IWAHORI-HECKE ALGEBRAS FOR p-ADIC LOOP GROUPS

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Abstract. This paper is a continuation of [3] in which the first two authors have introduced the *spherical Hecke algebra* and the Satake isomorphism for an untwisted affine Kac-Moody group over a non-archimedian local field. In this paper we develop the theory of the *Iwahori-Hecke algebra* associated to these same groups. The resulting algebra is shown to be closely related to Cherednik's double affine Hecke algebra. Furthermore, using these results, we give an explicit description of the affine Satake isomorphism, generalizing Macdonald's formula for the spherical function in the finite-dimensional case. The results of this paper have been previously announced in [4].

Contents

1. Introduction

Let $\mathcal{K}$ denote a non-archimedian local field with ring of integers $\mathcal{O}$. Pick $\pi \in \mathcal{O}$ a uniformizing element and denote the residue field $\mathcal{O}/\pi\mathcal{O}$ by $\mathbf{k}$. It is a finite field, whose cardinality we denote by q.

Usually we shall denote algebraic varieties over $\mathcal{K}$ (or a subring of $\mathcal{K}$) by boldface letters $\mathbf{X}, \mathbf{G}$ etc.; their sets of $\mathcal{K}$-points will then be denoted X, G etc.

1.1. **Finite Dimensional Case.** We shall first describe the finite-dimensional case of which this paper is an affine generalization.

1.1.1. *Notations on Groups.* Let $\mathbf{G}_o$ be a split, simple, and simply connected algebraic group (defined over $\mathbb{Z}$) and let $\mathfrak{g}_o$ be its Lie algebra. As agreed above, we set $G_o = \mathbf{G}_o(\mathcal{K})$. Let $\mathbf{A}_o \subset \mathbf{G}_o$ be a maximal split torus, which we assume is of rank ℓ; we denote its character lattice by Λ_o and its cocharacter lattice by $\Lambda_o^\vee$; note that since we have assumed that $\mathbf{G}_o$ is simply connected, $\Lambda_o^\vee$ is also the coroot lattice of $\mathbf{G}_o$. For any $x \in \mathcal{K}^*, \lambda^\vee \in \Lambda_o^\vee$ we set $x^{\lambda^\vee} = \lambda^\vee(x) \in A_o$.

Let us choose a pair $\mathbf{B}_o, \mathbf{B}_o^-$ of opposite Borel subgroups such that $\mathbf{B}_o \cap \mathbf{B}_o^- = \mathbf{A}_o$. We denote by R_o the set of roots of $\mathbf{G}_o$ and by $R_o^\vee$ the set of coroots. Similarly $R_{o,+}$ (resp. $R_{o,+}^\vee$) will denote the set of positive roots (resp. of positive coroots), and Π_o (resp. $\Pi_o^\vee$) the set of simple roots (resp. simple coroots). We shall also denote by $2\rho_o$ (resp. by $2\rho_o^\vee$) the sum of all positive roots (resp. of all positive coroots). We denote by $\Lambda_o^\vee$ (resp. $\Lambda_{o,+}^\vee$) the set of coweights (resp. dominant coweights). Let W_o be the Weyl group of G, which is a finite Coxeter group with generators simple reflections $w_1, \ldots, w_\ell$ corresponding to the simple roots Π_o.

For a reductive group $\mathbf{H}$ over $\mathcal{K}$ we denote by $H^\vee$ the Langlands dual group defined over $\mathbb{C}$ defined by exchanging the root and coroot data of $\mathbf{H}$.

1.1.2. *Hecke algebras.* Let $J_o \subset G_o$ be an open compact subgroup of G_o. Then one can consider the Hecke algebra $H(G_o, J_o)$ of J_o-bi-invariant compactly supported functions with respect to convolution. Studying the representation theory of G_o is essentially equivalent to studying the representation theory of the algebras $H(G_o, J_o)$ for various J_o.

There are two choices of open compact subgroups of G_o that will be of interest to us. The first is that of $K_o = \mathbf{G}_o(\mathcal{O})$; the corresponding Hecke algebra $H(G_o, K_o)$ is called the *spherical Hecke algebra.* The second is that of the Iwahori subgroup $I_o \subset K_o$, which is by definition equal to the preimage of $\mathbf{B}_o(\mathbf{k}) \subset \mathbf{G}(\mathbf{k})$ under the natural projection map $K_o \to \mathbf{G}_o(\mathbf{k})$. The corresponding Hecke algebra $H(G_o, I_o)$is called the *Iwahori-Hecke algebra.* Let us recall the description of the corresponding algebras in these two cases.

1.1.3. *Spherical Hecke Algebra and the Satake isomorphism.* The Cartan decomposition asserts that G_o is the disjoint union of double cosets $K_o\, \pi^{\lambda^\vee} K_o$, $\lambda^\vee \in \Lambda^\vee_{o,+}$; hence, $H(G_o, K_o)$ has a vector space basis corresponding to the characteristic functions of these double cosets $h_{\lambda^\vee}$, $\lambda^\vee \in \Lambda^\vee_{o,+}$. As an algebra, $H(G_o, K_o)$ is commutative, associative, and unital, with unit $\mathbf{1}_{K_o}$ equal to the characteristic function $h_{\lambda^\vee}$ with $\lambda^\vee = 0$ (i.e. the characteristic function of K_o).

Let $\mathbb{C}[\Lambda^\vee_o]$ denote the group algebra of $\Lambda^\vee_o$: it consists of finite $\mathbb{C}$-linear combinations in the symbols $e^{\lambda^\vee}$ with $\lambda^\vee \in \Lambda^\vee_o$, where $e^{\lambda^\vee} e^{\mu^\vee} = e^{\lambda^\vee + \mu^\vee}$, for $\lambda^\vee, \mu^\vee \in \Lambda^\vee_o$. The natural W_o-action on $\Lambda^\vee_o$ lifts to $\mathbb{C}[\Lambda^\vee_o]$; for $f \in \mathbb{C}[\Lambda^\vee_o]$ and $w \in W$ we denote by f^w the application of w to f. The Satake isomorphism S_o makes clear the algebra structure of $H(G_o, K_o)$: it provides a canonical isomorphism (see [31])

$$S_o : H(G_o, K_o) \overset{\cong}{\to} \mathbb{C}[\Lambda^\vee_o]^{W_o}, \tag{1.1}$$

where $\mathbb{C}[\Lambda^\vee_o]^{W_o}$ is the ring of W_o-invariant elements in $\mathbb{C}[\Lambda^\vee_o]$. The algebra $\mathbb{C}[\Lambda^\vee_o]^{W_o}$ admits other interpretations: it is isomorphic to the complexified Grothendieck ring $K_0(\mathrm{Rep}(G^\vee_o))$ of finite-dimensional representations of $G^\vee_o$ (the Langlands dual group of G_o); it is also isomorphic to the algebra $\mathbb{C}(A^\vee_o)^{W_o}$ of polynomial functions on the maximal torus $A^\vee_o \subset G^\vee_o$ which invariant under W_o.

For many purposes, it is desirable to have an explicit formula for the elements $S_o(h_{\lambda^\vee})$. Such a formula was given by Macdonald (and independently by Langlands [24, Chapter 3] in a slightly weaker form), and we shall present the answer below.

1.1.4. *The Iwahori-Hecke algebra.* As follows from the work of Iwahori and Matsumoto [19], the group G_o is the disjoint union of I_o-double cosets indexed by $\mathcal{W}_o := W_o \ltimes \Lambda^\vee_o$, the *affine Weyl group associated to* W_o[1]. It is well-known ([29]) that $\mathcal{W}_o$ is itself an infinite Coxeter group which has simple reflection generators $w_1, \dots, w_{\ell+1}$ where $w_1, \dots, w_\ell$ correspond to the previously introduced generators of W_o. Denote by $\ell : \mathcal{W}_o \to \mathbb{Z}$ the length function on $\mathcal{W}_o$ corresponding to this set of generators, and also let T_x be the characteristic function of the I_o-double coset corresponding to $x \in \mathcal{W}_o$. Then in *loc. cit*, it was shown that the algebra $H(G_o, I_o)$ has the following simple presentation: it is generated by $\{T_x\}_{x \in \mathcal{W}_o}$ and has relations

- **IM 1** $T_x T_y = T_{xy}$ for $x, y \in \mathcal{W}_o$ with $\ell(xy) = \ell(x) + \ell(y)$
- **IM 2** $T^2_{w_i} = qT_1 + (q-1)T_{w_i} = 0$ for $i = 1, \dots, \ell + 1$.

The algebra $H(G_o, I_o)$ has an important alternative description, the *Bernstein presentation* : it is generated by elements $\Theta_{\lambda^\vee}$ for $\lambda^\vee \in \Lambda^\vee_o$ and T_w for $w \in W_o$, with relations:

- **B 1** $T_w T_{w'} = T_{ww'}$ for $w, w' \in W_o$ with $\ell(ww') = \ell(w) + \ell(w')$;
- **B 2** $\Theta_{\lambda^\vee}\Theta_{\mu^\vee} = \Theta_{\lambda^\vee+\mu^\vee}$; in other words, the $\Theta_{\lambda^\vee}$'s generate a (commutative) subalgebra $\mathbb{C}[\Lambda^\vee_o]$ inside $H(G_o, I_o)$;

[1]recall that we have assumed that $\mathbf{G}_o$ is simply-connected, so that $\mathcal{W}_o = W_o \ltimes Q^\vee_o$ where $Q^\vee_o$ is the coroot lattice

B 3 For any $f \in \mathbb{C}[\Lambda_o^\vee]$ and any simple reflection w_i for $i = 1, \ldots, \ell$ we have

$$fT_{w_i} - T_{w_i} f^{w_i} = (q-1) \frac{f - f^{w_i}}{1 - \Theta_{-a_i^\vee}} \tag{1.2}$$

where $a_i^\vee$ are the simple coroots of G_o. Note that the right hand side of the above equation is an element of $\mathbb{C}[\Lambda_o^\vee]$.

1.1.5. *Explicit description of the Satake isomorphism.* For any subset $\Sigma_o \subset W_o$ we define

$$\Sigma_o(q^{-1}) = \sum_{w \in \Sigma_o} q^{-\ell(w)}. \tag{1.3}$$

For any $\lambda^\vee \in \Lambda_o^\vee$, let $W_{o,\lambda^\vee}$ denote the stabilizer of $\lambda^\vee$ in W_o. The following result is due to Macdonald.

Theorem. [26] *For any* $\lambda^\vee \in \Lambda_{o,+}^\vee$ *we have*

$$S_o(h_{\lambda^\vee}) = \frac{q^{\langle \rho_o, \lambda^\vee \rangle}}{W_{o,\lambda^\vee}(q^{-1})} \sum_{w \in W_o} w \left(e^{\lambda^\vee} \frac{\prod_{\alpha \in R_{o,+}} 1 - q^{-1} e^{-\alpha^\vee}}{\prod_{\alpha \in R_{o,+}} 1 - e^{-\alpha^\vee}} \right), \tag{1.4}$$

where recall that ρ_o *was defined as the half-sum of the positive roots of* G_o.

Note that it is not immediately clear that the right hand side of (1.4) belongs to $\mathbb{C}[\Lambda_o^\vee]$. Of course this follows from the theorem as the left hand side of (1.4) is in $\mathbb{C}[\Lambda_o^\vee]$.

Since S_o is an algebra map, it sends the identity in the Hecke algebra $\mathbf{1}_{K_o}$ (i.e., the characteristic function $h_{\lambda^\vee}$ with $\lambda^\vee = 0$) to the identity $1 \in \mathbb{C}[\Lambda_o^\vee]^{W_o}$. Specializing (1.4) to $\lambda^\vee = 0$ and noting that $S_o(h_0) = 1$ we obtain the combinatorial identity

$$1 = \frac{1}{W_o(q^{-1})} \sum_{w \in W_o} w \left(\frac{\prod_{\alpha \in R_{o,+}} 1 - q^{-1} e^{-\alpha^\vee}}{\prod_{\alpha \in R_{o,+}} 1 - e^{-\alpha^\vee}} \right), \tag{1.5}$$

(see [27]). We emphasize this point, as the naive analogue of the above identity *fails* in the affine setting.

1.2. **The Affine Case.** The main purpose of this paper is to extend the results described in §§ 1.1.4– 1.1.5 to the case of (untwisted) affine Kac-Moody groups, the results in § 1.1.3 having been generalized already to the affine setting by the first two authors in [3].

1.2.1. *Notations on Loop Groups.* As before, we start with a split, simple, simply connected group $\mathbf{G}_o$. Fix a symmetric, bilinear form $(\cdot, \cdot)$ on the coroot (or coweight) lattice of $\mathbf{G}_o$ (which is specified in §2.1.1). For this fixed choice of $(\cdot, \cdot)$ the polynomial loop group $\mathbf{G}_o[t, t^{-1}]$ (i.e. the functor whose points over a ring R are given by $\mathbf{G}_o(R[t, t^{-1}])$) admits a non-trivial central extension by $\mathbb{G}_m$ which we denote by $\widetilde{\mathbf{G}}$. The full affine Kac-Moody group is then $\mathbf{G} := \mathbb{G}_m \ltimes \widetilde{\mathbf{G}}$ [2], where $\mathbb{G}_m$ acts by rescaling the loop parameter t. Denote by

$$\eta : \mathbf{G} \to \mathbb{G}_m \tag{1.6}$$

the projection onto the rescaling parameter. We choose a pair $\mathbf{B}, \mathbf{B}_-$ of opposite Borel subgroups of $\mathbf{G}$ (see §2.2 for precise definition of this notion) whose intersection $\mathbf{A} = \mathbf{B} \cap \mathbf{B}_-$ is equal to the group $\mathbf{A} = \mathbb{G}_m \times \mathbf{A}_o \times \mathbb{G}_m$ where the first $\mathbb{G}_m$ corresponds to the central direction and the second to the rescaling parameter. Let R (resp. $R^\vee$) denote the set of roots (resp. coroots) and Π (resp. $\Pi^\vee$) the set of simple roots (resp. coroots) of $\mathbf{A}$. Recall that the roots (resp. coroots) come in two flavours: the real roots (resp. coroots) shall be denoted by R_{re} (resp. $R_{re}^\vee$) and the imaginary ones

[2] The reader should be warned that in the main body of this paper a slightly different (but equivalent) construction is adopted coming from the general theory of Kac-Moody groups

by R_{im} (resp. $R_{im}^\vee$). The minimal imaginary coroot shall be denoted by $\mathbf{c}$, and corresponds to the central extension. Also let $\Lambda^\vee$ denote the cocharacter lattice of $\mathbf{A}$ instead of A and $\Lambda_+^\vee$ the set of dominant cocharacters. We have that

$$\Lambda^\vee \cong \mathbb{Z} \oplus \Lambda_o^\vee \oplus \mathbb{Z}, \tag{1.7}$$

where again the first $\mathbb{Z}$-component corresponds to the center and the last to the loop rescaling. As before, let $\mathbb{C}[\Lambda^\vee]$ be the group algebra of $\Lambda^\vee$. We denote by W the Weyl group of G. The group W is an infinite Coxeter group, which leaves invariant the imaginary roots and coroots. We define the *Tits cone* X as the union $\cup_{w \in W} w(\Lambda_+^\vee)$. Unlike the finite-dimensional case, $X \neq \Lambda^\vee$; in terms of the decomposition (1.7), the Tits cone is characterized as the subset of all elements $(a, \lambda_o^\vee, k) \in \mathbb{Z} \oplus \Lambda_o^\vee \oplus \mathbb{Z}$ such that either $k > 0$ or $k = 0$ and $\lambda_o^\vee = 0$.

For the groups $\mathbf{G}$ or $\mathbf{A}$ we denote by $G^\vee$ and $A^\vee$ the Langlands dual group (defined over $\mathbb{C}$, see [3] for more details). If $\mathbf{G_o}$ is simply-laced, then the dual group to its affinization $\mathbf{G}$ is again an untwisted affine Kac-Moody group. But in general the dual group $G^\vee$ is a twisted affine Kac-Moody group.

Remark. To simplify matters, we assume that $\mathbf{G}_o$ simply-laced throughout this paper. In so doing, not only do we avoid having to leave the realm of untwisted affine Kac-Moody groups, but there is also some simplification in the final formula for the spherical function. Namely, we have the particularly simple expression (1.15) in this case. For a general affine root system, we refer to [1] for a corresponding formula. As far as we can see, there are no serious complications which arise when dropping the simply-laced assumption.

1.2.2. *p-adic Loop Groups.* Our main object of interest is the group $G = \mathbf{G}(\mathcal{K})$. It was observed in [3] that one should work with a certain semigroup $G_+ \subset G$. To describe G_+, recall the map $\eta : \mathbf{G} \to \mathbb{G}_m$ from (1.6) which induces the map

$$|\eta| : G \to \mathcal{K}^* \stackrel{\mathsf{val}}{\to} \mathbb{Z} \tag{1.8}$$

where the last map is the valuation map $\mathsf{val} : \mathcal{K}^* \to \mathbb{Z}$. We define $G_+ \subset G$ as the sub-semigroup of G generated by the following three types of elements:

(1) the central $\mathcal{K}^* \subset T \subset G$;
(2) the subgroup $\mathbf{G}(\mathcal{O}) \subset G$;
(3) All elements $g \in G$ such that $|\eta(g)| > 0$.

We define affine analogues of K_0 and I_0 as $K := \mathbf{G}(\mathcal{O}) \subset \mathbf{G}_+$ and $I := \pi_{\mathbf{k}}^{-1}(\mathbf{B}(\mathbf{k})) \subset K$ where $\pi_{\mathbf{k}} : \mathbf{G}(\mathcal{O}) \to \mathbf{G}(\mathbf{k})$ is the natural projection. The following result, proven in §3, generalizes the Cartan and Iwahori-Matsumoto decompositions from the theory of p-adic groups (see e.g., [26]),

Proposition. *There are bijective correspondences between the following sets*

(a) $\Lambda_+^\vee$ *and the set of double cosets* $K \setminus G_+ / K$
(b) $\mathcal{W}_X := W \ltimes X$ *and the set of double cosets* $I \setminus G_+ / I$ *(recall that* X *was defined to be the Tits cone).*

Part (a) of this Proposition follows form the results in §3.3 and Part (b) from those of §3.4.

1.2.3. *Spherical Hecke Algebras and the Satake Isomorphism.* Due to the infinite-dimensionality of K and I, one cannot resort to the usual techniques to define a convolution structure on the space of I or K-double cosets on the group. However, it was shown in [3] that an associative algebra structure can still be defined on a certain *completion* of the space of finite linear combinations of K-double cosets of G_+. We would like to emphasize that this claim is by no means trivial, and we refer to the remarks in §1.2.6 for more on this point.

Denote by $H_\leq(G_+, K)$ the *spherical Hecke algebra* of K-double cosets of G_+, where the subscript $\leq$ denotes that a certain completion (depending on the dominance order $\leq$ on $\Lambda^\vee$) of the space of finitely supported functions is necessary. The precise definition is reviewed in §5. The algebra

$H_{\leq}(G_+, K)$ is commutative and unital with $\mathbf{1}_K$ (the characteristic function of K) as the unit. The algebra $H_{\leq}(G_+, K)$ comes with two additional structures: a grading by non-negative integers coming from the map $|\eta|$ of (1.8) and a structure of an algebra over the field $\mathbb{C}((\mathsf{t}))$ of Laurent power series in a variable t (not to be confused with the loop variable 't') coming from the center of G.

The statement of the Satake isomorphism for G is very similar to that for G_o. First, in [3] the natural analogue of the algebra $\mathbb{C}[\Lambda_o^\vee]^{W_o}$ was defined (it had also made its appearance in the literature earlier by Looijenga [23]). The definition again involves a certain completion $\mathbb{C}_{\leq}[\Lambda^\vee]$ of group algebra of $\Lambda^\vee$. We shall denote the corresponding space of W-invariants here by $\mathbb{C}_{\leq}[\Lambda^\vee]^W$.

This algebra also has natural interpretations in terms of dual groups: either as the subring $\mathbb{C}_{\leq}[T^\vee]^W$ of W-invariants in a certain completion of the space of polynomial functions on $T^\vee$ or as a suitable category $\mathrm{Rep}(G^\vee)$ of representations of $G^\vee$ (stable under tensor product) so that $K_o(\mathrm{Rep}(G^\vee)) \cong \mathbb{C}_{\leq}[\Lambda^\vee]^W$ (see §2.1.7).

The algebra $\mathbb{C}_{\leq}[\Lambda^\vee]^W$ is also a finitely generated $\mathbb{Z}_{\geq 0}$-graded commutative algebra over the field $\mathbb{C}((\mathsf{t}))$ of Laurent formal power series in the variable t. The affine Satake isomorphism asserts that

Theorem. [3] *There is a natural isomorphism of graded $\mathbb{C}((\mathsf{t}))$-algebras*

$$S : H_{\leq}(G_+, K) \to \mathbb{C}_{\leq}[\Lambda^\vee]^W. \tag{1.9}$$

We shall present below a generalization of (1.1.5) giving an explicit formula for S on certain basis elements of $H_{\leq}(G_+, K)$.

1.2.4. *The Iwahori-Hecke algebra for G.* As we observed in Proposition 1.2.2(b), G_+ can be written as a disjoint union of I-double cosets paramterized by the semigroup $\mathcal{W}_X := W \ltimes X$ where X was the Tits cone of $\Lambda^\vee$. The semigroup $\mathcal{W}_X$ plays a role similar to $\mathcal{W}_o$ in the usual theory of p-adic groups; however, it is not a Coxeter group. Although we largely circumvent a systematic study of the combinatorics of this group, we do introduce here certain orders on this group and show how they arise naturally from a group theoretic point of view.

Denote by $H(G_+, I)$ the *Iwahori-Hecke algebra* associated to G, which is the space of I-bi-invariant functions on G_+ supported on a union of finitely many double cosets. It has a vector space basis T_x for $x \in \mathcal{W}_X$ where T_x is the characteristic function of the double coset IxI, $x \in \mathcal{W}_X$. In this paper we show the following result, which follows from the finiteness theorems in [3] or [2].

Theorem. *The space $H(G_+, I)$ can be naturally equipped with a convolution structure.*

Note the difference from the spherical case, where the convolution was only defined on a completion of some space of finitely-supported functions.

In the finite-dimensional case, one has two presentations for the algebra $H(G_o, I_o)$ as was described in §1.1.4, one in terms of the basis $\{T_x\}_{x \in \mathcal{W}_o}$ and the relations **IM 1, IM 2** , and the other in terms of a basis $\{\Theta_{\lambda^\vee}, T_w\}_{\lambda^\vee \in \Lambda_o^\vee, w \in W_o}$ subject to the relations **B1, B2, B3**. We do not know how to generalize the former presentation, but $H(G_+, I)$ does admit a description similar to the latter. To describe this it, recall that we have from (1.7) that $\Lambda^\vee = \mathbb{Z} \oplus \Lambda_o^\vee \oplus \mathbb{Z}$. Let $\mathbb{H}$ denote the algebra generated by elements $\Theta_{\lambda^\vee}$, $\lambda^\vee \in \Lambda^\vee$ and T_w for $w \in W \subset \mathcal{W}_X$ with relations **B 1**, **B 2**, **B 3** as in §1.1.4. The algebra $\mathbb{H}$ is $\mathbb{Z}$-graded; this grading is defined by setting

$$\deg T_w = 0; \quad \deg \Theta_{(a, \lambda_o^\vee, k)} = k \text{ where } (a, \lambda_o^\vee, k) \in \Lambda^\vee = \mathbb{Z} \oplus \Lambda_o^\vee \oplus \mathbb{Z}. \tag{1.10}$$

We denote by $\mathbb{H}_k$ the subspace of all elements of degree k in $\mathbb{H}$. Note that $\mathbb{H}_0$ is a subalgebra of $\mathbb{H}$, which is isomorphic to Cherednik's double affine Hecke algebra.

Let us now set,

$$\mathbb{H}_+ := \mathbb{C}\langle T_w \rangle_{w \in W} \oplus \left(\bigoplus_{k>0} \mathbb{H}_{\mathrm{aff},k} \right). \tag{1.11}$$

The following is one of the two main results of this paper:

Theorem. *The algebra $H(G_+, I)$ is isomorphic to the algebra $\mathbb{H}_+$.*

In particular, the algebra $H(G_+, I)$ is closely related to Cherednik's double affine Hecke algebra. We note that another relation between the double affine Hecke algebra and the group G was studied by Kapranov [22]. In *op. cit*, the double affine Hecke algebra was constructed as an algebra of intertwining operators on some space built from G. As such, it naturally comes equipped with a Bernstein type presentation. Our algebra $H(G_+, I)$ is first constructed as a convolution algebra of double cosets, and then shown to admit a Bernstein-type description by studying certain intertwining operators. Hence, by definition the algebra $H(G_+, I)$ is endowed with a natural basis corresponding to characteristic functions of double cosets of I on G_+. It is natural to conjecture that this basis has a purely algebraic or combinatorial description as well.

1.2.5. *Affine Macdonald formula.* As observed in Proposition 1.2.2 (see also Theorem), the semigroup G_+ is equal to the disjoint union of K-double cosets $K\pi^{\lambda^\vee}K$, $\lambda^\vee \in \Lambda_+^\vee$. Denote by $h_{\lambda^\vee} \in H(G,K)$ the characteristic function of the corresponding double coset and let $\mathbb{C}_v := \mathbb{C}((v))$ denote the ring of Laurent series in the formal variable v. Define $\mathbb{C}_{v,\leq}[\Lambda^\vee] := \mathbb{C}_v \otimes_{\mathbb{C}} \mathbb{C}_{\leq}[\Lambda^\vee]$. The element

$$\Delta := \prod_{a \in R_+} \left(\frac{1 - v^2 e^{-a^\vee}}{1 - e^{-a^\vee}} \right)^{m(a^\vee)} \tag{1.12}$$

where $m(a^\vee)$ is the multiplicity of the coroot $a^\vee$ may be regarded as an element in $\mathbb{C}_{v,\leq}[\Lambda^\vee]$ by expanding each rational function in *negative powers* of the coroots. For $w \in W$, we also define Δ^w to be the expansion of the product $\prod_{a \in R_+} \left(\frac{1 - v^2 e^{-wa^\vee}}{1 - e^{-wa^\vee}} \right)^{m(a^\vee)}$ in negative powers of the coroots. One can then show that the following element

$$H_{\lambda^\vee} = \frac{v^{-2\langle \rho, \lambda^\vee \rangle}}{W_{\lambda^\vee}(v^2)} \sum_{w \in W} \Delta^w e^{w\lambda^\vee} \tag{1.13}$$

lies in $\mathbb{C}_{v,\leq}[\Lambda^\vee]$, where $\rho \in \Lambda$ is the affine analogue of ρ_o (see (2.11) for a definition). The following is the other main new result of this paper,

Theorem. *The element $\frac{H_{\lambda^\vee}}{H_0}$ lies in the ring $\mathbb{C}[v^2, v^{-2}] \otimes_{\mathbb{C}} \mathbb{C}_{\leq}[\Lambda^\vee]$, and its specialization at $v^2 = q^{-1}$ is equal to $S(h_{\lambda^\vee})$.*

In the finite-dimensional case the second equality of the identity (1.5) ensures that the analogue of H_0 is equal to 1. However, in the affine setting

$$H_0 = \frac{1}{W(v^2)} \sum_{w \in W} \Delta^w \neq 1, \tag{1.14}$$

although the analogue of the relation $S(h_0) = 1$ must still hold. This explains the reason why we divide by H_0 in Theorem 1.2.5. Observe that the function H_0 was studied by Macdonald in [28] using the works of Cherednik. Macdonald has shown that H_0 has an infinite product decomposition. For example, when G_o, the underlying finite-dimensional group of G is of simply laced type, the Macdonald-Cherednik formula reads as follows:

$$H_0 = \prod_{i=1}^{\ell} \prod_{j=1}^{\infty} \frac{1 - v^{2m_i} e^{-j\mathbf{c}}}{1 - v^{2(m_i+1)} e^{-j\mathbf{c}}}, \tag{1.15}$$

where $\mathbf{c}$ was the minimal positive imaginary coroot and $m_1, \cdots, m_\ell$ are the exponents of G_o (defined for example via (7.39). A similar product decomposition for H_0 exists for not necessarily simply-laced groups (see [1]).

1.2.6. *Relations to Previous Literature.* There are two main new results in this work: (a) the computation of the spherical function (Theorem 1.2.5); and (b) the construction and identification of the Iwahori-Hecke algebra (see §1.2.4). Let us comment on each in turn, and then make some general remarks.

The first work that we know of to treat loop groups over local fields was Garland [15] which studied the Cartan decomposition on such groups. There it was noticed that one needed to work on half of the group (essentially our semigroup G_+ introduced above) in order for the Cartan decomposition to hold; this was an observation analogous to one previously made while studying the theory of loop arithmetic quotients (see [14]). Already, the work [15] was motivated by the desire to study a theory of spherical functions for loop groups. In the setting of a general finite-dimensional p-adic group , the theory of spherical functions was developed by Satake [31] [3], but it remained an open question how to explicitly compute these functions and determine the Plancherel measure. This question was taken up some years later by Langlands and Macdonald [4] for quite different reasons and using different techniques; both of their techniques and motivations were actually relevant for us. In [24], Langlands needed an explicit formula for spherical functions to prove the convergence of the L-functions he had just introduced (the now called automorphic L-functions). His strategy for computing the spherical function is reminiscent of Harish-Chandra's work in the real case– the answer is show to be a sum over the Weyl group with the coefficients in this sum asymptotically linked to certain Gindikin-Karpelevic integrals. Note that in this work, the Iwahori subgroups play no visible role. On the other hand, in [26] Macdonald developed an intricate strategy to compute the spherical function and thereby complete the work of Satake by using an Iwahori-level analysis (something which is absent in Satake's work). Note however that the actual Iwahori-Hecke algebra does not play an explicit role in Macdonald's work. However, Casselman [5] revisited Macdonald's work and gave a new proof of the formula for spherical functions which makes the connection to the representation theory of Iwahori-Hecke algebras explicit. The work of Macdonald has also been developed in a combinatorial direction, and we just note here that an important deformation of the p-adic spherical function are the so-called Macdonald polynomials. In this work (see (7.3.15)), a slightly different connection than what was observed earlier by Casselman is noted between the spherical function and the representation theory of the Iwahori-Hecke algebra: namely, the main recursion formula which Macdonald uses in [26] can be phrased in terms of the polynomial representation of the algebra which was introduced by Cherednik much later, and not coincidentally, in his study of Macdonald polynomials.

Leaving aside any issues of finiteness, an obstacle that had impeded the development a theory of spherical functions on loop groups was finding a way around the (absence of the) longest element of the Weyl group. In the finite dimensional case, this longest element plays a simple but persistent role, and it turns out to be a subtle point (essentially related to the Macdonald constant term conjectures) how to properly account for its absence. In our approach, the issue is addressed by first making a link between a certain Iwahori-level decomposition of the spherical function and the polynomial representation of the double affine Hecke algebra mentioned above. This provides a reduction to a purely algebraic/combinatorial question which was previously studied by Cherednik. Note that in our approach a certain normalization condition is natural and essential for pinpointing the mysterious factor H_0 (1.15) in the formula for spherical function. In hindsight, we recognize its complicated shape as the main obstacle to a direct approach to the calculation of the spherical function.

Prior to [15], Cherednik had introduced the double affine Hecke algebra as a new tool to study certain conjectures of Macdonald, one of which was the constant term conjecture mentioned in previous paragraph (essentially the equality (1.15) in our notation). It was a natural question (posed sometime in the mid 1990s) whether one could recover the double affine Hecke algebra

[3]for certain rank 1-groups, there was previous work by Mautner [30])

[4]Langlands work is in slightly less generality, and perhaps for this reason, the formula for the spherical function is nowadays just referred to as Macdonald's formula

from the p-adic loop group setting. The first affirmative answer to this question came with the work of Kapranov [22] where it was shown that, attached to a p-adic loop group G (not G_+), one could construct a family of operators acting on some natural p-adic symmetric space. The algebra generated by these operators was shown to be equal to Cherednik's double affine Hecke algebra. This work was then extended by the second-named author in several directions [9–11]. Note that in these works, the double affine Hecke algebra arises naturally through the Bernstein presentation and the issue of defining a convolution structure on the group itself is avoided. Thus, it remained an open question whether one could actually construct a convolution structure on the group. This problem was first resolved by the first and second-named authors in [3] where it was shown that if one restricts to working within G_+ (not G) one can in fact define a convolution structure on (a suitable completion of) the space K-double cosets. The main point of that work was to verify certain "spherical finiteness" conditions (see (7.6) below) and this was achieved by interpreting the problem geometrically. Subsequently, new proofs of this result have been found. One such proof is given in [2] which works in an "elementary" context (essentially using representation theory of affine Lie algebras and structure theory of the p-adic loop group). Note that in *loc. cit* a stronger result is proven (see [2, Theorem 1.9(1)]) which we refer to as "Gindikin-Karpelevic" finiteness and from which spherical finiteness follows immediately. This stronger Gindikin-Karpelevic finiteness is actually used in the present work when we construct the Iwahori-Hecke algebra. Another very different proof of the spherical finiteness which works for general Kac-Moody groups was found by Gaussent and Rousseau [16] by developing an analogue to the theory of buildings (called *hovels*). These techniques enabled them to attach a spherical Hecke algebra to any Kac-Moody group over a local field, and also to prove that the structure coefficients of the corresponding algebra are polynomials (in the size of the residue field). It would be interesting to understand whether the techniques of *loc. cit* extend to allow one to construct Iwahori-Hecke algebras on a general Kac-Moody group (one may also ask the related question: can the techniques of [16] allow one to prove the Gindikin-Karpelevic finiteness?). Let us note here that the techniques of this paper also allow one to prove the polynomiality of the structure coefficients of the spherical Hecke algebra (as we indicated in §7.3.16), but we do not study the corresponding problem for the Iwahori-Hecke algebras.

Having constructed a convolution structure on certain (finite) combinations of Iwahori double cosets, it remains to identify the corresponding algebra as a close variant of the double affine Hecke algebra. At this point, one could use (a slight extension of) the work of Kapranov [22] (which in turn uses the certain constructions from [17]). We opt here for a direct, if somewhat cumbersome, approach. The main difficulty is to construct a family of commuting operators $\Theta_{\lambda^\vee}$ for $\lambda^\vee \in X$ inside the Iwhori-Hecke algebra and we do this inductively on the length of the chamber in which $\lambda^\vee$ lies. One then needs to verify the construction is independent of certain choices made along the way, which we address in §6.3.

