\alpha(-1), z) Y(e^\alpha, z)$$

and

$$\frac{d}{dz} \widetilde{Y}(e^\alpha, z) = \widetilde{Y}(Te^\alpha, z) =: \widetilde{Y}(\alpha(-1), z) \widetilde{Y}(e^\alpha, z)$$

we see that $Y(e^\alpha, z)e^\beta = \widetilde{Y}(e^\alpha, z)e^\beta$.

Step 5: Using Step 2, we may compute $Y(e^\alpha, z)h_1(-n_1)\cdots h_k(-n_k)e^\beta$ and $\widetilde{Y}(e^\alpha, z)h_1(-n_1)\cdots h_k(-n_k)e^\beta$ and see that they are equal by Step 1 and 4.

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