Dedicated to Igor Frenkel on the occasion of his 60th birthday

AN AFFINE GINDIKIN-KARPELEVICH FORMULA

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Abstract. In this paper we give an elementary proof of certain finiteness results about affine Kac-Moody groups over a local non-archimedian field $\mathcal{K}$. Our results imply those proven earlier in [5],[4] and [16] using either algebraic geometry or a Kac-Moody version of the Bruhat-Tits building.

The above finiteness results allow one to formulate an affine version of the *Gindikin-Karpelevich formula*, which coincides with the one discussed in [4] in the case when $\mathcal{K}$ has positive characteristic. We deduce this formula from an affine version of the *Macdonald formula for the spherical function*, which will be proved in a subsequent publication.

1. Introduction

1.1. **Notations.** Let $\mathcal{K}$ denote a local non-archimedian field with ring of integers $\mathcal{O}$ and residue field $\Bbbk$. Let π denote a generator of the maximal ideal of $\mathcal{O}$, and q denote the size of the residue field $\mathcal{O}/\pi\mathcal{O}$.

Usually we shall denote algebraic varieties over $\mathcal{K}$ (or a subring of $\mathcal{K}$) by boldface letters $\mathbf{X}, \mathbf{G}$ etc.; their sets of $\mathcal{K}$-points will then be denoted X, G etc.

Let $\mathbf{G}$ be a split, semi-simple, and simply connected algebraic group (defined over $\mathbb{Z}$) and let $\mathfrak{g}$ be its Lie algebra. [1] As agreed above, we set $G = \mathbf{G}(\mathcal{K})$. In this Subsection we recall the usual Gindikin-Karpelevich formula for G. Let $\mathbf{T} \subset \mathbf{G}$ be a maximal split torus; we denote its character lattice by Λ and its cocharacter lattice by $\Lambda^\vee$; note that since we have assumed that $\mathbf{G}$ is simply connected, $\Lambda^\vee$ is also the coroot lattice of $\mathbf{G}$. Given an element $x \in \mathcal{K}^*$ we set $x^{\lambda^\vee} = \lambda^\vee(x) \in T$.

Let us choose a pair $\mathbf{B}, \mathbf{B}^-$ of opposite Borel subgroups such that $\mathbf{B} \cap \mathbf{B}^- = \mathbf{T}$. We denote by $\mathbf{U}, \mathbf{U}^-$ their unipotent radicals. We denote by R the set of roots of $\mathbf{G}$ and by $R^\vee$ the set of coroots. Similarly R_+ (resp. $R^\vee_+$) will denote the set of positive roots (resp. of positive coroots). We shall also denote by 2ρ (resp. by $2\rho^\vee$) the sum of all positive roots (resp. of all positive coroots). For any $\gamma \in \Lambda^\vee$ set $|\lambda^\vee| = \langle \lambda^\vee, \rho\rangle$. Note that if $\lambda^\vee = \sum n_i \alpha_i^\vee$ where $\alpha_i^\vee$ are simple coroots, then $|\lambda^\vee| = \sum n_i$. We let W denote the Weyl group of G.

In addition we let $K = \mathbf{G}(\mathcal{O})$. This is a maximal compact subgroup of G.

For a finite set X, we shall denote by $|X|$ the number of elements in X.

1.2. **Gindikin-Karpelevich formula.** It is well-known that

$$G = \bigsqcup_{\lambda^\vee \in \Lambda^\vee_+} K \pi^{\lambda^\vee} K.$$

[1]The reader should be warned from the very beginning that later we are going to change this notation; in particular, later in the paper $\mathbf{G}$ will denote the corresponding affine Kac-Moody group

For any $\lambda^\vee \in \Lambda^\vee$ we set

$$\mathbf{c}_{\mathfrak{g}} = \sum_{\mu^\vee \in \Lambda^\vee} |K \setminus K\pi^{\lambda^\vee} U^- \cap K\pi^{\lambda^\vee - \mu^\vee} U| e^{\lambda^\vee - \mu^\vee} q^{|\lambda^\vee - \mu^\vee|}. \tag{1.1}$$

It is easy to see that the right hand side is independent of $\lambda^\vee$; usually we shall take $\lambda^\vee = 0$.

Theorem 1.3. *[Gindikin-Karpelevich formula]*

$$\mathbf{c}_{\mathfrak{g}} = \prod_{\alpha \in R_+} \frac{1 - q^{-1} e^{-\alpha^\vee}}{1 - e^{-\alpha^\vee}}. \tag{1.2}$$

Let us note that classically, the Gindikin-Karpelevich formula computes the result of application of a certain intertwining operator to the spherical vector in a principal series representation of G. However, it is trivial to see that the classical Gindikin-Karpelevich formula is equivalent to the one presented above.

1.4. **The affine case: a preview.** The above formula (more precisely, some generalization of it) plays an important role in the theory of Eisenstein series (it allows one to compute the constant term of Eisenstein series through automorphic L-functions). In the recent years, some aspects of the theory of Eisenstein series for affine Kac-Moody groups have been developed (cf. [5], [14], [15], [22]), so having an analog of (1.2) in the affine case is highly desirable. While trying to generalize (1.2) to the case when $\mathbf{G}$ is replaced by an affine (or, more generally any symmetrizable) Kac-Moody group, one is immediately lead to the following questions:

1) Does the right hand side of (1.1) make sense? (A priori, the sets which appear in the right hand side of (1.1) might be infinite, when $\mathbf{G}$ is infinite-dimensional).

2) If the answer to question 1 is positive, is Theorem 1.3 true in the Kac-Moody setting?

It was shown in [4] that when $char(\mathcal{K}) > 0$, the answer to question 1) is positive; the proof of this fact given in [4] is algebro-geometric, and apparently it can't be adapted to p-adic local fields. Also (still in the case when $\mathcal{K}$ has positive characteristic) an answer to 2) was given: it was shown (modulo the assumption that the results of [3] hold in positive characteristic) that some *corrected* version of (1.2) holds when $\mathbf{G}$ is an untwisted affine Kac-Moody group.

The main purpose of this paper is to give a uniform approach to the above problems for all local non-archimedian fields. Namely, we show (cf. Theorem 1.9) that the answer to question 1) is positive for any local non-archimedian field and for any untwisted affine Kac-Moody group (the generalization to the twisted case is probably straightforward). The proof is elementary; as a byproduct we reprove certain similar (weaker) finiteness results, that were proven before by other means - cf. the discussion after Theorem 1.9.

In addition we also formulate an analog of the formula (1.2) in the affine case. We give a proof of this formula, based on a certain affine analog of the so called *Macdonald formula for the spherical function*, which is going to appear in our forthcoming paper [7]. Before we discuss the affine case in more detail, let us first recall this formula in the usual case (i.e. when $\mathbf{G}$ is a finite-dimensional semi-simple group as before).

1.5. **Macdonald formula.** For any $\lambda^\vee \in \Lambda_+^\vee$ let us set

$$\mathcal{S}(\lambda^\vee) := \sum_{\mu^\vee \in \Lambda^\vee} |K \setminus K\pi^{\lambda^\vee} K \cap K\pi^{\mu^\vee} U| q^{\langle \rho, \mu^\vee \rangle} e^{\mu^\vee}. \tag{1.3}$$

This is an element of the group algebra $\mathbb{C}[\Lambda^\vee]$. Actually, general facts about the Satake isomorphism (see Section 1.7) imply that $\mathcal{S}(\lambda^\vee) \in \mathbb{C}[\Lambda^\vee]^W$.

Theorem 1.6. *[Macdonald] Let us set*

$$\Delta = \prod_{\alpha \in R_+} \frac{1 - q^{-1} e^{-\alpha^\vee}}{1 - e^{-\alpha^\vee}}.$$

Then for any $\lambda^\vee \in \Lambda_+^\vee$ we have

$$\mathcal{S}(\lambda^\vee) = \frac{q^{\langle \rho, \lambda^\vee \rangle}}{W_{\lambda^\vee}(q^{-1})} \sum_{w \in W} w(\Delta) e^{w\lambda^\vee} \tag{1.4}$$

Here $W_{\lambda^\vee}$ is the stabilizer of $\lambda^\vee$ in W and

$$W_{\lambda^\vee}(q^{-1}) = \sum_{w \in W_{\lambda^\vee}} q^{-\ell(w)}.$$

It is easy to see that Macdonald formula implies (1.2). The reason is that (as is well-known) for any fixed $\mu^\vee$ we have

$$K\pi^{\lambda^\vee} K \cap K\pi^{\lambda^\vee - \mu^\vee} U = K\pi^{\lambda^\vee} U^- \cap K\pi^{\lambda^\vee - \mu^\vee} U \tag{1.5}$$

for any sufficiently dominant $\lambda^\vee$. Using (1.5) to take the limit of (1.3) as $\lambda^\vee$ is made more and more dominant (which we write as $\lambda^\vee \to +\infty$) one gets (1.2).

1.7. **Interpretation via Satake isomorphism.** Let us explain the meaning of (1.3) in terms of the *Satake isomorphism*. Let $\mathcal{H}_{\mathrm{sph}}$ denote the spherical Hecke algebra of G. By definition this is the convolution algebra of K-bi-invariant distributions with compact support on G; by choosing a Haar measure on G for which the volume is K is equal to 1, we may identify it with the space of K-bi-invariant functions with compact support. Thus the algebra $\mathcal{H}_{\mathrm{sph}}$ has a basis $h_{\lambda^\vee}$ where $h_{\lambda^\vee}$ is the characteristic function of the corresponding double coset. The Satake isomorphism is a canonical isomorphism between $\mathcal{H}$ and $\mathbb{C}[\Lambda^\vee]^W$. By its very definition this isomorphism sends $h_{\lambda^\vee}$ to $\mathcal{S}(\lambda^\vee)$. Thus (1.4) can be thought of as an explicit computation of the Satake isomorphism in terms of the above basis. In particular, since $h_0 = 1$, the right hand side of (1.4) for $\lambda^\vee = 0$ must be equal to 1; this is not completely obvious from the definition.