In §3, we develop some basic structure theory for p-adic loop groups. Aside from the Cartan decomposition, this is a quite standard extension of the finite-dimensional case, but we could not find a suitable reference for certain results we needed. Along the way, we realized that a criterion established earlier for detecting when unipotent elements are integral (see [2, Lemma 3.3]) yields a proof of the Cartan decomposition that may extend to a general Kac-Moody group (the proofs of the Cartan decomposition in [15] and [3] do not seem to extend). We decided to include a sketch of this proof in the affine case in Appendix A, but we do not pursue the general Kac-Moody case here. Finally, in Appendix B, we explain a certain pre-order on the double affine Weyl group (which, we recall is *not* a Coxeter group by any means) that naturally arises from the group theoretic point of view. It is analogous to the Bruhat order on the usual affine Weyl group when this group is viewed as a semi-direct product of a finite Weyl group and an infinite subset of translations. We expect a closer study of this order may play some role in understanding the questions posed at the end of §1.2.4.

We conclude by pointing out that a limit of the spherical function (see [2]) may be used to compute the Gindikin-Karpelevic integral. [5] This integral is the local input needed in a generalization of the Langlands-Shahidi method to loop groups– i.e. to compute the constant terms of certain Eisenstein series on loop groups and relate them to L-functions of cusp forms on *finite*-dimensional groups. This was one of the main motivations for us to undertake this study.

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2. Basic Notations on Groups and Algebras

2.1. Lie Algebras.

2.1.1. *Finite Dimensional Notations.* Let $\mathfrak{g}_o$ be a simple, simply-laced, split Lie algebra of rank ℓ over a field k. In general the index o will denote objects associated to a finite-dimensional root system. Choose a Cartan subalgebra $\mathfrak{h}_o \subset \mathfrak{g}_o$ and denote the set of roots with respect to $\mathfrak{h}_o$ by R_o. Choose a Borel subalgebra $\mathfrak{b}_o$ and denote by $\Pi_o = \{\alpha_1, \dots, \alpha_\ell\}$ the simple roots of R_o. Let $\mathfrak{h}_o^*$ denote the algebraic dual of $\mathfrak{h}_o$ and denote the natural pairing $\mathfrak{h}_o^* \times \mathfrak{h}_o \to k$ by $\langle \cdot, \cdot \rangle$. Let θ denote the highest root. Denote by $(\cdot, \cdot)$ the Killing form on $\mathfrak{h}_o$, which induces an isomorphism $\psi : \mathfrak{h}_o \to \mathfrak{h}_o^*$. We continue to denote the induced form on $\mathfrak{h}_o^*$ by $(\cdot, \cdot)$, normalized so that $(\theta, \theta) = 2$. For each root $\alpha \in R_o$ denote the corresponding coroot by $\alpha^\vee := \frac{2}{(\alpha,\alpha)} \psi^{-1}(\alpha) \in \mathfrak{h}_o$. For each $\alpha \in R_o$ we let $w_\alpha : \mathfrak{h}_o^* \to \mathfrak{h}_o^*$ be the corresponding reflection, and denote by W_o the Weyl group generated by the reflections w_{α_i} for $i = 1, \dots, \ell$. Let $\Lambda_o \subset \mathfrak{h}_o^*$ denote the weight lattice of $\mathfrak{g}_o$, defined as the set of $\lambda \in \mathfrak{h}_o^*$ such that $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}$ for $i = 1, \dots, \ell$. We define $\rho_o \in \Lambda$ by the condition that $\langle \rho_o, \alpha_i^\vee \rangle = 1$ for $i = 1, \dots, \ell$. We let $Q_o \subset \mathfrak{h}_o^*$ denote the root lattice, and observe that $Q_o \subset \Lambda_o$. Dual to these notions, we denote by $\Lambda_o^\vee$ and $Q_o^\vee$ the coweight and coroot lattice of $\mathfrak{g}_o$ in the usual way.

2.1.2. *Affine Lie Algebras.* For a field k, we denote by $\mathfrak{g}$ the affinization of the Lie algebra $\mathfrak{g}_o$. As a vector space $\mathfrak{g} := k\mathbf{d} \oplus \mathfrak{g}'$ where $\mathbf{d}$ is the degree derivation and $\mathfrak{g}'$ is the one-dimensional central extension of the the loop algebra $\mathfrak{g}_o \otimes_k k[t, t^{-1}]$ which is specified by the form $(\cdot, \cdot)$ defined in §2.1.1. Let $\mathfrak{h} \subset \mathfrak{g}$ denote a Cartan subalgebra containing the finite-dimensional Cartan $\mathfrak{h}_o$, the degree derivation $\mathbf{d}$, and the center of $\mathfrak{g}$. One has a direct sum decomposition

$$\mathfrak{h} := \mathfrak{h}_o \oplus \mathfrak{h}_{cen} \oplus k\mathbf{d} \tag{2.1}$$

where $\mathfrak{h}_{cen}$ is the one dimensional k-vector space containing the center, and we may equip $\mathfrak{h}$ with a symmetric, non-degenerate bilinear form on $(\cdot|\cdot)$ as in [20]. Let $\mathfrak{h}^*$ be the algebraic dual of $\mathfrak{h}$. As before we denote by

$$\langle \cdot, \cdot \rangle \, : \mathfrak{h}^* \times \mathfrak{h} \to k \tag{2.2}$$

the natural pairing.

Let R be the set of roots of $\mathfrak{g}$, and $R^\vee$ the set of coroots. We denote the set of simple roots of $\mathfrak{g}$ by

$$\Pi \quad = \quad \{a_1, \dots, a_{\ell+1}\} \quad \subset \quad \mathfrak{h}^* . \tag{2.3}$$

Similarly, we write

$$\Pi^\vee \quad = \quad \{a_1^\vee, \dots, a_{\ell+1}^\vee\} \quad \subset \quad \mathfrak{h} \tag{2.4}$$

[5] Note that this is essentially the inverse of the strategy employed by Langlands in his computation of the spherical function.

for the set of simple affine coroots. Note that the simple roots $a_i : \mathfrak{h} \to k$ satisfy the relations

$$\begin{aligned} \langle a_i, \mathbf{d} \rangle &= 0 \quad \text{for } i = 1, \ldots, \ell \\ \text{and} \qquad \langle a_{\ell+1}, \mathbf{d} \rangle &= 1\,. \end{aligned} \tag{2.5}$$

Each root $\alpha \in R_o$ extends to an element of $\mathfrak{h}^*$ which we denote by the same symbol α by requiring that $\langle \alpha, X \rangle = 0$, for $X \in \mathfrak{h}_{cen} \oplus k\mathbf{d}$. Let $\delta \in \mathfrak{h}^*$ be the minimal positive imaginary root defined by the conditions,

$$\langle \delta, X \rangle = 0 \text{ for } X \in \mathfrak{h}_{cen} \oplus \mathfrak{h}_o \quad \text{and } \langle \delta, \mathbf{d} \rangle = 1. \tag{2.6}$$

As is well known, we have $a_i = \alpha_i$ for $i = 1, \ldots, l$ and $a_{\ell+1} = -\theta + \delta$.

We define the affine root lattice as

$$Q = \mathbb{Z}a_1 + \cdots + \mathbb{Z}a_{\ell+1} \tag{2.7}$$

and the affine coroot lattice as

$$Q^\vee = \mathbb{Z}a_1^\vee + \cdots + \mathbb{Z}a_{\ell+1}^\vee\,. \tag{2.8}$$

We shall denote the subset of non-negative integral linear combinations of the affine simple roots (respectively, affine simple coroots) as Q_+ (respectively, $Q_+^\vee$). The integral weight lattice is defined by

$$\Lambda := \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, a_i^\vee \rangle \in \mathbb{Z} \text{ for } i = 1, \ldots, \ell+1 \text{ and } \langle \lambda, \mathbf{d} \rangle \in \mathbb{Z}\}\,. \tag{2.9}$$

The lattice Λ is spanned by δ and the fundamental affine weights $\Lambda_1, \ldots, \Lambda_{\ell+1}$, which are defined by the conditions that $\langle \Lambda_i, \mathbf{d} \rangle = 0$ for $i = 1, \ldots, \ell+1$ and

$$\langle \Lambda_i, a_j^\vee \rangle = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \qquad \text{for } 1 \le i, j \le \ell+1\,. \tag{2.10}$$

We define the element $\rho \in \Lambda$ by the conditions

$$\langle \rho, a_i^\vee \rangle = 0 \text{ for } i = 1, \ldots, \ell+1, \text{ and } \langle \rho, \mathbf{d} \rangle = 0. \tag{2.11}$$

The dual space $\mathfrak{h}^*$ is spanned by $a_1, a_2, \cdots, a_{\ell+1}, \Lambda_{\ell+1}$. We also let $\mathbf{c} \in \mathfrak{h}$ be the minimal imaginary coroot, characterized by the conditions,

$$\langle a, \mathbf{c} \rangle = 0,\ a \in R \quad \text{and} \quad \langle \Lambda_{\ell+1}, \mathbf{c} \rangle = 1\ . \tag{2.12}$$

We denote by $\Lambda^\vee$ the coweight lattice in $\mathfrak{g}$ which is defined as

$$\Lambda^\vee := \{\lambda^\vee \in \mathfrak{h}^* \mid \langle a_i, \lambda^\vee \rangle \in \mathbb{Z} \text{ for } i = 1, \ldots, \ell+1 \text{ and } \langle \Lambda_{\ell+1}, \lambda^\vee \rangle \in \mathbb{Z}\}. \tag{2.13}$$

In other words $\Lambda^\vee = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$. Let us denote by $\le$ the dominance partial order on $\Lambda^\vee$: i.e., for $\mu^\vee, \lambda^\vee \in \Lambda^\vee$

$$\mu^\vee \le \lambda^\vee \text{ if and only if } \lambda^\vee - \mu^\vee \in Q_+^\vee. \tag{2.14}$$

2.1.3. *Structure of Affine Roots.* For each $i = 1, \ldots, \ell+1$ we denote by $w_{a_i} : \mathfrak{h} \to \mathfrak{h}$ the reflection through the hyperplane $H_i := \{h \in \mathfrak{h} | (h|a_i^\vee) = 0\}$ and denote by $W \subset \mathrm{Aut}(\mathfrak{h})$ the group generated by the elements w_i for $i = 1, \ldots, \ell$. It is a Coxeter group. We denote by $\le_W$ the usual Bruhat order on the group W. The group W acts on $\mathfrak{h}^*$ in the usual way, i.e. $wf(X) = f(w^{-1}X)$ for $f \in \mathfrak{h}^*, X \in \mathfrak{h}, w \in W$. A root $a \in R$ is called a *real root* if there exists $w \in W$ such that $wa \in \Pi$. The set of such roots is denoted as R_{re}. Otherwise, a is called an *imaginary root*, and the set of all such imaginary roots is denoted R_{im}. We have decompositions,

$$R = R_{re} \sqcup R_{im} \tag{2.15}$$

and we may define $R_{re,\pm}$ and $R_{im,\pm}$ accordingly. The set of real roots admits the following description:

$$R_{re} = \{\alpha + m\delta \mid \alpha \in R_o, m \in \mathbb{Z}\}. \tag{2.16}$$

The set of positive real roots is then

$$R_{re,+} = \{\alpha + m\delta \mid \alpha \in R_{o,+},\, m \geq 0\} \cup \{\alpha + m\delta \mid \alpha \in R_{o,-},\, m > 0\}. \tag{2.17}$$

The set of imaginary roots is equal to

$$R_{im} = \{m\delta \mid m \in \mathbb{Z} \setminus 0\}. \tag{2.18}$$

Let $a \in R_{re}$ and choose some $w \in W$ such that $wa = a_i \in \Pi$. We define the corresponding coroot

$$a^\vee := w^{-1} a_i^\vee, \tag{2.19}$$

and note that this construction does not depend on the choice of w (see [20, §5.1]). Recalling that θ was the highest root of the underlying finite-dimensional root system, we now have

$$a_{\ell+1}^\vee = -\theta^\vee + \mathbf{c}. \tag{2.20}$$

The element $\mathbf{c}$ also spans the one-dimensional center of $\mathfrak{g}$ and we have a decomposition

$$\mathfrak{h} = k\mathbf{c} \oplus \mathfrak{h}_o \oplus k\mathbf{d}. \tag{2.21}$$

Parallel to the decomposition (2.15) we have a decomposition of the coroots

$$R^\vee = R_{re}^\vee \cup R_{im}^\vee \tag{2.22}$$

into real and imaginary coroots where

$$R_{re}^\vee = \{\alpha^\vee + m\mathbf{c} \mid \alpha \in R_o, m \in \mathbb{Z}\} \quad \text{and} \quad R_{im}^\vee = \{m\mathbf{c} \mid m \in \mathbb{Z} \setminus 0\}. \tag{2.23}$$

2.1.4. *The Tits Cone.* The group W also has an alternate description as

$$W = W_o \ltimes Q_o^\vee. \tag{2.24}$$

Elements in this group are sometimes denoted as wt_H where $w \in W_o$ and $H \in Q_o^\vee$. The coweight lattice $\Lambda^\vee$ can be written (see (2.21)) as $\Lambda^\vee = \mathbb{Z}\mathbf{c} \oplus \Lambda_o^\vee \oplus \mathbb{Z}\mathbf{d}$, where the elements in $\Lambda_o^\vee$ are regarded as elements of $\Lambda^\vee$ by defining their pairing with respect to δ and $\Lambda_{\ell+1}$ to be zero. One can check that for any $H \in Q_o^\vee, m, r \in \mathbb{Z}$ and $\lambda_o^\vee \in \Lambda_o^\vee$,

$$t_H(m\mathbf{c} + \lambda_o^\vee + r\mathbf{d}) = (m + (\lambda_o^\vee, H) - r\frac{(H,H)}{2})\mathbf{c} + \lambda_o^\vee - rH + r\mathbf{d}. \tag{2.25}$$

Moreover, one can also see that if $w \in W_o, m, r \in \mathbb{Z}$ and $\lambda_o^\vee \in \Lambda_o^\vee$, then

$$w(m\mathbf{c} + \lambda_o^\vee + r\mathbf{d}) = m\mathbf{c} + w(\lambda_o^\vee) + r\mathbf{d}. \tag{2.26}$$

Setting

$$\Lambda_r^\vee = \{\lambda^\vee \in \Lambda^\vee | \langle \delta, \lambda^\vee \rangle = r\}, \tag{2.27}$$

we see from the above two formulas that $\Lambda_r^\vee$ is W invariant and

$$\Lambda^\vee = \oplus_{r \in \mathbb{Z}} \Lambda_r^\vee. \tag{2.28}$$

The elements in $\Lambda_r^\vee$ are referred to as elements of level r. Let $\Lambda_+^\vee$ be the set of dominant coweights,

$$\Lambda_+^\vee := \{\lambda^\vee \in \Lambda^\vee \mid \langle a_i, \lambda^\vee \rangle \geq 0 \text{ for } i = 1, \ldots, \ell+1\}. \tag{2.29}$$

The *Tits cone* $X \subset \Lambda^\vee$ is defined as

$$X := \bigcup_{w \in W} w\Lambda_+^\vee. \tag{2.30}$$

One may then show that [21, Proposition 1.3(b)]

$$X = \{\lambda^\vee \in \Lambda^\vee | \langle a, \lambda^\vee \rangle < 0 \text{ for only finitely many } a \in R_{re,+}\}. \tag{2.31}$$

In terms of the description of $\Lambda^\vee$ given above, one has from [21, Proposition 1.9(a)],

$$X = \{\lambda^\vee \in \Lambda^\vee | \langle \delta, \lambda^\vee \rangle > 0\} \sqcup \mathbb{Z}\mathbf{c}. \tag{2.32}$$

In other words, the Tits cone just consists of elements of positive level, and the multiples of the imaginary coroot $\mathbf{c}$ (which are of level 0.)

2.1.5. *Ring of Affine Invariants.* We next define a completion $\mathbb{C}_{\leq}[\Lambda^\vee]$ of the group algebra of $\Lambda^\vee$ which is used to describe the image of the Satake isomorphism. This ring has also been introduced earlier by Looijenga [23] who calls it the *dual-weight algebra*.

To any $\lambda^\vee \in \Lambda^\vee$ we associate a formal symbol $e^{\lambda^\vee}$ and impose relations: $e^{\lambda^\vee} e^{\mu^\vee} = e^{\lambda^\vee + \mu^\vee}$. We define $\mathbb{C}_{\leq}[\Lambda^\vee]$ as the set of (possibly infinite) linear combinations

$$f = \sum_{\lambda^\vee \in X} c_{\lambda^\vee} e^{\lambda^\vee} \tag{2.33}$$

such that there exists finitely many elements $\lambda_1^\vee, \ldots, \lambda_r^\vee \in \Lambda_+^\vee$ so that

$$\mathrm{Supp}(f) := \{\lambda^\vee | c_{\lambda^\vee} \neq 0\} \subset \cup_{i=1}^r \mathfrak{c}(\lambda_i^\vee), \tag{2.34}$$

where with respect to the dominance order $\leq$ from (2.14), we set

$$\mathfrak{c}(\lambda^\vee) = \{\mu^\vee \in X | \mu^\vee \leq \lambda^\vee\}. \tag{2.35}$$

One may easily verify that $\mathbb{C}_{\leq}[\Lambda^\vee]$ is a unital, associative, commutative ring. Using the natural action of W on $\Lambda^\vee$ we may define the notation of W-invariant elements of $\mathbb{C}_{\leq}[\Lambda^\vee]$, namely

$$\mathbb{C}_{\leq}[\Lambda^\vee]^W := \{ \sum_{\mu^\vee \in \Lambda^\vee} c_{\mu^\vee} e^{\mu^\vee} \in \mathbb{C}_{\leq}[\Lambda^\vee] \mid c_{w\mu^\vee} = c_{\mu^\vee} \text{ for all } w \in W, \mu^\vee \in \Lambda^\vee\}. \tag{2.36}$$

The ring $\mathbb{C}_{\leq}[\Lambda^\vee]$ carries a natural grading, with graded pieces

$$\mathbb{C}_{\leq}[\Lambda^\vee]_r := \{f \in \mathbb{C}_{\leq}[\Lambda^\vee] \mid \mathrm{Supp}(f) \subset \Lambda_r^\vee \cap X\}. \tag{2.37}$$

As follows from (2.32) and (2.27), we have

$$\mathbb{C}_{\leq}[\Lambda^\vee] = \oplus_{r \geq 0} \mathbb{C}_{\leq}[\Lambda^\vee]_r. \tag{2.38}$$

It is easy to see that in fact $\mathbb{C}_{\leq}[\Lambda^\vee]$ is a graded module over the commutative ring $\mathbb{C}_{\leq}[\Lambda^\vee]_0$. Moreover, the piece $\mathbb{C}_{\leq}[\Lambda^\vee]_0$ is easy to describe explicitly.

Lemma. *Let* $\mathbf{c}$ *be the minimal imaginary positive coroot. Then* $\mathbb{C}_{\leq}[\Lambda^\vee]_0 = \mathbb{C}((e^{-\mathbf{c}}))$

Proof. First note that if $f \in \mathbb{C}_{\leq}[\Lambda^\vee]_0$ then its support is contained in the set of elements in the Tits cone of level 0 and such elements are of the form $\mathbb{Z}\mathbf{c}$. The Lemma will follow from the following claim, whose proof is given below.

Claim. *Let* $m \in \mathbb{Z}$ *and set* $\mu^\vee = m\mathbf{c}$. *If* $\lambda^\vee \in \Lambda_+^\vee$ *is such that* $\mu^\vee \leq \lambda^\vee$, *then* $\lambda^\vee = n\mathbf{c}$, *with* $n \geq m$.

Indeed, if $f \in \mathbb{C}_{\leq}[\Lambda^\vee]_0$ it follows from the claim that then there exists finitely many integers $n_1, \ldots, n_r$ such that every $\mu^\vee \in \mathrm{Supp}(f)$ is of the form $\mu = m_i\mathbf{c}$ with $m_i \leq n_i$. The Lemma follows from this. □

Proof of Claim. First note that if $\mu^\vee$ has level 0 and $\lambda^\vee \geq \mu^\vee$ then necessarily $\lambda^\vee$ also has level 0. But a dominant coweight of level 0 must be of the form $n\mathbf{c}$: if not, then $\lambda^\vee = n\mathbf{c} + \lambda_o^\vee$ where $\lambda_o^\vee \in \Lambda_o^\vee$. By the dominance property of $\lambda^\vee$ we must have

$$\langle a_i, \lambda^\vee \rangle = \langle a_i, \lambda_o^\vee \rangle \geq 0 \text{ for } i = 1, \ldots, \ell + 1. \tag{2.39}$$

Suppose there exists $i \in \{1, \ldots, \ell\}$ such that $\langle a_i, \lambda_o^\vee \rangle > 0$. Then we have

$$\langle a_{\ell+1}, \lambda_o^\vee \rangle = \langle -\theta + \delta, \lambda_o^\vee \rangle = -\langle \theta, \lambda_o^\vee \rangle < 0, \tag{2.40}$$

since θ is a positive sum of the elements a_i for $i = 1, \ldots, \ell$. So $\langle a_i, \lambda_o^\vee \rangle = 0$ for $i = 1, \ldots, \ell$ and in fact $\lambda^\vee = n\mathbf{c}$ for some $n \in \mathbb{Z}$. If in fact $\mu^\vee \leq \lambda^\vee$ then $\lambda^\vee - \mu^\vee = (n-m)\mathbf{c}$ must be a positive sum of coroots, and so in fact $n \geq m$. □

2.1.6. *Dual Root Systems.* The affine Kac-Moody algebra may also be defined as a Lie algebra associated to a generalized Cartan matrix. For the construction, we refer to [20]. Suppose that a generalized Cartan matrix M corresponds to the untwisted, affine Lie algebra $\mathfrak{g}$, then the transpose ${}^tM$ is also a generalized Cartan matrix corresponding to the dual Lie algebra $\mathfrak{g}^\vee$. In general $\mathfrak{g}^\vee$ is again an affine Lie algebra, but could be of twisted type. On the other hand, if M is of simply-laced type (i.e., the underlying finite-type root system attached to M is simply laced), then ${}^tM$ will again correspond to an untwisted affine Lie algebra. In fact, in this case, $\mathfrak{g}^\vee$ is the untwisted affine Lie algebra attached to $\mathfrak{g}_o^\vee$, the dual of the underlying finite-dimensional root system.

To avoid complications of twisted affine types, *we shall throughout restrict to the case that $\mathfrak{g}_o$ is of simply-laced type.*

2.1.7. *Modules for Affine Lie Algebras.* A $\mathfrak{g}$ module is called $\mathfrak{h}$-diagonalizable if $V = \bigoplus_{\lambda\in\mathfrak{h}^*} V(\lambda)$ where $V(\lambda)$ are the weight spaces $V(\lambda) := \{v \in V | h.v = \langle\lambda, h\rangle v\}$. We define the set $P(V)$ of *weights* of V by

$$P(V) := \{\lambda \in \mathfrak{h}^* | V(\lambda) \neq 0\}. \tag{2.41}$$

Given $\lambda, \mu \in \mathfrak{h}^*$ we define the dominance partial order on $P(V)$ by

$$\lambda \geq \mu \iff \lambda - \mu \in Q_+. \tag{2.42}$$

We define $\mathrm{Rep}(\mathfrak{g})$ as the category of $\mathfrak{g}$-modules Vsuch that

(1) V is $\mathfrak{h}$-diagonalizable
(2) $V(\lambda)$ are finite dimensional for each non-zero $\lambda \in P(V)$
(3) there exist finitely many $\lambda_1, \cdots, \lambda_r \in \mathfrak{h}^*$ such that $P(V) \subset \cup_{i=1}^r \mathfrak{c}(\lambda_i)$, where $\mathfrak{c}(\lambda_i)$ is defined analogously to (2.35).

One checks that $\mathrm{Rep}(\mathfrak{g})$ is an abelian category stable under tensor product. Therefore we may form its complexified Grothendieck ring $K_0(\mathfrak{g})$, and then easily check that the map which sends a representation to its character defines an isomorphism,

$$K_0(\mathfrak{g}^\vee) \overset{\cong}{\to} \mathbb{C}_\leq[\Lambda^\vee]^W. \tag{2.43}$$

2.2. **Loop Groups.**

2.2.1. *The Tits Group Functor.* We review in this part the construction of affine Kac-Moody group $\mathbf{G}$ due to Tits. [32]. A set $\Psi \subset R_{re}$ is *pre-nilpotent* if there exists $w, w' \in W$ such that $w\Psi \subset R_{re,+}$ and $w'\Psi \subset R_{re,-}$. If such a set Ψ also satisfies the condition:

$$\text{if } a, b \in \Psi, a + b \in R_{re}, \text{ then } a + b \in \Psi, \tag{2.44}$$

we say that Ψ is a nilpotent set. For any $a \in R_{re}$ we denote by $\mathbf{U}_a$ a corresponding one-dimensional additive group scheme, and fix an isomorphism $x_a : \mathbb{G}_a \to \mathbf{U}_a$. For any nilpotent set Ψ of roots, Tits has constructed (see [32, Proposition 1] a group schemes $\mathbf{U}_\Psi$ equipped with inclusions $\mathbf{U}_a \hookrightarrow \mathbf{U}_\Psi$ such that for any choice of an order on Ψ the map, $\prod_{a\in\Psi} \mathbf{U}_a \to \mathbf{U}_\Psi$ is an isomorphism of schemes.

Given any pre-nilpotent pair of roots $\{a, b\}$ we set $\theta(a, b) = (\mathbb{N}a + \mathbb{N}b) \cap R_{re}$. Tits has shown that for any total order on $\theta(a, b) \setminus \{a, b\}$, there exists a unique set $k(a, b; c)$ of integers such that for any ring S we have

$$(x_a(u), x_b(u')) = \prod_{c=ma+nb} x_c(k(a, b; c)u^m u'^n) \tag{2.45}$$

for all $u, u' \in S$ and where $c = ma + nb$ varies over $\theta(a, b) \setminus \{a, b\}$ and $(x_a(u), x_b(u'))$ is the commutator. We then define the Steinberg functor $\mathbf{St}$ to be the quotient of the free product of the groups $\mathbf{U}_a, a \in R_{re}$ by the normal subgroup generated by the above relations (2.45).

We let $\mathbf{A}$ be the functor which sends a ring S to

$$\mathbf{A}(S) = \mathrm{Hom}_\mathbb{Z}(\Lambda, S). \tag{2.46}$$

For $s \in S^*$ and $\lambda^\vee \in \Lambda^\vee = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ we write $s^{\lambda^\vee}$ for the element of $\mathbf{A}(S)$ map which sends each $\mu \in \Lambda$ to $s^{\langle \mu, \lambda^\vee \rangle} \in S$.

The Weyl group W of $\mathfrak{g}$ acts on functors $\mathbf{St}$ and $\mathbf{A}$. For $w \in W$ we denote by w^* the corresponding action on either of these functors. For each $i = 1, \dots, \ell+1$ we choose isomorphisms $x_{a_i} : \mathbb{G}_a \to \mathbf{U}_{a_i}$ and $x_{-a_i} : \mathbb{G}_a \to \mathbf{U}_{-a_i}$ and for each invertible element $r \in S^*$ and $i = 1, \dots, \ell+1$ denote by $\widetilde{w_i}(r)$ the image of the product

$$x_{a_i}(r) x_{-a_i}(-r^{-1}) x_{a_i}(r) \tag{2.47}$$

in $\mathbf{St}(S)$. We set $\widetilde{w_i} := \widetilde{w_i}(1)$.

The affine Kac-Moody group functor (Tits functor) is the functor $\mathbf{G}$ which associates to a given ring S the quotient of the free product $\mathbf{St}(S) \star \mathbf{A}(S)$ by the smallest normal subgroup containing the canonical images of the following relations, where $i = 1, \dots, \ell+1$, $r \in S$, and $t \in \mathbf{A}(S)$

$$t\, x_{a_i}(r)\, t^{-1} = x_{a_i}(t(a_i) r) \tag{2.48}$$

$$\widetilde{w_i}\, t\, \widetilde{w_i}^{-1} = w_i^*(t) \tag{2.49}$$

$$\widetilde{w_i}(r) \widetilde{w_i}^{-1} = r^{a_i^\vee} \text{ for } r \in S^* \tag{2.50}$$

$$\widetilde{w_i}\, u\, \widetilde{w_i}^{-1} = w_i^*(u) \text{ for } u \in \mathbf{U}_a(S), a \in R_{re}. \tag{2.51}$$

Note the following important, but simple identity holds in the group $\mathbf{G}(S)$,

$$x_{-a}(s^{-1}) = x_a(s)(-s)^{a^\vee} w_a(1) x_a(s), \tag{2.52}$$

where $s \in S^*$ and $a \in R_{re}$.

2.2.2. *Bruhat Decompositions.* Now we describe the structure of $G := \mathbf{G}(k)$ for any field k. For each $a \in R_{re}$ we define $U_a = \mathbf{U}_a(k)$, and $A = \mathbf{A}(k)$. Let U denote the subgroup generated by U_a for $a \in R_{re,+}$ and U^- the subgroup generated by U_a for $a \in R_{re,-}$. Define now B_a to be the subgroup of G generated by U_a and A. Also, set B and B^- to be the subgroups generated by all the B_a for $a \in R_{re,+}$ and $R_{re,-}$ respectively. We have semi-direct products $B = A \ltimes U$ and $B^- = A \ltimes U^-$. We let N be the group generated by A and the $\widetilde{w_i}$, where the elements $\widetilde{w_i}$ were defined above. There is a natural map

$$\zeta : N \to W \tag{2.53}$$

which sends $\widetilde{w_i} \mapsto w_i$ and which has kernel A. This map is surjective, and induces an isomorphism $\zeta : N/A \to W$. For each $w \in W$, we shall write $\dot{w}$ for any lift of w by ζ. If $w \in W$ has a reduced decomposition $w = w_{a_{i_1}} \cdots w_{a_{i_r}}$, with the $a_k \in \Pi$, we shall also sometimes write

$$\widetilde{w} := \widetilde{w}_{i_1} \cdots \widetilde{w}_{i_r} \tag{2.54}$$

for a specific lift of w (with respect to a given reduced decomposition of w) where the $\widetilde{w}_i$ were defined after (2.47).

Proposition ([32]). *One has the following Bruhat-type decompositions*

$$G = \sqcup_{w \in W} B\, \dot{w}\, B = \sqcup_{w \in W} B^- \dot{w}\, B^- \tag{2.55}$$

$$= \sqcup_{w \in W} B^-\, \dot{w}\, B = \sqcup_{w \in W} B\, \dot{w}\, B^-, \tag{2.56}$$

where $\dot{w}$ is any lift of $w \in W$ to N under the map (2.53) [6].

Note that it is important here that we are working with the so-called minimal Kac-Moody group in order to have Bruhat decompositions with respect to B and B^-. We shall also need the following claim whose proof we suppress (see the remarks after [8, Corollary 1.2].)

Claim. *Suppose that $w_1, w_2 \in W$ such that $B\dot{w}_1 B \cap B\dot{w}_2 B^- \neq \emptyset$. Then we have $w_2 \leq w_1$ in the Bruhat order on W.*

[6] Usually, for $w \in W$ we shall often just write BwB for the coset $B\dot{w}B$, as the choice of representative does not change the subset of the group.

2.2.3. *Integrable Modules.* Fix a Chevalley form $\mathfrak{g}_{\mathbb{Z}} \subset \mathfrak{g}$ as in [13, 32]. For any $\omega \in \Lambda_+$ we let V^ω denote the corresponding highest weight module for $\mathfrak{g}$. It is shown in [13] that V^ω can be equipped with a $\mathbb{Z}$-form $V^\omega_{\mathbb{Z}}$ which is stable under the elements of the Chevalley basis of $\mathfrak{g}_{\mathbb{Z}}$ and their divided powers. Moreover, the action integrates to an action of $\mathbf{G}(\mathbb{Z})$. We shall write V^ω_R for $V^\omega_{\mathbb{Z}} \otimes R$, which is naturally a $\mathbf{G}(R)$-module.

Let now $\mathcal{K}$ be a local field, so that we then have a representation the group $G := \mathbf{G}(\mathcal{K})$ on $V := V^\omega_{\mathcal{K}}$. Let $V_{\mathcal{O}} := \mathcal{O} \otimes_{\mathbb{Z}} V_{\mathbb{Z}} \subset V$, and denote by v_ω be a highest weight vector, i.e., $0 \neq v_\omega \in V(\omega)$. For $v \in V$ we can set

$$\mathrm{ord}_\pi(v) = \min\{n \in \mathbb{Z} | \pi^n v \in V_{\mathcal{O}}\} \tag{2.57}$$

and define

$$||v|| = q^{\mathrm{ord}_\pi(v)}, \tag{2.58}$$

where q is the size of the residue field of $\mathbf{k}$. If v, w are in different weight spaces, then

$$||v + w|| \geq ||v||. \tag{2.59}$$

Also, we can choose v_ω to be *primitive*, i.e., $||v_\omega|| = 1$. The elements of K preserve the norm $|| \cdot ||$. Note also that elements from U stabilize the highest weight vector v_ω. Moreover, an element $s^{\lambda^\vee} \in A$ with $s \in \mathcal{K}^*$ acts on an vector in the weight space $V(\mu)$ as the scalar $s^{\langle \mu, \lambda^\vee \rangle}$.

3. Basic Structure of p-adic Loop Groups

Recall the conventions for local fields $\mathcal{K}$ from the start of this paper. The goal of this section is to study the basic properties of the group $G := \mathbf{G}(\mathcal{K})$.

3.1. Subgroups of $\mathbf{G}(\mathcal{O})$.

In this part, we define various subgroups of the group of integral points $K := \mathbf{G}(\mathcal{O}) \subset G$ and establish some elementary properties of them.

3.1.1. *The integral torus.* Recall that we have set $A := \mathbf{A}(\mathcal{K}) \cong \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathcal{K}) \subset G$. Let

$$A_{\mathcal{O}} := \mathbf{A}(\mathcal{O}), \tag{3.1}$$

and note that we have an identification $A/A_{\mathcal{O}} = \Lambda^\vee$. Recall further that we have defined N as the group generated by A and the elements $\widetilde{w}_i$ in §2.2.2, and we have an isomorphism $\zeta : N/A \to W$ (2.53). We define the "affine" Weyl group as

$$\mathcal{W} := W \ltimes \Lambda^\vee, \tag{3.2}$$

and note that ζ can be naturally lifted to a homomorphism also denoted

$$N \to \mathcal{W}. \tag{3.3}$$

From now on we shall denote *this* homomorphism by ζ. The kernel of this map is $A_{\mathcal{O}}$, and we have $N_{\mathcal{O}} := N \cap K = \zeta^{-1}(W)$. If we write $\widetilde{w}$ for the representative of $w \in W$ as in (2.54), we have that

$$N_{\mathcal{O}} = \bigcup_{w \in W} A_{\mathcal{O}} \widetilde{w}. \tag{3.4}$$

Recall that for $\lambda^\vee \in \Lambda^\vee$, we have an element $\pi^{\lambda^\vee} \in A$. For each $x = (w, \lambda^\vee) \in \mathcal{W}$, or $w \in W$ we shall abuse notation and denote by $w\pi^{\lambda^\vee}$ the element $\widetilde{w}\pi^{\lambda^\vee} \in N$. Sometimes, we shall just write $w\pi^{\lambda^\vee}$ for any element in the set $\zeta^{-1}(w\pi^{\lambda^\vee})$, hoping no confusion will arise.

3.1.2. *Iwahori Subgroups.* For each $a \in R_{re}$, the elements of the one-dimensional group $U_a := \mathbf{U}_a(\mathcal{K}) \subset G$ will be written as $x_a(u)$ for $u \in \mathcal{K}$. For $m \in \mathbb{Z}$ we set

$$U_{(a,m)} := \{x_a(u) | \mathsf{val}(u) \geq m\}, \tag{3.5}$$

where $\mathsf{val} : \mathcal{K} \to \mathbb{Z} \cup \{\infty\}$ is the valuation map. As a shorthand, if $a \in R_{re}$ we write

$$U_{a,\mathcal{O}} := U_{(a,0)} = \mathbf{U}_a(\mathcal{O}), \quad U_{a,\pi} := U_{(a,1)}, \quad \text{and} \quad U_a[m] := U_{(a,m)} \setminus U_{(a,m-1)} \tag{3.6}$$

Let us also set U_π to be the group generated by $U_{a,\pi}$ with $a \in R_{re,+}$ and $U_{\mathcal{O}}^-$ the group generated by $U_{-a,\mathcal{O}}$ for $a \in R_{re,+}$. Similarly, we may define the groups $U_{\mathcal{O}}^+$ and U_π^-.

The group $K = \mathbf{G}(\mathcal{O})$ is generated by the subgroups $U_{a,\mathcal{O}}$ as one can see by referring back to the definition of the functor $\mathbf{G}$. Let

$$\kappa : K \to G_{\mathbf{k}} \tag{3.7}$$

denote the map induced from the natural reduction $\mathcal{O} \to \mathbf{k}$ We shall define the *Iwahori subgroup* $I \subset K$ to as

$$I := \{x \in K | \kappa(x) \in B_{\mathbf{k}}\}. \tag{3.8}$$

Similarly we can define the *opposite Iwahori subgroup*

$$I^- := \{x \in K \mid \kappa(x) \in B_{\mathbf{k}}^-\}. \tag{3.9}$$

The proof of the following is entirely analogous to the classical situation (see [19, §2]), so we suppress the details.

Proposition. *Keeping the notation above, we have the following decompositions*

(1) *The groups I and I^- admit the following decompositions,*

$$I = U_{\mathcal{O}} U_\pi^- A_{\mathcal{O}} = U_\pi^- U_{\mathcal{O}} A_{\mathcal{O}} \tag{3.10}$$

and

$$I^- = U_\pi U_{\mathcal{O}}^- A_{\mathcal{O}} = U_{\mathcal{O}}^- U_\pi A_{\mathcal{O}} \tag{3.11}$$

(2) *Choose representatives $\widetilde{w} \in K$ for $w \in W$ as in (2.54). Then there exist disjoint unions,*

$$K = \sqcup_{w \in W} I \, \widetilde{w} \, I = \sqcup_{w \in W} I^- \widetilde{w} \, I^- \tag{3.12}$$
$$= \sqcup_{w \in W} I^- \, \widetilde{w} \, I = \sqcup_{w \in W} I \, \widetilde{w} \, I^-. \tag{3.13}$$

3.2. **Iwasawa Decompositions.** Recall that we have defined subgroups $K, A, U \subset G$, together with a subgroup $A_{\mathcal{O}} \subset A$. Setting

$$A' := \{\pi^{\lambda^\vee} : \lambda^\vee \in \Lambda^\vee\}, \tag{3.14}$$

we have a direct product decomposition, $A = A' \times A_{\mathcal{O}}$ which gives the identification $A/A_{\mathcal{O}} = \Lambda^\vee$ The Iwasawa decomposition in this context states,

Theorem. *Every $g \in G$ has a decomposition $g = k\pi^{\lambda^\vee} u$ where $k \in K$, $\pi^{\lambda^\vee} \in A'$, $u \in U$, and $\lambda^\vee$ is uniquely determined by g. Furthermore, we also have an opposite Iwasawa decomposition: every $g \in G$ may be written as $g = k\pi^{\lambda^\vee} u^-$ where $k \in K$, $\pi^{\lambda^\vee} \in A'$, $u \in U^-$, and $\lambda^\vee$ is uniquely determined by g.*

The existence of the Iwasawa decomposition can be deduced via a standard manner (see [14, §16] and the references therein) from the Bruhat decomposition and a rank one computation. Note that in order to have both Iwasawa decompositions with respect to the groups U and U^- it is important that we are working in the minimal Kac-Moody group.

As for the uniqueness in Theorem 3.2, let us just show uniqueness in $A'/A'_{\mathbf{d}}$ where $A'_{\mathbf{d}}$ is the subgroup generated by $\pi^{n\mathbf{d}}$ with $n \in \mathbb{Z}$. Fix notations as in §2.2.3. If $g = k\pi^{\lambda^\vee} u$ then

$$||gv_\omega|| = ||\pi^{\lambda^\vee} v_\omega|| = q^{\langle \omega, \lambda^\vee \rangle}. \tag{3.15}$$

One then sees that $\lambda^\vee$ is uniquely determined (modulo $\mathbb{Z}\mathbf{d}$) from g by varying ω over a set of fundamental weights of $\mathfrak{g}$.

For the uniqueness statement with respect to U^-, we argue slightly differently. Suppose that we have $K\pi^{\lambda^\vee}U^- \cap K\pi^{\mu^\vee}U^- \neq \emptyset$, i.e., $K\pi^{\lambda^\vee}U^- = K\pi^{\mu^\vee}U^-$. Then $K\pi^{\lambda^\vee}U^- \cap K\pi^{\lambda^\vee}U \neq \emptyset$, and we also have $K\pi^{\mu^\vee}U^- \cap K\pi^{\lambda^\vee}U \neq \emptyset$, and hence from [2, Theorem 1.9 (2)] , $\mu^\vee \geq \lambda^\vee$ and $\lambda^\vee \geq \mu^\vee$, and so $\lambda^\vee = \mu^\vee$.

3.3. **Cartan Decomposition and the semigroup G_+.** Unlike the finite-dimensional case, the Cartan decomposition does not hold for the group G, i.e., not every element $g \in G$ can be written as $g = k_1\pi^{\lambda^\vee}k_2$ with $k_1, k_2 \in K$ and $\lambda^\vee \in \Lambda^\vee$. On the other hand, this property does hold for the semi-group G_+ introduced in § 1.2.2 (see [3, 15]). Recall that G_+ was defined to be the semigroup generated by K, the central $\mathcal{K}^*$, and the elements $g \in G$ such that $|\eta(g)| > 0$ where the map $|\eta|$ was constructed in (1.8).

Theorem ([3, 15])**.** *The semi-group G_+ can be written as a disjoint union,*

$$G_+ = \sqcup_{\lambda^\vee \in \Lambda^\vee_+} K\pi^{\lambda^\vee}K \tag{3.16}$$

In lieu of the above result, we shall often refer to G_+ as the *Cartan semigroup.* The above result implies in particular that the right hand side of (3.16) is a semi-group, a non-trivial fact. In appendix A, we shall give another proof of this theorem based in part on the argument from ([15]).

3.4. **Iwahori-Matsumoto Decomposition.**

3.4.1. *"Affine" Weyl (semi)-group.* We would now like to study another descriptions of G_+, which is the analogue of the Iwahori-Matsumoto decomposition for a classical p-adic group into cosets indexed by the the affine Weyl group. The set indexing Iwahori double cosets of G_+ will be called the "affine" Weyl semi-group $\mathcal{W}_X$ [7] which is defined as follows: recall that $X \subset \Lambda^\vee$ was defined to the Tits cone, which carries a natural action of W the Weyl group of G. We have already defined the "affine" Wey group $\mathcal{W} = W \ltimes \Lambda^\vee$ in (3.2), and we now set

$$\mathcal{W}_X := W \ltimes X. \tag{3.17}$$

For an element $x = (w, \lambda^\vee) \in \mathcal{W}$, with $w \in W$ and $\lambda^\vee \in \Lambda^\vee$ we shall abuse notation and just write $w\pi^{\lambda^\vee}$ for the corresponding lift $\widetilde{w}\pi^{\lambda^\vee}$ in N, where $\widetilde{w} \in K$ was defined in (2.54). Sometimes we shall also just write $x = w\pi^{\lambda^\vee} \in \mathcal{W}$ to mean the pair $x = (w, \lambda^\vee) \in \mathcal{W}$ hoping this will not cause any confusion in the sequel.

3.4.2. *Iwahori-Matsumoto Semigroup.* Let us define, following Iwahori and Matsumoto [19],

$$G_+^{\mathsf{IM}} := \bigcup_{x \in \mathcal{W}_X} IxI, \tag{3.18}$$

where for each $x \in \mathcal{W}_X$ we denote by the same letter a corresponding lift to an element of G_+.

Proposition. *We have an equality of semigroups $G_+^{\mathsf{IM}} = G_+$.*

Proof. If $x = w\pi^{\lambda^\vee} \in \mathcal{W}_X$, with $w \in W$ and $\lambda^\vee \in X$ then clearly $Iw\pi^{\lambda^\vee}I \subset K\pi^{\lambda^\vee}K \subset G_+$. Conversely, since we also have a decomposition $K = \bigcup_{w \in W} IwI$ from (3.12), we obtain

$$K\pi^{\lambda^\vee}K \subset \bigcup_{w, w' \in W} IwI\pi^{\lambda^\vee}Iw'I \subset IW\pi^{\lambda^\vee}WI \subset I\mathcal{W}_X I, \tag{3.19}$$

where the second inclusion is a consequence of Lemma 3.4.2 below. So the two sets in question are equal, and by Theorem 3.3 the semigroup property of G_+^{IM} follows. □

In the proof of the previous Proposition, we used the following simple result,

[7] We use the term "affine" Weyl group to refer to what some other others call the double affine Weyl group. Our present notation is meant to emphasize the analogy with the usual theory of groups over a local field.

Lemma. *Let $\lambda^\vee \in \Lambda^\vee$. Then,*

(1) *For each $w_1, w_2 \in W$ we have $w_1 I w_2 \pi^{\lambda^\vee} I \subset \cup_{w' \in W} I w' \pi^{\lambda^\vee} I$. Symbolically, we write $WIW\pi^{\lambda^\vee} I \subset IW\pi^{\lambda^\vee} I$*

(2) *Similarly we have that $I\pi^{\lambda^\vee} WIW \subset I\pi^{\lambda^\vee} WI$.*

Proof. Let us establish the first statement, the proof of the second being similar. Notice that it suffices to the consider the case when $w_1 = w_a$ is a simple reflection ($a \in \Pi$). Now, given any $w \in W$, it is easy to see (see e.g. [19, Proposition 2.6] for an essentially similar argument) that

$$wI \subset IwU_w[0] \text{ where } U_w[0] = \prod_{a \in R_+ : w \cdot a < 0} U_a[0], \tag{3.20}$$

where the notation $U_a[0]$ was introduced in (3.6). In particular, if $w = w_a$ for $a \in \Pi$ a simple root, we have,

$$w_a I w_2 \pi^{\lambda^\vee} I \subset I w_a U_a[0] w_2 \pi^{\lambda^\vee} I. \tag{3.21}$$

Letting $x = w_2 \pi^{\lambda^\vee}$, we have two possibilities

a. If $x^{-1} U_a[0] x \in I$ then

$$w_a I w_2 \pi^{\lambda^\vee} I \subset I w_a w_2 \pi^{\lambda^\vee} I. \tag{3.22}$$

b. If $x^{-1} U_a[0] x \notin I$, then one can check that $x^{-1} U_{-a}[0] x \in I$. Now, a rank one computation as in (2.52) shows that we can write

$$w_a U_a[0] w_2 \pi^{\lambda^\vee} I \subset w_a U_{-a}[0] w_a U_{-a}[0] w_2 \pi^{\lambda^\vee} I. \tag{3.23}$$

Now, $w_a U_{-a}[0] w_a \in I$ and also $x^{-1} U_{-a}[0] x \in I$ by our assumption. Thus,

$$w_a U_{-a}[0] w_a U_{-a}[0] w_2 \pi^{\lambda^\vee} I \quad \subset \quad I w_2 \pi^{\lambda^\vee} I. \tag{3.24}$$

□

3.4.3. *Disjointness of I-double cosets.* We would like to show that the decomposition (3.18) above is disjoint. In other words,

Lemma. *Let $x, y \in \mathcal{W}_X$. Then $IxI \cap IyI \neq \emptyset$ implies that $x = y$.*

Proof. Suppose that $x = w\pi^{\lambda^\vee}$ and $y = v\pi^{\mu}$ with $w, v \in W$ and $\lambda^\vee, \mu^\vee \in X$. If the above intersection is non-empty, we have that in fact $Iw\pi^{\lambda^\vee} I = Iv\pi^{\mu^\vee} I$. Hence we also have $K\pi^{\lambda^\vee} IU = K\pi^{\mu} IU$. Now using that $I = A_{\mathcal{O}} U_\pi^- U_{\mathcal{O}}$ we obtain $K\pi^{\lambda^\vee} A_{\mathcal{O}} U_\pi^- U = K\pi^{\mu} A_{\mathcal{O}} U_\pi^- U$. Thus $\pi^{\lambda^\vee} \in K\pi^{\mu^\vee} U_\pi^- U$ and so $K\pi^{\lambda^\vee} U \cap K\pi^{\mu^\vee} U^- \neq \emptyset$. Using [2, Theorem 1.9(3)], we conclude that $\mu^\vee \geq \lambda^\vee$. A similar argument shows that $\mu^\vee \leq \lambda^\vee$. So $\mu^\vee = \lambda^\vee$.

So we may assume now that $x = w\pi^{\lambda^\vee}$ and $y = v\pi^{\lambda^\vee}$. If $Iw\pi^{\lambda^\vee} I \cap Iv\pi^{\lambda^\vee} I \neq \emptyset$, then again using the decomposition $I = U_{\mathcal{O}} U_\pi^- A_{\mathcal{O}}$ we obtain

$$Iw\pi^{\lambda^\vee} U_{\mathcal{O}} \cap Iv\pi^{\lambda^\vee} U_\pi^- \neq \emptyset. \tag{3.25}$$

As $\pi^{\lambda^\vee}$ normalizes both U and U^- we may conclude that $IwU \cap IvU^- \neq \emptyset$. Suppose that we take some element in this intersection

$$iwu^+ = i'vu^- \text{ where } i, i' \in I, \, u^\pm \in U^\pm. \tag{3.26}$$

Then rearranging this last equation, we find that $u^- \in KU$. By [2, Lemma 3.3] (or see (A.7) below), this implies that $u^- \in U_{\mathcal{O}}^- = U^- \cap K$ and so it follows form (3.26) that we may assume that $u^+ \in U_{\mathcal{O}}$ as well. Thus, we have produced an element which lies in the intersection of $IwI \cap IvI^-$. From here, we may proceed by a simple induction to conclude that $w \geq v$ in the usual Bruhat order on W (see Claim 2.2.2). Reversing the role of v and w in the above argument, we may also conclude that $v \geq w$ and so $v = w$. □

3.4.4. *An order on $\mathcal{W}$.* An analysis of the argument above suggests the following order on $\mathcal{W}$.

Definition. *Let $x, y \in \mathcal{W}$, which we write as $x = \pi^{\lambda^\vee} w$ and $y = \pi^{\mu^\vee} v$ with $w, v \in W$ and $\lambda^\vee, \mu^\vee \in \Lambda^\vee$. We say that $x \preceq y$ if either*

- *$\lambda^\vee < \mu^\vee$, where $\leq$ is the dominance order on $\Lambda^\vee$;*
- *$\lambda^\vee = \mu^\vee$ and $w \leq v$ with respect to the Bruhat order on W.*

It is easy to see that $\preceq$ is a partial order on $\mathcal{W}$. For us, this order will be important due to the following result, which translates into showing the the Iwahori-Hecke algebra $H(G_+, I)$ acts faithfully on its generic principal series $M(G, I)$.

Proposition. *Let $x \in \mathcal{W}_X$ and $y \in \mathcal{W}$. If $IxI \cap UyI \neq \emptyset$, then $x \preceq y$.*

Proof. Write $x = \pi^{\lambda^\vee} w$ and $y = \pi^{\mu^\vee} v$ with $w, v \in W$ and $\lambda^\vee \in X$, $\mu^\vee \in \Lambda^\vee$. Then if $U\pi^{\mu^\vee} vI \cap I\pi^{\lambda^\vee} wI \neq \emptyset$, we also have that

$$U\pi^{\mu^\vee} K \cap K\pi^{\lambda^\vee} K \neq \emptyset. \tag{3.27}$$

It follows from this (see [2, Theorem 1.9(1)]) that $\mu^\vee \geq \lambda^\vee_+$ where $\lambda^\vee_+$ is the dominant element in the W-orbit of $\lambda^\vee$. Hence also $\mu^\vee \geq \lambda^\vee$ since $\lambda^\vee_+ \geq \lambda^\vee$.

Thus, we shall assume $\mu^\vee = \lambda^\vee$, and so $I\pi^{\lambda^\vee} wI \cap U\pi^{\lambda^\vee} vI \neq \emptyset$. Using the decomposition $I = U_{\mathcal{O}} U^-_\pi A_{\mathcal{O}}$ we obtain that $U^-_\pi \pi^{\lambda^\vee} wI \cap U\pi^{\lambda^\vee} vI \neq \emptyset$, or in other words, $U^-_\pi \pi^{\lambda^\vee} \cap U\pi^{\lambda^\vee} vIw^{-1} \neq \emptyset$. From here it follows that $U^- \cap UvIw^{-1} \neq \emptyset$. From [2] (see also (A.7)), we may conclude that $U^- \cap UK \subset U^- \cap K$, and so we may conclude that $U^-_{\mathcal{O}} \cap UvIw^{-1} \neq \emptyset$. From this last statement, we conclude that $U^-_{\mathcal{O}} \cap U_{\mathcal{O}} vIw^{-1} \neq \emptyset$. Thus we have produced an element in the intersection of $I^- wI \cap IvI$, which implies that $w \leq v$.

□

3.4.5. *A double coset decomposition.* In the sequel, we shall also need to understand the double cosets of G under the left action of the group $A_{\mathcal{O}} U$ and under the right action of the groups I or I^-. The following result follows immediately from the Iwasawa decomposition and (3.12) and we suppress the proof.

Lemma. *The maps $\mathcal{W} \to A_{\mathcal{O}} U \setminus G/I$ and $\mathcal{W} \to A_{\mathcal{O}} U \setminus G/I^-$ which send $x \in \mathcal{W}$ to $A_{\mathcal{O}} UxI$ and $A_{\mathcal{O}} UxI^-$ are bijections.*

4. Generalities on Convolution Algebras

In this section, we describe some axiomatic patterns which our construction of Hecke algebras and their modules in the next two sections will follow. The notation is independent of the previous sections. Throughout this section we shall fix the following notation: Γ will be a group and $\Gamma_+ \subset \Gamma$ will be a sub-semigroup. All constructions will take this pair (Γ, Γ_+) as data, though we shall often omit Γ from our notation.

4.1. **Convolution of Finitely Supported Functions.**

4.1.1. *Basic Notations on Spaces of Double Cosets.* Let $L, R \subset \Gamma_+$ be sub*groups* of Γ, and further assume that there exists a set $\Lambda_{L,R}$ equipped with a bijection to the set of (L, R) double cosets of Γ_+,

$$X_{L,R} : \Lambda_{L,R} \to L \setminus \Gamma_+/R, \quad \lambda \mapsto X^\lambda \tag{4.1}$$

When there is no danger of confusion, we shall often omit the subscripts L and R from our notation and just write $X : \Lambda \to L \setminus \Gamma_+/R$.

Given a function $f : \Gamma_+ \to \mathbb{C}$ which is left L-invariant and right R-invariant, there exists a subset $\Lambda_f \subset \Lambda$ defined as

$$\Lambda_f := \{\lambda \in \Lambda \mid f(x) \neq 0 \text{ for any } x \in X^\lambda\} \tag{4.2}$$

We shall often write $f = \sum_{\mu \in \Lambda_f} c_\mu X^\mu$ where $c_\mu = f(x)$ for any $x \in X^\mu$. We shall say that f has *finite support* if $\Lambda_f \subset \Lambda$ is finite. Denote by $\mathcal{F}(L \setminus \Gamma_+ / R)$ the set of finitely supported left L and right R-invariant functions. It is clear that $\mathcal{F}(L \setminus \Gamma_+ / R)$ is a $\mathbb{C}$-vector space with basis indexed by the characteristic functions X^λ for $\lambda \in \Lambda$.

4.1.2. *Fiber Products.* Let $R \subset \Gamma_+$ be a subgroup. Given a right R-set A (i.e., a set with a right R-action), a left R-set B, and any set S, we say that a map $m : A \times B \to S$ is *R-linear* if

$$m(ai, i^{-1}b) = m(a, b) \text{ for } a \in A, b \in B, i \in R.$$

Equivalently, if we endow the set $\mathrm{Hom}(B, S)$ with a left R-action via

$$i\varphi(b) = \varphi(i^{-1}b) \text{ for } i \in R, \varphi \in \mathrm{Hom}(B, S), \tag{4.3}$$

then an R-linear map from $A \times B \to S$ is a map $\psi : A \to \mathrm{Hom}(B, S)$ such that

$$\psi(ai^{-1}) = i.\psi(a) \text{ for } i \in R, a \in A. \tag{4.4}$$

Indeed, for each $m : A \times B \to S$ the map $\psi_m : A \to \mathrm{Hom}(B, S)$ given by $\psi_m(a) = m(a, \cdot)$ satisfies (4.4); and it is easy to see that all such maps come from R-linear maps m. Using either of these two descriptions, it is easy to verify that the functor

$$F_{A,B}(S) = \{R\text{-linear maps } A \times B \to S\} \tag{4.5}$$

is corepresented by the quotient set

$$A \times_R B := A \times B / \equiv \tag{4.6}$$

where $(a, b) \equiv (ai, i^{-1}b)$ for $a \in A, b \in B, i \in R$: i.e., $\mathrm{Hom}(A \times_R B, S) = F_{A,B}(S)$.

A variant of the above construction is as follows: let A_1 be a right R-set, $A_2, \ldots, A_{r-1}$ be left-right R-sets, and A_r a left R set. Then for any set S, an R^{r-1}-linear map $m : A_1 \times \cdots \times A_r \to S$ is one such that

$$m(a_1 i_1, i_1^{-1} a_2 i_2, i_2^{-1}, \ldots, i_{r-1}^{-1} a_r) = m(a_1, \ldots, a_r) \text{ for } a_k \in A_k, i_j \in I.$$

One can then check that the functor

$$F_{A_1,\ldots,A_r}(S) = \{R^{r-1} - \text{linear maps } m : A_1 \times \cdots \times A_r \to S\}$$

is represented by

$$A_1 \times_R A_2 \times \cdots \times_R A_r := A_1 \times A_2 \cdots \times A_r / \equiv$$

where $(a_1, \ldots a_r) \equiv (a_1 i_1, i_1^{-1} a_2 i_2, i_2^{-1}, \ldots, i_{r-1}^{-1} a_r)$ for $(i_1, \ldots, i_{r-1}) \in I^{r-1}$ generates the equivalence relation. Suppose in addition that A_1, A_2, A_3 carry both a left and right R-action. Then $A_1 \times_R A_2$ inherits a left and right R-set structure, and similarly for $A_2 \times_R A_3$. Hence we may form the sets $(A_1 \times_R A_2) \times_R A_3$ and $A_1 \times_R (A_2 \times_R A_3)$. It is then easy to verify the following associativity of functors,

$$F_{A_1,A_2,A_3} = F_{A_1 \times_R A_2, A_3} = F_{A_1, A_2 \times_R A_3}, \tag{4.7}$$

which will be used below in the proof of Proposition 4.1.5 below.

4.1.3. *Explicit Description of Fibers.* Let $R \subset \Gamma_+$ be a subgroup, and let A be a right R-coset and B a left R-coset of Γ. Define the multiplication $m_{A,B} : A \times_R B \to \Gamma_+$ which sends $(a, b) \mapsto ab$. For any $x \in \Gamma_+$, we can easily verify that there is a bijection of sets

$$m_{A,B}^{-1}(x) = R \setminus A^{-1}x \cap B. \tag{4.8}$$

Variant 1: Suppose that A, B are R-double cosets of Γ_+. Then we can describe the fibers $m^{-1}(x)$ in terms of left or right R-cosets:

$$m^{-1}(x) = R \setminus A^{-1}x \cap B = A \cap xB^{-1}/R. \tag{4.9}$$

In this case, we see that $m^{-1}(x)$ only depends, up to bijection, on the coset of x in $R \setminus \Gamma_+ / R$.

Variant 2: Let L, R, H are three subgroups of Γ_+, and A be a (L, R) double coset and B a (R, H) double coset. Then we have

$$m^{-1}(x) = R \setminus A^{-1}x \cap B \tag{4.10}$$

and we see that $m^{-1}(x)$ only depends, up to bijection, on the class of x in $L \setminus G/H$.

4.1.4. *Finite Hecke Datum.* Let $R \subset \Gamma_+$ be as above. Let us write $\Lambda_+ := \Lambda_{R,R}$, the set indexing R-double cosets of Γ_+, and $X : \Lambda_+ \to \Gamma_+$. For $\lambda, \mu \in \Lambda_+$ let $m_{\lambda,\mu} : X^\lambda \times_R X^\mu \to \Gamma_+$ denote the multiplication map. Note that by (4.9) the fibers $m^{-1}(x)$ for $x \in \Gamma_+$ only depend, up to bijection, on the class of x in $R \setminus \Gamma_+/R = \Lambda$. We thus write $|m^{-1}(\mu)|$ to denote the cardinality of $m^{-1}(x)$ for any $x \in X^\mu$.

Definition. *We shall say that* (R, Λ) *(we drop the map* X *from our notation) is a* finite Hecke datum *for the semi-group* $\Gamma_+ \subset \Gamma$ *if it satisfies the following two conditions,*

1. ***(H1)*** *For any* $\lambda, \mu, \nu \in \Lambda$ *we have* $|m^{-1}_{\lambda,\mu}(\nu)|$ *is finite.*
2. ***(H2)*** *Given any* $\lambda, \mu \in \Lambda$*, the set of* $\nu \in \Lambda$ *such that* $|m^{-1}_{\lambda,\mu}(\nu)| \neq 0$ *is finite.*

Let (R, Λ_+) be a finite Hecke datum. Recall that we have the space of finitely supported R-binvariant functions which we shall just denote by $H(\Gamma_+, R) := \mathcal{F}(R \backslash \Gamma_+/R)$. Given any $\lambda, \mu \in \Lambda_+$, the assumptions (H1) and (H2) above allow us to make sense of the sum

$$\sum_{\nu \in \Lambda_+} |m^{-1}_{\lambda,\mu}(\nu)| \; X^\nu \tag{4.11}$$

as an element of $H(\Gamma_+, R)$ and so we can define a map $\star : H(\Gamma_+, R) \times H(\Gamma_+, R) \to H(\Gamma_+, R)$ by linearly extending the above map,

$$X^\lambda \star X^\mu = \sum_{\nu \in \Lambda_+} |m^{-1}_{\lambda,\mu}(\nu)| \; X^\nu. \tag{4.12}$$

4.1.5. *Finite Convolution Hecke Algebras.* The importance of the notion of finite Hecke datum is provided by the following result, which is certainly well-known. We sketch a few points of the proof as we shall need several variants of it in the sequel.

Proposition. *Let* (R, Λ_+) *be a finite Hecke datum. Then the map* $\star$ *in (4.12) equips* $H(\Gamma_+, R)$ *with the structure of an associative* $\mathbb{C}$*-algebra with a unit. The unit is given by convolution with the characteristic function of* R*.*

Proof. Let us show the associativity of $\star$: let $\lambda, \mu, \nu \in \Lambda_+$. Given $\xi \in \Lambda_+$, and $x \in X^\xi$ we would like to see that

$$(X^\lambda \star X^\mu) \star X^\nu(x) = X^\lambda \star (X^\mu \star X^\nu)(x). \tag{4.13}$$

Let $m_{\lambda,\mu,\nu} : X^\lambda \times_R X^\mu \times_R X^\nu \to \Gamma_+$ be the map induced by multiplication. Consider the R-bilinear sets $X^\lambda \times_R X^\mu$ and $X^\mu \times_R X^\nu$. From (4.7), we have

$$F_{(X^\lambda \times_R X^\mu) \times_R X^\nu}(\Gamma_+) = F_{X^\lambda \times_R X^\mu \times_R X^\nu}(\Gamma_+) = F_{X^\lambda \times_R (X^\mu \times_R X^\nu)}(\Gamma_+). \tag{4.14}$$

The map $m_{\lambda,\mu,\nu} \in F_{X^\lambda \times_R X^\mu \times_R X^\nu}(\Gamma_+)$ and hence defines corresponding maps which we denote by $m_{(\lambda,\mu),\nu} \in F_{(X^\lambda \times_R X^\mu) \times_R X^\nu}(\Gamma_+)$ and $m_{\lambda,(\mu,\nu)} \in F_{X^\lambda \times_R (X^\mu \times_R X^\nu)}(\Gamma_+)$. Then one can show that the left hand side of (4.13) is equal to $|m^{-1}_{(\lambda,\mu),\nu}(x)|$ and the right hand side is equal to $|m^{-1}_{\lambda,(\mu,\nu)}(x)|$; both of which are in turn equal to $|m^{-1}_{\lambda,\mu,\nu}(x)|$ which proves the associativity. We supress the proof of this fact, as well as that of the rest of the Proposition. □

4.1.6. *Finite Hecke Modules.* Let $L, R \subset \Gamma_+$ be as in §4.1.1. Let us write Λ_+ for $\Lambda_{R,R}$, and continue to write $H(\Gamma_+, R)$ for $\mathcal{F}(R \setminus \Gamma_+/R)$. Now suppose that $\Omega := \Lambda_{L,R}$ is a set of representatives for the (L, R)-double cosets of Γ (*note:* we want to really consider Γ here instead of Γ_+, as this is the case which comes up in the sequel). For each $\lambda \in \Omega$ we denote the corresponding double coset by Y^λ. Let us write $M(L, \Gamma, R)$ for the set $\mathcal{F}(L \setminus \Gamma/R)$. Given $\lambda \in \Omega, \mu \in \Lambda_+$ we have a map

$$a_{\lambda,\mu} : Y^\lambda \times_R X^\mu \to \Gamma$$

induced from multiplication.

Definition. *We say that the collection $(R, \Lambda_+; L, \Omega)$ is a* finite Hecke module datum *if*

1. ***(M0)*** *(R, Λ_+, X) is a finite Hecke datum*
2. ***(M1)*** *For any $\lambda \in \Omega$ and $\mu \in \Lambda_+$ the map $a_{\lambda,\mu}$ has finite fibers.*
3. ***(M2)*** *For any $\lambda \in \Omega$ and $\mu \in \Lambda_+$, the there are only finitely many $\nu \in \Omega$ such that $a^{-1}_{\lambda,\mu}(\nu)$ is non-empty.*

Using the properties (M1) and (M2) we can make sense, for each $\lambda \in \Omega, \mu \in \Lambda_+$ of the sum,

$$\sum_{\nu \in \Omega} |a^{-1}_{\lambda,\mu}(\nu)| Y^\nu \tag{4.15}$$

as an element of $M(L, \Gamma, R)$. Thus, we can define a map $\star_a : M(L, \Gamma, R) \times H(\Gamma_+, R) \to M(L, \Gamma, R)$ by linearly extending the following formula,

$$Y^\lambda \star_a X^\mu := \sum_{\nu \in \Omega} |a^{-1}_{\lambda,\mu}(\nu)| Y^\nu. \tag{4.16}$$

We omit the proof of the following since it is very similar to that of Proposition 4.1.5.

Proposition. *Let $(R, \Lambda_+; L, \Omega)$ be a finite Hecke module datum. Then $\star_a$ defines an associative, unital right $H(\Gamma_+, R)$-module structure on $M(L, \Gamma, R)$, where the unit element of $H(\Gamma_+, R)$ acts as the identity map on $M(L, \Gamma, R)$.*

4.1.7. *A Simplified Critrerion.* In what follows, we shall use the following variant of the above results, which reduces the number of conditions one needs to verify for $(R, \Lambda_+; L, \Omega)$ to be a finite Hecke module datum. We leave the proof to the reader (see Proposition 4.2.4 for more details in a slightly more complicated context).

Proposition. *The quadruple $(R, \Lambda_+; L, \Omega)$ is a Hecke module datum if (R, Λ_+) satisfy (H2) and $(R, \Lambda_+; L, \Omega)$ satisfy (M1) and (M2).*

4.2. **Completions of Convolution Algebras.**

4.2.1. *Semi-infinite Support.* Let Λ be an abelian group whose underlying set is equipped with a partial order $\leq$ that is compatible with the group structure: i.e, if $\lambda \leq \mu$ then $\lambda + \nu \leq \mu + \nu$ for $\lambda, \mu, \nu \in \Lambda$. For elements $\lambda \geq \mu$ we shall write

$$[\lambda, \mu] := \{\nu \in \Lambda | \lambda \leq \nu \leq \mu\}. \tag{4.17}$$

For an element $\lambda \in \Lambda$ we shall set

$$\mathfrak{c}(\lambda) = \{\mu \in \Lambda : \mu \leq \lambda\}.$$

A subset $\Xi \subset \Lambda$ is said to be *semi-infinite* if there exists a finite set of elements $\lambda_1, \ldots, \lambda_r$ such that

$$\Xi \subset \bigcup_{i=1}^{r} \mathfrak{c}(\lambda_i). \tag{4.18}$$

If $\Lambda^+ \subset \Lambda$ is a sub-semigroup, it inherits an order which we shall continue to denote by $\leq$ and all of the constructions above can be repeated with Λ^+ in place of Λ.

4.2.2. *Semi-infinite Convolution Hecke Algebras.* Let $\Gamma_+ \subset \Gamma, R, X$ be as in §4.1.1. As before set $\Lambda_+ := \Lambda_{R,R}$. Let $\Lambda \supset \Lambda_+$ be an ordered abelian group with order denoted $\leq$ (the restriction of $\leq$ to Λ_+ makes it into an ordered semi-group; the restricted order is also denoted by $\leq$). Let $f : \Gamma_+ \to \mathbb{C}$ be a function which is R-binvariant. We may define its support $\Lambda_f \subset \Lambda_+$ as in (4.2). We shall say that f has semi-infinite support if $\Lambda_f \subset \Lambda_+$ is a semi-infinite subset. Denote the set of R-binvariant semi-infinitely supported functions on Γ_+ by $H_{\leq}(\Gamma_+, R)$. For $\lambda, \mu \in \Lambda_+$ denote by $m_{\lambda,\mu} : X^\lambda \times_R X^\mu \to \Gamma_+$ the natural multiplication map.