1.8. **Affine Kac-Moody groups.** Our aim in this paper is to discuss a generalization of the Gindkin-Karpelevich formula to the case when $\mathbf{G}$ is an untwisted affine Kac-Moody group. When $\mathcal{K}$ has positive characteristic this was already done in [4] (modulo the assumption that the results of [3] hold in positive characteristic).

Let $\mathbf{G}_o$ be a simple, simply connected group. Then one can form the polynomial loop group $\mathbf{G}_o[t, t^{-1}]$ which admits a central extension by $\mathbb{G}_m$ which we denote by $\widetilde{\mathbf{G}}$. The full affine Kac-Moody group is then $\mathbf{G} = \widetilde{\mathbf{G}} \rtimes \mathbb{G}_m$ where $\mathbb{G}_m$ acts by rescaling the loop parameter

t. We shall be interested in the group $G = \mathbf{G}(\mathcal{K})$. In this setting, we may define analogues of the groups K, U, U^-.

Theorem 1.9. *With the above notation we have:*

(1) *For $\lambda^\vee, \mu^\vee \in \Lambda^\vee$ the set $K \setminus K\pi^{\lambda^\vee}U^- \cap K\pi^{\mu^\vee}U$ is finite. Moreover, it is empty unless $\lambda^\vee \geq \mu^\vee$.*

(2) *For any $\lambda^\vee \in \Lambda^\vee_+$, $\mu^\vee \in \Lambda^\vee$ as above the set*

$$K\backslash K\pi^{\lambda^\vee}K \cap K\pi^{\mu^\vee}U$$

is finite. Moreover, it is empty unless $\lambda^\vee \geq \mu^\vee$.

(3) *The set $K\pi^{\lambda^\vee}K \cap K\pi^{\mu^\vee}U^-$ is empty unless $\lambda^\vee \geq \mu^\vee$.*

(4) *For $\lambda^\vee$ sufficiently dominant and fixed $\mu^\vee$ we have*

$$K\pi^{\lambda^\vee}K \cap K\pi^{\lambda^\vee-\mu^\vee}U = K\pi^{\lambda^\vee}U^- \cap K\pi^{\lambda^\vee-\mu^\vee}U \tag{1.6}$$

The third statement of Theorem 1.9 is easy; it is also easy to see that (1) and (3) imply (2). Statement (4) is also not difficult, once we know (1) (details can be found in Section 6). Thus essentially it is enough to prove (1). This is the main result of this paper. To prove it, we develop some techniques for analyzing the infinite dimensional group U^- : the main idea is to break up the group U^- into pieces U^-_w indexed by the Weyl group of $\mathbf{G}$. These pieces are not finite-dimensional, but if we replace U^- in the above intersection by U^-_w the resulting set is finite. We can then also show, using a certain result from representation theory, that only finitely many w contribute to the above intersection.

Theorem 1.9(1) is apparently new. On the other hand, Theorem 1.9(2) appears in [5]. In addition, while this paper was in preparation, the paper [16] has appeared, where an analog of Theorem 1.9(2) is proved in a greater generality (for $\mathbf{G}$ being any almost split symmetrizable Kac-Moody group). However, the proofs in both [5] and [16] are quite cumbersome and use some rather complicated machinery. From the results of this paper we get a new elementary proof of Theorem 1.9(2). Let us also add that if one assumes Theorem 1.9(4) that clearly Theorem 1.9(1) follows from (the earlier known) Theorem 1.9(2). However, we don't know how to prove Theorem 1.9(4) without using Theorem 1.9(1).

Using the above theorem, we can then make sense in our infinite dimensional setting of the formal Gindikin-Karpelevich sum $\mathbf{c}_{\mathfrak{g}}$. Let us now formulate the statement of the affine Gindikin-Karpelevich formula. For this we first want to discuss the affine Macdonald formula.

1.10. **Affine Macdonald formula.** Let us define $\mathcal{S}(\lambda^\vee)$ as in (1.3). Note that it makes sense due to Theorem 1.9(2).

Let us also set

$$H_{\lambda^\vee} = \frac{\mathsf{t}^{\langle\rho,\lambda^\vee\rangle}}{W_{\lambda^\vee}(\mathsf{t}^{-1})} \sum_{w\in W} w(\Delta)e^{w\lambda^\vee}. \tag{1.7}$$

Here t is a formal variable (which should not be confused with t!), Δ is defined as in the finite-dimensional case (with q replaced by t) and ρ is the element of Λ such that $\langle\rho, \alpha_i^\vee\rangle = 1$ for any simple coroot $\alpha_i^\vee$.

Theorem 1.11. *$\mathcal{S}(\lambda^\vee)$ is equal to the specialization of $\frac{H_\lambda}{H_0}$ at $\mathsf{t} = q$.*

This theorem will be proved in [7].

Note that contrary to the finite dimensional case, $H_0 \neq 1$. In fact, the function H_0 was studied by Macdonald in [26] using the works of Cherednik. Macdonald has shown that H_0 (which a priori is defined as an infinite sum) has an infinite product decomposition. For example, when $\mathfrak{g}_o$ is simply laced, Macdonald's formula reads as follows:

$$H_0 = \prod_{i=1}^{\ell} \prod_{j=1}^{\infty} \frac{1 - \mathsf{t}^{-m_i} e^{-j\delta}}{1 - \mathsf{t}^{-m_i - 1} e^{-j\delta}}. \tag{1.8}$$

Here δ is the minimal positive imaginary coroot of $\mathfrak{g}$ and $m_1, \cdots, m_\ell$ are the exponents of $\mathfrak{g}_o$.

A similar product decomposition for H_0 exists for any $\mathfrak{g}_o$ (cf. [26]).

1.12. **Affine Gindikin-Karplevich formula.** We are now ready to formulate the affine version of the Gindikin-Karpelevich formula.

Theorem 1.13. *For any $\mathfrak{g}$ as above, we have*

$$\mathbf{c}_{\mathfrak{g}} = \frac{1}{H_0} \prod_{\alpha \in R_+} \left(\frac{1 - q^{-1} e^{-\alpha^\vee}}{1 - e^{-\alpha^\vee}} \right)^{m_\alpha}. \tag{1.9}$$

Here the product is taken over all positive roots (or, equivalently, coroots) of $\mathfrak{g}$ and m_α denotes the multiplicity of the coroot $\alpha^\vee$.

The proof is very simple, and we shall state it here.

Proof. It follows from Theorem 1.9(4) that the coefficient of some $e^{\gamma^\vee}$ in $\mathbf{c}_{\mathfrak{g}}$ is equal to the coefficient of $e^{\lambda^\vee - \gamma^\vee}$ in $\mathcal{S}(\lambda^\vee)$ divided by $q^{\langle \rho, \lambda^\vee \rangle}$ for sufficiently large $\lambda^\vee$. However, it is clearly equal to the right hand side of (1.9) (since only the term with $w = 1$ in the formula (1.7) can contribute to the coefficient $e^{\lambda^\vee - \gamma^\vee}$ for sufficiently large $\lambda^\vee$; also if $\lambda^\vee$ is sufficiently large, then it is also regular and thus $W_{\lambda^\vee}$ is trivial). $\square$

Let us stress that Theorem 1.13 depends on Theorem 1.11 which will be proved in another publication. Also, when $\mathcal{K} = \mathbb{F}_q((t))$, Theorem 1.13 was proved in [4] modulo an assumption that certain geometric results from [3] hold in positive characteristic ([3] deals only with characteristic 0).

We conclude with a final remark regarding the constant term conjecture of Macdonald (Cherednik's Theorem), which is essentially the statement that H_0^{-1} is the imaginary root part (or constant term in the terminology of [23], [24] of Δ^{-1}): using the techniques of [7] and this paper, we can show (without resorting to the work of Cherednik) that the constant term of Δ^{-1} can be related to the Gindikin-Karpelevich sum $\mathbf{c}_{\mathfrak{g}}$. On the other hand, in the case when the local field $\mathcal{K}$ has positive characteristic, the Gindikin-Karpelevich sum $\mathbf{c}_{\mathfrak{g}}$ has been computed in [4] using a geometric argument which is independent of Cherednik's work. Thus the combination of the works of [7, 4] and this paper also yield an independent proof of the constant term conjecture of Macdonald.

1.14. **Organization of the paper.** In Section 2 we recall some basic facts about affine Kac-Moody group schemes. Section 3, Section 4 and Section 5 are devoted to the proof of the first assertion of Theorem 1.9. In Section 6 we prove the other assertions of Theorem 1.9 and explain how to deduce Theorem 1.13 from Theorem 1.9 and Theorem 1.11.

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2. Affine Kac-Moody groups

2.1. **Affine Kac-Moody group functor.** Let $\mathbf{G}_o$ be a split simple simply connected group over $\mathbb{Z}$; let $\mathfrak{g}_o$ denote its Lie algebra. To this group one can attach a group ind scheme $\mathbf{G}$ (called the affinization of $\mathbf{G}_o$) in the following way.

First, one considers the polynomial loop group functor $\mathbf{G}_o[t,t^{-1}]$, which by definition sends a commutative ring R to $\mathbf{G}_o(R[t,t^{-1}]$. We also may define the formal loop group functor $\mathbf{G}_o[[t]]$ which sends a ring R to $\mathbf{G}_o(R((t)))$. In this paper, we will mostly work with the polynomial loop group, but will occasionally need to use the formal version in the course of our proofs. The multiplicative group $\mathbb{G}_m$ acts on $\mathbf{G}_o[t,t^{-1}]$ by rescaling the loop parameter t and we denote by $\mathbf{G}'$ the corresponding semi-direct product. It is well-known (cf. e.g. [9]) that any invariant integral bilinear form on $\mathfrak{g}_o$ gives rise to a central extension of $\mathbf{G}'$ by means of $\mathbb{G}_m$. In the case, when the bilinear form in question is the minimal one (equal to the Killing form divided by $2h^\vee$ where $h^\vee$ is the dual Coxeter number of $\mathbf{G}_o$) we denote the corresponding central extension by $\mathbf{G}$. Thus, by definition we have a short exact sequence of group ind-schemes

$$1 \to \mathbb{G}_m \to \mathbf{G} \to \mathbf{G}' \to 1.$$

The above central extension is known to be split over $\mathbf{G}_o[t]$ and in what follows we choose such a splitting.