Definition. *A collection $(R, \Lambda_+, \Lambda, \leq)$ will be said to be a* semi-infinite Hecke algebra datum *if the following conditions are met,*

(1) ***(F)*** *For every pair $\lambda, \mu \in \Lambda$, $\lambda \leq \mu$, the set $[\lambda, \mu]$ is finite.*
(2) ***(SH1)*** *For any $\lambda, \mu \in \Lambda_+$ the fibers of the map $m_{\lambda,\mu}$ are finite.*
(3) ***(SH2)*** *For any $\lambda, \mu \in \Lambda_+$, if $|m^{-1}_{\lambda,\mu}(\nu)| \neq 0$, then $\nu \leq \lambda + \mu$.*

Given any $\lambda, \mu \in X$, the assumptions (SH1) and (SH2) allow us to make sense of the sum

$$\sum_{\nu \in \Lambda_+} |m^{-1}_{\lambda,\mu}(\nu)|\ X^\nu$$

as an element of $H_{\leq}(\Gamma_+, R)$. Setting

$$X^\lambda \star X^\mu = \sum_{\nu \in \Lambda_+} |m^{-1}_{\lambda,\mu}(\nu)|\ X^\nu,$$

the Proposition below tells us in particular that we may extend this multiplication to a map

$$\star : H_{\leq}(\Gamma_+, R) \times H_{\leq}(\Gamma_+, R) \to H_{\leq}(\Gamma_+, R) \tag{4.19}$$

using the assumptions (F), (SH1), and (SH2).

Proposition. *Let$(R, \Lambda_+, \Lambda, \leq)$ be a semi-infinite Hecke datum. Then the product $\star$ endows $H_{\leq}(\Gamma_+, R)$ with the structure of an associative, unital algebra with unit being given by the trivial R-double coset R.*

Proof. The proof is very similar to that of Proposition 4.1.5, and so let us just verify that the above formula extends to give a well-defined map as in (4.19). Then let $f_1, f_2 \in H_{\leq}(\Gamma_+, R)$ which are of the form

$$f_1 = \textstyle\sum_{\mu \leq \lambda_1} a_\mu X^\mu \quad \text{and} \quad f_2 = \sum_{\nu \leq \lambda_2} b_\nu X^\nu$$

Suppose that

$$X^\mu \star X^\nu = \sum_{\eta \leq \mu+\nu} c^{\mu,\nu}_\eta X^\eta.$$

Then we write

$$f_1 \star f_2 = \sum_{\mu \leq \lambda_1, \nu \leq \lambda_2} a_\mu b_\nu X^\mu \star X^\nu \tag{4.20}$$

$$= \sum_{\mu \leq \lambda_1, \nu \leq \lambda_2} a_\mu b_\nu \Big(\sum_{\eta \leq \mu+\nu} c^{\mu,\nu}_\eta X^\nu \Big) \tag{4.21}$$

$$= \sum_{\mu \leq \lambda_1, \nu \leq \lambda_2, \eta \leq \mu+\nu} a_\mu b_\nu c^{\mu,\nu}_\eta X^\eta \tag{4.22}$$

For fixed η we have that $\eta \leq \mu + \nu \leq \lambda_1 + \lambda_2$ and by property (F) there are only finitely many values which $\mu + \nu$ can take. Again for fixed η, since $\mu \leq \lambda_1$ and $\nu \leq \lambda_2$ there are only finitely many terms in the sum,

$$\sum_{\mu \leq \lambda_1, \nu \leq \lambda_2, \eta \leq \mu+\nu} a_\mu b_\nu c^{\mu,\nu}_\eta$$

and hence the formula for $f_1 \star f_2$ is well-defined. To show that $f_1 \star f_2$ has semi-infinite support, we note that

$$\Lambda_{f_1\star f_2} \subset \bigcup_{\mu\leq\lambda_1,\nu\leq\lambda_2} \Lambda_{X^\mu\star X^\nu} \subset \bigcup_{\mu\leq\lambda_1,\nu\leq\lambda_2,\eta\leq\mu+\nu} \mathfrak{c}(\eta) \subset \bigcup_{\eta\leq\lambda_1+\lambda_2} \mathfrak{c}(\eta)$$

□

4.2.3. *Semi-infinite Hecke Modules.* Let $L \subset \Gamma_+$ be a subgroup of Γ. Denote by $\Omega := \Lambda_{L,R}$ the set parametrizing (L, R)-double cosets of Γ. Assume that Ω is an ordered abelian group with order $\leq$. Furthermore, we assume this construction is compatible with the constructions from the previous paragraph in the following sense: we are given a fixed embedding $\Lambda_+ \subset \Omega$ such that the order on Λ_+ is the restriction of the order on Ω. For a function $f : \Gamma \to \mathbb{C}$ which is (L, R)-binvariant, we can define its support $\Omega_f \subset \Omega$ as in (4.2), and we say that f is semi-infinitely supported if $\Omega_f \subset \Omega$ is a semi-infinite subset. Denote as in §4.1.6 the map $Y : \Omega \to \Gamma$ which sends $\nu \in \Omega$ to the corresponding coset (L, R)-double coset Y^ν. Given $\lambda \in \Omega, \mu \in \Lambda_+$ we have a map $a_{\lambda,\mu} : Y^\lambda \times_R X^\mu \to \Gamma$, defined via multiplication.

Definition. *We say that the collection $(R, \Lambda_+; L, \Omega, \leq)$ as above is a* semi-infinite Hecke module datum *if*

1. ***(M0)*** *$(R, \Lambda_+, \Omega, \leq)$ is a semi-infinite Hecke module datum*
2. ***(SM1)*** *For any $\lambda \in \Omega$ and $\mu \in \Lambda_+$ the map $a_{\lambda,\mu}$ has finite fibers.*
3. ***(SM2)*** *For any $\lambda \in \Omega$ and $\mu \in \Lambda_+$, if $a^{-1}_{\lambda,\mu}(\nu) \neq \emptyset$ then $\nu \leq \lambda + \mu$.*

Using the properties (SM1) and (SM2) we can make sense, for each $\lambda \in \Omega, \mu \in \Lambda_+$ of the sum,

$$\sum_{\nu\in\Omega} |a^{-1}_{\lambda,\mu}(\nu)| Y^\nu$$

as an element of $M_\leq(\Gamma; L, R)$. Thus, we can define $\star_a$ via the formula

$$Y^\lambda \star_a X^\mu = \sum_{\nu\in\Omega} |a^{-1}_{\lambda,\mu}(\nu)| Y^\nu. \tag{4.23}$$

The following proposition, whose proof we omit, then shows that $\star_a$ extends to a map

$$M_\leq(\Gamma; L, R) \times H_\leq(\Gamma_+, R) \to M_\leq(\Gamma; L, R). \tag{4.24}$$

Proposition. *Let $(R, \Lambda_+; L, \Omega, \leq)$ be a semi-infinite Hecke module datum. Then $\star_a$ defines a right $(H_\leq(\Gamma_+, R), \star)$-module structure on $M_\leq(\Gamma; L, R)$.*

4.2.4. *A Simplified Criterion.* In practice, one has the following slight but useful strengthening of the above result.

Proposition. *Let $(R, \Lambda_+; L, \Omega, \leq)$ satisfy all of the conditions of being a semi-infinite Hecke module datum except (SH1). Then (SH1) follows and $(R, \Lambda_+, L, \Omega, \leq)$ is in fact a semi-infinite Hecke module datum.*

Proof. Let $\lambda, \mu \in \Lambda_+$, $\nu \in \Omega$, and consider the map,

$$a_{\nu,\lambda,\mu} : Y^\nu \times_R X^\lambda \times_R X^\mu \to \Gamma \tag{4.25}$$

induced by multiplication, i.e., which sends $(x, y, z) \mapsto xyz$. It suffices to show that the fibers of this map are finite. Indeed, if (SH1) were not satisfied, then there would exist some $\lambda, \mu \in \Lambda_+$ and $\xi \in \Lambda_+$ such that $m^{-1}_{\lambda,\mu}(\xi)$ is infinite. Choose any element z which lies in the image of the multiplication map $a_{\nu,\xi} : Y^\nu \times_R X^\xi \to \Gamma$. Then $a^{-1}_{\nu,\lambda,\mu}(z)$ will be infinite.

By condition (SM2) the image of $a_{\nu,\lambda}$ lies in the union $\bigcup_{\eta\leq\lambda+\nu} Y^\eta$. For any such η, if Y^ξ lies in the image of $a_{\eta,\mu}$, we must have $\xi \leq \nu + \mu$. Hence, for each such ξ we have inequalities,

$$\xi \leq \eta + \mu \leq \lambda + \mu + \nu.$$

From this it follows that there can be only finitely many η for fixed ξ, λ, μ, ν. The finiteness of the fibers of $a_{\nu,\lambda,\mu}$ follows from associativity using (SM1): for any element z in the image of this map, we must have $z \in Y^{\xi}$ as above for fixed ξ, λ, μ, ν as above. □

5. Iwahori Theory I: "Affine" Hecke Algebras and Convolution Hecke algebras

Fix the notations of §3 in this section. We shall now apply the axiomatics developed in the previous section to construct a convolution algebra $H(G_+, I)$ on the space of I-double cosets of G_+. This algebra may be identified with an "affine" Hecke algebra (a slight variant of Cherednik's DAHA) as we show in Theorem 5.3.3. The proof of this theorem rests on the construction of a certain family of commuting elements $H(G_+, I)$ (see Proposition 5.3.3) whose proof is deferred to §6.

5.1. "Affine" Hecke Algebras.

We begin by defining the precise variant of Cherednik's DAHA which will arise from our group theoretic convolution algebra.

5.1.1. *Garland-Grojnowski Algebras.* Fix the notations as in §2.1. The Weyl group W of our Kac-Moody Lie algebra $\mathfrak{g}$ is a Coxeter group with length function, $\ell : W \to \mathbb{Z}$. As such, we can associate a Hecke algebra to it as follows: first, define the *Braid group* B_W as the group with generators $\mathbb{T}_w$ for $w \in W$ subject to the relations,

$$\mathbb{T}_{w_1}\mathbb{T}_{w_2} = \mathbb{T}_{w_1 w_2} \text{ if } \ell(w_1) + \ell(w_2) = \ell(w_1 w_2). \tag{5.1}$$

Let v be an indeterminate, and consider $F = \mathbb{C}(v)$, the field of rational functions in the intermediate v. [8] The *Hecke algebra* $\mathbb{H}_W$ associated to W is then the quotient of the group algebra $F[B_W]$ by the ideal generated by the relations

$$(\mathbb{T}_a + 1)(\mathbb{T}_a - v^{-2}) = 0 \text{ for } a \in \Pi^\vee. \tag{5.2}$$

Recall that $\Lambda^\vee$ was the weight lattice of $\mathfrak{g}$, and consider the group algebra $R = \mathbb{C}[\Lambda^\vee]$. For $\lambda^\vee \in \Lambda^\vee$, denote by $\Theta_{\lambda^\vee}$ the corresponding element of R subject to the relations $\Theta_{\lambda^\vee}\Theta_{\mu^\vee} = \Theta_{\lambda^\vee+\mu^\vee}$ if $\lambda^\vee, \mu^\vee \in \Lambda^\vee$. Following Garland and Grojnowski [12] we can then define $\mathbb{H}$, the *"affine" Hecke algebra associated to* W as the algebra generated by $\mathbb{H}_W$ and $\mathbb{C}[\Lambda^\vee]$ subject to the relations

$$\mathbb{T}_a\Theta_{\lambda^\vee} - \Theta_{w_a(\lambda^\vee)}\mathbb{T}_a = (v^{-2} - 1)\frac{\Theta_{\lambda^\vee} - \Theta_{w_a\lambda^\vee}}{1 - \Theta_{-a^\vee}} \tag{5.3}$$

Expanding the right hand side of (5.3) as a series in $\Theta_{-a^\vee}$, it is seen to be an element in $F[\Lambda^\vee]$, i.e.,

$$(5.3) = \begin{cases} 0 & \text{if } \langle a, \lambda^\vee \rangle = 0 \\ (v^{-2} - 1)(\Theta_{\lambda^\vee} + \Theta_{\lambda^\vee - a^\vee} + \cdots + \Theta_{w_a\lambda^\vee + a^\vee}) & \text{if } \langle a, \lambda^\vee \rangle > 0 \\ (1 - v^{-2})(\Theta_{w_a\lambda^\vee} + \Theta_{w_a\lambda^\vee - a^\vee} + \cdots + \Theta_{w_a\lambda^\vee - (\langle a, w_a\lambda^\vee \rangle - 1)a^\vee}) & \text{if } \langle a, \lambda^\vee \rangle < 0 \end{cases} \tag{5.4}$$

Recall from §2.1.4 that $\Lambda^\vee$ is equipped with a grading $\Lambda^\vee = \oplus_{r\in\mathbb{Z}}\Lambda^\vee_r$. Hence, $F[\Lambda^\vee]$ is also equipped with a grading

$$F[\Lambda^\vee] = \bigoplus_{r\in\mathbb{Z}} F[\Lambda^\vee_r] \tag{5.5}$$

where $F[\Lambda^\vee_r]$ is the F span of $\Theta_{\lambda^\vee}$ for $\langle \delta, \lambda^\vee \rangle = r$. The algebra $\mathbb{H}$ also inherits a $\mathbb{Z}$-grading in this way, and we denote by $\mathbb{H}_r$ the typical graded piece for $r \in \mathbb{Z}$. As each graded piece $\Lambda^\vee$ is W-invariant, so too are the pieces $F[\Lambda^\vee_r]$. Using (5.3) we can easily deduce that for each $r \in \mathbb{Z}$, the subspace $F[\Lambda^\vee_r]$ is a $\mathbb{H}_W$-module. In general $\mathbb{H}_r$ is not a subalgebra since $\Lambda^\vee_r$ is not closed under addition. However, $\mathbb{H}_0$ is a subalgebra and is essentially the Double Affine Hecke Algebra (DAHA) of Cherednik . Let $\mathbb{H}_0' \subset \mathbb{H}$ be the subalgebra generated by $\mathbb{H}_W$ and $\Theta_{n\mathbf{c}}$ for $n \in \mathbb{Z}$ and $\mathbf{c}$ the

[8] One could also consider the ring $\mathbb{C}[v, v^{-1}]$ in what follows

minimal positive imaginary coroot. As W fixes $\mathbf{c}$ and $\mathbb{Z}\mathbf{c}$ is closed under addition, the subspace $\mathbb{H}_0'$ is also a subalgebra of $\mathbb{H}$. What will be important for us is that the following subspace of $\mathbb{H}$,

$$\mathbb{H}_+ := \mathbb{H}_0' \oplus \bigoplus_{r>0} \mathbb{H}_r \tag{5.6}$$

is also a subalgebra. Indeed, this follows from the following simple result whose proof we omit

Lemma. *The algebra $\mathbb{H}_+$ is the subalgebra of $\mathbb{H}$ generated by $\mathbb{H}_W$ and $F[X]$, the group algebra of the Tits cone $X \subset \Lambda^\vee$.*

5.1.2. *A module for $\mathbb{H}$.* Let $\mathbb{M}$ be the F-vector space with basis $\mathbf{v}_x$ for $x \in \mathcal{W}$ (denote by $\mathbf{v}_1$ the basis element corresponding to the identity element of $\mathcal{W}$.) Then we set

$$\Theta_{\lambda^\vee}\mathbf{v}_1 = \mathbf{v}_{\pi^{\lambda^\vee}} \text{ for } \lambda^\vee \in \Lambda^\vee \tag{5.7}$$

$$\mathbb{T}_w \mathbf{v}_{\pi^{\mu^\vee}} = \mathbf{v}_{\pi^{\mu^\vee} w} \text{ for } w \in W, \mu^\vee \in \Lambda^\vee. \tag{5.8}$$

It is easy to see that this defines a representation of $\mathbb{H}$ on $\mathbb{M}$. Let $\mathbb{M}_+$ denote the F-subspace of $\mathbb{M}$ which is generated by $\mathbf{v}_x$ for $x \in \mathcal{W}_X$. Then one has that $\mathbb{M}_+ \subset \mathbb{H}_+ \cdot \mathbf{v}_1$. Indeed, if $x = \pi^{\mu^\vee} w \in \mathcal{W}_X$, then

$$\mathbb{T}_w \Theta_{\mu^\vee}\mathbf{v}_1 = \mathbf{v}_{\pi^{\mu^\vee} w}. \tag{5.9}$$

We shall show below (see Step 2 of the proof of Theorem 5.3.3) the reverse inclusion: $\mathbb{H}_+ \cdot \mathbf{v}_1 \subset \mathbb{M}_+$.

5.2. **Convolution Algebras of I-double cosets.** We would like to show that the set of finite linear combinations of I-double cosets on the semigroup G_+ can be equipped with the structure of a convolution algebra. We shall moreover construct a natural action of this convolution algebra on the space of functions on G with are right I-invariant and left invariant by the subgroup $A_{\mathcal{O}}U$.

5.2.1. *Constructing $H(G_+, I)$ and $M(G, I)$.* From (3.18) and Proposition 3.4.2 , the space of I-double cosets of G_+ is parametrized by the semi-group $\mathcal{W}_X$ (see (3.17). Denote by $T : \mathcal{W}_X \to G_+$, $x \mapsto T_x$ the map which assigns $x \in \mathcal{W}_X$ to the corresponding I-double coset $T_x := IxI$. Consider the subgroup $A_{\mathcal{O}}U \subset G$ and note that from Lemma 3.4.5 the set of $(A_{\mathcal{O}}U, I)$-double cosets of G are parametrized by the set $\mathcal{W}$. Denote by $\mathbf{v} : \mathcal{W} \to G$, $y \mapsto \mathbf{v}_y$ the map which assigns to each $x \in \mathcal{W}$ the corresponding double coset $\mathbf{v}_y := A_{\mathcal{O}}UxI \subset G$. We then have the following,

Theorem. *The data $(I, \mathcal{W}_X)$ forms a finite Hecke datum for G_+ (in the sense of Definition 4.1.4), and the the triple $(I, \mathcal{W}_X; A_{\mathcal{O}}U, \mathcal{W})$ forms a finite Hecke module datum for G_+ (in the sense of Definition 4.1.6).*

We denote by $H(G_+, I)$ and $M(G, I) := M(A_{\mathcal{O}}U, G, I)$ the corresponding convolution algebra and module, noting that we have an action

$$\star : M(G, I) \times H(G_+, I) \to M(G, I) \tag{5.10}$$

given by right-convolution as in (4.24) . The remainder of §5.2 is devoted to the proof of Theorem 5.2.1.

5.2.2. *Reformulating Theorem 5.2.1.* The following result will be shown to imply Theorem 5.2.1.

Proposition. *We have the following,*

(1) *Let $x, y \in \mathcal{W}_X$. Then there exist finitely many $z_i \in \mathcal{W}_X, i = 1, \dots, n$ such that*

$$IxIyI \subset \bigcup_{i=1}^n Iz_iI.$$

(2) *Let $x, y, z \in \mathcal{W}_X$. Then*

$$I \setminus Ix^{-1}Iz \cap IyI \tag{5.11}$$

is a finite set.

(3) *Let* $x \in \mathcal{W}_X$ *and* $y \in \mathcal{W}$. *Then there exists finitely many* $z_i \in \mathcal{W}, i = 1, \ldots, \ell$ *such that*

$$A_{\mathcal{O}} U y I x I \subset \bigcup_{i=1}^{\ell} A_{\mathcal{O}} U z_i I \tag{5.12}$$

(4) *Let* $x \in \mathcal{W}_X$ *and* $y, z \in \mathcal{W}$. *Then*

$$I \setminus I y A_{\mathcal{O}} U z \cap I x I \tag{5.13}$$

is a finite set.

Proof of Theorem 5.2.1 from Proposition 5.2.2. For $x, y \in \mathcal{W}_X$ consider the multiplication map $m_{x,y} : T_x \times_I T_y \to G_+$. The image is clearly equal to $IxIyI$, and hence (H1) is equivalent to Part (1) of the above proposition. The fiber over $z \in \mathcal{W}_X$ is equal to $I \setminus Ix^{-1}Iz \cap IyI$ by (4.8) and hence (H2) follows from part (2) of the above proposition. Similarly, one checks that Part (3) is equivalent to (M1) and Part (4) to condition (M2). □

We are now reduced to showing Proposition 5.2.2, parts (1), (3), (4) , since by invoking Proposition 4.1.7, part (2) will then follow automatically.

5.2.3. *Proof of Proposition 5.2.2, part (1).* Recall that $I = U_{\mathcal{O}} U_{\pi}^{-} A_{\mathcal{O}}$ and so

$$IxIyI = IxU_{\mathcal{O}} U_{\pi}^{-} yI$$

since $\mathcal{W}$ normalizes $A_{\mathcal{O}}$. The following result is the group theoretic analogue of Proposition B.1 from Appendix B.

Lemma. *For* $x, y \in \mathcal{W}_X$ *the following spaces of cosets are finite,*

$$I \setminus IxU_{\mathcal{O}} \text{ and } U_{\pi}^{-} yI/I. \tag{5.14}$$

Proof. Let us write $x = w\pi^{\mu^\vee}$ where $w \in W$ and $\mu^\vee \in X$. Furthermore, as $\mu^\vee \in X$ we may write it as $\mu^\vee = \sigma^{-1}\lambda^\vee$ with $\lambda^\vee \in \Lambda_+^\vee$ and $\sigma \in W$.

Decompose $U_{\mathcal{O}}$ into a semi-direct product $U_{\mathcal{O}} = U_{\mathcal{O}}^{\sigma} U_{\sigma,\mathcal{O}}$, where as before

$$U_{\sigma,\mathcal{O}} = \{u \in U_{\mathcal{O}} \mid \sigma u \sigma^{-1} \in U^-\} \quad \text{and} \quad U_{\mathcal{O}}^{\sigma} = \{u \in U_{\mathcal{O}} \mid \sigma u \sigma^{-1} \in U\}, \tag{5.15}$$

and note that $U_{\sigma,\mathcal{O}}$ is a finite product of one-dimensional groups. So $Iw\pi^{\mu^\vee} U_{\mathcal{O}} = Iw\pi^{\mu^\vee} U_{\mathcal{O}}^{\sigma} U_{\sigma,\mathcal{O}}$. On the other hand, we know that $\pi^{\mu^\vee} U_{\mathcal{O}}^{\sigma} \pi^{-\mu^\vee} \subset U_{\mathcal{O}}^{\sigma}$ since

$$\pi^{\mu^\vee} U_{\mathcal{O}}^{\sigma} \pi^{-\mu^\vee} = \sigma^{-1} \pi^{\lambda^\vee} \sigma U_{\mathcal{O}}^{\sigma} \sigma^{-1} \pi^{-\lambda^\vee} \sigma \subset U_{\mathcal{O}}^{\sigma} \tag{5.16}$$

and $\pi^{\lambda^\vee}$, $\lambda^\vee \in \Lambda_+^\vee$ normalizes $U_{\mathcal{O}}$. Hence $Iw\pi^{\mu^\vee} U_{\mathcal{O}} \subset IwU_{\mathcal{O}} \pi^{\mu^\vee} U_{\sigma,\mathcal{O}}$. It is easy to see that $I \setminus IwU_{\mathcal{O}}$ is finite, and so it follows that that $I \setminus IwU_{\mathcal{O}} \pi^{\mu^\vee} U_{\sigma,\mathcal{O}}$ is finite, by the finite-dimensionality of U_{σ}.

A similar argument shows that $U_{\pi}^{-} yI/I$ is finite. □

From the Lemma 5.2.3, there exist $u_1, \ldots, u_n \in U_{\mathcal{O}}$ and $u_1^-, \ldots, u_l^- \in U_{\pi}^-$ such that

$$IxIyI \subset \bigcup_{i,j} Ixu_i u_j^- yI. \tag{5.17}$$

Part (1) of Proposition 5.2.2 then follows from the semi-group property of G_+.

5.2.4. *Proof of Proposition 5.2.2, part (3).* Given $x \in \mathcal{W}_X$ and $y = w\pi^{\lambda^\vee} \in \mathcal{W}$ with $w \in W$ and $\lambda^\vee \in \Lambda^\vee$, we have that

$$\begin{aligned} A_{\mathcal{O}}UyIxI &= A_{\mathcal{O}}Uw\pi^{\lambda^\vee}IxI = A_{\mathcal{O}}U\pi^{w\lambda^\vee}wU_{\mathcal{O}}U_\pi^- xI & (5.18)\\ &= A_{\mathcal{O}}U\pi^{w\lambda^\vee}wU_{w,\mathcal{O}}U_\pi^- xI & (5.19)\\ &\subset A_{\mathcal{O}}U\pi^{w\lambda^\vee}I^- wxI & (5.20)\end{aligned}$$

where $U_{w,\mathcal{O}}$ is as in (5.16). Thus, we are reduced to showing that for any $\mu^\vee \in \Lambda^\vee$ and $x \in \mathcal{W}_X$ the set $A_{\mathcal{O}}U\pi^{\mu^\vee}I^- xI$ is contained in a finite union of $(A_{\mathcal{O}}U, I)$-double cosets. Since $I^- = U_\pi U_{\mathcal{O}}^- A_{\mathcal{O}}$ we have

$$A_{\mathcal{O}}U\pi^{\mu^\vee}I^- xI = A_{\mathcal{O}}U\pi^{\mu^\vee}U_\pi U_{\mathcal{O}}^- A_{\mathcal{O}}xI = A_{\mathcal{O}}U\pi^{\mu^\vee}U_{\mathcal{O}}^- xI. \quad (5.21)$$

Arguing as in Lemma 5.2.3 we see that $U_{\mathcal{O}}^- xI/I$ is finite, and part (3) of the Proposition follows.

5.2.5. *Proof of Proposition 5.2.2, part (4).* first, we claim that we may choose $z \in \mathcal{W}$ to lie in $W \subset \mathcal{W}$. Indeed, suppose that $z = w\pi^{\lambda^\vee} = \pi^{w\lambda^\vee}w$ with $w \in W$ and $\lambda^\vee \in \Lambda^\vee$. Then

$$IxI \cap IyA_{\mathcal{O}}Uz = IxI \cap IyA_{\mathcal{O}}U\pi^{w\lambda^\vee}w = IxI \cap Iy\pi^{w\lambda^\vee}A_{\mathcal{O}}Uw. \quad (5.22)$$

Replacing y by $y\pi^{w\lambda^\vee}$ we may assume that $z \in W$.

Next, we claim that in fact we may choose $z = 1$. Indeed, for $w \in W$, we have that $IxI \cap IyA_{\mathcal{O}}Uw$ consists of finitely many left I-cosets if and only if $IxIw^{-1} \cap IyA_{\mathcal{O}}U$ consists of finitely many left I-cosets. Since $Iw^{-1}I/I$ is finite, an application of the semi-group property of G_+ shows that $IxIw^{-1}I$ is contained in finitely many I-double cosets. Thus, we are reduced to showing that for $x \in \mathcal{W}_X$ and $y \in \mathcal{W}$ that

$$I \setminus IxI \cap IyA_{\mathcal{O}}U \quad (5.23)$$

is finite. Note that $IxI = IxU_{\mathcal{O}}U_\pi^-$, and by Lemma 5.2.3 there exists finitely many x_i for $i = 1, \dots, n$ such that $IxI \subset \bigcup_{i=1}^n Ix_iU_\pi^-$. Use the Iwasawa decomposition to write Ix_i as $Ix_i'U^-$ with $x_i' \in \mathcal{W}$ for each $i = 1, \dots, n$. Noting that $Ix_iU_\pi^- \subset Ix_i'U^-$ we are reduced to showing the following,

Lemma. *Let $x, y \in \mathcal{W}$. Then the set $I \setminus IxU \cap IyU^-$ is finite.*

Proof of Lemma. For fixed $x, y \in \mathcal{W}$, consider the set

$$K_{x,y} = \{k \in K \mid k(IxU \cap IyU^-) \subset (IxU \cap IyU^-)\}. \quad (5.24)$$

Note that the set $K_{x,y}$ is invariant by left multiplication by I. Moreover, if we can show that $I \setminus K_{x,y}$ has finitely many elements, the Lemma will follow since we know from [2, Theorem 1.9(1)] that $K \setminus K(IxU \cap IyU^-)$ is finite.

To show that $I \setminus K_{x,y}$ is finite, it suffices to show that $K_{x,y}$ is contained in finitely many I-double cosets. This proof of the lemma is thus concluded using the following result. □

Claim. *Let $x \in \mathcal{W}$. There are only finitely many $w' \in W$ such that*

$$Iw'I \cdot IxU \cap IxU \neq \emptyset. \quad (5.25)$$

Proof of Claim. Let $x = w\pi^{\lambda^\vee}$ with $w \in W$ and $\lambda^\vee \in \Lambda^\vee$. Then $Iw'I \cdot IxU = Iw'Iw\pi^{\lambda^\vee}U$ so that if (5.25) holds, we must have $Iw'IwU \cap IwU \neq \emptyset$. Hence, we have $Iw'Iw \cap IwU \neq \emptyset$. As $Iw'Iw \in K$ we may assume the intersection takes place in K, i.e., $Iw'IwI \cap IwI \neq \emptyset$. Over the residue field this implies

$$\mathbf{B}(\mathbf{k})w'\mathbf{B}(\mathbf{k})w\mathbf{B}(\mathbf{k}) \cap \mathbf{B}(\mathbf{k})w\mathbf{B}(\mathbf{k}) \neq \emptyset. \quad (5.26)$$

Let $\mathbf{U}_w(\mathbf{k}) := \mathbf{U}(\mathbf{k}) \cap w^{-1}\mathbf{U}^-(\mathbf{k})w$ and $\mathbf{U}_{-,w}(\mathbf{k}) = w\mathbf{U}_w(\mathbf{k})w^{-1}$. Since $\mathbf{B}(\mathbf{k})w\mathbf{B}(\mathbf{k}) = \mathbf{B}(\mathbf{k})w\mathbf{U}_w(\mathbf{k})$ we conclude from (5.26) that

$$\mathbf{U}_{-,w}(\mathbf{k}) \cap \mathbf{B}(\mathbf{k})w'\mathbf{B}(\mathbf{k}) \neq \emptyset. \quad (5.27)$$

For a fixed $w \in W$, there can be at most finitely many w' which satisfy this condition. Indeed, the left hand side of (5.27) is a finite set and the double cosets $\mathbf{B}(\mathbf{k})w'\mathbf{B}(\mathbf{k})$ are disjoint for varying $w' \in W$. □

5.2.6. *A faithfulness Result.* Finally we record here an important property of the action of $H(G_+, I)$ on $M(G, I)$, which follows from the results of §3.4.4.

Proposition. *Let $h_1, h_2 \in H(G_+, I)$. If $\mathbf{v}_1 \star h_1 = \mathbf{v}_1 \star h_2$ then we have $h_1 = h_2$.*

Proof. Recall $\mathbf{v}_1 \in M(G, I)$ is the coset corresponding to $A_{\mathcal{O}}UI$ and let T_y be the characteristic function of the double coset IyI, with $y \in \mathcal{W}_X$. Then from the definition of convolution we have $\mathbf{v}_1 \star T_y := \sum_{z \in \mathcal{W}} |m_{1,y}^{-1}(z)| \mathbf{v}_z$ where $m_{1,y} : A_{\mathcal{O}}UI \times_I IyI \to G$ is the multiplication map. If $m_{1,y}^{-1}(z) \neq \emptyset$, then by definition we have $A_{\mathcal{O}}UIyI \cap A_{\mathcal{O}}UzI \neq \emptyset$, which implies that $UzI \cap IyI \neq \emptyset$. By Proposition 3.4.4 this implies that $y \preceq z$. Moreover, it is easy to see that $m_{1,y}^{-1}(y) \neq \emptyset$ so that

$$\mathbf{v}_1 \star T_y = c_y \mathbf{v}_y + \sum_{y \preceq z} c_z \mathbf{v}_z,\ c_y \neq 0. \tag{5.28}$$

As $\preceq$ is a partial order, a simple triangularity argument yields the Proposition, since every $h \in H(G_+, I)$ is a (finite) linear combination of T_y with $y \in \mathcal{W}_X$. □

5.3. **Convolution Iwahori-Hecke algebras as DAHAs.** We would now like to state our main result identifying $H(G_+, I)$ with $\mathbb{H}_+$.

5.3.1. *On the subspace H_W.* We have constructed the convolution algebra of $H(G_+, I)$ as well as the module of $(A_{\mathcal{O}}U, I)$ invariant functions $M(G, I)$ over this algebra in §5.2. Consider the subspace $H_W \subset H(G_+, I)$ spanned by the double cosets $T_w := IwI$ with $w \in W \subset \mathcal{W}$. In the same manner as in [19], we can show

Proposition. *The subspace $H_W \subset H(G_+, I)$ is a subalgebra. Moreover, it is spanned by the elements T_w, $w \in W$ together with the following relations,*

$$T_{w_1} T_{w_2} = T_{w_1 w_2} \textit{ if } \ell(w_1 w_2) = \ell(w_1) + \ell(w_2) \tag{5.29}$$

$$(T_a + 1)(T_a - q) = 0 \textit{ for } a \in \Pi. \tag{5.30}$$

5.3.2. *Some basic Properties of $M(G, I)$.* The module $M(G, I)$ has a basis $\mathbf{v}_x$ with $x \in \mathcal{W}$. We then have the following simple properties,

Proposition. *Let $w \in W$ and $\lambda^\vee \in \Lambda_+^\vee$. Then we have*

(1) $\mathbf{v}_1 \star T_w = \mathbf{v}_w$,
(2) $\mathbf{v}_1 \star T_{\pi^{\lambda^\vee}} = \mathbf{v}_{\pi^{\lambda^\vee}}$.

Proof. The first result is straightforward. Let us sketch a proof of the second. First, let us first determine the support of $\mathbf{v}_1 \star T_{\pi^{\lambda^\vee}}$. To do this, we note that

$$A_{\mathcal{O}}UI\pi^{\lambda^\vee}I = A_{\mathcal{O}}UU_{\mathcal{O}}U_\pi^- A_{\mathcal{O}}\pi^{\lambda^\vee}I = A_{\mathcal{O}}UU_\pi^-\pi^{\lambda^\vee}I = A_{\mathcal{O}}U\pi^{\lambda^\vee}I, \tag{5.31}$$

where in the last line we have used the fact that $\pi^{-\lambda^\vee}$ normalizes U_π^- for $\lambda^\vee \in \Lambda_+^\vee$. The above computation shows that $\mathbf{v}_1 \star T_{\pi^{\lambda^\vee}}$ is just a multiple of $\mathbf{v}_{\pi^{\lambda^\vee}}$; moreover, the constant is equal by definition to the cardinality of the set of left I-cosets of the intersection,

$$I\pi^{\lambda^\vee}I \cap I\pi^{\lambda^\vee}A_{\mathcal{O}}U = I\pi^{\lambda^\vee}U_\pi^- \cap I\pi^{\lambda^\vee}U. \tag{5.32}$$

From (A.7) below we have that $I\pi^{\lambda^\vee}U_\pi^- \cap I\pi^{\lambda^\vee}U = I\pi^{\lambda^\vee}$ since

$$IU^- \cap IU \subset I(KU^- \cap U) = I(K \cap U) = I; \tag{5.33}$$

thus, this cardinality is equal to 1. □

5.3.3. *Identification of* $\mathbb{H}_+$ *and* $H(G_+, I)$. In Proposition 6.3.2 we shall generalize part (2) of Proposition 5.3.2 as follows.

Proposition. *For any* $\mu^\vee \in X$ *there exists an element* $\theta_{\mu^\vee} \in H(G_+, I)$ *which satisfies the condition*

$$\mathbf{v}_1 \star \theta_{\mu^\vee} = e^{\mu^\vee} \circ \mathbf{v}_1. \tag{5.34}$$

We may now state our main result on Iwahori-Hecke algebras.

Theorem. *There exists an isomorphism of algebras*

$$\varphi : \mathbb{H}_+ \stackrel{\cong}{\to} H(G_+, I), \tag{5.35}$$

where the parameter v *in* $\mathbb{H}$ *is specialized to* $v = q^{-1/2}$.

Proof. Step 1: Let $H' \subset H(G_+, I)$ be the subalgebra of $H(G_+, I)$ generated by T_w for $w \in W$ and $\theta_{\mu^\vee}$ for $\mu^\vee \in X$. Consider the map $\varphi : \mathbb{H}_+ \to H'$ which sends $\mathbb{T}_w \mapsto T'_w$ for $w \in W$ and $\Theta_{\mu^\vee} \mapsto \theta_{\mu^\vee}$ for $\mu^\vee \in X$. It is easy to see that this map is actually an algebra isomorphism. We can see this for the restriction to $\mathbb{H}_W$ using Proposition 5.3.1. Then using the fact that $\mathbf{v}_1 \star \theta_{\mu^\vee} = e^{\mu^\vee} \circ \mathbf{v}_1$, the algebra generated by $\theta_{\mu^\vee}$ is commutative. Using (5.34) and the basic properties of the intertwining operators from §6.1, one verifies that the $\theta_{\mu^\vee}$ satisfy the Bernstein relations (5.4). Hence the map φ is an algebra map. It is clearly surjective by the definition of $\mathbb{H}_+$. It remains to see that φ is injective. For this it suffices to show that for example the elements $\{\theta_{\mu^\vee} T_w\}_{\mu^\vee \in X, w \in W}$ are linearly independent. However this follows by exactly the same argument as in [25, Lemma 3.4]

Step 2: It remains to show that $H' = H(G_+, I)$. This will follow if we show that $\mathbf{v}_1 \star H' = \mathbf{v}_1 \star H(G_+, I)$. It is easy to see that $\mathbb{M}_+ = M(G_+, I) \subset \mathbf{v}_1 \star H'$ On the other hand, for $x \in \mathcal{W}_X$ we write $\mathbf{v}_1 \star T_x = \sum_{y \in \mathcal{W}} c_y \mathbf{v}_y$. If $c_y \neq 0$ then we must have that $A_{\mathcal{O}} U I x I \cap A_{\mathcal{O}} U y I \neq \emptyset$, or in other words, $UyI \cap IxI \neq \emptyset$. From Proposition B.3, we see that $y \leq_B x$ (in the notation of Definition B.2). Writing $x \in \mathcal{W}_X$ as $x = \pi^{\lambda^\vee} \sigma$ where $\lambda^\vee \in X$ and $\sigma \in W$, from (2.32) either $\langle \delta, \lambda^\vee \rangle = 0$ in which case $\lambda^\vee = n\mathbf{c} \in \Lambda^\vee_+$, or $\langle \delta, \lambda^\vee \rangle > 0$. In the former case, $\lambda^\vee \in \Lambda^\vee_+$ and we apply the remark after Proposition B.3 to conclude that $y = x$ and so in particular $y \in \mathcal{W}_X$. On the other hand, if $\langle \delta, \lambda^\vee \rangle > 0$, then from the definition of the order $\leq_B$ we see that $y \in \mathcal{W}_X$ as well.Thus,

$$\mathbf{v}_1 \star H(G_+, I) \subset M(G_+, I) \subset \mathbf{v}_1 \star H'. \tag{5.36}$$

Thus $H' = H(G_+, I)$ and the Theorem is proven. □

6. Iwahori Theory II: Intertwiners and Construction of $\theta_{\mu^\vee}$

6.1. **Intertwining Operators.** The aim of this subsection is to develop some basic properties of intertwining operators. The arguments are mostly analogous to the finite-dimensional setting, so we only sketch the proofs.

6.1.1. *Algebraic Convolution.* Let $A_1, A_2, A_3 \subset G$ be three subgroups. Let X, Y and Z denote the sets which parametrize (A_1, A_2), (A_2, A_3) and (A_1, A_3)-double cosets of G. For $x \in X$ we denote by $\mathbf{v}_x$ the corresponding double coset $A_1 x A_2$. Similarly, we define $\mathbf{v}_y, \mathbf{v}_z$ for $y \in Y$ and $z \in Z$. Denote by $M(X), M(Y), M(Z)$ the spaces of *all* functions on G which are (A_1, A_2), (A_2, A_3), and (A_1, A_3)-binvariant. One can formally attempt to define a convolution structure, $M(X) \star M(Y) \to M(Z)$ which is defined on characteristic functions by

$$\mathbf{v}_x \star \mathbf{v}_y = \sum_{z \in Z} |m^{-1}_{x,y}(z)| \mathbf{v}_z \tag{6.1}$$

where $m_{x,y} : \mathbf{v}_x \times_{A_2} \mathbf{v}_y \to G$ is the map induced by multiplication. To make sense of the formula (6.1), one needs to impose certain finiteness conditions of course.

6.1.2. *Some Completions.* Let $A_1 = A_{\mathcal{O}}U$ and let A_2, A_3 be arbitrary subgroups as above and X, Y, Z also as above. Let $M_{fin}(X) \subset M(X)$ denote the space of all function $\phi \in M(X)$ which are supported on finitely many double cosets. Let $V = \mathbb{C}[\Lambda^\vee]$ be the group algebra of the coweight lattice of $\mathfrak{g}$, which has generators $e^{\lambda^\vee}$ with $\lambda^\vee \in \Lambda^\vee$ subject to the relation $e^{\lambda^\vee} e^{\mu^\vee} = e^{\lambda^\vee + \mu^\vee}$ with $\lambda^\vee, \mu^\vee \in \Lambda^\vee$. The algebra V acts on $M(X)$ on the left via the formula,

$$e^{\lambda^\vee} \phi(x) = q^{-\langle \rho, \lambda^\vee \rangle} \phi(\pi^{-\lambda^\vee} x) \text{ for } \phi \in M_{fin}(X), \tag{6.2}$$

where ρ was defined in (2.11).

We shall also need the following completions of $M_{fin}(X)$ which are defined with respect to the left V-action as follows. Let $J \subset R_{re}$ be a finite subset of real roots, and consider the subalgebra of V defined as $B_J = \mathbb{C}[e^{a^\vee}]_{a \in J}$. We can complete B_J with respect to the maximal ideal spanned by $e^{a^\vee}$ for $a \in J$, and we denote the corresponding completion by $\hat{B}_J$. Let us set $V_J := \hat{B}_J \otimes_{B_J} V$ and define

$$M_J(X) := V_J \otimes_V M_{fin}(X). \tag{6.3}$$

For each $w \in W$ we let $D_w = R_{re,+} \cap w R_{re,-}$ and we set

$$V_w := V_{D_w} \text{ and } M_w(X) := M_{D_w}(X) \ . \tag{6.4}$$

6.1.3. *Definition of Intertwiners.* Keep the conventions of the previous paragraph. Fix $w \in W$, and consider the group

$$U_w := \prod_{a \in D_w} \mathbf{U}_a(\mathcal{K}) \tag{6.5}$$

It is a finite product of root groups, and carries a natural Haar measure du_w which assigns to $U_w \cap K$ measure 1. We may consider the integral operator $\mathfrak{T}_w$ defined via the formula,

$$\mathfrak{T}_w(\phi)(x) = \int_{U_w} \phi(w^{-1} u_w x) du_w. \tag{6.6}$$

where $\phi \in M(X)$ is a function which is left $A_{\mathcal{O}}U$-invariant. Note that by the way $\mathfrak{T}_w$ is defined, if $\phi \in M(X)$ then $\mathfrak{T}_w(\phi) \in M(X)$ as well. Of course, in order to use the above formula one needs to verify certain finiteness criterion. We shall return to this point below, after considering the following simple result.

Lemma. *Let $\phi_1 \in M(X)$, $\phi_2 \in M(Y)$, $w \in W$, and $\lambda^\vee \in \Lambda^\vee$.*

(1) *Suppose that $\phi_1 \star \phi_2$, $\mathfrak{T}_w(\phi_1) \star \phi_2$, and $\mathfrak{T}_w(\phi_1 \star \phi_2)$ are all well-defined elements in $M(Z)$. Then we have*

$$\mathfrak{T}_w(\phi_1 \star \phi_2) = \mathfrak{T}_w(\phi_1) \star \phi_2. \tag{6.7}$$

(2) *Suppose that $\mathfrak{T}_w(e^{\lambda^\vee} \circ \phi_1)$ and $\mathfrak{T}_w(\phi_1)$ are well-defined elements on $M(X)$, then*

$$\mathfrak{T}_w(e^{\lambda^\vee} \circ \phi_1) = e^{w\lambda^\vee} \circ \mathfrak{T}_w(\phi_1). \tag{6.8}$$

(3) *Suppose that $w = w_{a_1} \cdots w_{a_r}$ is a reduced decomposition for $w \in W$ where w_{a_i} for $i = 1, \ldots, r$ are reflections through simple roots $a_i \in \Pi$. If $\mathfrak{T}_w(\phi_1)$ and the composition $\mathfrak{T}_{w_{a_1}} \circ \cdots \circ \mathfrak{T}_{w_{a_r}}(\phi_1)$ are well-defined, then*

$$\mathfrak{T}_w(\phi_1) = \mathfrak{T}_{w_{a_1}} \circ \cdots \circ \mathfrak{T}_{w_{a_r}}(\phi_1). \tag{6.9}$$

The verification of the above Lemma is straightforward, and we suppress the details here.

6.1.4. *Intertwiners for* I. Let $A_1 = A_{\mathcal{O}}U$ as above, and $A_2 = I$. Then from Lemma 3.4.5, $X = \mathcal{W}$ is the set parametrizing $(A_{\mathcal{O}}U, I)$ double cosets, and we write $M(G,I)$ for $M_{fin}(X)$ and $M_w(G,I)$ for $M_w(X)$, etc.

The following result is necessary to formally make sense of $\mathfrak{T}_w : M(G,I) \to M(G,I)$,

Lemma. *Let* $w \in W$ *and* $x, y \in \mathcal{W}$. *Let* μ *denote the Haar measure on* U_w *which assigns* $U_w \cap K$ *measure* 1. *Then* $\mu(\{u \in U_w | w^{-1}uy \in A_{\mathcal{O}}UxI\}) < \infty$.

Finally, one can verify the following simple formula for the action of $\mathfrak{T}_a := \mathfrak{T}_{w_a} : M_{fin}(X) \to M_{w_a}(X)$ for $a \in \Pi$.

Lemma. *The map* $\mathfrak{T}_a : M(G,I) \to M(G,I)_{w_a}$ *is given by the following formula on* $\mathbf{v}_1$,

$$\mathfrak{T}_a(\mathbf{v}_1) = q^{-1}\mathbf{v}_{w_a} + (1-q^{-1})\sum_{j=1}^{\infty} e^{ja^\vee} \circ \mathbf{v}_1 = q^{-1}\mathbf{v}_{w_a} + \frac{(1-q^{-1})e^{a^\vee}}{1-e^{a^\vee}} \circ \mathbf{v}_1, \tag{6.10}$$

where the fraction in the last expression is formally expanded in the completion V_{w_a}.

6.1.5. *Intetwiners for* K. Finally we turn to the case that $A_2 = K$, so that $X = \Lambda^\vee$. Let us write $M(G,K)$ for $M(X)$. We again have maps $\mathfrak{T}_w : M(G,K)_w \to M(G,K)_{ww'}$ for each $w, w' \in W$. We write $\mathbf{v}_{\lambda^\vee}$ to refer to the double coset $A_{\mathcal{O}}U\pi^{\lambda^\vee}K$ for $\lambda^\vee \in \Lambda^\vee$, and if $\lambda^\vee = 0$ we write $\mathbf{1}_K := \mathbf{v}_0$ and call this element the *spherical vector*.

Lemma. [18, Lemma 1.13.1] *The map* $\mathfrak{T}_a : M(G,K) \to M(G,K)_{w_a}$ *is given by the following formula on* $\mathbf{1}_K$,

$$\mathfrak{T}_a(\mathbf{1}_K) = \frac{1-q^{-1}e^{a^\vee}}{1-e^{a^\vee}}\mathbf{1}_K \tag{6.11}$$

where the fraction in the last expression is formally expanded in the completion V_{w_a}.

6.1.6. *Normalized Intertwiners.* We shall find it convenient to renormalize $\mathfrak{T}_a$ as follows. Define

$$\mathfrak{K}_a := \frac{1-e^{a^\vee}}{1-q^{-1}e^{a^\vee}}\mathfrak{T}_a, \tag{6.12}$$

and one can again verify that $\mathfrak{K}_a : M(G,K)_w \to M(G,K)_{ww_a}$ for any $w \in W$ such that $\ell(ww_a) = \ell(w)+1$. Moreover, by Lemma 6.1.3 above, we have that

$$\mathfrak{K}_w = \mathfrak{K}_{w_{a_1}} \circ \cdots \circ \mathfrak{K}_{w_{a_r}} \tag{6.13}$$

for any reduced decomposition $w = w_{a_1} \cdots w_{a_r}$ of w. This precise normalization is chosen so that we have

$$\mathfrak{K}_w(\mathbf{1}_K) = \mathbf{1}_K. \tag{6.14}$$

6.2. **A Construction of Elements in** $H(G_+, I)$**.** We now begin the construction of elements $\theta_{\mu^\vee} \in H(G_+, I)$ for $\mu^\vee \in X$ which are specified in Proposition 5.3.3. To each $\mu^\vee \in X$ we give an algorithm for producing certain elements $\theta^\bullet_{\mu^\vee} \in H(G_+, I)$ which depends on various choices (to be specified below). In the §6.3, we shall show that the outcome of our construction does not actually depend on the choices made, and so we can unambiguously refer to an element $\theta_{\mu^\vee} \in H(G_+, I)$. Note that these elements will satisfy

$$\theta_{\mu^\vee} \star \theta_{\lambda^\vee} = \theta_{\mu^\vee+\lambda^\vee} \text{ for } \mu^\vee, \lambda^\vee \in X, \tag{6.15}$$

as follows from Propositions 5.3.3, 5.2.6, and the associativity of the action $\circ$.

6.2.1. *Basic Strategy.* For $\lambda^\vee \in \Lambda_+^\vee$, we set $\theta_{\lambda^\vee} := q^{-\langle \rho, \lambda^\vee \rangle} T_{\pi^{\lambda^\vee}}$, and note from Proposition 5.3.2 (2) that

$$e^{\lambda^\vee} \circ \mathbf{v}_1 = \mathbf{v}_1 \star \theta_{\lambda^\vee}, \text{ for } \lambda^\vee \in \Lambda_+^\vee. \tag{6.16}$$

We would like to extend this construction and define elements $\theta_{\mu^\vee}$ for any $\mu^\vee \in X$. We refer to the elements $\mu^\vee \in w(\Lambda_+^\vee)$ as the shifted w-chamber of X, or just a shifted chamber for short. The length $\ell(w)$ of the Weyl group element defining the shifted chamber will be called the length of the chamber. Note that within each shifted chamber, we may consider the elements with respect to the dominance order $\leq$. Our construction below shall proceed based on both the length of the chamber and on this dominance order.

We begin with the case of chamber length 1, i.e., those $\mu^\vee \in X$ such that there exists a simple root $a \in \Pi$ such that $w_a\mu^\vee \in \Lambda_+^\vee$. We proceed by induction on the quantity $\langle a, \mu^\vee \rangle < 0$, and would like to use the formula

$$\text{“}\theta_{\mu^\vee} = T_a \theta_{w_a\mu^\vee} T_a^{-1} - (q-1)(\theta_{w_a\mu^\vee} + \theta_{w_a\mu^\vee - a^\vee} + \cdots + \theta_{w_a\mu^\vee + (\langle a, \mu^\vee \rangle + 1)a^\vee}) T_a^{-1}\text{”} \tag{6.17}$$

in a manner to be made precise below.

6.2.2. *Step 1.* If $\langle a, \mu^\vee \rangle = -1$ then the right hand side of (6.17) reduces to

$$T_a \theta_{w_a\mu^\vee} T_a^{-1} - (q-1)\theta_{w_a\mu^\vee} T_a^{-1}. \tag{6.18}$$

As $w_a\mu^\vee \in \Lambda_+$ the term $\theta_{w_a\mu^\vee}$ has already been defined as elements in $H(G_+, I)$. Moreover, the elements T_a and T_a^{-1} have also be defined in $H(G_+, I)$. Thus we may take (6.18) to be the *definition* of $\theta_{\mu^\vee}$ for any $\mu^\vee \in w_a(\Lambda_+^\vee)$ with $\langle a, \mu^\vee \rangle = -1$ i.e.,

$$\theta_{\mu^\vee} = T_a \theta_{w_a\mu^\vee} T_a^{-1} - (q-1)\theta_{w_a\mu^\vee} T_a^{-1} \text{ if } \mu^\vee \in w_a(\Lambda_+^\vee) \text{ and } \langle a, \mu^\vee \rangle = -1. \tag{6.19}$$

6.2.3. *Step 2.* We next observe the following simple result, which is useful for our inductive construction.

Lemma. *Let $\mu^\vee \in \Lambda^\vee$ be such that $w_a\mu^\vee \in \Lambda_+^\vee$. Set $d := -\langle a, \mu^\vee \rangle > 0$, and assume $d > 1$.*

(1) *If $b \in \Pi$ with $b \neq a$, then*

$$\langle b, w_a\mu^\vee - ja^\vee \rangle \geq 0 \text{ for } j = 1, 2, \ldots, d-1, \tag{6.20}$$

i.e., the elements $w_a\mu^\vee - ja^\vee$ for $j = 1, 2, \ldots, d-1$ lie in $w_a(\Lambda_+^\vee) \cup \Lambda_+^\vee$.

(2) *For $j = 1, 2, \ldots, d-1$, we have*

$$-\langle a, w_a\mu^\vee - ja^\vee \rangle < d. \tag{6.21}$$

Proof. Part (1) follows immediately from the following two facts: (i) the inner product $\langle b, a^\vee \rangle \leq 0$ for b a simple root not equal to a; and (ii) $\langle b, w_a\mu^\vee \rangle \geq 0$ since $w_a\mu^\vee \in \Lambda_+^\vee$ and b is a positive root.

As for (2), we compute,

$$-\langle a, w_a\mu^\vee - ja^\vee \rangle = \langle a, \mu^\vee \rangle + \langle a, ja^\vee \rangle = -d + 2j. \tag{6.22}$$

However, $-d + 2j < d$ since $j < d$.

□

6.2.4. *Step 3.* Fix $d > 1$, and suppose now that we have defined expressions $\theta_{\mu^\vee}$ for $\mu^\vee \in w_a(\Lambda_+^\vee)$ such that $-\langle a, \mu^\vee \rangle < d$. Choose now $\mu^\vee \in w_a(\Lambda_+^\vee)$ with $-\langle a, \mu^\vee \rangle = d$. Then the right hand side of the expression (6.17) takes the form,

$$T_a \theta_{w_a\mu^\vee} T_a^{-1} - (q-1)(\theta_{w_a\mu^\vee} + \theta_{w_a\mu^\vee - a^\vee} + \cdots + \theta_{w_a\mu^\vee - (d-1)a^\vee}) T_a^{-1}. \tag{6.23}$$

From Lemma 6.2.3 (1) we know that the elements $w_a\mu^\vee - ja^\vee$ for $j = 1, \ldots, d-1$ are all in $w_a(\Lambda_+^\vee)$ or $\Lambda_+^\vee$. Let $\xi^\vee$ be one of these elements. If $\xi^\vee \in \Lambda_+^\vee$ we know how to define $\theta_{\xi^\vee}$. On the other hand, if $\xi^\vee \in w_a(\Lambda_+^\vee)$ we know from Lemma 6.2.3(2) that $-\langle a, \xi^\vee \rangle < d$, and so $\theta_{\xi^\vee}$ has been defined inductively. Continuing in this way, we can define $\theta_{\mu^\vee}$ for any $\mu^\vee \in w_a(\Lambda_+^\vee)$.

Proceeding again by induction on the length of the chamber, and then by a second induction based on dominance, we may construct elements $\theta_{\mu^\vee}$ for every $\mu^\vee \in X$.

6.2.5. *Step 4.* It is important to note that in this construction a number of choices have been made to define each $\theta_{\mu^\vee}$. We denote by $\theta^\bullet_{\mu^\vee}$ *any* element associated to a given $\mu^\vee \in X$ which can be constructed by the procedure described above. It will be shown below that the construction is independent of the choices made. i.e., that $\theta^\bullet_{\mu^\vee}$ only depends on $\mu^\vee$. Our strategy will be to show that for any of the elements $\theta^\bullet_{\mu^\vee} \in H(G_+, I)$ constructed above, we have a relation of the form (6.16), i.e., $\mathbf{v}_1 \star \theta^\bullet_{\mu^\vee} = e^{\mu^\vee} \circ \mathbf{v}_1$. Proposition 5.2.6 will then imply that $\theta^\bullet_{\mu^\vee}$ depends only on $\mu^\vee \in \Lambda^\vee$. Our proof proceeds in an inductive manner and rests on the following fact which is obvious from our construction.

Lemma. *Let $\theta^\bullet_{\mu^\vee}$, $\mu \in X$ constructed as above. If $\mu^\vee \notin \Lambda^\vee_+$, there exists a simple root $a \in \Pi$ such that $w_a\mu^\vee > \mu^\vee$ and a sequence of elements $\theta^\bullet_{w_a\mu^\vee}, \theta^\bullet_{w_a\mu^\vee - a^\vee}, \ldots, \theta^\bullet_{w_a\mu^\vee - (\langle a, w_a\mu^\vee\rangle - 1)a^\vee}$ such that*

$$T_a \star \theta^\bullet_{w_a\mu^\vee} = \theta^\bullet_{\mu^\vee} \star T_a + (q-1)(\theta^\bullet_{w_a\mu^\vee} + \theta^\bullet_{w_a\mu^\vee - a^\vee} + \cdots + \theta^\bullet_{w_a\mu^\vee - (\langle a, w_a\mu^\vee\rangle - 1)a^\vee}). \tag{6.24}$$

6.3. **Proof of Independence of Construction.** We now show that the elements $\theta^\bullet_{\mu^\vee}$ defined in §6.2 do not depend on the various choices made in their construction.

6.3.1. *Preliminary Computations.* Let $\mu^\vee \in X$ and let $\theta^\bullet_{\mu^\vee} \in H(G_+, I)$ be any of the elements constructed in §6.2. Recall that we have defined the intertwining operators $\mathfrak{T}_a$, $a \in \Pi$ for some completion of the $(V, H(G_+, I))$ bimodule $M(G, I)$ in §6.1.4.

Lemma. *Let $a \in \Pi$ and $\theta^\bullet_{\mu^\vee}$ as above. Then we have*

$$\mathbf{v}_1 \star T_a \star \theta^\bullet_{w_a\mu^\vee} = q\,\mathfrak{T}_a(\mathbf{v}_1 \star \theta^\bullet_{w_a\mu^\vee}) + \frac{q-1}{1 - e^{-a^\vee}}\mathbf{v}_1 \star \theta^\bullet_{w_a\mu^\vee} \tag{6.25}$$

Proof. Note that the right hand side a priori only lives in some completion of $M(G, I)$. However, if we can prove the equality (6.25) holds in some completion, it holds in $M(G, I)$ itself since the left hand side lies in $M(G, I)$. The proof of the Lemma is a simple computation, using the properties of interwining operators established earlier. Indeed, using the fact (see Lemma 6.1.3 (1)) that $\mathfrak{T}_a$ commutes with the right convolution action and the explicit formula (6.10), we find

$$\begin{aligned} \mathfrak{T}_a(\mathbf{v}_1 \star \theta^\bullet_{w_a\mu^\vee}) &= \mathfrak{T}_a(\mathbf{v}_1) \star \theta^\bullet_{w_a\mu^\vee} && (6.26)\\ &= (q^{-1}\mathbf{v}_{w_a} + \frac{(1-q^{-1})e^{a^\vee}}{1 - e^{a^\vee}}\mathbf{v}_1) \star \theta^\bullet_{w_a\mu^\vee} && (6.27)\\ &= q^{-1}\mathbf{v}_1 \star T_a \star \theta^\bullet_{w_a\mu^\vee} + \frac{(1-q^{-1})e^{a^\vee}}{1 - e^{a^\vee}}(\mathbf{v}_1 \star \theta^\bullet_{w_a\mu^\vee}), && (6.28) \end{aligned}$$

where in the last line we have used the fact (see Proposition 5.3.2) that $\mathbf{v}_{w_a} = \mathbf{v}_1 \star T_{w_a}$. ☐

6.3.2. *Proof of Independence of Construction.* Let us now consider the following

Proposition. *For any $\mu^\vee \in X$ and for any of the elements $\theta^\bullet_{\mu^\vee}$ constructed in §6.2, we have*

$$\mathbf{v}_1 \star \theta^\bullet_{\mu^\vee} = e^{\mu^\vee} \circ \mathbf{v}_1. \tag{6.29}$$

In particular, $\theta^\bullet_{\mu^\vee}$ only depends on $\mu^\vee \in X$.

Proof. Consider the statement for each $\mu^\vee \in X$,

$\mathbb{P}(\mu^\vee)$**:** For any $\theta^\bullet_{\mu^\vee}$ constructed as in §6.2, we have $\mathbf{v}_1 \star \theta^\bullet_{\mu^\vee} = e^{\mu^\vee} \circ \mathbf{v}_1$

If $\mathbb{P}(\mu^\vee)$ is true, then we can define $\theta^\bullet_{\mu^\vee}$ unambiguously according to the faithfulness result, Proposition 5.2.6. We shall just write $\theta_{\mu^\vee}$ in this case. We know that $\mathbb{P}(\mu^\vee)$ is true for $\mu^\vee \in \Lambda^\vee_+$. Given any $\mu^\vee \in X$, assume by induction that $\mathbb{P}(\xi^\vee)$ is true for all $\xi^\vee > \mu^\vee$. Let us show that it holds for $\mu^\vee$ as well.

Step 1: Given $\mu^\vee \in X$, from Lemma 6.2.5, there exits a simple root $a \in \Pi$ such that $w_a\mu^\vee > \mu^\vee$ and such that we have a relation of the form

$$T_a \star \theta^\bullet_{w_a\mu^\vee} = \theta^\bullet_{\mu^\vee} \star T_a + (q-1)(\theta^\bullet_{w_a\mu^\vee} + \theta^\bullet_{w_a\mu^\vee - a^\vee} + \cdots + \theta^\bullet_{w_a\mu^\vee - (\langle a, w_a\mu^\vee\rangle - 1)a^\vee}). \tag{6.30}$$

As $w_a\mu^\vee > \mu^\vee$ we must have $\langle a, \mu^\vee \rangle = -\langle a, w_a\mu^\vee \rangle < 0$. Let us next note that

$$w_a\mu^\vee - ja^\vee > \mu^\vee \text{ for } j = 0, \ldots, \langle a, w_a\mu^\vee\rangle - 1. \tag{6.31}$$

Indeed for these values of j we find

$$w_a\mu^\vee - ja^\vee - \mu^\vee = (-\langle a, \mu^\vee\rangle - j)a^\vee = (\langle a, w_a\mu^\vee\rangle - j)a^\vee \geq 0 \tag{6.32}$$

By the inductive hypothesis, we now have that $\mathbb{P}(w_a\mu^\vee - ja^\vee)$ is true for j as above, and in particular the elements $\theta_{w_a\mu^\vee - ja^\vee}$ are well-defined. So (6.30) may be rewritten as

$$T_a\theta_{w_a\mu^\vee} = \theta^\bullet_{\mu^\vee}T_a + (q-1)(\theta_{w_a\mu^\vee} + \theta_{w_a\mu^\vee - a^\vee} + \cdots + \theta_{w_a\mu^\vee - (\langle a, w_a\mu^\vee\rangle - 1)a^\vee}), \tag{6.33}$$

where only the element $\theta^\bullet_{\mu^\vee}$ may in fact depend on its construction (and not only on $\mu^\vee$).

Step 2: From Lemma 6.3.1 we have

$$\mathbf{v}_1 \star T_a \star \theta_{w_a\mu^\vee} = q\,\mathfrak{T}_a(\mathbf{v}_1 \star \theta_{w_a\mu^\vee}) + \frac{q-1}{1-e^{-a^\vee}}\mathbf{v}_1 \star \theta_{w_a\mu^\vee}, \tag{6.34}$$

which may be rewritten using the assumption $\mathbb{P}(w_a\mu^\vee)$ and Lemma 6.1.3 and (6.10) as follows,

$$\begin{aligned}
\mathbf{v}_1 \star T_a \star \theta_{w_a\mu^\vee} &= q\,\mathfrak{T}_a(e^{w_a\mu^\vee} \circ \mathbf{v}_1) + \frac{q-1}{1-e^{-a^\vee}}e^{w_a\mu^\vee} \circ \mathbf{v}_1 && (6.35)\\
&= q\,e^{\mu^\vee}\mathfrak{T}_a(\mathbf{v}_1) + \frac{q-1}{1-e^{-a^\vee}}e^{w_a\mu^\vee} \circ \mathbf{v}_1 && (6.36)\\
&= qe^{\mu^\vee} \circ (q^{-1}\mathbf{v}_{w_a} + \frac{(1-q^{-1})e^{a^\vee}}{1-e^{a^\vee}} \circ \mathbf{v}_1) + \frac{q-1}{1-e^{-a^\vee}}e^{w_a\mu^\vee} \circ \mathbf{v}_1 && (6.37)\\
&= e^{\mu^\vee} \circ \mathbf{v}_{w_a} - (q-1)\frac{e^{\mu^\vee} - e^{w_a\mu^\vee}}{1-e^{-a^\vee}} \circ \mathbf{v}_1 && (6.38)\\
&= e^{\mu^\vee} \circ \mathbf{v}_1 \star T_a && (6.39)\\
&+ (q-1)(e^{w_a\mu^\vee} + e^{w_a\mu^\vee - a^\vee} + \cdots + e^{w_a\mu^\vee - (\langle a, w_a\mu^\vee\rangle - 1)a^\vee}) \circ \mathbf{v}_1. && (6.40)
\end{aligned}$$

Step 3: We may also compute $v_1 \star T_a \star \theta_{w_a\mu^\vee}$ in another way using the expression (6.33):

$$\begin{aligned}
&\mathbf{v}_1 \star \theta^\bullet_{\mu^\vee} \star T_a + \mathbf{v}_1 \star (q-1)(\theta_{w_a\mu^\vee} + \theta_{w_a\mu^\vee - a^\vee} + \cdots + \theta_{w_a\mu^\vee - (\langle a, w_a\mu^\vee\rangle - 1)a^\vee}) && (6.41)\\
= \; &\mathbf{v}_1 \star \theta^\bullet_{\mu^\vee} \star T_a + (q-1)(e^{w_a\mu^\vee} + e^{w_a\mu^\vee - a^\vee} + \cdots + e^{w_a\mu^\vee - (\langle a, w_a\mu^\vee\rangle - 1)a^\vee}) \circ \mathbf{v}_1 && (6.42)
\end{aligned}$$

where in the second line we have used the fact $\mathbb{P}(w_a\mu^\vee - ja^\vee)$ for $j = 0, \ldots, \langle a, w_a\mu^\vee\rangle - 1$. Comparing with (6.39) we immediately conclude that

$$e^{\mu^\vee} \circ \mathbf{v}_1 \star T_a = \mathbf{v}_1 \star \theta^\bullet_{\mu^\vee} \star T_a \tag{6.43}$$

and hence the claim $\mathbb{P}(\mu^\vee)$ follows from Proposition 5.2.6 since T_a is invertible.

□

7. Spherical Theory

In this section, we shall first review the construction by the first two authors [3] of the *spherical Hecke algebra*, i.e., the convolution algebra of certain infinite collections of K-double cosets. The main technical step in the construction is the verification of certain finiteness properties of the fibers of convolution. This was achieved in *op. cit* by interpreting these fibers geometrically. Here we sketch an alternative construction which uses in an essential way the main finiteness result of [2]. While this work was in preparation, there has appeared yet another approach to proving these finiteness results by S. Gaussent and G. Rousseau [16], which works in the setting of general Kac-Moody groups.

The main new result in this section is Theorem 7.2.3, which gives an explicit formula for the image under the Satake isomorphism of the characteristic function of a K-double coset. This generalizes the formula of Macdonald [26] (see also Langlands [24]) in the finite-dimensional setting. Its generalization to the general Kac-Moody setting is not known to us; more precisely, although it seems that large portions of the proof of Theorem 7.2.3 will hold in the general Kac-Moody setting, we do not know what is the correct analogue of the formula (7.38) for the quantity defined by (7.37).

We fix the notation of §3 in this chapter: so G will be an affine Kac-Moody group over a local field $\mathcal{K}$, which has subgroups I, K, U etc.

7.1. **Spherical Hecke Algebras and the Satake Isomorphism.**

7.1.1. *Verification of property (F).* Recall from Theorem 3.3 that $\Lambda_+^\vee$ was in bijective correspondence with the set of K-double cosets of the semi-group G_+. Furthermore, from Theorem 3.2, we see that $\Lambda^\vee$ is in bijective correspondence with the space of $(A_{\mathcal{O}}U, K)$ double cosets of G. The abelian group $\Lambda^\vee$ equipped with the dominance order $\leq$ becomes an ordered abelian group in the sense of §4.2, and $\Lambda_+^\vee$ inherits the dominance order to become an ordered abelian semi-group.

Lemma. *The abelian group $\Lambda^\vee$ equipped with the dominance order $\leq$ satisfies the following condition: for each $\lambda^\vee, \mu^\vee \in \Lambda^\vee$ the set*

$$[\lambda^\vee, \mu^\vee] = \{\xi^\vee \in \Lambda^\vee | \lambda^\vee \leq \xi^\vee \leq \mu^\vee\} \tag{7.1}$$

is finite.