We choose a pair of opposite Borel subgroups $\mathbf{B}_o, \mathbf{B}_o^-$ with unipotent radicals $\mathbf{U}_o, \mathbf{U}_o^-$. The intersection $\mathbf{T}_o = \mathbf{B}_o \cap \mathbf{B}_o^-$ is a maximal torus of $\mathbf{G}_o$.

Let $\mathbf{T} = \mathbb{G}_m \times \mathbf{T}_o \times \mathbb{G}_m$. Let $\mathbf{G}_o[t]_{\mathbf{B}}$ denote the preimage of $\mathbf{B}_o$ under the natural map $\mathbf{G}_o[t] \to \mathbf{G}_o$. We let $\mathbf{B}$ to be the preimage in $\mathbf{G}$ of $\mathbf{G}_o[t]_{\mathbf{B}} \rtimes \mathbb{G}_m \subset \mathbf{G}'$. This is a group-scheme, which is endowed with a natural map to $\mathbf{T}$. We denote by $\mathbf{U}$ the kernel of this map. This is the pro-unipotent radical of $\mathbf{B}$.

Similarly, let $\mathbf{G}_o[t^{-1}]_{\mathbf{B}^-}$ be the preimage of $\mathbf{B}_o^-$ under the map $\mathbf{G}_o[t^{-1}] \to \mathbf{G}_o$ coming from evaluating t to ∞. We let $\mathbf{B}^- \subset \mathbf{G}$ to be the preimage of $\mathbf{G}_o[t^{-1}]_{\mathbf{B}^-} \rtimes \mathbb{G}_m \subset \mathbf{G}'$. This is a group ind-scheme, which (similarly to $\mathbf{B}$) is endowed with a natural map to $\mathbf{T}$ and we denote its kernel by $\mathbf{U}^-$.

In addition, the intersection $\mathbf{B} \cap \mathbf{B}^-$ is naturally isomorphic to $\mathbf{T}$.

2.2. **The affine root system.** We denote the Lie algebras of $\mathbf{G}, \mathbf{T}, \mathbf{B}, \mathbf{B}^-, \mathbf{U}, \mathbf{U}^-$ respectively by $\mathfrak{g}, \mathfrak{t}, \mathfrak{b}, \mathfrak{b}^-, \mathfrak{n}, \mathfrak{n}^-$. We shall denote by R_o the set of roots of $\mathfrak{g}_o$ and by R the set of roots of $\mathfrak{g}$. We denote by W the Weyl group of R. Although there is a natural embedding $R_o \subset R$, in the future it will be convenient to use different notations for elements of R_o and R: we shall typically denote elements of R_o $\alpha, \beta, ...$ and elements of R by $a, b, ...$

Let $\Pi = \{a_1, \ldots, a_{l+1}\}$ be the set of simple roots such that for $i \leq l$ the element a_i is a simple root in R_o (which we shall also denote by α_i). We define the *height* of a root $\alpha \in R_o$ as follows: if $\alpha = \sum_{i=1}^{l} p_i \alpha_i$, then

$$\mathrm{ht}(\alpha) = \sum_{i=1}^{l} p_i.$$

Recall that $a_{l+1} = -\theta + \delta$ where θ is the classical root of maximal height and δ is the minimal positive imaginary root. Let Q be the (affine) root lattice. Any $\gamma \in Q$ we may write it uniquely as

$$\gamma = \sum_{i=1}^{l+1} p_i a_i.$$

We define the *depth of* γ to be

$$\mathrm{dep}(\gamma) = -p_{l+1}.$$

If we have $\mathrm{dep}(\gamma) = 0$ (i.e. $p_{l+1} = 0$), then we define the *order of* γ to be

$$\mathrm{ord}(\gamma) = -\mathrm{ht}(\gamma).$$

Note that if we write $a \in R$ as $a = \alpha + n\delta$ for $\alpha \in R_o$, then $\mathrm{dep}(a) = -n$.

We denote by $R_{o,+}$ (resp. $R_{o,-}$ the set of positive (resp. negative) roots of $\mathfrak{g}_o$. For each $\alpha \in R_{o,+}$ (resp. $\alpha \in R_{o,-}$ we denote by $\chi_\alpha : \mathbb{G}_a \to \mathbf{U}_o$ (resp. $\chi_\alpha : \mathbb{G}_a \to \mathbf{U}_o^-$) the corresponding homomorphism. We shall use similar notation for real roots of $\mathfrak{g}$.

2.3. **The Chevalley basis and the integral form.** Let us pick a Chevalley basis for $\mathfrak{g}$ which we write as follows: if $a = \alpha - n\delta \in R_{re,-}$ then the basis element in $\mathfrak{g}^a$ is denoted by $\xi_a = \xi_{-n,\alpha}$. Let $\{\xi_i\}_{i=1}^{l}$ denote a Chevalley basis for $\mathfrak{t}_o$ and if $a = -n\delta$ then we denote by $\{\xi_{n,i}\}_{i=1}^{l}$ the Chevalley basis for $\mathfrak{g}^{-n\delta} = \mathfrak{t}_o \otimes t^{-n}$.

One may also construct an integral form $U_{\mathbb{Z}}(\mathfrak{g})$ of the universal enveloping algebra of $\mathfrak{g}$ (see [12, 28]) which admits a triangular decomposition

$$U_{\mathbb{Z}}(\mathfrak{g}) = U_{\mathbb{Z}}(\mathfrak{n}^-) \otimes U_{\mathbb{Z}}(\mathfrak{t}) \otimes U_{\mathbb{Z}}(\mathfrak{n}).$$

2.4. **Integrable modules.** We let Λ denote the lattice of characters of $\mathbf{T}$; let also $\Lambda^\vee$ denote the dual lattice. Let Λ^+ denote the set of dominant weights (and $(\Lambda^\vee)^+$ the set of dominant coweights). For any $\lambda \in \Lambda^+$ we let $V_{\mathbb{Z}}^\lambda$ denote the corresponding integrable highest weight $\mathbf{G}(\mathbb{Z})$-module (a.k.a. Weyl module). It is defined over $\mathbb{Z}$ and we shall write V_R^λ for $V_{\mathbb{Z}}^\lambda \otimes R$ (this is a $\mathbf{G}(R)$-module). We shall usually set $V^\lambda = V_{\mathcal{K}}^\lambda$.

The module $V_{\mathbb{Z}}^\lambda$ has a weight decomposition

$$V_{\mathbb{Z}}^\lambda = \oplus V_{\mathbb{Z}}^\lambda(\mu).$$

For every $w \in W$ the weight module $V_{\mathbb{Z}}^\lambda(w\lambda)$ is of rank 1. For every such w we set $V_{w,\mathbb{Z}}^\lambda = U_{\mathbb{Z}}(\mathfrak{b}) \cdot V_{\mathbb{Z}}^\lambda(w\lambda)$. This is a free $\mathbb{Z}$-module of finite rank inside $V_{\mathbb{Z}}^\lambda$ (usually called "the Demazure module"). Hence V_w^λ is a finite-dimensional $\mathcal{K}$-vector space (endowed with an action of B).

In addition, let us set $V^\lambda(m)$ to be the direct sum of all the $V^\lambda(\mu)$ for which $\mathrm{dep}(\mu) > m$ and

$$V^\lambda[m] = V^\lambda / V^\lambda(m), \tag{2.1}$$

which is a finite-dimensional U^--submodule.

2.5. **The Heisenberg subalgebra.** The algebra $\mathfrak{g}$ also contains a rank l infinite-dimensional *Heisenberg subalgebra* Heis $\subset \mathfrak{g}$,

$$\mathrm{Heis} := \mathcal{K}\mathsf{c} \oplus \bigoplus_{n \in \mathbb{Z}} \mathfrak{t}_o \otimes t^n$$

where the element c is the central element in $\mathfrak{t}$. We shall say that an element in $\mathfrak{t}_o \otimes t^n$ is of *degree* n.

2.6. **Some decompositions.** Let now $\mathcal{K}$ be a local non-archimedian field with ring of integers $\mathcal{O}$ and residue field $\Bbbk$. We let q denote the cardinality of $\Bbbk$; we also choose a generator of the maximal ideal of $\mathcal{O}$ which we shall denote by π.

For a scheme $\mathbf{X}$ over $\mathcal{O}$ we set $X = \mathbf{X}(\mathcal{K}), X_{\mathcal{O}} = \mathbf{X}(\mathcal{O}), X_{\Bbbk} = \mathbf{X}(\Bbbk)$.

Let $K = \mathbf{G}(\mathcal{O})$ and let us set I (resp. I^-) to be the preimage of $B_{\Bbbk}$ (resp. $B_{\Bbbk}^-$) under the natural map $K \to G_{\Bbbk}$. We shall choose a lift of all the elements of W to K and we are going to call this set of representatives $\dot{W}$. We shall sometimes identify $\dot{W}$ with W (when it doesn't lead to a confusion).

We have the Bruhat decompositions

$$\mathbf{G}_{\Bbbk} = B_{\Bbbk} \dot{W} B_{\Bbbk}; \quad G_{\Bbbk} = B_{\Bbbk} \dot{W} B_{\Bbbk}^-$$

which lift to give decompositions of the form

$$K = I\dot{W}I \text{ and } K = I^- \dot{W} I$$

In addition we have the Iwasawa decomposition

$$G = K \cdot B$$

(cf. e.g. Section 2 of [4] for the proof).

3. Finiteness of the Gindikin-Karpelevich Sum, Part I

The main goal of this paper will be to prove the following result which shows that the sets appearing in the expression (1.1) are finite.

Theorem 3.1. *Let $\mu^\vee \in Q_+$. Then we have,*

$$|K \backslash KU^- \cap K\pi^{-\mu^\vee} U| < \infty. \tag{3.1}$$

The proof will be carried out in three steps:

(1) We shall decompose the set U^- into disjoint subsets $U^- = \bigcup_{w \in W} U_w^-$;
(2) We shall show that there are only finitely many $w \in W$ such that U_w^- can contribute to (3.1)
(3) We shall show that the sets (3.1) with U^- replaced by U_w^- are finite;

Steps 1 and 2 will be performed in this Section. Step 3 will be performed in Section 5.
Step 1: We have an injection $U^- \hookrightarrow G/B$. By the Iwasawa decomposition, we have that $G/B \cong K/K \cap B$ and so we have an embedding,

$$\mathsf{Iw}_K : U^- \hookrightarrow K/K \cap B.$$

More explicitly, if we write $u^- = khu$ in terms of its Iwasawa coordinates, then the above map sends $u^- \mapsto k \pmod{K \cap B}$.

Since $K \setminus KU^- = U^-_{\mathcal{O}} \setminus U^-$, the map Iw_K descends to a map

$$\mathsf{Iw}_K : K \setminus KU^- \hookrightarrow U^-(\mathcal{O}) \setminus K/K \cap B.$$

Let us denote by ϖ the reduction modulo π map $\varpi : K \to G_{\Bbbk}$.

By composing the map $\mathsf{Iw}_K : K \setminus KU^- = U^-_{\mathcal{O}} \setminus U^- \to U^-_{\mathcal{O}} \setminus K/K \cap B$ with the the natural map $U^-_{\mathcal{O}} \setminus K/K \cap B \to U^-_{\Bbbk} \setminus G_{\Bbbk}/B_{\Bbbk} = W$ we get a map $\varphi : K \setminus KU^- \to W$. Letting $\psi : U^- \to U^-_{\mathcal{O}} \setminus U^-$ denote be the projection map, we define

$$U^-_w = \psi^{-1}\varphi^{-1}(w) \subset U^-.$$

We can give a more explicit description of the elements in U^-_w as follows. First set

$$G_\pi = \{k \in K | \varpi(k) = 1.\}$$

We can define U_π, U^-_π and T_π in the same way.

Then $G_\pi \subset I$ and in fact we have a direct product decomposition $G_\pi = U_\pi U^-_\pi T_\pi$. We have

$$\varpi^{-1}(w) = U^-_{\mathcal{O}} G_\pi w B_{\mathcal{O}} = U^-_{\mathcal{O}} U_{w,\pi} w B_{\mathcal{O}}$$

where

$$U_{w,\pi} = \prod_{a>0, w^{-1}a<0} U_{a,\pi}$$

where $U_{a,\pi}$ is the first congruence subgroup in $U_{a,\mathcal{O}}$.

So every $u^- \in U^-_w$ has an expression of the form

$$u^- = u^-_{\mathcal{O}} u_{w,\pi} w b_{\mathcal{O}} \pi^\xi u \tag{3.2}$$

where lowercase elements are in the corresponding uppercase subgroups and $\xi \in \Lambda^\vee$.

Lemma 3.2. *For $u_{w,\pi} \in U_{w,\pi}$, there exist $u^- \in U^-, u \in U, h \in T$ such that*

$$u_{w,\pi} w = u^- h u.$$

Moreover, u^-, u, h are uniquely defined.

Proof. If

$$u_{w,\pi} = u_1^- h_1 u_1 = u_2^- h_2 u_2 \;\text{ where } u_1^\pm, u_2^\pm \in U^\pm,\; h_1, h_2 \in T \tag{3.3}$$

then it follows that $(u_2^-)^{-1} u_1^- \in B$, which is a contradiction unless $u_1^- = u_2^-$. Similarly, we conclude that $u_1 = u_2$ and $h_1 = h_2$. □

Step 2: We now prove the following inequality which is the main result of this Section.

Lemma 3.3. *Fix $\mu^\vee \in Q_+^\vee$ Then if*

$$KU_w^- \cap K\pi^{-\mu^\vee} U \neq \emptyset,$$

we must have

$$\frac{l(w)}{2} \leq |\mu^\vee| := \langle \rho, \mu^\vee \rangle.$$

Proof of Lemma. Suppose $u_w^- \in U_w^-$ is such that $Ku_w^- \subset KU_w^- \cap K\pi^{-\mu^\vee} U$. Then we can write

$$u_w^- = u_{\mathcal{O}}^- u_{w,\pi} w b_{\mathcal{O}} \pi^{-\mu^\vee} u,$$

which produces a relation of the form

$$u^- = u_{w,\pi} w b_{\mathcal{O}} \pi^{-\mu^\vee} u. \tag{3.4}$$

Applying both sides of (3.4) to v_ρ, a primitive highest weight vector in V^ρ :

$$u^- v_\rho = u_{w,\pi} w b_{\mathcal{O}} \pi^{-\mu^\vee} u v_\rho. \tag{3.5}$$

The left hand side of (3.5) is of the form

$$v_\rho + \text{lower terms}, \tag{3.6}$$

whereas the right hand side is of the form

$$\delta \pi^{-\langle \rho, \mu^\vee \rangle} u_{w,\pi} v_{w\rho}, \tag{3.7}$$

where $\delta \in \mathcal{O}^*$ and $v_{w\rho}$ is again a primitive vector in $V_{\mathcal{O}}^\rho$.

Consider an element $u_{w,\pi} \in U_{w,\pi}$ and keep the notation of Step 2. So the element $u_{w,\pi}$ acts via a sum of the form,

$$\sum_{n_1,\ldots,n_r} \sigma_1^{n_1} \cdots \sigma_r^{n_r} \xi_{\beta_1}^{(n_1)} \cdots \xi_{\beta_r}^{(n_r)}, \tag{3.8}$$

where the β_i, for $i = 1, \ldots, r$ are the positive real roots $a \in R_{re,+}$ such that $w^{-1}a < 0$, and $\xi_{\beta_i}^{(n_i)}$ are the divided powers of our fixed Chevalley basis elements. Let us now consider the element v_ρ in the highest weight space of V^ρ in the expression (3.7).

Let us now use the following result due to A. Joseph (cf. [3], Lemma 18.2):

Proposition 3.4. *Suppose that $v_\rho \in F^m(U(\mathfrak{n}))v_{w\rho}$. Then,*

$$m \geq l(w)/2,$$

where $l(w)$ is the length of w.

By Proposition 3.4 we must apply at least $l(w)/2$ operators from $\mathfrak{n}$ to $v_{w\rho}$ in order to obtain an element in V_ρ. This corresponds, in an expression of the form (3.8), to terms in which $n_1 + \cdots + n_r = m \geq l(w)/2$. Since the $\xi_{\beta_i}^{(n)}$ map $V^\rho(\mathcal{O})$ into itself, such an expression will introduce a zero of order at least $n_1 + \cdots + n_r$ into the resulting element in V^ρ. However, since from (3.6) the element produced in V^ρ as a result of applying $u_{w,\pi}$ must be primitive, we obtain the desired equality:

$$|\mu^\vee| \geq l(w)/2$$

in light of (3.7).

□

Thus to prove Theorem 1.13(1) we need to show that the quotient $K\backslash KU_w^- \cap K\pi^{-\mu^\vee}U$ is finite for every w and $\mu^\vee$. In other words, we need to prove the following

Theorem 3.5. *The set $K\backslash KU^-(w,\mu^\vee)$ is finite, where*

$$U^-(w,\mu^\vee) = U^- \cap U_{w,\pi}T_{\mathcal{O}}w\pi^{-\mu^\vee}U. \tag{3.9}$$

This will be done in Section 5. But first we need to introduce some notation related to the group U^- and its completion.

4. Completion of U^-

The purpose of this section is to discuss some sort of coordinates on a formal completion of U^- which will be used later.

4.1. **The completion.** Recall that the group ind-scheme $\mathbf{U}^-$ is the the preimage of $\mathbf{U}_o^-$ under the natural (evaluation at ∞) map $\mathbf{G}_o[t^{-1}] \to \mathbf{G}_o$. Let $\widehat{\mathbf{U}}^-$ denote the preimage of $\mathbf{U}_o^-$ in $\mathbf{G}_o[[t^{-1}]]$, where $\mathbf{G}_o[[t^{-1}]]$ is the formal loop group functor in the variable t^{-1}. This is a group-scheme; we have a natural map $\mathbf{U}^- \to \widehat{\mathbf{U}}^-$ which induces an injection

$$i : U^- \hookrightarrow \widehat{U}^-.$$

We shall often identify U^- with its image in $\widehat{U}^-$.