Proof. Let $\lambda^\vee, \mu^\vee \in \Lambda$, and assume that $\lambda^\vee \leq \mu^\vee$ so that $\mu^\vee - \lambda^\vee = \sum_{i=1}^{\ell+1} n_i a_i^\vee$ with $n_i \geq 0$. Then every $\xi \in [\lambda^\vee, \mu^\vee]$ is of the form $\xi = \lambda^\vee + \sum_{i=1}^{\ell+1} m_i a_i^\vee$ with $0 \leq m_i \leq n_i$. The finiteness of $[\lambda^\vee, \mu^\vee]$ follows. □

7.1.2. *Constructing $H_{\leq}(G_+, K)$.* Recall the setup of §4.2. Our aim in the remainder of §7.1 is to prove the following result which was first shown in ([3]) with slightly different terminology.

Theorem ([3])**.** *The quadruple $(K, \Lambda_+^\vee; A_{\mathcal{O}}U, \Lambda, \leq)$ is a semi-infinite Hecke module datum in the sense of Definition 4.2.3.*

Thus, as in §4.2.2-4.2.3, we may define a convolution algebra structure on the space of K-double cosets $H_{\leq}(\Gamma_+, K)$, as well as an action (on the right) of this algebra on the module structure on

$$M_{\leq}(G; A_{\mathcal{O}}U, K) := M_{\leq}(G, K). \tag{7.2}$$

Proof. To prove the theorem, we shall utilize Proposition 4.2.4. Thus we need to verify that $(K, \Lambda_+^\vee, \Lambda^\vee, \leq)$ satisfies conditions (F), (SH2), and $(K, \Lambda_+^\vee; A_{\mathcal{O}}U, \Lambda^\vee, \leq)$ satisfies (SM1), and (SM2). Condition (F) follows from Lemma 7.1.1, so we just focus on (SH2), (SM1), and (SM2). Let

$$\mathbf{v} : \Lambda^\vee \to A_{\mathcal{O}}U \setminus G/K, \quad \lambda^\vee \mapsto \mathbf{v}_{\lambda^\vee} \tag{7.3}$$

$$h : \Lambda_+^\vee \to K \setminus G_+/K, \quad \lambda^\vee \mapsto h_{\lambda^\vee} \tag{7.4}$$

be the bijections induced respectively from the Iwasawa and Cartan decompositions. The properties (SM1) and (SM2) are expressed in terms of the fiber of the map $m_{\mu^\vee,\lambda^\vee} : \mathbf{v}_{\mu^\vee} \times h_{\lambda^\vee} \to G$, for $\mu^\vee \in \Lambda^\vee, \lambda^\vee \in \Lambda_+^\vee$. Using (4.8),

$$m_{\mu^\vee,\lambda^\vee}^{-1}(\xi^\vee) = K \setminus K\pi^{\lambda^\vee} K \cap K\pi^{-\mu^\vee} U \pi^{\xi^\vee} \text{ where } \lambda^\vee \in \Lambda_+^\vee \text{ and } \mu^\vee, \xi^\vee \in \Lambda^\vee. \tag{7.5}$$

Now the results (SM1) and (SM2) follow respectively from the next two facts: let $\lambda^\vee \in \Lambda_+$ and $\mu^\vee \in \Lambda$, then we have the following results from [2, Theorem 1.9(2)]

$$|K \setminus K\pi^{\lambda^\vee} K \cap K\pi^{\mu^\vee} U| \quad < \quad \infty; \text{ and} \tag{7.6}$$

$$\text{if } K\pi^{\lambda^\vee} K \cap K\pi^{\mu^\vee} U \quad \neq \quad \emptyset \text{ then } \mu^\vee \leq \lambda^\vee. \tag{7.7}$$

Finally, we address the property (SH2), which involves the fibers of the map $m_{\lambda^\vee,\mu^\vee} : h_{\lambda^\vee} \times h_{\mu^\vee} \to G_+$ where $\lambda^\vee, \mu^\vee \in \Lambda^\vee_+$. If $m^{-1}_{\lambda^\vee,\mu^\vee}(\xi^\vee) \neq \emptyset$, then it follows from the definitions that

$$K\pi^{\lambda^\vee} K \pi^{\mu^\vee} K \cap K\pi^{\xi^\vee} K \neq \emptyset. \tag{7.8}$$

Using the decomposition (3.12) we may write $K = IWI$, for the Iwahori I and the Weyl group W. Thus

$$K\pi^{\lambda^\vee} K\pi^{\mu^\vee} K = \bigcup_{w \in W} K\pi^{\lambda^\vee} IwI\pi^{\lambda^\vee} K = \bigcup_{w\in W} K\pi^{w\lambda^\vee} U_{\mathcal{O}} \pi^{\mu^\vee} K \tag{7.9}$$

where we have used (3.10) to write $I = U_{\mathcal{O}} U^-_\pi A_{\mathcal{O}}$ as well as the dominance condition on $\lambda^\vee, \mu^\vee$ which implies that $\pi^{\lambda^\vee} U_{\mathcal{O}} \pi^{-\lambda^\vee} \subset K$ and $\pi^{-\mu^\vee} U^-_\pi \pi^{\mu^\vee} \subset K$. So if (7.8) is satisfied, one obtains from (7.9) that for some $w \in W$,

$$K\pi^{w\lambda^\vee} U_{\mathcal{O}} \pi^{\mu^\vee} K \cap K\pi^{\xi^\vee} K \neq \emptyset. \tag{7.10}$$

Trivially we thus obtain that $K\pi^{w\lambda^\vee+\mu^\vee} U \cap K\pi^{\xi^\vee} K \neq \emptyset$. From (7.7) we thus obtain that $\xi^\vee \leq w\lambda^\vee + \mu^\vee \leq \lambda^\vee + \mu^\vee$ where we have $w\lambda^\vee \leq \lambda^\vee$ since $\lambda^\vee$ was assumed dominant. ☐

7.1.3. *$M_\leq(G,K)$ as a right $H_\leq(G_+,K)$-module.* Describing the structure of $H_\leq(G_+,K)$ along the lines of Satake ([31]) is our next goal. Recall the construction of Looijenga's coweight algebra $\mathbb{C}_\leq[\Lambda^\vee]$ from §2.1.5 [9]. We can define an action of the elements $e^{\lambda^\vee} \in \mathbb{C}_\leq[\Lambda^\vee]$ on $\mathbf{v}_{\mu^\vee} \in M_\leq(G,K)$ via the formula (6.2) and obtain

$$e^{\lambda^\vee} \circ \mathbf{v}_{\mu^\vee} = q^{-\langle \rho, \lambda^\vee\rangle} \mathbf{v}_{\mu^\vee+\lambda^\vee}. \tag{7.11}$$

This action extends to give an action of the completion $\mathbb{C}_\leq[\Lambda^\vee]$ on $M_\leq(G,K)$, and we can easily verify the following result using the Iwasawa decomposition and the definition of the completions involved.

Lemma. *As a $\mathbb{C}_\leq[\Lambda^\vee]$-module, $M_\leq(G,K)$ is free of rank one with generator the* spherical vector $\mathbf{1}_K := \mathbf{v}_0$, *i.e., the characteristic function of the subset $A_{\mathcal{O}}UK$. Further, the action (7.11) is a right $H_\leq(G_+,K)$-module map, i.e.,*

$$e^{\lambda^\vee} \circ (\mathbf{v}_{\mu^\vee} \star h) = (e^{\lambda^\vee} \circ \mathbf{v}_{\mu^\vee}) \star h, \tag{7.12}$$

where $h \in H_\leq(G_+,K)$ and $\star$ denotes the convolution action of $H_\leq(G_+,K)$ on $M_\leq(G,K)$ as in (4.23).

7.1.4. *Affine Satake Isomorphism.* Using Lemma 7.1.3, we obtain a map,

$$S : H_\leq(G_+,K) \to \mathbb{C}_\leq[\Lambda^\vee],\ h \mapsto S(h) \tag{7.13}$$

defined by the expression

$$\mathbf{1}_K \star h = S(h) \circ \mathbf{1}_K. \tag{7.14}$$

Explicitly, if $h_{\lambda^\vee}$ is as in (7.4), then

$$S(h_{\lambda^\vee}) = \sum_{\mu^\vee \in \Lambda^\vee} |K \setminus K\pi^{\mu^\vee} U \cap K\pi^{\lambda^\vee} K| e^{\mu^\vee} q^{\langle \rho, \mu^\vee \rangle}. \tag{7.15}$$

The above map is called *the Satake homorphism:* it is a homomorphism of algebras since for $h_1, h_2 \in H_\leq(G_+,K)$, we have

$$\begin{aligned} \mathbf{1}_K \star (h_1 \star h_2) &= (S(h_1) \circ \mathbf{1}_K) \star h_2 = S(h_2) \circ (S(h_1) \circ \mathbf{1}_K) && (7.16)\\ &= (S(h_2)\, S(h_1)) \circ \mathbf{1}_K, && (7.17) \end{aligned}$$

In fact, we have the following analogue of the classical Satake isomorphism,

[9] We may also regard $\mathbb{C}_\leq[\Lambda^\vee]$ as the (semi-infinite)Hecke algebra associated to the datum $(A_{\mathcal{K}}, A_{\mathcal{O}}, \Lambda^\vee, \leq)$.

Theorem. *The map S is an isomorphism of algebras,*

$$S : H_{\leq}(G_+, K) \stackrel{\cong}{\to} \mathbb{C}_{\leq}[\Lambda^\vee]^W \tag{7.18}$$

where $\mathbb{C}_{\leq}[\Lambda^\vee]^W$ is the ring of W-invariant elements of $\mathbb{C}_{\leq}[\Lambda^\vee]$.

The proof will be divided into three parts

7.1.5. *Proof of Theorem 7.1.4, Part 1: W-invariance of the image of S.* The fact that the image of the Satake map lies in $\mathbb{C}_{\leq}[\Lambda^\vee]^W$ follows as in the classical case from the properties of certain intertwining maps. Recall that have described the basic properties of spherical intertwiners in §6.1.5, and we freely use now same notation introduced there . In particular, we have defined the rings V_w and the spaces $M_w := V_w \otimes_V M(G, K)$ in (6.4) as well as the normalized intertwiners $\mathfrak{K}_w : M(G, K) \to M(G, K)_w$ for each $w \in W$ in (6.12-6.13). We can extend these definitions easily to define the spaces $M_{\leq}(G, K)_w := V_w \otimes_V M_{\leq}(G, K)$ and corresponding maps $\mathfrak{K}_w : M_{\leq}(G, K) \to M_{\leq}(G, K)_w$. Moreover, these maps $\mathfrak{K}_w$ are compatible with the right $H_{\leq}(G_+, K)$ action as follows,

Proposition. *In the notation above, we have*

(a) *For any $w \in W$, $\mathfrak{K}_w$ is a right $H(G_+, K)$-module maps i.e.,*

$$\mathfrak{K}_w(\phi \star h) = \mathfrak{K}_w(\phi) \star h \tag{7.19}$$

for any $\phi \in M_{\leq}(G, K)_w$ and $h \in H_{\leq}(G_+, K)$.

(b) *With respect to the $\circ$ action of $\mathbb{C}_{\leq}[\Lambda^\vee]$ on $M_{\leq}(G, K)_w$ defined as in (7.11), the maps $\mathfrak{K}_w$ satisfy*

$$\mathfrak{K}_w \circ e^{\lambda^\vee} = e^{w\lambda^\vee} \circ \mathfrak{K}_w \text{ for } w \in W,\ \lambda^\vee \in \Lambda^\vee. \tag{7.20}$$

(c) *The maps $\mathfrak{K}_w$ fix the spherical vector, i.e., $\mathfrak{K}_w(\mathbf{1}_K) = \mathbf{1}_K$ for any $w \in W$.*

From the previous Proposition 7.1.5, we find that for any $w \in W$,

$$S(h_{\lambda^\vee}) \circ \mathbf{1}_K = \mathbf{1}_K \star h_{\lambda^\vee} = \mathfrak{K}_w(\mathbf{1}_K) \star h_{\lambda^\vee} = \mathfrak{K}_w(\mathbf{1}_K \star h_{\lambda^\vee}) \tag{7.21}$$

$$= \mathfrak{K}_w(S(h_{\lambda^\vee}) \circ \mathbf{1}_K) = S(h_{\lambda^\vee})^w \circ \mathbf{1}_K. \tag{7.22}$$

It follows that $S(h_{\lambda^\vee})^w = S(h_{\lambda^\vee})$ for any $w \in W$.

7.1.6. *Proof of Theorem 7.1.4 Part 2: Injectivity of S.* To show that S is injective, we shall verify that

$$S(h_{\lambda^\vee}) = q^{\langle \rho, \lambda^\vee \rangle} e^{\lambda^\vee} + \sum_{\mu^\vee < \lambda^\vee} c_{\mu^\vee} e^{\mu^\vee}, \tag{7.23}$$

where $\mu^\vee < \lambda^\vee$ means that $\mu^\vee$ is strictly less than $\lambda^\vee$ in the dominance order, and $c_{\mu^\vee} \in \mathbb{Q}_{\geq 0}$. Indeed, this follows from (7.7) and the following,

Lemma. *Let $\lambda^\vee \in \Lambda^\vee_+$. Then*

$$K\pi^{\lambda^\vee} U \cap K\pi^{\lambda^\vee} K = K\pi^{\lambda^\vee} \tag{7.24}$$

Proof of Lemma. Indeed, from decompositions (3.12) and (3.10), we obtain

$$K\pi^{\lambda^\vee} K = \bigcup_{w \in W} K\pi^{w\lambda^\vee} U^-_\pi U_{\mathcal{O}}. \tag{7.25}$$

If there exist $w \in W$ such that $K\pi^{w\lambda^\vee} U^- \cap K\pi^{\lambda^\vee} U \neq \emptyset$. By [2, Theorem 1.9 (3)] we must have $\lambda^\vee \leq w\lambda^\vee$. But since $\lambda^\vee \in \Lambda^\vee_+$, it follows that $w\lambda^\vee = \lambda^\vee$ for any such w. Hence from (7.25) we conclude that

$$K\pi^{\lambda^\vee} K \cap K\pi^{\lambda^\vee} U \subset K\pi^{\lambda^\vee} I \cap K\pi^{\lambda^\vee} U \subset K\pi^{\lambda^\vee} U^-_\pi \cap K\pi^{\lambda^\vee} U. \tag{7.26}$$

Let us now show this latter intersection is $K\pi^{\lambda^\vee}$. Indeed, $\pi^{\lambda^\vee} u \in K\pi^{\lambda^\vee} U^-$ implies that $\pi^{\lambda^\vee} u \pi^{-\lambda^\vee} \in KU^-$. So by [2, Lemma 3.3] or (A.7) below, we have that $\pi^{\lambda^\vee} u \pi^{-\lambda^\vee} \in U_{\mathcal{O}}$. $\square$

7.1.7. *Proof of Theorem 7.1.4 Part 3: Surjectivity of S.* Finally we need to show that the map S is surjective. The proof (in a slightly different context) is essentially contained in [23, p.25]. Let

$$x = \sum_{\xi^\vee \in \Lambda^\vee} c_{\xi^\vee} e^{\xi^\vee} \in \mathbb{C}_{\leq}[\Lambda^\vee]^W \tag{7.27}$$

and let $\Xi \subset \Lambda^\vee_+$ be a finite set of elements such that if $\lambda^\vee \in \mathrm{Supp}(x)$ then $\lambda^\vee \leq \mu^\vee$ for some $\mu^\vee \in \Xi$. For any integer n we let $\Upsilon(n) \subset \Lambda^\vee_+$ be the set of elements of the form $\mu^\vee - n_1 a_1^\vee - \cdots - n_{\ell+1} a_{\ell+1}^\vee$ where $\mu^\vee \in \Xi$, $n_i \geq 0$ for $i = 1, \ldots, \ell+1$, and $n_1 + \cdots + n_{\ell+1} \geq n$. A subset $\Sigma \subset \Lambda^\vee$ is said to be dominated by a subset $\Sigma' \subset \Lambda^\vee_+$ if for every $\tau^\vee \in \Sigma$ there exists $\lambda^\vee \in \Sigma'$ such that $\tau^\vee \leq \lambda^\vee$. We now construct a family of elements $h_n \in H_{\leq}(G_+, K)$ for $n \in \mathbb{Z}_{\geq 0}$ such that: (a) $\mathrm{Supp}(h_n)$ is dominated by Ξ; and (b) $\mathrm{Supp}(x - S(h_n))$ is dominated by $\Upsilon(n)$. Indeed, let $h_0 = 0$; suppose inductively

$$S(h_n) = \sum_{\xi^\vee \in \Lambda^\vee} c_{n,\xi^\vee} e^{\xi^\vee}, \tag{7.28}$$

we then define

$$h_{n+1} = h_n + a_n; \text{ where } a_n = \sum_{\mu^\vee \in \Upsilon(n) \setminus \Upsilon(n+1)} q^{-\langle \rho, \mu^\vee \rangle} (c_{\mu^\vee} - c_{n,\mu^\vee}) h_{\mu^\vee}. \tag{7.29}$$

Condition (a) is immediately verified since $\Upsilon(n)$ is dominated by Ξ; and condition (b) follows from (7.23). Note that we have $h_n = a_1 + \cdots + a_{n-1}$. Thus, setting $h_\infty := \sum_{n \geq 0} a_n$, we see from the definition of the completion that $h_\infty \in H_{\leq}(G_+, K)$. Furthermore, from (b), we see that $x - S(h_\infty)$ has support dominated by $\cap_{n=0}^{\infty} \Upsilon(n)$, and is hence 0.

7.2. **Explicit Formula for the Satake Isomorphism.** We now present an explicit formula (7.42) in $\mathbb{C}_{\leq}[\Lambda^\vee]^W$ for the image of $h_{\lambda^\vee}$ with $\lambda^\vee \in \Lambda^\vee_+$ under the Satake isomorphism S of Theorem 7.1.4.

7.2.1. *v-finite elements.* Let v be a formal variable, and consider the ring of Laurent series $\mathbb{C}_v := \mathbb{C}((v))$. Let $\mathbb{C}_{\leq,v}[\Lambda^\vee]$ be the ring of collections as in (2.33) where the coefficients are now taken to lie in $\mathbb{C}_v$. Recall that $Q^\vee$ was the coroot lattice of $\mathfrak{g}$, which has a $\mathbb{Z}$-basis $\{a_1^\vee, \ldots, a_{\ell+1}^\vee\}$, and that $Q^\vee_-$ denoted the negative coroots. Denote by

$$\mathcal{Q}_v = \mathbb{C}_v[[Q^\vee_-]], \tag{7.30}$$

the ring of formal Taylor series in the variables $e^{-a_i^\vee}$ with coefficients in $\mathbb{C}_v$, where as usual

$$e^{-a_i^\vee} e^{-a_j^\vee} = e^{-a_i^\vee - a_j^\vee} \text{ for } i, j = 1, \ldots, \ell+1. \tag{7.31}$$

Note that $\mathcal{Q}_v \subset \mathbb{C}_{\leq,v}[\Lambda^\vee]$ since for each $x \in \mathcal{Q}_v$ we have that $\mathrm{Supp}(x) \subset \mathfrak{c}(0)$, where $0 \in \Lambda^\vee_+$ denotes the zero coweight and the notation is as in (2.34) and (2.35).

Definition. *The elements $\mathbb{C}_{\leq}[v, v^{-1}][\Lambda^\vee] \subset \mathbb{C}_{v,\leq}[\Lambda^\vee]$ are said to be v-finite. Similarly, an element of $\mathcal{Q}_v$ is said to be v-finite if it is v-finite as an element of $\mathbb{C}_{v,\leq}[\Lambda^\vee]$, i.e., it lies in $\mathbb{C}[v, v^{-1}][[Q^\vee_-]]$.*

Remark: Not all $f \in \mathbb{C}_{v,\leq}[\Lambda^\vee]$ may be "specialized" at $v^2 = q^{-1}$ but this is certainly possible for v-finite elements.

7.2.2. *Macdonald-Cherednik Constant Term Formula.* For each $a \in R_+$, we set

$$c(a^\vee) = \left(\frac{1 - v^2 e^{-a^\vee}}{1 - e^{-a^\vee}} \right), \tag{7.32}$$

which we regard as an element of $\mathcal{Q}_v$ by formally expanding the rational function as a series in $e^{-a^\vee}$. Letting $m(a^\vee)$ denote the multiplicity of the coroot $a^\vee$, we now set

$$\Delta := \prod_{a \in R_+} c(a^\vee)^{m(a^\vee)} = \prod_{a \in R_{re,+}} c(a^\vee) \prod_{n \in \mathbb{Z}_{>0}} c(n\mathbf{c})^\ell, \tag{7.33}$$

which is again regarded as an element of $\mathcal{Q}_v$ by formally expanding the above series. In fact, it is easy to see (see [29]) that the element Δ is invertible in $\mathcal{Q}_v$.

For $a \in R_+$ and $w \in W$, we shall write $c(wa^\vee)$ to denote the expansion of (7.32) in $\mathcal{Q}_v$. For example, if a is a real root and $w = w_a$ then

$$c(w_a a^\vee) = c(-a^\vee) = \frac{1 - v^2 e^{a^\vee}}{1 - e^{a^\vee}} = \frac{e^{-a^\vee} - v^2}{e^{-a^\vee} - 1} = v^2 + (v^2 - 1)e^{-a^\vee} + (v^2 - 1)e^{-2a^\vee} + \cdots . \quad (7.34)$$

Now, for each $w \in W$, we may consider the element

$$\Delta^w := \prod_{a \in R_+} c(wa^\vee)^{m(a^\vee)} \in \mathcal{Q}_v. \quad (7.35)$$

The same argument as in [28, p.199] shows that the sum $\sum_{w \in W} \Delta^w$ is a well-defined element in $\mathcal{Q}_v$ which again is invertible.

For any subset $\Sigma \subset W$ we write its Poincare polynomial as

$$\Sigma(v) = \sum_{w \in \Sigma} v^{\ell(w)}. \quad (7.36)$$

Consider now the following expression, which takes its values in $\mathcal{Q}_v$,

$$H_0 := \frac{\sum_{w \in W} \Delta^w}{W(v^2)}. \quad (7.37)$$

In fact, it is not hard to see ([28, (3.8)]) that $H_0 \in \mathbb{C}_v[[e^{-\mathbf{c}}]]$ where $\mathbf{c}$ is the minimal imaginary coroot of $\mathfrak{g}$. One may give a formula for H_0 as an infinite product of expressions in the variables v^2 and $e^{\mathbf{c}}$ (see [1, Theorem 1.7]). In the case when $\mathfrak{g}$ is simply-laced, the formula takes the following form, which is the result of [28, (3.8)] and the work of Cherednik [6] on Macdonald's Constant Term Conjecture.

Theorem. *Let $\mathfrak{g}$ be a simply-laced untwisted affine Kac-Moody algebra. Then we have that*

$$H_0 = \prod_{j=1}^{\ell} \prod_{i=1}^{\infty} \frac{1 - v^{2m_j} e^{-i\mathbf{c}}}{1 - v^{2(m_j+1)} e^{-i\mathbf{c}}}, \quad (7.38)$$

where the integers m_j for $j = 1, \ldots, \ell$ are the exponents of $\mathfrak{g}_o$ defined by the relation that

$$W_o(v^2) = \prod_{j=1}^{\ell} \frac{1 - v^{2(m_j+1)}}{1 - v^2} \quad (7.39)$$

where $W_o \subset W$ is the finite Weyl group. In particular, we see that H_0 is v-finite (and so can be evaluated at $v^2 = q^{-1}$).

7.2.3. *Statement of Main Formula.* We can now state precisely the formula for $S(h_{\lambda^\vee})$ with $\lambda^\vee \in \Lambda_+^\vee$. For any such $\lambda^\vee \in \Lambda_+^\vee$, define

$$W_{\lambda^\vee} = \{w \in W | w\lambda^\vee = \lambda^\vee\} \subset W, \quad (7.40)$$

and recall from (1.13) that we have defined the element

$$H_{\lambda^\vee} = \frac{v^{-2\langle \rho, \lambda^\vee \rangle}}{W_{\lambda^\vee}(v^2)} \sum_{w \in W} \Delta^w e^{w\lambda^\vee}, \quad (7.41)$$

which we will argue is in $\mathbb{C}_{v,\leq}[\Lambda^\vee]$.

Theorem. *Let $\lambda \in \Lambda_+^\vee$. The ratio $\frac{H_{\lambda^\vee}}{H_0} \in \mathbb{C}_{v,\leq}[\Lambda^\vee]$ is v-finite, and its value at at $v^2 = q^{-1}$ is equal to $S(h_{\lambda^\vee})$.*

Informally, we shall write the above theorem as follows:

$$S(h_{\lambda^\vee}) = \frac{1}{H_0} \frac{q^{-\langle \rho, \lambda^\vee \rangle}}{W_{\lambda^\vee}(q^{-1})} \sum_{w \in W} \Delta^w e^{w\lambda^\vee}, \tag{7.42}$$

where Δ is as in (7.33) and H_0 is defined in (7.37) (with $v^2 = q^{-1}$). The proof will occupy §7.3 and will be broken up into four parts.

Step 1: Disassembly: We first break up the computation of $S(h_{\lambda^\vee})$ into pieces indexed by a set of minimal length representatives in $W/W_{\lambda^\vee}$. This is done via a passage to the Iwahori subgroup I^-.

Step 2: Recursion: We show a certain recursion relation between the pieces introduced in the previous step corresponding to $w, w' \in W$ where w and w' differ by a simple reflection.

Step 3: Algebraic Identities: We recall some purely formal algerbaic identities from [7] involving the affine symmetrizers and polynomial representation of Cherednik.

Step 4: Rephrasal and Reassembly: Reinterpreting the recursion of Step (2) using the polynomial representation of Step (3), we can rewrite the disassembly from Step (1) using the affine symmetrizers of Cherednik. The argument is then concluded by applying an algebraic proportionality principle from Step (3).

7.3. **Proof of Theorem 7.2.3.**

Step 1: Disassembly.

7.3.1. *Minimal Length Representatives.* Recall that we have defined the stabilizer $W_{\lambda^\vee}$ for each $\lambda^\vee \in \Lambda_+^\vee$ in (7.40). If $\lambda^\vee = 0$, then $W_{\lambda^\vee} = W$. In the opposite extreme, if $W_{\lambda^\vee} = 1$ we say that $\lambda^\vee$ is *regular.* Choose a set $W^{\lambda^\vee} \subset W$ of minimal length representatives for the set $W/W_{\lambda^\vee}$, and note that for this choice, we have

$$\ell(w w_{\lambda^\vee}) = \ell(w) + \ell(w_{\lambda^\vee}) \text{ for } w \in W^{\lambda^\vee}, w_{\lambda^\vee} \in W_{\lambda^\vee}. \tag{7.43}$$

Also note that every $w \in W$ has a unique factorization

$$w = w_1 w_2 \text{ where } w_1 \in W^{\lambda^\vee}, w_2 \in W_{\lambda^\vee}. \tag{7.44}$$

The following observation will be useful for us later: if $w \in W^{\lambda^\vee}$ and $w_a, a \in \Pi$ is a simple reflection such that $\ell(w_a w) < \ell(w)$ then w and $w_a w$ cannot be in the same coset $W/W_{\lambda^\vee}$ (since their lengths differ and w is the minimal length element it its coset), and also $w_a w \in W^{\lambda^\vee}$. Indeed, if $w_a w$ were not in $W^{\lambda^\vee}$, there would exist some reduced decomposition for it ending in a simple reflection which lies in $W_{\lambda^\vee}$. However, this would give a reduced decomposition of w (by premultiplying by w_a) which ends with simple reflection that lies in $W_{\lambda^\vee}$, contradicting (7.43).

7.3.2. *The subgroup P.* Recall that I^- was the opposite Iwahori subgroup defined in (3.9). We define

$$P_{\lambda^\vee} := \sqcup_{w \in W_{\lambda^\vee}} I^- w I^-, \tag{7.45}$$

and note that it is a group: indeed, $W_{\lambda^\vee}$ is generated by the simple reflections $w_a, a \in \Pi$ which it contains, so one may now argue as in the classical case, see [19]. Often we drop the subscript $\lambda^\vee$ and just denote this group by P when $\lambda^\vee$ is fixed in the discussion. If $\lambda^\vee$ is regular $P = I^-$ and in the opposite extreme, if $\lambda^\vee = 0$ then $P = K$. Note that

$$UwP = Uw_1P \tag{7.46}$$

for w as in (7.44), and hence we have a disjoint union,

$$G = \sqcup_{w \in W^{\lambda^\vee}} UwP. \tag{7.47}$$

7.3.3. *Decomposition into Iwahori Pieces.* Let us define for $w \in W, \mu^\vee \in \Lambda^\vee$, and $\lambda^\vee \in \Lambda_+^\vee$ the following two sets,

$$F^P_{w,\lambda^\vee}(\mu^\vee) := \{(a,c) \in UwP \times_P P\pi^{\lambda^\vee} K | ac = \pi^{\mu^\vee}\} \tag{7.48}$$

$$F^K_{\lambda^\vee}(\mu^\vee) := \{(x,y) \in UK \times_K K\pi^{\lambda^\vee} K | xy = \pi^{\mu^\vee}\}. \tag{7.49}$$

In our previous notation, we may write this as follows: consider the multiplication maps

$$m_{w,\lambda^\vee} : UwP \times_P P\pi^{\lambda^\vee} K \to G \tag{7.50}$$

$$m_{\lambda^\vee} : UK \times_K K\pi^{\lambda^\vee} K \to G, \tag{7.51}$$

then we have

$$F^P_{w,\lambda^\vee}(\mu^\vee) = m^{-1}_{w,\lambda^\vee}(\pi^{\mu^\vee}) \quad \text{and} \quad F^K_{\lambda^\vee}(\mu^\vee) = m^{-1}_{\lambda^\vee}(\pi^{\mu^\vee}). \tag{7.52}$$

Consider the map $\varphi_w : F^P_{w,\lambda^\vee}(\mu^\vee) \to F^K_{\lambda^\vee}(\mu^\vee)$ which sends $(a,c) \mapsto (a,c)$ where $a \in UwP$ and $c \in P\pi^{\lambda^\vee}K$. It clearly induces a well-defined map: i.e., if $j \in P$ then (a,c) and $(aj, j^{-1}c)$ are equivalent in $F^K_{\lambda^\vee}(\mu^\vee)$. Let us now set,

$$F^P_{\lambda^\vee}(\mu^\vee) := \sqcup_{w \in W^{\lambda^\vee}} F^P_{w,\lambda^\vee}(\mu^\vee) \subset \sqcup_{w \in W^{\lambda^\vee}} UwP \times_P P\pi^{\lambda^\vee} K \tag{7.53}$$

and define φ to be the map from $F^P_{\lambda^\vee}(\mu^\vee) \to F^K_{\lambda^\vee}(\mu^\vee)$ which restricts to φ_w on $F^P_{w,\lambda^\vee}(\mu^\vee)$.

Lemma. *The map* $\varphi : F^P_{\lambda^\vee}(\mu^\vee) \to F^K_{\lambda^\vee}(\mu^\vee)$ *is bijective.*

Proof. Let us first see that φ is surjective: each $(x,y) \in F^K_{\lambda^\vee}(\mu^\vee)$ has representative of the form $x \in UK$ and $y \in \pi^{\lambda^\vee}K$. Write $x = uk$ with $u \in U, k \in K$. Decomposing K according to (I, I^-)-cosets (see (3.12)), we may assume $k \in IwI^-$ for $w \in W$. Note that $UIwI^- = UwI^-$ for $w \in W$. Writing $w = w_1 w_2$ with $w_1 \in W^{\lambda^\vee}$ and $w_2 \in W_{\lambda^\vee}$ as in (7.44), we further observe that $UIwP = Uw_1P$. Surjectivity follows from this.

Injectivity will follow from the following ,

Claim. *Let* $(a,c), (a',c') \in \sqcup_{w \in W^{\lambda^\vee}} F^P_{w,\lambda^\vee}(\mu^\vee)$, *and suppose that* $\varphi(a,c) = \varphi(a',c')$. *Then* $c = vc'$ *with* $v \in I^- w I^-$ *and* $w \in W_{\lambda^\vee}$.

Proof of Claim. Given (a,c) and (a',c') as above such that $\varphi(a,c) = \varphi(a',c')$, there must exist by definition an element $v \in K$ such that $c = vc'$. Suppose $c = l\pi^{\lambda^\vee}k$ and $c' = l'\pi^{\lambda^\vee}k'$, where $l, l' \in P$ and $k, k' \in K$. We may assume that $l, l' \in I^-$ actually. Indeed, if $\sigma \in W_{\lambda^\vee}$ then

$$I^-\sigma I^- \pi^{\lambda^\vee} K = I^- \sigma \pi^{\lambda^\vee} K = I^- \pi^{\lambda^\vee} K \tag{7.54}$$

where in the first equality we use the fact that $\lambda^\vee \in \Lambda^\vee_+$ and in the second that $\sigma \in W_{\lambda^\vee}$. Thus we have a relation of the form

$$i\pi^{\lambda^\vee} k = vi'\pi^{\lambda^\vee} k', \tag{7.55}$$

with $i, i' \in I^-$. We may suppose that $v \in I^- w I^-$ with $w \in W$. So, we have found an element in the intersection of

$$I^-\pi^{\lambda^\vee} K \cap I^- w I^- \pi^{\lambda^\vee} K. \tag{7.56}$$

Again, since $\lambda^\vee$ is dominant, we have that $I^- w I^- \pi^{\lambda^\vee} K = I^- w \pi^{\lambda^\vee} K$. However, (7.56) can only be non-empty if $w \in W_{\lambda^\vee}$: indeed, if the intersection was non-empty, then we would actually have

$$I^- w \pi^{\lambda^\vee} K = I^- \pi^{\lambda^\vee} K. \tag{7.57}$$

Consider the multiplication maps

$$m : UI^- \times_{I^-} I^- \pi^{\lambda^\vee} K \to G \tag{7.58}$$

$$m : UI^- \times_{I^-} I^- w \pi^{\lambda^\vee} K \to G. \tag{7.59}$$

The image of the former is supported on $U\pi^{\lambda^\vee}K$ whereas the image of the latter contains the element $\pi^{w\lambda^\vee}$. This is a contradiction unless $w \in W_{\lambda^\vee}$. □

This also concludes the proof of the Lemma. □

7.3.4. $J_w(\lambda^\vee)$. For each $w \in W^{\lambda^\vee}$, define the sum

$$J_w(\lambda^\vee) := \sum_{\mu^\vee \in \Lambda^\vee} |F^P_{w,\lambda^\vee}(\mu^\vee)| q^{\langle \rho, \mu^\vee \rangle} e^{\mu^\vee}. \tag{7.60}$$

The preceeding Lemma yields the following disassembly of $S(h_{\lambda^\vee})$

$$S(h_{\lambda^\vee}) = \sum_{w \in W^{\lambda^\vee}} J_w(\lambda^\vee). \tag{7.61}$$

Let us note that if $w \in W$ is written as $w = w_1 w_2$ where $w_1 \in W^{\lambda^\vee}$ and $w_2 \in W_{\lambda^\vee}$, then

$$J_w(\lambda^\vee) = J_{w_1}(\lambda^\vee) \tag{7.62}$$

since $UwP = Uw_1P$ in this case. We take (7.62) as the *definition* of $J_w(\lambda^\vee)$ for any $w \in W$ (the previous definition was for $w \in W^{\lambda^\vee}$).