For every $m \geq 0$ we shall set $U^-(m)$ to be the subgroup of U^- consisting of elements which are equal to 1 modulo t^{-m}. We set $U^-[m] = U^-/U^-(m)$ and we shall denote by ω_m the natural projection,

$$U^- \to U^-[m]. \tag{4.1}$$

4.2. **Some infinite products.** For $\alpha \in R_{o,+}$ and $\beta \in R_{o,-}$ let us set

$$\sigma_\alpha = \sum_{i=1}^{\infty} c_i t^{-i} \in t^{-1}\mathcal{K}[[t^{-1}]] \text{ and } \sigma_\beta = \sum_{i=0}^{\infty} c_i t^{-i} \in \mathcal{K}[[t^{-1}]].$$

Then we may consider the following products as elements of $\widehat{U}^-$

$$\chi_\alpha(\sigma_\alpha) = \prod_{i=1}^{\infty} \chi_{\alpha-i\delta}(c_i) \text{ and } \chi_\beta(\sigma_\beta) = \prod_{i=0}^{\infty} \chi_{\beta-i\delta}(c_i).$$

Suppose we are given a unit $\sigma \in \mathcal{K}[[t^{-1}]]^*$. If $\sigma \equiv 1 (\operatorname{mod} t^{-1})$, we have a factorization,

$$\sigma = \prod_{j\geq 1}(1 + c_j t^{-j}), \tag{4.2}$$

where the c_j are uniquely determined. For $i = 1, \ldots, l$ we form the expressions,

$$h_i(\sigma) := \chi_{\alpha_i}(\sigma)\chi_{-\alpha_i}(-\sigma^{-1})\chi_{\alpha_i}(\sigma)\chi_{\alpha_i}(1)\chi_{-\alpha_i}(-1)\chi_{\alpha_i}(1)$$

which again define elements of $\widehat{U}^-$. With respect to the factorization (4.2) we then have

$$h_i(\sigma) = \prod_{j\geq 1} h_i(1 + c_j t^{-j}).$$

Fix a positive integer $m \geq 1$ and consider now an element of $\widehat{U}^-$ of the form

$$u^-[m] := \prod_{\alpha \in R_{o,+}} \chi_\alpha(t^{-m} s_{m,\alpha}) \prod_{i=1}^{l} h_i(1 + c_{i,m} t^{-m}) \prod_{\alpha \in R_{o,+}} \chi_{-\alpha}(t^{-m} \tilde{s}_{m,\alpha}), \tag{4.3}$$

where the products are with respect to a fixed ordering on $R_{o,+}$ and $s_{m,\alpha},\ \tilde{s}_{m,\alpha},\ c_{i,m} \in \mathcal{K}$. If $m = 0$, consider elements of the form

$$u^-[0] := \prod_{\alpha \in R_{o,+}} \chi_{-\alpha}(\tilde{s}_{0,\alpha}) \ \text{ with } \tilde{s}_{0,\alpha} \in \mathcal{K}. \tag{4.4}$$

4.3. **"Coordinates" on U^-.** Though there is no easy way to put coordinates on U^-, one can use the above infinite products to define coordinates on $\widehat{U}^-$. Restricted to U^-, we have the following,

Theorem 4.4. *Every element $u^- \in U^- \subset \widehat{U}^-$ has a unique expression of the form*

$$u^- = \prod_{m=0}^{\infty} u^-[m] = \cdots u^-[m] u^-[m-1] \cdots u^-[0]$$

(the product is considered in decreasing *order of m) where the $u^-[m]$ are of the form* (4.3) *or* (4.4)*. Furthermore, with respect to the map $\omega_m : \widehat{U}^- \to U^-[m]$ we have*

$$\omega_m(u^-) = \omega_m(u^-[m]\ u^-[m-1]\ \cdots\ u^-[0]).$$

4.5. For future use, we shall need to understand the action of the various components of $u^-[m]$ on a highest weight vector $v_\lambda \in V^\lambda$. This is essentially contained in the following,

Lemma 4.6. *Let $m \in \mathbb{Z}_{\geq 0}$ be a positive integer and $a = \alpha \otimes t^{-m} \in R_{-,re}$ be a root of depth m. Assume also that λ is regular and dominant. Then in the highest weight module V^λ we have*

(1) $\xi_{-m,\alpha} v_\lambda \neq 0$
(2) *If $h \in$ Heis is of degree $-m$, then $h v_\lambda \neq 0$. Moreover, the family of vectors $\{\xi_{-m,i} v_\lambda\}$, $i = 1, \ldots, l$, are linearly independent.*
(3) $\chi_a(s) v_\lambda = v_\lambda + s\xi_{-m,\alpha} v_\lambda +$ *terms of greater depth*

Proof. Parts (1) and (3) are clear. Let us prove (2). For any element of degree n in Heis, say $h_n := h \otimes t^n$ with $h \in \mathfrak{t}_o$, we may find an element $h_{-n} := h' \otimes t^{-n} \in$ Heis with $h' \in \mathfrak{t}_o$ such that

$$[h_n, h_{-n}] = \kappa \mathsf{c}$$

with $\kappa \neq 0$. Indeed, the this follows immediately from the commutation relation,

$$[h_n, h_{-n}] = n(h|h')\mathsf{c}$$

and the non-degeneracy of $(\cdot|\cdot)$ when restricted to $\mathfrak{t}_o$ But we also have,

$$[h_n, h_{-n}] v_\lambda = h_n h_{-n} v_\lambda$$

from which it must follow that $h_{-n} v_\lambda \neq 0$. The linear independence of $\{\xi_{-m,i} v_\lambda\}$, for $i = 1, \ldots, l$ follows easily. $\square$

4.7. A reformulation. In the theory of vertex operators, one encounters expressions of the following form for $i \in \{1, \ldots, l\}$ and $p \in \mathbb{Z}_{\geq 0}$,

$$\mathcal{P}_i(s,p) := \exp(-\sum_{k>0} \frac{\xi_{pk,i} s^k}{k}).$$

It is easy to see that that $\mathcal{P}_i(s,p)$ is a well defined operator in $\mathrm{Aut}_{\mathcal{K}}(V^\lambda)$ in the case when $\mathcal{K}$ has characteristic 0. Moreover, we know from [12, Theorem 5.8, Remark (i)] that if we define the element $\Lambda_k(\xi_i(p))$ via the relation,

$$\mathcal{P}_i(s,p) = \sum_{k \geq 0} \Lambda_k(\xi_i(p)) s^k$$

then the elements $\Lambda_k(\xi_i(p))$ will actually lie in $U_{\mathbb{Z}}(\mathfrak{g})$, our fixed $\mathbb{Z}$-form of the enveloping algebra. We then have the following,

Proposition 4.8. *As elements of the corresponding completion of $U_{\mathbb{Z}}(\mathfrak{g})$ we have an equality,*

$$\mathcal{P}_i(s,p) = h_i(1 - st^p).$$

Proof. We are going to check that the left hand side and the right hand side act in the same way in any irreducible integrable representation. Let us first note that from [13, Lemma 12.2], it follows that $h_i(1 - st^p)$ is in U, and hence fixes any highest weight vector. From the definition of $\mathcal{P}_i(s,p)$, we see that it has the same property.Hence, by the analogue of Schur's Lemma [13, Lemma 9.1], in order to show that $\mathcal{P}_i(s,p)$ and $h_i(1 - st^p)$ are equal, we need only to consider their action on the adjoint representation. But we may use the perfectness of $\mathfrak{g}$ as a Lie algebra and [13, Lemma 8.11] to reduce to a computation in the adjoint representation of the loop algebra $\mathfrak{g}_o \otimes \mathcal{K}((t))$. To complete the proof we know that, working in $\mathfrak{g}_o \otimes \mathcal{K}((t))$, one has (in the notation above) that

$$\Lambda_k(\xi_i(p)) x_\beta = (-1)^k \binom{\langle \beta, \xi \rangle}{k} x_\beta \otimes t^{pk}$$

where $x_\beta \in {\mathfrak{g}_o}^\beta$ for $\beta \in R_o$. By the usual binomial formula, one now has that

$$\mathcal{P}_i(s,p) x_\beta = \sum_k (-1)^k \binom{\langle \beta, \xi \rangle}{k} x_\beta \otimes t^{pk} s^k = (1 - t^p s)^{\langle \xi, \beta \rangle} x_\beta.$$

This agrees with the action of $h_i(1 - st^p)$ on x_β and so the lemma follows. $\square$

An entirely similar argument to the above shows the following,

Proposition 4.9. *The element $\mathcal{P}_i^-(s,p) := \exp(-\sum_{k>0} \frac{\xi_{-pk,i} s^k}{k})$ defines an element of $\widehat{U}^-$ and we have an equality of elements of $\widehat{U}^-$,*

$$\mathcal{P}_i^-(s,p) = h_i(1 - st^{-p}).$$

Corollary 4.10. *Let $c_i \in k$ for $i = 1, \ldots, l$. Then there exists a positive integer m_0 so that for all $m \geq m_0$ we have we have an equality in $V[m]$,*

$$\prod_{i=1}^{l} h_i(1 - c_i t^{-m}) v_\lambda = v_\lambda + \sum_{i=1}^{l} c_i \xi_{m,i} v_\lambda + \text{ terms of greater depth.}$$

Moreover, $\sum_{i=1}^{l} c_i \xi_{m,i} v_\lambda \neq 0$*, provided not all* $c_i = 0$ *for* $i = 1, \dots, l$*.*

The proof of the following theorem is parallel to that for U described in [13].

5. Finiteness of the Gindikin-Karpelevich Sum, Part 2

Notation: For a constant M, we write $M(x, y, z, ...)$ to indicate that the choice of M depends only on $x, y, z, ...$

5.1. Fix some highest weight module V^λ with primitive highest weight v_λ. Let us define a norm $|| \cdot ||$ on V^λ in the following way. For any $v \in V^\lambda$ let us set

$$\mathrm{ord}(v) = \min \{n \in \mathbb{Z} \mid \pi^n v \in V^\lambda_{\mathcal{O}}\}.$$

Then we set

$$||v|| = q^{\mathrm{ord}(v)}.$$

For an operator $g : V^\lambda \to V^\lambda$ and a positive constant $C > 0$ we shall say that g is *bounded by* C if $||g v_\lambda|| < C$. We say that g is *bounded by* C *at depth* j if when we write

$$g v_\lambda = \sum_{\mu \in P^\lambda} v_\mu,$$

then

$$|| \sum_{\mu \in P^\lambda(j)} v_\mu || < C.$$

where P^λ and $P^\lambda(j)$ denote the set of weights and weights of depth greater than j of the module V^λ respectively.

We say that a family of elements $\Xi \subset G$ is bounded (bounded at depth j) if there exists some $C > 0$ such that every element of Ξ is bounded by C (respectively, bounded by C at depth j).

We shall need a finer notion of boundedness in the sequel. Recall that each $u^-[j]$ can be explicitly written in terms coordinates (see (4.3) if $j > 0$ and (4.4) if $j = 0$.) We say that an element $u^-[j]$ is *componentwise bounded* by a positive number C if $j > 0$ and in an expression (4.3) we have

$$||s_{j,\alpha}|| < C, ||\tilde{s}_{j,\alpha}|| < C, ||c_{j,m}|| < C \text{ where } \alpha \in R_o, \ m \geq 0$$

or $j = 0$ and in an expression (4.4) we have

$$||s_{0,\alpha}|| < C \text{ where } \alpha \in R_o.$$

A family of elements of the form $u^-[j]$ will be said to be componentwise bounded if there exists a constant C such that all elements in this family are componentwise bounded by C.

5.2. The relation between an element in $\widehat{U}^-$ being bounded and being componentwise bounded is partially explained by the following,

Lemma 5.3. *Suppose that* $u^-[j]$ *is bounded at depth* j *by* C*. Then there exists a constant* $D = D(C)$ *such that* $u^-[j]$ *is componentwise bounded by* D*.*

Proof. We will consider two cases:

5.4. Case 1: $j = 0$**.** Let us write

$$u^-[0] = \prod_{\alpha \in R_{o,-}} \chi_\alpha(s_{0,\alpha})$$

where the product is ordered according to decreasing height from left to right. Consider now the following statement for each $t = 1, \dots, \mathrm{ht}(\theta)$.

$\mathcal{H}(t)$: Suppose there exists $C > 0$ such that for any

$$u_t^- := \prod_{\alpha \in R_{o,-},\ \mathrm{ht}(a) \geq t} \chi_\alpha(s_{0,\alpha}), \quad s_{0,\alpha} \in \mathcal{K} \tag{5.1}$$

that satisfies

$$||u_t v_\lambda|| < C$$

then there exists a constant $D = D(C) > 0$ (depending only on C) such that $||s_{0,\alpha}|| < D$ for each $\alpha \in R_{o,-}$.

We shall show that $\mathcal{H}(t)$ is true for $t = 1$ by decreasing induction on t. For $t = \mathrm{ht}(\theta)$ this follows from Lemma 4.6. So we need to argue that that if $\mathcal{H}(t+1)$ is true, then also $\mathcal{H}(t)$ is true. So given u_t^- as in (5.1) we may write

$$u_t^- = u_{t+1}^- \chi_{\gamma_1}(s_{\gamma_1}) \cdots \chi_{\gamma_r}(s_{\gamma_r})$$

where $s_{\gamma_j} \in \mathcal{K}$, $\mathrm{ht}(\gamma_j) = t$ for $j = 1, \cdots r$ and u_{t+1}^- is a product of elements corresponding to roots of height at least $t+1$. We then have

$$u_t^- v_\lambda = v_\lambda + \sum_{i=1}^r s_{\gamma_i} \xi_{\gamma_i} v_\lambda + \text{ terms of lower order },$$

By Lemma 4.6 each of the $\xi_{\gamma_i} v_\lambda \neq 0$, and furthermore they lie in different weight spaces. Hence by the hypothesis of boundedness in $\mathcal{H}(t)$ we see that there exists a constant D only depending on C such that $||s_{\gamma_j}|| < D$ for $j = 1, \dots, r$. Now, if we set

$$\tilde{u}_{t+1}^- := (\chi_{\gamma_1}(s_{\gamma_1}) \cdots \chi_{\gamma_r}(s_{\gamma_r}))^{-1} u_{t+1}^- \chi_{\gamma_1}(s_{\gamma_1}) \cdots \chi_{\gamma_r}(s_{\gamma_r}),$$

then there exists a constant $\tilde{C}$ such that $||\tilde{u}_{t+1}^- v_\lambda|| < \tilde{C}$ for all u_t^-. On the other hand, the element $\tilde{u}_{t+1}^-$ is a product over roots of height at least $t+1$ and so we may apply inductively the hypothesis $\mathcal{H}(t+1)$ to the $\tilde{u}_{t+1}^-$ and conclude that it is componentwise bounded by a constant depending only on C. As $\tilde{u}_{t+1}^-$ and u_t^- are conjugates by a componentwise bounded expression, we see that the u_t^- are also componentwise bounded.

5.5. Case 2: $j > 0$**.** Suppose we write

$$u^-[j] = \prod_{\alpha \in R_{o,+}} \chi_\alpha(t^{-j} s_{j,\alpha}) \prod_{i=1}^l h_i(1 + c_{i,j} t^{-j}) \prod_{\alpha \in R_{o,+}} \chi_{-\alpha}(t^{-j} \tilde{s}_{j,\alpha}) \text{ with } s_{j,\alpha},\ \tilde{s}_{j,\alpha},\ c_{i,j} \in \mathcal{K}.$$

Then by Lemma 4.6 we have

$$u^-[j] v_\lambda = v_\lambda + \sum_{i=1}^l c_{i,j} \xi_{j,i} v_\lambda + \sum_{\alpha \in R_{o,+}} s_{m,\alpha} \xi_{m,\alpha} v_\lambda + \sum_{\alpha \in R_{o,+}} \tilde{s}_{m,-\alpha} \xi_{m,-\alpha} v_\lambda + \text{ terms or lower depth.}$$

Again by Lemma 4.6 we have that $\xi_{j,i}v_\lambda$ are linearly independent and so we may bound each of the coefficients $c_{i,j}$. Furthermore, each of the vectors $\xi_{m,\alpha}v_\lambda$ and $\xi_{m,-\alpha}v_\lambda$ are non-zero and lie in different weight spaces, so we obtain a bound on the coefficients $s_{m,\alpha}$ and $\tilde{s}_{m,-\alpha}$. $\square$

5.6. An extension of Lemma 5.3 is given by the following,

Proposition 5.7. *Let m be a positive integer, and C a positive constant. Suppose that $\mathcal{F} \subset U^-$ is a family such that*
 (a) $||u^- v_\lambda|| < C$ for all $u^- \in \mathcal{F}$
 (b) every $u^- \in \mathcal{F}$ may be written as a product $u^-[m] \cdots u^-[0]$.
Then there exists a $D = D(C, m) > 0$ such that $u^-[j]$ is componentwise bounded by D, for $j = 0, \dots, m$.

Proof. The proof will consist of a decreasing induction on j for the following statement denoted by $\mathcal{P}(j)$:

$\mathcal{P}(j)$: Suppose that $u^-[m]u^-[m-1] \cdots u^-[j]$ is bounded by C. Then there exists $D = D(C)$ such that each $u^-[k]$ for $k = j, \cdots, m$ is componentwise bounded by D.

The statement $\mathcal{P}(m)$ follows from Lemma 5.3. So let us assume that $\mathcal{P}(j+1)$ is true, and let us then argue that $\mathcal{P}(j)$ then follows. Let us write

$$u^- := u^-[m] \cdots u^-[j+1]u^-[j]v_\lambda = v_\lambda + \text{ terms of depth } j \ + \text{ terms of higher depth} .$$

We then have that

$$(u^- - u^-[j])v_\lambda = \text{ terms of depth} \geq j+1.$$

As u^- belongs to a bounded family, we must have that the $u^-[j]$ is bounded at depth j. Hence it is bounded and componentwise bounded by Lemma 5.3. Now consider

$$\tilde{u}^- := u^-[j]^{-1}u^-[m] \cdots u^-[j] = \tilde{u}^-[m] \cdots \tilde{u}^-[j+1]$$

for some elements $\tilde{u}^-[m], \dots, \tilde{u}^-[j+1]$. The expression $\tilde{u}^-$ is bounded by some constant $D = D(C)$ and so applying statement $\mathcal{P}(j+1)$ we conclude that $\tilde{u}^-$ is componentwise bounded by $E = E(D) = E(C)$. But then there exists a constant $F = F(E) = F(D) = F(C)$ such that

$$u^- = u^-[j]\tilde{u}^- u[j]^{-1}$$

is componentwise bounded (since $u^-[j]$ was componentwise bounded). $\square$

5.8. In the future we are going to need the following elementary result, which is proved in the appendix to this paper.

Proposition 5.9. *Let n be a positive integer and let $C > 0$. There exists $r = r(C, n) \geq 0$ such that for any $A, B \in GL(n, \mathcal{K})$ such that*
 (1) *The entries of A, B are bounded by C*
 (2) *$A - B \equiv 0 \pmod{\pi^r}$,*
Then $AB^{-1} \in GL(n, \mathcal{O})$.

5.10. We now proceed to the proof of Theorem 3.5, which proceeds in several steps.

5.11. Step 1. Let $\lambda^\vee$ be a regular dominant weight. Then the natural map $\zeta : U^- \to V^\lambda$ which sends $u^- \mapsto u^- v_\lambda$ is injective. We claim that for m sufficiently large, that the same is true for the map$\zeta_m : U^-(w, \mu^\vee) \to V^\lambda[m]$ the map obtained by composing ζ with the projection $V^\lambda \to V^\lambda[m]$. Indeed, since $U(w, \mu^\vee) \subset BwB$, it follows that the image of ζ lies in V^λ_w. Since the projection $V^\lambda_w \to V^\lambda[m]$ is injective for large m, the statement follows.