7.3.5. $J_w(\lambda^\vee)$ *as a convolution, I.* Let $\mathbf{v}^P_w$ be the characteristic function of UwP for $w \in W$, $\theta^P_{\lambda^\vee, K}$ be the characteristic function of $P\pi^{\lambda^\vee}K$, and recall that we have defined $\mathbf{v}_{\mu^\vee}$ in (7.3) to be the characteristic function of $U\pi^{\mu^\vee}K$. Using the map

$$m_{w,\lambda^\vee} : UwP \times_P P\pi^{\lambda^\vee}K \to G, \tag{7.63}$$

we define

$$\mathbf{v}^P_w \star \theta^P_{\lambda^\vee, K} := \sum_{\mu^\vee \in \Lambda^\vee} |m^{-1}_{w,\lambda^\vee}(\pi^{\mu^\vee})| \mathbf{v}_{\mu^\vee}. \tag{7.64}$$

By definition (7.60) we have

$$\mathbf{v}^P_w \star \theta^P_{\lambda^\vee, K} = \sum_{\mu^\vee \in \Lambda^\vee} |m^{-1}_{w,\lambda^\vee}(\pi^{\mu^\vee})| e^{\mu^\vee} q^{\langle \rho, \mu^\vee \rangle} \circ \mathbf{1}_K = J_w(\lambda^\vee) \circ \mathbf{1}_K \text{ for } w \in W^{\lambda^\vee}. \tag{7.65}$$

By the finiteness (7.6) and Lemma 7.3.3 above, we conclude that each coefficient in (7.64) is finite. Also from Lemma 7.3.3 and (7.7) we conclude that if $\mu^\vee \in \Lambda^\vee$ lies in the support of $J_w(\lambda^\vee)$, then $\mu^\vee \leq \lambda^\vee$. Hence $J_w(\lambda^\vee) \in \mathbb{C}_{\leq}[\Lambda^\vee]$. In Proposition 7.3.7 below, we argue that in fact $J_w(\lambda^\vee) \in \mathbb{C}[\Lambda^\vee]$ (i.e. that it has finite support).

7.3.6. $J_w(\lambda^\vee)$ *as a convolution, II.* We will also need a slight variant on the above formula (7.65). To state it, for each $w \in W$ let Y_w and $\mathbf{v}^P_w$ be the characteristic function of the subset IwP and UwP respectively. Recall that we have also defined $\mathbf{v}_w$ for $w \in W$ to be the characteristic function of $A_{\mathcal{O}}UwI$. Then for $w, w' \in W$ we note that $UwIw'P$ is a (finite) union of cosets $U\sigma P$ with $\sigma \in W$. The convolution $\mathbf{v}_w \star Y_{w'}$ for w, w' can be defined in the usual way: i.e., if $m_{w,w'} : UwI \times_I Iw'P \to G$ denotes the multiplication map, then

$$\mathbf{v}_w \star Y_{w'} = \sum_{\sigma \in W} |m^{-1}_{w,w'}(\sigma)| \mathbf{v}^P_\sigma \tag{7.66}$$

Lemma. *Let $w \in W$. Then*

(1) $\mathbf{v}_1 \star Y_w = \mathbf{v}^P_w$
(2) *If $w \in W^{\lambda^\vee}$ and $w = w_a w'$ where $a \in \Pi$ is a simple root and $\ell(w) = \ell(w') + 1$, then*

$$\mathbf{v}_{w_a} \star Y_{w'} = (q-1)\, \mathbf{v}^P_{w'} + q\, \mathbf{v}^P_w. \tag{7.67}$$

Proof. For part (1), since $UIwP = UwP$, it suffices to verify that

$$|I \setminus IUw \cap IwP| = 1, \tag{7.68}$$

which follows easily from the fact that if $u \in U$ and $Iuw \in IwP$, then in fact $u \in U \cap K$. Hence, for any such u, we have $Iuw = Iw$. As for part (2), first note that $Uw_aIw'P = Uw_aU_a[0]w'P$ and clearly the support of the left hand side of (7.67) is contained in the union of $Uw_aw'P$ and $Uw'P$. By our assumptions that $w \in W^{\lambda^\vee}$ and $\ell(w_aw') > \ell(w')$ we have that $Uw_aw'P$ and $Uw'P$ are disjoint (as follows from the remarks at the end of §7.3.1 and the disjointness (7.47)). Now, to verify the multiplicities in question, we need

$$|I \setminus Iw_aU_a[0]w_aw' \cap Iw'P| = q \quad \text{and} \quad |I \setminus Iw_aU_a[0]w' \cap Iw'P| = q-1, \tag{7.69}$$

which we leave to the reader. ☐

Remark. Defining the triple convolution $\mathbf{v}_1 \star Y_w \star \theta^P_{\lambda^\vee,K} \in M(G,K)$ in a natural way, part (1) of the above Lemma gives the formula,

$$J_w(\lambda^\vee) \circ \mathbf{1}_K = \mathbf{v}_1 \star Y_w \star \theta^P_{\lambda^\vee,K} \tag{7.70}$$

which will be used in the next section.

Step 2: Recursion.

7.3.7. *Macdonald's Recursion Formula.* Now we verify a recursion relation for $J_w(\lambda^\vee)$ introduced in the previous step. The result is an analogue of the recursion relations obtained by Macdonald in [26, Theorem 4.4.5].

Proposition. *Let $\lambda^\vee \in \Lambda^\vee_+, w \in W$.*

(1) *The expression $J_w(\lambda^\vee)$ actually lies in $\mathbb{Q}_{\geq 0}[\Lambda^\vee]$. In other words,*

$$J_w(\lambda^\vee) = \sum_{\mu^\vee \in \Lambda^\vee} \Phi_{w,\mu^\vee} e^{\mu^\vee} \tag{7.71}$$

with $\Phi_{w,\mu^\vee} \in \mathbb{Q}_{\geq 0}$ and non-zero for only finitely many $\mu^\vee \in \Lambda^\vee$.

(2) *If $w \in W_{\lambda^\vee}$, then we have*

$$J_w(\lambda^\vee) = J_1(\lambda^\vee) = q^{\langle \rho, \lambda^\vee \rangle} e^{\lambda^\vee}. \tag{7.72}$$

(3) *Let $w \in W^{\lambda^\vee}, a \in \Pi$ and $w = w_aw'$ with $\ell(w) = 1 + \ell(w')$. Then we have the recursion relation,*

$$J_w(\lambda^\vee) = \frac{1 - q^{-1}e^{a^\vee}}{1 - e^{a^\vee}} \cdot J_{w'}(\lambda^\vee)^{w_a} + \frac{q^{-1} - 1}{1 - e^{a^\vee}} \cdot J_{w'}(\lambda^\vee), \tag{7.73}$$

where by $J_{w'}(\lambda^\vee)^{w_a}$ we mean the termwise application of w_a to the expression $J_{w'}(\lambda^\vee)$, and the rational functions which appear on the right hand side are expanded in the ring V_{w_a} of (6.4).

(4) *For $w \in W$ and $\lambda^\vee \in \Lambda^\vee_+$, there exist polynomials* [10] *$\Phi_{w,\mu^\vee}(v^2) \in \mathbb{Q}[v^2, v^{-2}]$ (actually in $v^{-2\langle \rho, \lambda^\vee \rangle}\mathbb{Q}[v^2]$) such that*

$$J_w(\lambda^\vee) = \sum_{\mu^\vee \in \Lambda^\vee} \Phi_{w,\mu^\vee}(q^{-1}) e^{\mu^\vee}. \tag{7.74}$$

[10] In Proposition 7.3.15 we connect the polynomials $\Phi_{w,\mu^\vee}(v^2)$ with the polynomial representation of the DAHA.

7.3.8. *Remarks on Part (3).* From the way in which statement (3) is written, it is not clear that the expression $J_w(\lambda^\vee)$ is actually a finite sum. However, there is a cancellation which occurs when one expands the rational functions appearing in (7.73). Let us illustrate this with a simple example: suppose $\lambda^\vee \in \Lambda_+^\vee$ and $a \in \Pi$ is such that $w_a \notin W_{\lambda^\vee}$. Then we may write $w_a\lambda^\vee = \lambda^\vee - ka^\vee$ for $k := \langle \lambda^\vee, a\rangle > 0$. From (2) we find that $J_1(\lambda^\vee) = q^{\langle \rho, \lambda^\vee\rangle} e^{\lambda^\vee}$. Hence, from (3) we find that

$$\begin{aligned} J_{w_a}(\lambda^\vee) &= \frac{1-q^{-1}e^{a^\vee}}{1-e^{a^\vee}} \cdot q^{\langle\rho,\lambda^\vee\rangle} e^{w_a\lambda^\vee} + \frac{q^{-1}-1}{1-e^{a^\vee}} \cdot q^{\langle\rho,\lambda^\vee\rangle} e^{\lambda^\vee} && (7.75)\\ &= q^{\langle\rho,\lambda^\vee\rangle}\{(1+(1-q^{-1})e^{a^\vee} + (1-q^{-1})e^{2a^\vee} + \cdots)e^{\lambda^\vee - ka^\vee} && (7.76)\\ &+ ((q^{-1}-1) + (q^{-1}-1)e^{a^\vee} + \cdots)e^{\lambda^\vee}\} && (7.77)\\ &= q^{\langle\rho,\lambda^\vee\rangle}(e^{w_a\lambda^\vee} + (1-q^{-1})e^{w_a\lambda^\vee + a^\vee} + \cdots + (1-q^{-1})e^{\lambda^\vee - a^\vee}). && (7.78)\end{aligned}$$

Hence, one can see directly in this this example that $J_{w_a}(\lambda^\vee)$ actually lies in $\mathbb{C}[\Lambda^\vee]$. Moreover, note that the sum $J_1(\lambda^\vee) + J_{w_a}(\lambda^\vee)$ is easily seen to agree with the usual spherical function for $SL(2)$.

7.3.9. *Proof of Proposition 7.3.7.* The proofs of (1) and (2) are quite straightforward, and we just sketch the argument here. From the definition of $J_w(\lambda^\vee)$ the support (i.e., the set of $\mu^\vee \in \Lambda^\vee$ such that $F^P_{w,\lambda^\vee}(\mu^\vee) \neq \emptyset$) consists of those $\mu^\vee$ such that $UwP\pi^{\lambda^\vee}K \cap U\pi^{\mu^\vee}K \neq \emptyset$. If $W_{\lambda^\vee}$ is infinite, then $W = W_{\lambda^\vee}$ and $P = K$. Moreover, in this case,

$$UK\pi^{\lambda^\vee}K = UIWI^-\pi^{\lambda^\vee}K = UIWU^-_{\mathcal{O}}\pi^{\lambda^\vee}K = UU^-_\pi W\pi^{\lambda^\vee}K = U\pi^{\lambda^\vee}K \tag{7.79}$$

since $\lambda^\vee \in \Lambda_+^\vee$ and W fixes $\lambda^\vee$. So the only possibility is that $\mu^\vee = \lambda^\vee$ in this case. Next, assume that $W_{\lambda^\vee}$ is finite so P is a finite union of I^--double cosets. Hence, it suffices to verify that for fixed $\sigma \in W$ there are only finitely many $\mu^\vee \in \Lambda^\vee$ such that $U\sigma I^-\pi^{\lambda^\vee}K \cap U\pi^{\mu^\vee}K$ is nonempty. Using the decomposition $I^- = U_\pi U^-_{\mathcal{O}} A_{\mathcal{O}}$ together with the dominance of $\lambda^\vee \in \Lambda_+^\vee$ we can immediately deduce that

$$U\sigma I^-\pi^{\lambda^\vee}K \cap U\pi^{\mu^\vee}K \neq \emptyset \iff U_{-\sigma,\pi}\sigma\pi^{\lambda^\vee}K \cap U\pi^{\mu^\vee}K \neq \emptyset \tag{7.80}$$

where $U_{-\sigma,\pi} := \{u \in U^-_\pi \mid \sigma^{-1}u\sigma \in U\}$ is a finite product of one-dimensional groups . From this we may deduce that only finitely many $\mu^\vee$ can occur.

As for part (2), we have already remarked in §7.3.4 that $J_w(\lambda^\vee) = J_1(\lambda^\vee)$ if $w \in W_{\lambda^\vee}$. The verification that $J_1(\lambda^\vee) = e^{\lambda^\vee} q^{\langle\rho,\lambda^\vee\rangle}$ can be deduced from Lemma 7.1.6 and Lemma 7.3.3.

As for (3), we use the intertwining operators $\mathfrak{T}_w$ defined in §6. From Lemmas (6.1.4) and (6.1.5) we have that

$$\mathfrak{T}_a(\mathbf{v}_1) = q^{-1}\mathbf{v}_{w_a} + \tfrac{(1-q^{-1})e^{a^\vee}}{1-e^{a^\vee}} \circ \mathbf{v}_1 \quad \text{and} \quad \mathfrak{T}_a(\mathbf{1}_K) = \tfrac{1-q^{-1}e^{a^\vee}}{1-e^{a^\vee}} \circ \mathbf{1}_K, \tag{7.81}$$

where the rational functions are expanded in positive powers of $e^{a^\vee}$. We now compute the action of $\mathfrak{T}_a(J_{w'}(\lambda^\vee) \circ \mathbf{1}_K)$ in two different ways. On the one hand, from (6.8) we have

$$\mathfrak{T}_a(J_{w'}(\lambda^\vee) \circ \mathbf{1}_K) = (J_{w'}(\lambda^\vee))^{w_a}\mathfrak{T}_a(\mathbf{1}_K) = \frac{1-q^{-1}e^{a^\vee}}{1-e^{a^\vee}} J_{w'}(\lambda^\vee)^{w_a} \circ \mathbf{1}_K. \tag{7.82}$$

On the other hand, from (7.70) $J_{w'}(\lambda^\vee) \circ \mathbf{1}_K = \mathbf{v}_1 \star Y_{w'} \star \theta^P_{\lambda^\vee,K}$. So we have

$$\mathfrak{T}_a(J_{w'}(\lambda^\vee) \circ \mathbf{1}_K) = \mathfrak{T}_a(\mathbf{v}_1 \star Y_{w'} \star \theta^P_{\lambda^\vee,K}) \quad = \quad \mathfrak{T}_a(\mathbf{v}_1) \star Y_{w'} \star \theta^P_{\lambda^\vee,K} \tag{7.83}$$

using the fact that $\mathfrak{T}_a$ commutes with convolutions on the right. Using (7.81) we find that the previous expression is equal to

$$\begin{aligned} &(q^{-1}\mathbf{v}_{w_a} + \frac{(1-q^{-1})e^{a^\vee}}{1-e^{a^\vee}} \circ \mathbf{v}_1) \star Y_{w'} \star \theta^P_{\lambda^\vee,K} && (7.84)\\ =\ & q^{-1}\mathbf{v}_{w_a} \star Y_{w'} \star \theta^P_{\lambda^\vee,K} + \frac{(1-q^{-1})e^{a^\vee}}{1-e^{a^\vee}} \circ \mathbf{v}_1 \star Y_{w'} \star \theta^P_{\lambda^\vee,K}. && (7.85)\end{aligned}$$

Using Lemma (7.3.6), we have that

$$\mathbf{v}_{w_a} \star Y_{w'} = (q-1)\mathbf{v}^P_{w'} + q\mathbf{v}^P_w = (q-1)\mathbf{v}_1 \star Y_{w'} + q\mathbf{v}_1 \star Y_w. \tag{7.86}$$

Hence, the previous expression is also equal to

$$= \mathbf{v}_1 \star Y_w \star \theta^P_{\lambda^\vee,K} + (1-q^{-1})\mathbf{v}_1 \star Y_{w'} \star \theta^P_{\lambda^\vee,K} + \frac{(1-q^{-1})e^{a^\vee}}{1-e^{a^\vee}} \circ \mathbf{v}_1 \star Y_{w'} \star \theta^P_{\lambda^\vee,K} \tag{7.87}$$

$$= J_w(\lambda^\vee) \circ \mathbf{1}_K + \frac{(1-q^{-1})}{1-e^{a^\vee}} J_{w'}(\lambda^\vee) \circ \mathbf{1}_K. \tag{7.88}$$

Comparing with (7.82), Part (3) of the Proposition follows.

As for Part (4): if $w = w_a$ this follows from the remarks in §7.3.8. The general case follows by induction on the length of w using the recursion relation (7.73).

Step 3: Algebraic Identities.

7.3.10. *The Polynomial Representation of Cherednik.* In this part, we shall establish a purely algebraic identity (7.104) which will be used in the subsequent step 4 to establish Theorem 7.2.3. Let v and X be formal variables, and set

$$\mathbf{c}(X) = \tfrac{vX-v^{-1}}{X-1} \text{ and } \mathbf{b}(X) = \tfrac{v-v^{-1}}{1-X}. \tag{7.89}$$

The following identities are easy to verify, see [29, p. 58, 4.2.3(i) -(iv)]

$$\mathbf{c}(X) = v - \mathbf{b}(X) = v^{-1} + \mathbf{b}(X^{-1}) \tag{7.90}$$

$$\mathbf{c}(X) + \mathbf{c}(X^{-1}) = v + v^{-1} \tag{7.91}$$

$$\mathbf{c}(X)\mathbf{c}(X^{-1}) = 1 + \mathbf{b}(X)\mathbf{b}(X^{-1}). \tag{7.92}$$

Recall the ring $\mathcal{Q}_v$ from (7.30) anove. For each $a \in R$ we may consider elements of $\mathcal{Q}_v$ $c(a) := \mathbf{c}(e^{a^\vee})$ and $b(a) := \mathbf{b}(e^{a^\vee})$ obtained by substituting $e^{a^\vee}$ for X in (7.89) and then formally expanding.

Denote by $\mathcal{Q}_v[W]$ the vector space consisting of elements

$$f = \sum_{i=1}^n c_i[\sigma_i], \text{ where } c_i \in \mathcal{Q}_v, \sigma_i \in W. \tag{7.93}$$

Let $g \in \mathbb{C}_v(Q^\vee)$ be such that g and g^σ for $\sigma \in W$ admit expansions in $\mathcal{Q}_v$. Then we set

$$[\sigma]g = g^\sigma[\sigma]. \tag{7.94}$$

Warning: Let $\sigma \in W$. For a general $g \in \mathcal{Q}_v$, it may not be the case that $g^\sigma \in \mathcal{Q}_v$ (so $\mathcal{Q}_v[W]$ is not a twisted group algebra).

For each $a \in \Pi$, we consider elements

$$\mathbf{T}_a = v[w_a] + b(a)(1-[w_a]) = c(a)[w_a] + b(a)[1] \in \mathcal{Q}_v[W], \tag{7.95}$$

where [1] denotes the element corresponding to the identity of W. One checks immediately that

$$v\mathbf{T}_a = \frac{1-v^2e^{a^\vee}}{1-e^{a^\vee}}[w_a] + \frac{v^2-1}{1-e^{a^\vee}}[1] \tag{7.96}$$

For $w \in W$ with a reduced decomposition $w = w_{a_1} \cdots w_{a_n}$ where $a_1, \ldots, a_n \in \Pi$ we set

$$\mathbf{T}_w := \mathbf{T}_{a_1} \cdots \mathbf{T}_{a_n}. \tag{7.97}$$

We view $\mathbf{T}_w \in \mathcal{Q}_v[W]$ as follows: first substitute for each $\mathbf{T}_{a_i}$ the expression $c(a_i)[w_{a_i}] + b(a_i)[1]$ from (7.95), move all the rational functions to the left so that we may write

$$\mathbf{T}_w = \sum_{\sigma \in W} A_\sigma(w)[\sigma], \tag{7.98}$$

where $A_\sigma(w)$ is some sum of products of rational functions in $c(\cdot)$ and $b(\cdot)$. It is easy to verify that $A_\sigma(w) = 0$ unless $\sigma \leq w$ in the Bruhat order. After now expanding the rational functions involved in $A_\sigma(w)$ in $\mathcal{Q}_v$, we may view $\mathbf{T}_w \in \mathcal{Q}_v[W]$. One may further verify that the above definition of $\mathbf{T}_w$ does not depend on the reduced decomposition of w. Moreover, one can check the following Hecke identity using (7.90, 7.91, 7.92),

$$(\mathbf{T}_a + v^{-1})(\mathbf{T}_a - v) \quad = \quad 0 \text{ for } a \in \Pi. \tag{7.99}$$

We may summarize the above by writing,

$$\mathbf{T}_w \mathbf{T}_a \quad = \quad \mathbf{T}_{ww_a} \text{ if } \ell(ww_a) > \ell(w) \tag{7.100}$$

$$\mathbf{T}_w \mathbf{T}_{w_a} \quad = \quad \mathbf{T}_{ww_a} + (v - v^{-1})\mathbf{T}_w \text{ if } \ell(ww_a) < \ell(w). \tag{7.101}$$

7.3.11. *The expression $\mathcal{P}_v$.* In order to consider certain infinite sums of elements from $\mathcal{Q}_v[W]$ we introduce the formal dual $\mathcal{Q}_v[W]^\vee$ as the set of all possibly formal infinite sums $F = \sum_{w \in W} f_w[w]$ where $f_w \in \mathcal{Q}_v$. One again implements a multiplication rule as in (7.94).

Consider now the formal expression

$$\mathcal{P}_v := \sum_{w \in W} v^{\ell(w)} \mathbf{T}_w, \tag{7.102}$$

and let us now argue that it lies in $\mathcal{Q}_v[W]^\vee$. Using (7.98), we write

$$\mathcal{P}_v = \sum_{w \in W} v^{\ell(w)} \mathbf{T}_w = \sum_{w \in W} \sum_{\sigma \leq w} v^{\ell(w)} A_\sigma(w)[\sigma] = \sum_{\tau \in W} C_\tau[\tau], \tag{7.103}$$

where $C_\tau := \sum_{w \in W} v^{\ell(w)} A_\tau(w)$ is an infinite sum of rational functions. We may then attempt to expand C_τ in the domain $\mathcal{Q}_v$, but it is *not* the case that, for a fixed $\tau \in W$ only finitely many $A_\tau(w)$ will be non-zero. However, the following result of Cherednik ensures that C_τ is not just well-defined as an element on $\mathcal{Q}_v$ but that is a v-finite quantity.

Lemma. [7, Lemma 2.19(e)] *The element $\mathcal{P}_v$ lies in $\mathcal{Q}_v[W]^\vee$ and has v-finite coefficients. In other words, for each $\tau \in W$, the expansion of the sum $C_\tau := \sum_{w \in W} v^{\ell(w)} A_\tau(w)$ is a well-defined, v-finite element in $\mathcal{Q}_v$.*

We refer the reader to [7] for the proof, but just remark here that the essence of the argument is to show the following: for any fixed $\tau \in W$, the contributions $A_\tau(w)$ will all arise with a factor $e^{n\mathbf{c}}$ with $n \to \infty$ as $\ell(w) \to \infty$. Hence, if one fixes a value $e^{b^\vee}$ with $b^\vee \in Q^\vee$ and $\tau \in W$, there are only finitely many $w \in W$ such that $e^{b^\vee}$ will occur in $A_\tau(w)$.

7.3.12. *Cherednik's Identity.* For each $w \in W$, we have the element $\Delta^w \in \mathcal{Q}_v$ as defined in (7.35). Thus we have the element $\sum_{w \in W} \Delta^w[w] \in \mathcal{Q}_v[W]^\vee$. We then have the following algebraic identity in $\mathcal{Q}_v[W]^\vee$ due to Cherednik (see also ([29, §5.5] for a finite-dimensional analogue).

Proposition ([7])**.** *As elements of $\mathcal{Q}_v[W]^\vee$ we have an equality,*

$$\mathcal{P}_v = \sum_{w \in W} v^{\ell(w)} \mathbf{T}_w = \mathfrak{m} \sum_{w \in W} \Delta^w[w], \tag{7.104}$$

where $\mathfrak{m} \in \mathcal{Q}_v$ is some W-invariant factor.

Proof. Step 1: First we would like to explain how to make sense, for any $a \in \Pi$, of the expressions $\mathbf{T}_a \, \mathcal{P}_v$ and $\mathcal{P}_v \, \mathbf{T}_a$ as elements in $\mathcal{Q}_v[W]^\vee$. To make sense of $\mathbf{T}_a \, \mathcal{P}_v$, we may proceed in two equivalent ways: (a) we compute $\mathbf{T}_a \mathcal{P}_v$ using the Hecke relations (7.100, 7.101); (b) we can write $\mathbf{T}_a = c(a)[w_a] + b(a)[1]$ as in (7.95), and $\mathcal{P}_v = \sum_{\tau \in W} C_\tau[\tau]$ as in (7.103), with the C_τ some (infinite) sum of rational functions. Then $\mathbf{T}_a \, \mathcal{P}_v$ is defined to be the expansion in $\mathcal{Q}_v[W]^\vee$ of

$$\sum_{\tau \in W} c(a) C_\tau^{w_a}[w_a \tau] + b(a) C_\tau[\tau], \tag{7.105}$$

where $C_\tau^{w_a}$ is the application of w_a to C_τ. It is easy to see that the procedure (a) gives the following relation (see [29, 5.5.9, p. 113]),

$$\mathbf{T}_a \, \mathcal{P}_v \quad = \quad v\mathcal{P}_v, \tag{7.106}$$

which shows that the expansion of (7.105) is well-defined (one could also proceed as in the proof of Lemma 7.3.11 to show that this expansion was well-defined).

To define $\mathcal{P}_v \, \mathbf{T}_a$ we proceed similarly. We can use the Hecke relations (7.100 7.101) from (a) above to conclude as in *op. cit* that

$$\mathcal{P}_v \, \mathbf{T}_a \quad = \quad v\mathcal{P}_v. \tag{7.107}$$

Or, we may proceed as in (b) and expand $\mathbf{T}_a$ as in (7.95) and $\mathcal{P}_v$ as in (7.103). In this case, we compute $\mathcal{P}_v \, \mathbf{T}_a$ as the expansion in $\mathcal{Q}_v[W]^\vee$ of the expression

$$\sum_{\tau \in W} C_\tau \, c(\tau a)[\tau w_a] + C_\tau \, b(\tau a)[\tau]. \tag{7.108}$$

The remainder of the proof will draw conclusions between, on the one hand (7.107) and (7.108), and on the other (7.106) and (7.105).

Step 2: First, let us say that an element of $\mathcal{Q}_v[W]^\vee$ is W-invariant if it is of the form

$$\sum_{w \in W} f^w[w] \tag{7.109}$$

for some $f \in \mathcal{Q}_v$ (in the sequel, f and f^w will be sums of rational functions which can be expanded in $\mathcal{Q}_v$). We want to show that $\mathcal{P}_v$ is W-invariant. Indeed, combining (7.106) and (7.105) we have the following equality in $\mathcal{Q}_v[W]^\vee$,

$$\sum_{\tau \in W} c(a) C_\tau^{w_a}[w_a \tau] + \sum_{\tau \in W} b(a) C_\tau[\tau] = \sum_{\tau \in W} v C_\tau[\tau]. \tag{7.110}$$

From (7.90), we have that $c(a) = v - b(a)$, and hence we may conclude that

$$\sum_{\tau \in W} c(a) C_\tau^{w_a}[w_a \tau] = \sum_{\tau \in W} c(a) C_\tau[\tau], \tag{7.111}$$

and hence

$$\sum_{\tau \in W} C_\tau^{w_a}[w_a \tau] = \sum_{\tau \in W} C_\tau[\tau]. \tag{7.112}$$

Letting $\Gamma = C_1$ (regarded as an infinite sum of rational functions) we conclude that $C_\tau = \Gamma^\tau$ for any $\tau \in W$. We write this formally as

$$\mathcal{P}_v = \sum_{w \in W} [w] \, \Gamma, \tag{7.113}$$

bearing in mind the above is to be interpreted as $\sum_{w \in W} \Gamma^w[w]$ where each Γ^w is expanded as an expression in $\mathcal{Q}_v$.

Step 3: From the definition of $\mathbf{T}_a$ we have an equality (in $\mathcal{Q}_v[W]$) $v\mathbf{T}_a + 1 = ([w_a] + 1)vc(-a)$. Hence, using (7.113) we may write

$$\begin{aligned} \mathcal{P}_v \, (v\mathbf{T}_a + 1) \quad &= \quad (\sum_{w} [w] \, \Gamma) \, (v\mathbf{T}_a + 1) = (\sum_{w \in W} [w] \, \Gamma) \, ([w_a] + 1) \, vc(-a) && (7.114) \\ &= \quad (\sum_{w \in W} [w] \, \Gamma) \, vc(-a) + (\sum_{w \in W} [w] \, \Gamma) \, [w_a] \, vc(-a) && (7.115) \\ &= \quad (\sum_{w \in W} [w] \, \Gamma) \, vc(-a) + (\sum_{w} [w][w_a] \, \Gamma^{w_a}) \, vc(-a) && (7.116) \end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathcal{P}_v(v^2+1) &= (\sum_{w\in W}[w]\,\Gamma)(v^2+1) && (7.117)\\
&= (\sum_{w\in W}[w]\,\Gamma)\,v(c(-a)+c(a)) && (7.118)\\
&= (\sum_{w\in W}[w]\,\Gamma)\,vc(-a)+(\sum_{w\in W}[w]\,\Gamma)\,vc(a) && (7.119)\\
&= (\sum_{w\in W}[w]\,\Gamma)\,vc(-a)+(\sum_{w\in W}[w][w_a]\,\Gamma)\,vc(a). && (7.120)
\end{aligned}$$

From (7.107), we see that (7.116) is equal to (7.120), and so we conclude that

$$\Gamma^{w_a}c(-a)=\Gamma c(a)\in\mathcal{Q}_v. \tag{7.121}$$

On the other hand, from the definition (7.33), we have that

$$\frac{\Delta^{w_a}}{\Delta}=\frac{c(a)}{c(-a)}. \tag{7.122}$$

And so we obtain that

$$\frac{\Gamma^{w_a}}{\Gamma}=\frac{\Delta^{w_a}}{\Delta}. \tag{7.123}$$

An induction then gives that for any $w\in W$ we have

$$\frac{\Gamma^{w}}{\Gamma}=\frac{\Delta^{w}}{\Delta}, \tag{7.124}$$

or in other words the element $\Gamma\Delta^{-1}\in\mathcal{Q}_v$ is W-invariant. Now, we may write

$$\mathcal{P}_v\,\Delta^{-1}=(\sum_{w\in W}[w]\,\Gamma)\,\Delta^{-1}=\sum_{w\in W}w\,(\Gamma\Delta^{-1}). \tag{7.125}$$

As $\Gamma\,\Delta^{-1}$ is W-invariant, we obtain that

$$\mathcal{P}_v\,\Delta^{-1}=\mathfrak{m}\sum_{w}[w] \tag{7.126}$$

where $\mathfrak{m}\in\mathcal{Q}_v$ is some W-invariant factor. □

Step 4: Rephrasal and Reassambly.

7.3.13. *Factorization of* $\mathcal{P}_v$. For each $\lambda^\vee\in\Lambda_+^\vee$, in analogy to the element $\mathcal{P}_v$ defined in (7.102), define the following two elements,

$$\mathcal{P}_{v,\lambda^\vee}=\textstyle\sum_{w\in W_{\lambda^\vee}}v^{\ell(w)}\mathbf{T}_w \quad\text{and}\quad \mathcal{P}_v^{\lambda^\vee}=\sum_{w\in W^{\lambda^\vee}}v^{\ell(w)}\mathbf{T}_w. \tag{7.127}$$

If $W_{\lambda^\vee}$ is infinite, then in fact it is equal to W and so $\mathcal{P}_{v,\lambda^\vee}=\mathcal{P}_v$, and $\mathcal{P}_v^{\lambda^\vee}=1$ in this case. On the other hand, if $W_{\lambda^\vee}$ is finite, then $\mathcal{P}_{v,\lambda^\vee}\in\mathcal{Q}_v[W]$, and $\mathcal{P}_v^{\lambda^\vee}$ (an infinite sum) is seen to be a v-finite element of $\mathcal{Q}_v[W]^\vee$ using Lemma 7.3.11. Using the defining property of $W^{\lambda^\vee}$ (7.44) and (7.100) we obtain factorizations in $\mathcal{Q}_v[W]^\vee$

$$\mathcal{P}_v=\mathcal{P}_v^{\lambda^\vee}\mathcal{P}_{v,\lambda^\vee} \tag{7.128}$$

where if $W_{\lambda^\vee}$ is infinite the above equality is a tautology.

7.3.14. *Applying $\mathcal{P}_v$.* Using the natural action of W on $\Lambda^\vee$ we may use formula (7.95) to define an action of $\mathbf{T}_a, a \in \Pi$ (and hence also $\mathbf{T}_w$ for $w \in W$ by induction) on $\mathbb{C}_v[\Lambda^\vee]$. For example, when $a \in \Pi$ we have that

$$\mathbf{T}_a(e^{\lambda^\vee}) = c(a)e^{w\lambda^\vee} + b(a)e^{\lambda^\vee}. \tag{7.129}$$

Using the above and (7.90), we see that if $w \in W_{\lambda^\vee}$ then

$$\mathbf{T}_w(e^{\lambda^\vee}) = v^{\ell(w)}e^{\lambda^\vee}. \tag{7.130}$$

Consider the formal applications of the elements $\mathcal{P}_v^{\lambda^\vee}$ and $\mathcal{P}_v$ to $e^{\lambda^\vee}$,

$$\mathcal{P}_v^{\lambda^\vee}(e^{\lambda^\vee}) := \sum_{w \in W^{\lambda^\vee}} v^{\ell(w)}\mathbf{T}_w(e^{\lambda^\vee}). \tag{7.131}$$

and

$$\mathcal{P}_v(e^{\lambda^\vee}) := \sum_{w \in W} v^{\ell(w)}\mathbf{T}_w(e^{\lambda^\vee}). \tag{7.132}$$

We have the following,

Proposition. *Let $\lambda^\vee \in \Lambda_+^\vee$. Then*

(1) *The element $\mathcal{P}_v^{\lambda^\vee}(e^{\lambda^\vee})$ defined in (7.131) is a v-finite expression in $\mathbb{C}_{v,\leq}[\Lambda^\vee]$.*
(2) *The element $\mathcal{P}_v(e^{\lambda^\vee})$ is a well-defined element in $\mathbb{C}_{v,\leq}[\Lambda^\vee]$ satisfying the relation,*

$$\mathcal{P}_v(e^{\lambda^\vee}) = W_{\lambda^\vee}(v^2)\mathcal{P}_v^{\lambda^\vee}(e^{\lambda^\vee}). \tag{7.133}$$

Proof. For part (1), we show that the coefficient of each $e^{\mu^\vee}$ with $\mu^\vee \in \Lambda^\vee$ is a polynomial in v^2. If $W_{\lambda^\vee} = W$ this is obvious. So suppose that $W_{\lambda^\vee}$ is finite. We have already argued that we may write $\mathcal{P}_v^{\lambda^\vee} = \sum_{\sigma \in W} C_\sigma[\sigma]$ with $C_\sigma \in \mathcal{Q}_v$ a v-finite element. Applying this to $e^{\lambda^\vee}$ we obtain

$$\mathcal{P}_v^{\lambda^\vee}(e^{\lambda^\vee}) = \sum_{\sigma \in W} C_\sigma e^{\sigma\lambda^\vee}. \tag{7.134}$$

For any fixed $\mu^\vee$, since $\lambda^\vee$ is dominant and $W_{\lambda^\vee}$ is finite, if $\ell(\sigma)$ is sufficiently large we have $\sigma\lambda^\vee < \mu^\vee$. Since each C_σ is an expansion in negative powers of the coroots, it follows that only finitely many terms in the above sum can contribute to each $e^{\mu^\vee}$, and the v-finiteness of (7.131) follows from that of C_σ.