Moreover, we claim that if m is sufficiently large, then both ζ_m and the map

$$\begin{aligned} U^-(w, \mu^\vee) \times U^-(w, \mu^\vee) &\to V^\lambda[m] & (5.2)\\ (u_1^-, u_2^-) &\mapsto u_2^-(u_1^-)^{-1} v_\lambda \ (\mathrm{mod}\ V^\lambda(m)) & (5.3) \end{aligned}$$

are injective. Indeed, if $u_1^-, u_2^- \in U^-(w, \mu)$ then the product

$$u_2^-(u_1^-)^{-1} \in BwBw^{-1}B \subset \bigcup_{w' \in \Omega} Bw'B$$

where Ω is a finite set. Then any $m \geq \max_{w' \in \Omega} m_{w'}$ will satisfy the second injectivity requirement.

5.12. Step 2. The second step is the following simple lemma:

Lemma 5.13. *The set $U(w, \mu^\vee)$ is bounded. Equivalently, there exists $i > 0$ such that $\zeta(U^-(w, \mu^\vee)) \subset \pi^{-i} V^\lambda_{\mathcal{O}}$.*

Proof. Take $i = \langle \mu^\vee, \lambda \rangle$. Then for every $g \in K\pi^{-\mu^\vee} U$ we have

$$\zeta(g) \subset \pi^{-i} V^\lambda_{\mathcal{O}}.$$

Hence Lemma 5.13 follows since $U^-(w, \mu^\vee) \subset K\pi^{-\mu^\vee} U$. □

5.14. Step 3. Let m be greater than the depth of $w(\lambda)$. Then for every $u^- \in U(w, \mu^\vee)$ in the decomposition

$$u^- v_\lambda = \sum_{\mu \in P_\lambda} v_\mu, \text{ where } v_\mu \in V^\lambda(\mu), \tag{5.4}$$

we must have $v_\mu = 0$ for all μ of $\mathrm{dep}(\mu) > m$, and where we recall again that P_λ denotes the weight lattice of the representation V^λ.

Using Theorem 4.4 let us write $u^- = \prod_{j=0}^\infty u^-[j]$. Then for every m as above we see immediately that $\omega_m(u^-[j]) = 1$ if $j \geq m$. Hence for sufficiently large m (independent of u^- we have

$$u^- = u^-[m]u^-[m-1] \cdots u^-[0] \tag{5.5}$$

5.15. Step 4. Let us choose m to satisfy the conditions of Step 1 and Step 3. From now on let us set $V_w = V^\lambda_w, V_{w,\mathcal{O}} = V^\lambda_{w,\mathcal{O}}, v = v_\lambda$. We claim that there exists $c > 0$ such that for any $u^- \in U^-(w, \mu^\vee)$ we have

$$u^-(V_{\mathcal{O}}) \subset \pi^{-c} V_{\mathcal{O}}.$$

In other words, we claim that if we choose an $\mathcal{O}$-basis for $V_{w,\mathcal{O}}$ then the image of the natural embedding $\zeta_w : U(w, \mu^\vee) \to GL(V_w)$ consists of lower-triangular matrices (with respect to some natural basis) whose entries are bounded by some constant C depending only on c and w. This immediately follows from Proposition 5.7.

5.16. Step 5. We claim that there exists a finite set $\mathcal{F} \subset U^-(w, \mu^\vee)$ such that for any $u^- \in U^-(w, \mu^\vee)$ there exists $u^-_\mathcal{O} \in U^-_\mathcal{O}$ and $u^-_f \in \mathcal{F}$ such that

$$u^- = u^-_\mathcal{O} u^-_f.$$

First, we construct, for every positive integer $l \geq 0$ we construct a set $\mathcal{F}_l$ as follows. We have already constructed in the previous step an embedding ζ_w of $U^-(w, \mu^\vee)$ into the group of unipotent lower triangular matrices whose entries are uniformly bounded by some constant C. For a given $l \geq 0$ there are only finitely many such matrices $\operatorname{mod} \pi^l$, and we denote this finite set by $\mathcal{A}_l$. Let $\mathcal{F}_l \subset U^-(w, \mu^\vee)$ then be a set of representatives of $\zeta_w(U^-(w, \mu))$ mod π^l. In other words, for every element $u^- \in \mathcal{F}_l$, we may write

$$\omega_l(u^-) = A_0 + \epsilon \;\text{ where } \epsilon \equiv 0 \pmod{\pi^l},\; A_0 \in \mathcal{A}_l. \tag{5.6}$$

We claim that if l is sufficiently large, then the set $\mathcal{F}_l$ satisfies our requirements. First we choose l satisfying the condition of Proposition 5.9. Then for every $u^- \in U^-(w, \mu)$, there exists $u^-_f \in \mathcal{F}_l$ such that $\xi((u^-))^{-1}\xi(u^-_f)$ has integral entries. So, we know that $(u^-)^{-1}u^-_f v_\lambda \in V^\lambda_\mathcal{O}[m]$ and hence also $(u^-)^{-1}u^-_f v_\lambda \in V^\lambda_\mathcal{O}$ by the second injectivity requirement from Step 1. Hence, we note that it is enough to prove the following

Lemma 5.17. *Let λ be a regular dominant weight. Assume that $u^- v_\lambda \in V^\lambda_\mathcal{O}$ for some $u^- \in U^-$. Then $u^- \in U^-_\mathcal{O}$.*

Proof. For simplicity let us assume that $\lambda = \rho$ (the general case is similar). We know that $u^- \in K\pi^{-\mu^\vee}U$ for some $\mu^\vee \in Q^\vee_+$. The fact that $u^- v_\lambda \in V^\lambda_\mathcal{O}$ implies that $\mu^\vee = 0$. Hence $u^- \in U^-_\mathcal{O}$ by Lemma 3.3. □

6. Proof of Theorem 1.9 and Theorem 1.13

6.1. Proof of Theorem 1.9(1) and Theorem 1.9(3). Let $\lambda \in \Lambda^+$. Then any $g \in K\pi^{\lambda^\vee}K$ satisfies

$$g^{-1}(V^\lambda_\mathcal{O}) \subset \pi^{-\langle \lambda^\vee, \lambda \rangle} V^\lambda_\mathcal{O}. \tag{6.1}$$

Indeed, the condition (6.1) is clearly K-bi-invariant and it is trivially satisfied by $g = \pi^{\lambda^\vee}$.

On the other hand, any $g \in K\pi^{\mu^\vee}U^-$ satisfies

$$\eta_\lambda(g^{-1}(V^\lambda_\mathcal{O})) \subset \pi^{-\langle \mu^\vee, \lambda \rangle}\mathcal{O}, \tag{6.2}$$

where $\eta_\lambda : V^\lambda \to \mathcal{K}$ is the projection to the highest weight line (normalized by the condition that $\eta_\lambda(v_\lambda) = 1$). This is true because the set of all g that satisfy (6.2) is clearly invariant under U^--action on the right and the K-action on the left and it is satisfied by $g = \pi^{\mu^\vee}$.

Thus we see that if

$$K\pi^{\lambda^\vee}K \cap K\pi^{\mu^\vee}U^- \neq \emptyset,$$

then (6.1) and (6.2) imply that $\langle \lambda^\vee, \lambda \rangle \geq \langle \mu^\vee, \lambda \rangle$ for every $\lambda \in \Lambda^+$. Hence $\lambda^\vee \geq \mu^\vee$ which proves Theorem 1.9(3).

Similarly, if $g \in K\pi^{\mu^\vee}U$ then

$$g^{-1}(v_\lambda) \in \pi^{-\langle \mu^\vee, \lambda \rangle} V^\lambda_\mathcal{O}. \tag{6.3}$$

Hence (6.2) and (6.3) imply that if

$$K\pi^{\lambda^\vee}U^- \cap K\pi^{\mu^\vee}U \neq \emptyset,$$

then $\langle \lambda^\vee, \lambda \rangle \geq \langle \mu^\vee, \lambda \rangle$ for every $\lambda \in \Lambda^+$. Hence $\lambda^\vee \geq \mu^\vee$. This proves the second assertion of Theorem 1.9(1) and we already know the first assertion.

6.2. Proof of Theorem 1.9(2). The second assertion of Theorem 1.9(2) follows immediately from (6.1) and (6.3). Now we need to prove that every $K\pi^{\lambda^\vee}K \cap K\pi^{\mu^\vee}U$ is finite. But we can write

$$K\pi^{\lambda^\vee}K \cap K\pi^{\mu^\vee}U = \bigcup_{\lambda^\vee \geq \nu^\vee \geq \mu^\vee} K\pi^{\lambda^\vee}K \cap K\pi^{\nu^\vee}U^- \cap K\pi^{\mu^\vee}U. \tag{6.4}$$

Each $K\pi^{\lambda^\vee}K \cap K\pi^{\nu^\vee}U^- \cap K\pi^{\mu^\vee}U$ is finite since it is a subset in the finite set $K\pi^{\nu^\vee}U^- \cap K\pi^{\mu^\vee}U$. Since there are finitely many $\nu^\vee$ such that $\lambda^\vee \geq \nu^\vee \geq \mu^\vee$, it follows that the right hand side of (6.4) is finite.

6.3. Proof of Theorem 1.9(4). This statement is well-known in the finite case and the proof in the affine case is essentially similar. We include it here for completeness.

We need to prove the following

Proposition 6.4. *Let us fix $\mu^\vee \in Q_+^\vee$. Then for sufficiently dominant $\lambda^\vee$ we have*
(1) $K\pi^{\lambda^\vee}K \cap K\pi^{\lambda^\vee-\mu^\vee}U \subset K\pi^{\lambda^\vee}U^-$
(2)$K\pi^{\lambda^\vee}U^- \cap K\pi^{\lambda^\vee-\mu^\vee}U \subset K\pi^{\lambda^\vee}K$

Proof. If $\lambda^\vee$ is dominant, then $K\pi^{\lambda^\vee}I \subset K\pi^{\lambda^\vee}U^-$. So in order to prove (1) it is enough to prove the following

Lemma 6.5. *For sufficiently dominant $\lambda^\vee$ we have*

$$K\pi^{\lambda^\vee}K \cap K\pi^{\lambda^\vee-\mu^\vee}U \subset K\pi^{\lambda^\vee}I$$

Proof. Suppose we have an element $x \in K\pi^{\lambda^\vee}K \cap K\pi^{\lambda^\vee-\mu^\vee}U$. So we may write

$$x = k_1\pi^{\lambda^\vee}k_2 = k_3\pi^{\lambda^\vee-\mu^\vee}u$$

where $k_i \in K, u \in U$. This then implies that there exists $k_4 \in K$ such that

$$\pi^{\lambda^\vee}k_2 = k_4\pi^{\lambda^\vee-\mu^\vee}u. \tag{6.5}$$

We need to show if $\lambda^\vee$ is sufficiently dominant, then $k_2 \in I$. In other words, we need to show that if $k_2 \in IwI$ for $w \in W$ then $w = 1$.

Let us choose a dominant weight λ and let V^λ, v_λ be as before. Then for sufficiently dominant $\lambda^\vee$ the following condition is satisfied:

$$\text{if } w \neq 1, \text{ and } w\lambda^\vee = \lambda^\vee - \beta^\vee, \beta^\vee \in Q_+, \text{ then } \langle \beta^\vee, \lambda \rangle > \langle \mu^\vee, \lambda \rangle. \tag{6.6}$$

Rewriting (6.5) under the assumption that $k_2 \in IwI$, we have

$$\pi^{\lambda^\vee}i_1wi_2 \in K\pi^{\lambda^\vee-\mu^\vee}U, \text{ for } i_1, i_2 \in I$$

Since $I = U_{\mathcal{O}}^+U_\pi^-T_{\mathcal{O}}$ and $\pi^{\lambda^\vee}U_{\mathcal{O}}^+T_{\mathcal{O}}\pi^{-\lambda^\vee} \subset U_{\mathcal{O}}^+T_{\mathcal{O}}$ and $wU_\pi^-w^{-1} \subset I$ we have that

$$\pi^{\lambda^\vee}IwI = \pi^{\lambda^\vee}U_{\mathcal{O}}^+U_\pi^-T_{\mathcal{O}}wI \subset K\pi^{\lambda^\vee}wU_\pi^-U_{\mathcal{O}}^+$$

So if (6.5) holds, then we can conclude that there exists $u^- \in U_\pi^-$, $k \in K$ and $u \in U$ such that

$$\pi^{\lambda^\vee}wu^- = k\pi^{\lambda^\vee-\mu^\vee}u \tag{6.7}$$

Now, apply (6.7) to the highest weight vector v_λ :

$$\pi^{\lambda^\vee} w u^- v_\lambda = k\pi^{\lambda^\vee - \mu^\vee} u v_\lambda \tag{6.8}$$

Consider the left hand side of (6.8) we obtain,

$$||\pi^{\lambda^\vee} w u^- v_\lambda|| \geq ||\pi^{\lambda^\vee} w v_\lambda|| = q^{-\langle \lambda^\vee, w\lambda\rangle}.$$

From the right hand side of (6.7) we obtain,

$$||k\pi^{\lambda^\vee - \mu^\vee} u v_\lambda|| = q^{-\langle \lambda^\vee - \mu^\vee, \lambda\rangle}.$$

Writing $w^{-1}\lambda^\vee = \lambda^\vee - \beta^\vee$ for $\beta^\vee \in Q_+$, we have that

$$q^{-\langle \lambda^\vee, w\lambda\rangle} = q^{-\langle w^{-1}\lambda^\vee, \lambda\rangle} = q^{-\langle \lambda^\vee - \beta^\vee, \lambda\rangle} \leq q^{-\langle \lambda^\vee - \mu^\vee, \lambda\rangle}.$$

This implies that

$$q^{\langle \beta^\vee, \lambda\rangle} \leq q^{\langle \mu^\vee, \lambda\rangle}$$

which contradicts the fact that we have chosen $\lambda^\vee$ to satisfy (6.6).

□

6.6. Proof of Proposition 6.4(2). There exists a finite set $\Omega \subset U^-$ such that

$$K\pi^{\lambda^\vee - \mu^\vee} U \cap K\pi^{\lambda^\vee} U^- \subset \bigcup_{u^- \in \Omega} K\pi^{\lambda^\vee} u^-.$$

If all such $u^- \in \Omega$ actually lie in K we are done. Let $\Omega_0 \subset \Omega$ be the (finite) subset of $u^- \in \Omega$ such that $u^- \notin K$. Let $u^- \in \Omega_0$. Since $u^- \notin K$ there exists $w \in W$ with $l(w) > 1$ such that $u^- \in U_w^-$, where U_w^- is as in Section 3. Let us write

$$u^- v_\rho = v_\rho + \sum_{\gamma \in Q_+} v_{\rho - \gamma}.$$

Then there are only finitely many γ which appear in such expressions as $u^- \in \Omega$. By Lemma 3.3, we have that $||u^- v_\rho|| > 0$, so there exists $\gamma \neq 0$ such that $v_{\rho-\gamma} \notin V^\rho_{\mathcal{O}}$.

By the hypothesis, we have an expression of the form

$$\pi^{\lambda^\vee} u^- = k\pi^{\lambda^\vee - \mu^\vee} u^+.$$

Applying the right hand side to v_ρ we find that

$$||k\pi^{\lambda^\vee - \mu^\vee} u^+ v_\rho|| = q^{-\langle \lambda^\vee - \mu^\vee, \rho\rangle}$$

whereas when we apply the same expression to the left hand side, we obtain that

$$||\pi^{\lambda^\vee} u^- v_\rho|| \geq ||\pi^{\lambda^\vee} v_{\rho-\gamma}|| \geq q^{-\langle \lambda^\vee, \rho - \gamma\rangle}.$$

By choosing $\lambda^\vee$ sufficiently dominant so that $\langle \lambda^\vee, \gamma\rangle > \langle \rho, \mu^\vee\rangle$ for the finitely many γ which can occur, we obtain a contradiction.

□

7. Appendix: Proof of Proposition 5.9

7.1. Let $A = (a_{ij})$ be an $n \times n$ lower triangular, unipotent matrix with coefficients bounded by $||a_{i,j}|| < C$. Then A^{-1} is also a bounded matrix whose entries are bounded by some $C' = C'(C, n)$. For $k \geq 0$, assume we have an expression

$$A = A_0 + \epsilon \tag{7.1}$$

where A_0 is lower triangular unipotent and ϵ are $n \times n$ matrices such that the coefficients of ϵ all lie in $\pi^k \mathcal{O}$.

Lemma 7.2. *Let A and A_0 be $n \times n$ unipotent lower triangular matrices whose coefficients have norm bounded by some number C. Then given any $m \in \mathbb{Z}_{\geq 0}$, there exists $l_0 = l_0(C, n, m)$ such that if $A = A_0 + \epsilon$ with $\epsilon \equiv 0 (\operatorname{mod} \pi^{l_0})$ then*

$$A^{-1} = A_0^{-1} + \delta$$

with $\delta \equiv 0 (\operatorname{mod} \pi^m)$.

Proof. We first note that the following two facts:

(1) Given $p \geq 0$, there there exists a positive integer $d_0 = d_0(C, n, p)$ such that if $\epsilon \equiv 0 \ (\operatorname{mod} \pi^{d_0})$, then $\epsilon A^{-1} \equiv 0 \ (\operatorname{mod} \pi^p)$.

(2) The Lemma is true when $A_0 = \mathbb{I}_n$, the $n \times n$ identity matrix.

Indeed, (1) follows from the fact that if the coefficients of A are bounded by C then those of A^{-1} must be bounded by some C'. Also, (2) follows from the identity of matrices,

$$(\mathbb{I}_n + \epsilon)^{-1} = \mathbb{I}_n - \epsilon + \epsilon^2 + \cdots .$$

Now if we write

$$A = A_0 + \epsilon, \tag{7.2}$$

then we have

$$A^{-1} = (\mathbb{I}_n + A_0^{-1}\epsilon)^{-1} A_0^{-1}.$$

Choosing ϵ sufficiently small (in the π-adic topology), then thanks to (1) and (2) we can assume that

$$(\mathbb{I}_n + \epsilon A_0^{-1})^{-1} = \mathbb{I}_n + \tilde{\delta}$$

where $\tilde{\delta}$ is sufficiently small. Hence,

$$A^{-1} = A_0^{-1}(\mathbb{I}_n + \tilde{\delta}) = A_0^{-1} + A_0^{-1}\tilde{\delta}$$

and by stipulating that $\tilde{\delta}$ is sufficiently small we may assume that

$$\delta = A_0^{-1}\delta$$

is arbitrarily small. In sum, choosing ϵ sufficiently small in (7.2) we have $A^{-1} = A_0^{-1} + \delta$ with δ arbitrarily small (uniformly for all A, A_0 lower triangular unipotent matrices with coefficients bounded by C.) $\square$

7.3. Proof of Proposition 5.9. Let us write

$$A = X_0 + \epsilon_1, B = X_0 + \epsilon_2, \epsilon_i \equiv 0 \pmod{\pi^r} \text{ for } i = 1, 2.$$

Given a $p \geq 0$, we may use Lemma 7.2 to choose $r = r(p, C, n)$ sufficiently large so that

$$B^{-1} = X_0^{-1} + \delta \text{ for } \delta \equiv 0 (\operatorname{mod} \pi^p).$$

For this choice of r we have

$$AB^{-1} = (X_0 + \epsilon_1)(X_0^{-1} + \delta) = 1 + X_0\delta + \epsilon_1 X_0^{-1} + \epsilon_1\delta.$$

Pick p sufficiently large so that $X_0\delta$ (and also trivially $\epsilon_1\delta$) is integral. Increasing the value of r if necessary, we can also arrange that $\epsilon_1 X_0^{-1}$ is also always integral. The proposition follows.

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