As for part (2), note that

$$\mathcal{P}_{v,\lambda^\vee}(e^{\lambda^\vee}) := \sum_{w \in W_{\lambda^\vee}} v^{\ell(w)}\mathbf{T}_w(e^{\lambda^\vee}) = W_{\lambda^\vee}(v^2)e^{\lambda^\vee}. \tag{7.135}$$

If $W_{\lambda^\vee}$ is infinite so $W_{\lambda^\vee} = W$ and $W^{\lambda^\vee} = \{1\}$, then both sides of (7.133) are equal to $W(v^2)e^{\lambda^\vee}$. Suppose then that $W_{\lambda^\vee}$ is finite, then from (7.128) we have

$$\mathcal{P}_v(e^{\lambda^\vee}) = \mathcal{P}_v^{\lambda^\vee}\mathcal{P}_{v,\lambda^\vee}(e^{\lambda^\vee}) = \mathcal{P}_v^{\lambda^\vee}(W_{\lambda^\vee}(v^2)e^{\lambda^\vee}), \tag{7.136}$$

and so the result follows.

□

7.3.15. *p-adic connection.* The following result is the key to linking the algebraic and p-adic treatments of the spherical function.

Proposition. *Let $\lambda^\vee \in \Lambda_+^\vee$, and $w \in W^{\lambda^\vee}$. Then $J_w(\lambda^\vee)$ is the specialization of the elements $v^{-2\langle\rho,\lambda^\vee\rangle}v^{\ell(w)}\mathbf{T}_w(e^{\lambda^\vee})$ at $v^2 = q^{-1}$ i.e.,*

$$q^{\langle\rho,\lambda^\vee\rangle}q^{-\ell(w)/2}\mathbf{T}_w(e^{\lambda^\vee}) = J_w(\lambda^\vee). \tag{7.137}$$

Proof. We may proceed by an induction on $\ell(w)$. If $\ell(w) = 0$ then $w = 1$ and $J_w(\lambda^\vee) = e^{\lambda^\vee} q^{\langle \rho, \lambda^\vee \rangle}$ by Proposition 7.3.7 (2) and the result follows. Assume then that $\ell(w) > 0$, and choose some simple reflection $a \in \Pi$ such that $w' := w_a w$ has length less than w. Then by the remarks at the end of §7.3.1, we also have $w' \in W^{\lambda^\vee}$. The result now follows from an induction using the definition of $\mathbf{T}_w$ and Proposition 7.3.7 (3). □

7.3.16. *Relating* $\mathcal{P}_v^{\lambda^\vee}(e^{\lambda^\vee})$ *and* $S(h_{\lambda^\vee})$. Using (7.96), we may write

$$v^{\ell(w)} \mathbf{T}_w(e^{\lambda^\vee}) = \sum_{\mu^\vee \in \Lambda^\vee} b_{w,\mu^\vee}(v^2) e^{\mu^\vee}, \tag{7.138}$$

with $b_{w,\mu^\vee}(v^2) \in \mathbb{C}[v^2]$. For fixed $\mu^\vee$, set

$$b_{\mu^\vee}(v^2) := \sum_{w \in W^{\lambda^\vee}} b_{w,\mu^\vee}(v^2). \tag{7.139}$$

Part (1) of Proposition 7.3.14 states that $b_{\mu^\vee}(v^2) \in \mathbb{C}[v^2]$. We thus write

$$\mathcal{P}_v^{\lambda^\vee}(e^{\lambda^\vee}) = \sum_{w \in W^{\lambda^\vee}} v^{\ell(w)} \mathbf{T}_w(e^{\lambda^\vee}) = \sum_{\mu^\vee \in \Lambda^\vee} b_{\mu^\vee}(v^2) e^{\mu^\vee}. \tag{7.140}$$

Proposition. *The expression* $q^{-\langle \rho, \lambda^\vee \rangle} S(h_{\lambda^\vee})$ *is the specialization of* $\mathcal{P}_v^{\lambda^\vee}(e^{\lambda^\vee})$ *at* $v^2 = q^{-1}$. *In other words, for each* $\mu^\vee \in \Lambda^\vee$, $b_{\mu^\vee}(q^{-1})$ *is finite and equal to the* $e^{\mu^\vee}$*-coefficient of* $q^{-\langle \rho, \lambda^\vee \rangle} S(h_{\lambda^\vee})$.

Proof. Recall the elements $\Phi_{w,\mu^\vee} \in \mathbb{Q}_{\geq 0}$ and the polynomials $\Phi_{w,\mu^\vee}(\cdot)$ from Proposition 7.3.7 (1) and (4) respectively. From Proposition 7.3.15, we know that

$$b_{w,\mu^\vee}(q^{-1}) = q^{-\langle \rho, \lambda^\vee \rangle} \Phi_{w,\mu^\vee} \tag{7.141}$$

for any q a power of a prime. Hence from Proposition 7.3.7 (4), we may conclude that

$$b_{w,\mu^\vee}(q^{-1}) = q^{-\langle \rho, \lambda^\vee \rangle} \Phi_{w,\mu^\vee}(q^{-1}) \tag{7.142}$$

holds for every q. Thus as polynomials, we have

$$b_{w,\mu^\vee}(v^2) = v^{2\langle \rho, \lambda^\vee \rangle} \Phi_{w,\mu^\vee}(v^2). \tag{7.143}$$

Setting,

$$\Phi_{\mu^\vee}(v^2) = \sum_{w \in W^{\lambda^\vee}} \Phi_{w,\mu^\vee}(v^2) \in \mathbb{C}[v^2, v^{-2}], \tag{7.144}$$

we have from (7.139) that

$$b_{\mu^\vee}(v^2) = v^{2\langle \rho, \lambda^\vee \rangle} \Phi_{\mu^\vee}(v^2) \tag{7.145}$$

as elements in $\mathbb{C}[[v^2]]$. But in fact, the above is an equality in $\mathbb{C}[v^2]$ since the left hand side lies in this smaller ring.

Now, from Step 2 (see (7.61)) we have written $S(h_{\lambda^\vee}) = \sum_{w \in W^{\lambda^\vee}} J_w(\lambda^\vee)$ as elements in $\mathbb{C}_{\leq}[\Lambda^\vee]$, which can be further written using Proposition 7.3.7 (4) as

$$S(h_{\lambda^\vee}) = \sum_{w \in W^{\lambda^\vee}} \sum_{\mu^\vee \in \Lambda^\vee} \Phi_{w,\mu^\vee}(q^{-1}) e^{\mu^\vee}. \tag{7.146}$$

Since each $\Phi_{w,\mu^\vee}(q^{-1}) \in q^{\langle \rho, \mu^\vee \rangle} \mathbb{Z}_{\geq 0}$ and $q^{-\langle \rho, \mu^\vee \rangle}$ times the coefficient of each $e^{\mu^\vee}$ in $S(h_{\lambda^\vee})$ is a finite positive number, we may conclude that for any fixed $\mu^\vee$ there are only finitely many $w \in W^{\lambda^\vee}$ such that $\Phi_{w,\mu^\vee}(q^{-1}) \neq 0$. [11] Thus, the sum $\Phi_{\mu^\vee}(q^{-1}) = \sum_{w \in W^{\lambda^\vee}} \Phi_{w,\mu^\vee}(q^{-1})$ is finite for any $\mu^\vee$, and we may write

$$S(h_{\lambda^\vee}) = \sum_{\mu^\vee \in \Lambda^\vee} \Phi_{\mu^\vee}(q^{-1}) e^{\mu^\vee}. \tag{7.147}$$

[11] Note that we could also conclude this from the fact that only finitely many $b_{w,\mu^\vee}(v^2) \neq 0$.

On the other hand, using (7.145), we may also write

$$S(h_{\lambda^\vee}) = \sum_{\mu^\vee \in \Lambda^\vee} \Phi_{\mu^\vee}(q^{-1}) e^{\mu^\vee} = q^{\langle \rho, \lambda^\vee \rangle} \sum_{\mu^\vee \in \Lambda^\vee} b_{\mu^\vee}(q^{-1}) e^{\mu^\vee}. \tag{7.148}$$

□

From (7.133) and Cherednik's identity Proposition 7.3.12, we note the following equality in $\mathbb{C}_{v,\leq}[\Lambda^\vee]$

$$\mathcal{P}_v^{\lambda^\vee}(e^{\lambda^\vee}) = \frac{1}{W_{\lambda^\vee}(v^2)} \mathcal{P}_v(e^{\lambda^\vee}) = \frac{\mathfrak{m}}{W_{\lambda^\vee}(v^2)} \sum_{w \in W} \Delta^w e^{w\lambda^\vee}. \tag{7.149}$$

Hence we obtain from the previous Proposition that the expression $q^{-\langle \rho, \lambda^\vee \rangle} S(h_{\lambda^\vee})$ is the specialization at $v^2 = q^{-1}$ of

$$\frac{\mathfrak{m}}{W_{\lambda^\vee}(v^2)} \sum_{w \in W} \Delta^w e^{w\lambda^\vee}. \tag{7.150}$$

We shall simply write this as

$$S(h_{\lambda^\vee}) = q^{\langle \rho, \lambda^\vee \rangle} \frac{\mathfrak{m}}{W_{\lambda^\vee}(q^{-1})} \sum_{w \in W} \Delta^w e^{w\lambda^\vee}. \tag{7.151}$$

7.3.17. *Determining* $\mathfrak{m}$. Finally it remains to determine $\mathfrak{m}$ (which is independent of $\lambda^\vee$). This is achieved by evaluating both sides of (7.151) at $\lambda^\vee = 0$ (and $v^2 = q^{-1}$.) By definition of convolution, we must have $S(h_0) = S(\mathbf{1}_K) = 1$, and so (7.151) reduces to

$$1 = \frac{\mathfrak{m}}{W(q^{-1})} \sum_{w \in W} \Delta^w. \tag{7.152}$$

We then have $\mathfrak{m} = H_0^{-1}$ and using (7.37), and the proof of the theorem is concluded.

A. The Cartan Semigroup

This appendix is devoted to the study of the semigroup G_+ of §1.2.2, and we will give a proof of Theorem 3.3 here. We begin by defining the following subset of G, which we aim to show is equal to G_+,

$$G'_+ := \cup_{\lambda^\vee \in \Lambda^\vee_+} K \pi^{\lambda^\vee} K. \tag{A.1}$$

In order to work effectively with this set one needs to verify that G'_+ is in fact a semi-group. In the process of showing this, we shall also see that G'_+ is in fact equal to G_+. The techniques which we employ here are based on Garland [15], with an simplification stemming from the work in [2, Lemma 3.3].

A.1. **The Semigroup of Bounded Elements.** Fix the notation as in §2.2.3, and consider the subset of G defined as follows

$$G_b := \{ g \in G \mid \max_{v \in V^\omega_{\mathcal{O}}} ||gv|| < \infty \text{ for every highest weight representation } V^\omega, \omega \in \Lambda_+ \}. \tag{A.2}$$

Note that if $g \in G_b$ and v_ω is a primitive highest weight vector of V^ω then we also have that

$$\max_{k \in K} ||gkv_\omega|| < \infty \tag{A.3}$$

since $kv_\omega \in V^\omega_{\mathcal{O}}$. This shows that G_b is right K-invariant. In fact, G_b is K bi-invariant, as the left K-invariance follows from the way we have defined the norms $|| \cdot ||$ on V^ω. We call G_b the set of *bounded* elements of G, and observe the following simple result.

Lemma. *The set G_b is a semigroup.*

Proof. Indeed, let $g_1, g_2 \in G_b$. Fix a highest weight representation V^ω and suppose that there exists an integer M such that $||g_2 v|| \leq q^M$ for every $v \in V^\omega_{\mathcal{O}}$. In other words, $g_2 V^\omega_{\mathcal{O}} \subset \pi^{-M} V^\omega_{\mathcal{O}}$, and hence $||g_1 g_2 v|| \leq q^M ||g_1 v||$ for any $v \in V^\omega_{\mathcal{O}}$. □

A.2. **Relation of G_b to the Tits Cone.** Let us next record the relation between G_b and the Tits cone.

Proposition. *Let $\lambda^\vee \in \Lambda^\vee$. We have that $\pi^{\lambda^\vee} \in G_b$ if and only if $\lambda^\vee \in -X$.*

Proof. Suppose first that $\lambda^\vee = -\lambda^\vee_+$ for $\lambda^\vee_+ \in \Lambda^\vee_+$. Fix a highest weight representation V^ω with weight lattice P_ω. Then for any $v \in V^\omega_{\mathcal{O}}$ belonging to the weight space $\mu \in P_\omega$ we have

$$||\pi^{\lambda^\vee} v|| \leq q^{-\langle \mu, \lambda^\vee \rangle}. \tag{A.4}$$

On the other hand, every weight $\mu \in P_\omega$ is of the form $\mu = \omega - \beta$ where $\beta \in Q_+$. Hence,

$$\langle \mu, \lambda^\vee \rangle = -\langle \omega, \lambda^\vee_+ \rangle + \langle \beta, \lambda^\vee_+ \rangle. \tag{A.5}$$

Thus $\langle \mu, \lambda^\vee \rangle$ is bounded below and so $\pi^{-\lambda^\vee_+} \in G_b$. By K-binvariance, we also have that $\pi^{-w\lambda^\vee_+} \in G_b$ for any $w \in W$. Conversely, if $\pi^{\lambda^\vee} \in G_b$ then the same argument as above shows that $-\langle \beta, \lambda^\vee \rangle$ is bounded below as $\beta \in Q_+$ varies over the same set as above. From this one can conclude that $\lambda^\vee$ must be in $-X$. □

From the fact that G_b is K-binvariant, we obtain

Corollary. *Let $\lambda^\vee \in \Lambda^\vee$. Then the coset $K\pi^{\lambda^\vee} K \in G_b$ if and only if $\lambda^\vee \in -X$.*

A.3. **Relation between G_b and G'_+.** Our next goal is to relate G_b and the set G'_+ defined in (A.1). To do so, it is convenient to also define

$$G'_- := (G'_+)^{-1} := \cup_{\lambda^\vee \in \Lambda^\vee_+} K \pi^{-\lambda^\vee} K. \tag{A.6}$$

Proposition. *We have an equality of sets (and hence semigroups) $G'_- = G_b$, and hence also $G'_+ = G_b^{-1}$.*

Proof. The second statement follows immediately from the first. Note that as both G'_- and G_b are K-invariant sets, the fact that $G'_- \subset G_b$ follows immediately from Lemma A.2. It remains to show that $G_b \subset G'_-$. To do this, it suffices to show that every element of $g \in G_b$ has an expression $g = k_1 \pi^{\mu^\vee} k_2$. Indeed, if this were the case, then by Lemma A.2 above (and using K-binvariance), we necessarily have that $\mu^\vee \in -X$. So the proof of the Proposition can be concluded from the following result, an alternate proof of which can be found in [15, (2.8)].

Lemma. *Let $g \in G_b$. Choose $k \in K$ which maximizes the norm $||gkv_\omega||$ for some fixed representation V_ω with primitive highest weight vector v_ω. Then if we write $gk = k_1 a u$ in terms of its Iwasawa coordinates, we must have $u \in K$.*

Proof of Lemma A.3. Recall from [2, §3, and especially (3.2)] that we had a decomposition of U^- into *disjoint* subsets U^-_w where $U^-_1 = U^-_{\mathcal{O}}$ and in general, if $u^- \in U^-_w$ then it has an expression

$$u^- = k\pi^{-\mu^\vee} u, \quad \mu^\vee \geq 0, \quad |\mu^\vee| \geq l(w)/2. \ ,$$

where $|\mu^\vee| = \langle \rho, \mu^\vee \rangle$. From this it follows that

$$U^- \cap KU \subset K \cap U^- \quad \text{and} \quad U \cap KU^- \subset K \cap U. \tag{A.7}$$

Now given g and k as in the lemma, assume that $gk = k_1 a u$ with $u \notin K$. Then write an opposite Iwasawa decomposition (in terms of $G = U^- AK$) for u : i.e.,

$$u = u^- \pi^{\xi^\vee} k_2. \tag{A.8}$$

By the above, it follows that $\xi^\vee \neq 0$. In fact $\xi^\vee < 0$ actually: we also know from [2, Theorem 1.9 (3)] that

$$KU^- \cap K\pi^{\lambda^\vee}U \neq \emptyset$$

implies that $\lambda^\vee \leq 0$, just take inverses in this expression. So we have

$$\begin{aligned} ||gkk_2^{-1}v_\omega|| &= ||k_1auk_2^{-1}v_\omega|| && \text{(A.9)} \\ &= ||au^-\pi^{\xi^\vee}v_\omega|| && \text{(A.10)} \\ &\geq q^{-\langle \omega, \xi^\vee \rangle}||av_\omega|| = q^{-\langle \omega, \xi^\vee \rangle}||gkv_\omega|| && \text{(A.11)} \end{aligned}$$

contradicting the original choice of k, since $\langle \omega, \xi^\vee \rangle < 0$. Hence $u \in K$. □

The proof of the Proposition is thus also completed. □

A.4. **Relating G'_+ and G_+ (Proof of Theorem 3.3).** In the introduction §1.2.2, we defined a semi-group G_+. Recall again that we have defined a map $|\eta| : G \to \mathbb{Z}$ in (1.8). Recall that η was defined with respect to the description of G as a semi-direct product $G = \mathcal{K}^* \ltimes G'$ by projection onto the $\mathcal{K}^*$ factor and then composing with the valuation map $\mathcal{K}^* \to \mathbb{Z}$. Writing an element $g \in G$ with respect to the Iwasawa decomposition $g = u\pi^{\lambda^\vee}k$ where $u \in U, \lambda^\vee \in \Lambda^\vee, k \in K$ one can easily verify that

$$|\eta(g)| = |\eta(\pi^{\lambda^\vee})| = \langle \delta, \lambda^\vee \rangle, \tag{A.12}$$

where δ was the minimal positive imaginary root.

Proposition. *The semigroup G'_+ is equal to G_+, the semigroup defined in (1.2.2).*

Proof. If $\lambda^\vee \in \Lambda_+$, then using (2.32) one can verify that $\langle \delta, \lambda^\vee \rangle \geq 0$, and that if $\langle \delta, \lambda^\vee \rangle = 0$, then $\lambda^\vee = n\mathbf{c}$. From this it follows easily that $G'_+ \subset G_+$.

As for the opposite inclusion, it is clear that $K \subset G'_+$. To show that central $\mathcal{K}^* \subset G$ is contained in G'_+ we proceed as follows. If $\sigma^{\mathbf{c}}$ is such a central element with $\sigma \in \mathcal{K}^*$, $\sigma = \pi^m u$ with u a unit and $m \in \mathbb{Z}$, then $u^{\mathbf{c}} \in K$ and $m\mathbf{c} \in \Lambda_+$. Thus $\sigma^{\mathbf{c}} = u^{\mathbf{c}}\pi^{m\mathbf{c}} \in G'_+$.

It remains to show that every $g \in G$ with $|\eta(g)| > 0$ lies in G'_+. For any such g write an Iwasawa decomposition $g = u\pi^{\mu^\vee}k$ with $k \in K$, $\mu^\vee \in \Lambda^\vee$ and $u \in U$. As observed above, $|\eta(g)| = |\eta(\pi^{\mu^\vee})| = \langle \delta, \mu^\vee \rangle > 0$. Consider now the element $g^{-1} = k^{-1}\pi^{-\mu^\vee}u^{-1}$. We may conclude that $g^{-1} \in G_b$ from the following result of Garland,

Theorem. [15, Theorem 1.7] *The set of element $K\pi^{-\mu^\vee}U$ with $\langle \delta, \mu^\vee \rangle > 0$ lie in G_b.*

Thus for each $g \in G$ with $|\eta(g)| > 0$ we have shown that $g^{-1} \in G_b$ and so $g \in G'_+$ using Proposition A.3.

□

B. The "affine" Root System and the Bruhat pre-order on $\mathcal{W}$

The goal of this appendix is to introduce a notion of an "affinized" root system attached to a Kac-Moody root system and study some of its basic properties.

B.1. **"Affine" Roots.** Recall that R_{re} was the set of real roots of $\mathfrak{g}$. Let us define four subsets of the set of *"affine" roots* $\mathcal{R} := R_{re} \times \mathbb{Z}$ as follows,

$$\begin{aligned} \mathcal{R}^+_+ &:= \{(a,k) \in R_{re} \times \mathbb{Z} : a > 0, k \geq 0\} && \text{(B.1)} \\ \mathcal{R}^+_- &:= \{(a,k) \in R_{re} \times \mathbb{Z} : a > 0, k < 0\} && \text{(B.2)} \\ \mathcal{R}^-_+ &:= \{(a,k) \in R_{re} \times \mathbb{Z} : a < 0, k > 0\} && \text{(B.3)} \\ \mathcal{R}^-_- &:= \{(a,k) \in R_{re} \times \mathbb{Z} : a < 0, k \leq 0\} && \text{(B.4)} \end{aligned}$$

Note that upper indices shall refer to Kac-Moody parameters, and lower ones to the local field. We also define the set of *positive and negative "affine" roots* as

$$\mathcal{R}_+ := \mathcal{R}_+^+ \cup \mathcal{R}_+^- \quad \text{and} \quad \mathcal{R}_- := \mathcal{R}_-^+ \cup \mathcal{R}_-^-. \tag{B.5}$$

We shall sometimes write write $\alpha = a + k\pi$, with $a \in R_{re}$ and $k \in \mathbb{Z}$ to denote the pair $(a, k) \in \mathcal{R}$.

There is a left action of $\mathcal{W}$ (see (3.17)) on $\mathcal{R}$ defined through the formula

$$w\pi^{\lambda^\vee}.(a + k\,\pi) = w \cdot a + (\langle \lambda^\vee, a\rangle + k)\pi \tag{B.6}$$

where $a \in R_{re}, k \in \mathbb{Z}$ and $w\pi^{\lambda^\vee} \in \mathcal{W}$.

In the usual setting of Coxeter groups, the length of an element $x \in \mathcal{W}$ could be defined as the size of the set $x\mathcal{R}_+ \cap \mathcal{R}_-$ or $x\mathcal{R}_- \cap \mathcal{R}_+$. In our setting, however these sets are not finite in general. On the other hand, if we restrict to $\mathcal{W}_X$ the following is true.

Proposition. *Let $x \in \mathcal{W}_X$. Then the following sets are finite*

$$x\mathcal{R}_+^+ \cap \mathcal{R}_- \ \ and \quad \mathcal{R}_+^- \cap \quad x\mathcal{R}_- \tag{B.7}$$

Proof. Let us prove the first statement, the proof of the second being similar. Write $x = w\pi^{\lambda^\vee}$ for $\lambda^\vee \in X$, and let $a + k\pi \in \mathcal{R}_+^+$ (so $a > 0, k \geq 0$). If $x.(a + k\pi) \in \mathcal{R}_-$ then we must have from (B.6) that either,

$$wa > 0, \quad \text{and} \quad \langle \lambda^\vee, a\rangle + k < 0 \tag{B.8}$$

$$wa < 0 \quad \text{and} \quad \langle \lambda^\vee, a\rangle + k \leq 0 \tag{B.9}$$

Since $\lambda^\vee \in X$ for any integer n, the number of roots $a > 0$ such that $\langle \lambda^\vee, a\rangle < n$ is finite in number, and so there are only finitely many a which satisfy either of the above two conditions. On the other hand, for any such a fixed a there are only finitely many k such that either equation will be satisfied. The finiteness required follows. □

B.2. **Another pre-order on $\mathcal{W}$.** We shall also consider an action of $\mathcal{W}$ on the right on $\mathcal{R}$,

$$(a + n\pi).\pi^{\lambda^\vee} w := w^{-1}(a) + (n - \langle \lambda^\vee, a\rangle)\pi, \tag{B.10}$$

where $a \in R_{re}, n \in \mathbb{Z}$. Let us also introduce the following simple elements in $\mathcal{W}$, attached to $\alpha = a + n\pi \in \mathcal{R}$,

$$w_\alpha := w_a(n) := \pi^{na^\vee} w_a \tag{B.11}$$

which satisfy

$$w_\alpha \pi^{\lambda^\vee} w = w_a(n)\pi^{\lambda^\vee} w = \pi^{w_a\lambda^\vee + na^\vee} w_a w. \tag{B.12}$$

Definition. *Given $x \in \mathcal{W}$ and $\alpha = a + n\pi \in \mathcal{R}$ we say that α is x*-positive *or x*-negative *if $\alpha.x \in \mathcal{R}_\pm$. For $x, y \in \mathcal{W}$, we shall say that $y \leq_B x$ if there exists $\alpha_i := a_i + n_i\pi \in \mathcal{R}$ for $i = 1, \ldots, k$ such that $y = w_{\alpha_k} \cdots w_{\alpha_1} x$, and*

$$\alpha_1 \textit{ is } x\textit{-negative and} \alpha_j \textit{ is } w_{\alpha_{j-1}} \cdots w_{\alpha_1} x\textit{-negative for } j = 2, \ldots, k. \tag{B.13}$$

Remarks: We do not know whether the relation $\leq_B$ is an order: it is clear that it is a pre-order, namely if $x \leq_B y$ and $y \leq_B z$, then $x \leq_B z$ for $x, y, z \in \mathcal{W}$. However, we do not know whether $x \leq_B y$ and $y \leq_B x$ implies that $x = y$. It would also be interesting to understand: a) the structure of the set of elements which are $\leq_B$ to a fixed $x \in \mathcal{W}$; and b) the relation of $\leq_B$ and $\preceq$ of Definition 3.4.4.

B.3. **The order $\leq_B$ and Iwahori intersections.** Let $x \in \mathcal{W}_X$ and $y \in \mathcal{Y}$. Then we have seen in Proposition 3.4.4 that $\leq$ arises naturally when one considers the intersection $UyI \cap IxI$. We also have the following result which relates these same intersections to $\leq_B$,

Proposition. *Let $x \in \mathcal{W}_X$ and $y \in \mathcal{W}$ be such that $IxI \cap UyI \neq \emptyset$. Then $y \leq_B x$.*

Proof. As $I = U_{\mathcal{O}} U_\pi^- A_{\mathcal{O}}$ we first note

$$IxI \cap UyI \neq \emptyset \iff U_\pi^- xI \cap UyI \neq \emptyset. \tag{B.14}$$

Let us write $x = \pi^{\lambda^\vee}\sigma$, $\sigma \in W$ and $\lambda^\vee = w\lambda_+^\vee$ with $\lambda_+^\vee \in \Lambda_+^\vee$ and $w \in W$ (we may do this since we assumed that $x \in \mathcal{W}_X$.)

Step 1: Reduction to a Finite Dimensional Problem. The first thing we would like to show is that in analyzing the intersection (B.14) above, we may actually replace U_π^- by a subgroup which is a product of finitely many root groups. To explain this, we break up

$$R_{-,re} = R_{-,w} \sqcup R_-^w \tag{B.15}$$

where $R_{-,w}$ and R_-^w are characterized by the conditions

$$w^{-1}R_{-,w} \subset R_+ \quad \text{and} \quad w^{-1}R_-^w \subset R_- \ . \tag{B.16}$$

Corresponding to the above decomposition, we have a product decomposition $U^- = U_{-w}U_-^w$, where

$$U_{-w} := \textstyle\prod_{\beta \in R_{-,w}} U_\beta \quad \text{and} \quad U_-^w := U^- \cap w^{-1}U^- w \ . \tag{B.17}$$

This in turn implies a decomposition,

$$U_\pi^- = U_{-w,\pi}U_{-,\pi}^w. \tag{B.18}$$

Now suppose we are given $\beta \in R_-^w$ so that $\chi_\beta(s) \in U_{-,\pi}^w$ with $s \in \mathcal{K}$ such that its valuation $\mathsf{val}(s) = \ell$ (so necessarily $\ell \geq 1$.) Then

$$\sigma^{-1}\pi^{-\lambda^\vee}\chi_\beta(s)\pi^{\lambda^\vee}\sigma = \chi_{\sigma^{-1}(\beta)}(\pi^{\langle \lambda^\vee, -\beta\rangle}s), \tag{B.19}$$

where

$$\mathsf{val}(\pi^{\langle\lambda^\vee, -\beta\rangle}s) = \langle \lambda^\vee, -\beta\rangle + \ell. \tag{B.20}$$

Recalling that $\lambda^\vee = w\lambda_+^\vee$ we have

$$\langle \lambda^\vee, -\beta\rangle = -\langle \lambda_+^\vee, w^{-1}\beta\rangle \geq 0 \tag{B.21}$$

since $w^{-1}\beta \in R_-$ (as $\beta \in R_-^w$). Hence $\chi_{-\sigma^{-1}(\beta)}(\pi^{\langle -\lambda^\vee, \beta\rangle}s) \in I$ since $-\langle \lambda^\vee, \beta\rangle + \ell \geq \ell \geq 1$. So, we have seen that

$$UyI \cap U_\pi^- xI \neq \emptyset \iff U_{-w,\pi}\pi^{\lambda^\vee}\sigma I \cap UyI \neq \emptyset. \tag{B.22}$$

We shall now study the intersection problem (B.22) where we replace the integral group $U_{-w,\pi}$ with the larger group $U_{-w,\mathcal{K}} = U_{-w}$, i.e., we are now analyzing the problem of when

$$U_{-w}\pi^{\lambda^\vee}\sigma I \cap UyI \neq \emptyset. \tag{B.23}$$

This is a problem more tractable to a "Gindikin-Karpelevic"-type induction.

Step 2: Some finite "Gindikin-Karpelevic"-combinatorics. Before proceeding further, we recall some simple combinatorial facts about the group U_{-w}. It is easy to see that $R_{-,w} = R_- \cap wR_+$. Further, if $w = w_{a_1}\cdots w_{a_r}$ is a reduced decomposition with $a_i \in \Pi$, then

$$\begin{aligned} R_{-,w} &= \{-a_1, -w_{a_1}(a_2), \cdots, -w_{a_1}\cdots w_{a_{r-1}}(a_r)\} && \text{(B.24)} \\ &= R_{-,w'} \cup \{-w_{a_1}\cdots w_{a_{r-1}}(a_r)\} && \text{(B.25)} \end{aligned}$$

where $w' = ww_{a_r} = w_{a_1}\cdots w_{a_{r-1}}$. Let $\gamma = w_{a_1}\cdots w_{a_{r-1}}(a_r)$. Then we have,

Claim. *Suppose that* $x_\gamma \in U_\gamma$ *and* $u_{w'} \in U_{w'}$*. Then*

$$x_\gamma^{-1} u_{-w'} x_\gamma \in U(w') U_{-,w'}$$

where $U(w') := w' U w'^{-1} \cap U \subset U$

Proof. Note that $w'Uw'^{-1} = U(w')U^-(w')$ where we set $U^\pm(w') := w'Uw'^{-1} \cap U^\pm$. Now $x_\gamma \in U(w')$ since $\gamma = w'(\alpha_r)$. Furthermore, $U^-(w') = U_{-,w'}$ as is easily verified. □

Step 3: Relation to positivity. We may write $u_{-w} \in U_{-w,k}$ as

$$u_{-w} = u_{-w'} u_{-\gamma} \tag{B.26}$$

where $u_{-w'} \in U_{-w'}$ and $u_{-\gamma} = x_{-\gamma}(s) \in U_{-\gamma}$ with $\mathsf{val}(s) = \ell$. Then there are two possibilities,

- **(a):** We have $\sigma^{-1}\pi^{-\lambda^\vee} x_{-\gamma}(s) \pi^{\lambda^\vee}\sigma \in I$. Equivalently, $-\gamma + \ell\pi$ is $\pi^{\lambda^\vee}\sigma$-positive.
- **(b):** We have $\sigma^{-1}\pi^{-\lambda^\vee} x_{-\gamma}(s) \pi^{\lambda^\vee}\sigma \notin I$. Equivalently, $-\gamma + \ell\pi$ is $\pi^{\lambda^\vee}\sigma$-negative.

If we are in case (b), we may rewrite using (2.52)

$$x_{-\gamma}(s) = x_\gamma(s^{-1})(-s)^{\gamma^\vee}\, w_\gamma\, x_\gamma(s^{-1}) \tag{B.27}$$

and so

$$u_{-w'} x_{-\gamma}(s)\pi^{\lambda^\vee}\sigma I \;=\; u_{-w'} x_\gamma(s^{-1})(-s)^{\gamma^\vee} w_\gamma x_\gamma(s^{-1})\pi^{\lambda^\vee}\sigma I \tag{B.28}$$

$$=\; u_{-w'} x_\gamma(s^{-1})\pi^{\ell\gamma^\vee} w_\gamma \pi^{\lambda^\vee}\sigma I \tag{B.29}$$

where we have used the condition (b) in the last line. But using the Claim above, the last expression may be written in the form

$$U(w')\tilde{u}_{-w'} w_\gamma(l)\pi^{\lambda^\vee}\sigma I, \tag{B.30}$$

where we recall our notation that $w_\gamma(\ell) = \pi^{\ell\gamma^\vee} w_\gamma$. In summary, we have shown the following,

Lemma. *In the notation above, if* $u_{-w} \in U_{-w}$ *is such that* $u_{-w}\pi^{\lambda^\vee}\sigma I \in UyI$*, then either:*

- **a:** *we have* $u_{-w'}\pi^{\lambda^\vee}\sigma I \in UyI$*, or*
- **b:** *we have* $u_{-w'} w_\gamma(\ell)\pi^{\lambda^\vee}\sigma I \in UyI$*.*

Furthermore, the case (b) occurs if $-\gamma + l\pi$ *is* $\pi^{\lambda^\vee}\sigma$*-negative.*

The proof of Proposition B.3 follows from an easy induction using this Lemma.

□

Remark. Actually, the proof above shows that if $x = \pi^{\lambda^\vee}\sigma$ with $\lambda^\vee \in \Lambda_+^\vee$ and $\sigma \in W$, then $IxI \cap UyI \neq \emptyset$ implies that $x = y$. Indeed, in this case $w = 1$ (in the notation of the proof), so $R_{-w} = \emptyset$ and so $U_{-w} = \{1\}$ and (B.23) implies that $y = \pi^{\lambda^\vee}\sigma$.

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