LOCAL BIRKHOFF DECOMPOSITIONS FOR LOOP GROUPS AND A FINITENESS RESULT

by

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Abstract. — Let $\mathbf{G}$ denote an affine Kac–Moody group, and G its points over the local field $\mathbb{F}_q((s))$. We establish a local Birkhoff decomposition for a subset of G in terms of a pair of subgroups roughly of the form $\mathbf{G}(\mathbb{F}_q[[s]])$ and $\mathbf{G}(\mathbb{F}_q[s^{-1}])$. Our techniques are global-to-local and use the reduction theory for loop groups due to H. Garland. Building on these ideas, we establish the finiteness of a set whose cardinality is related to spherical R-polynomials in D. Muthiah's conjectural double-affine Kazhdan–Lusztig theory.

Contents

1. Introduction

Let $\mathbb{F}_q$ denote the finite field with q elements, $F = \mathbb{F}_q(s)$ the function field of $\mathbb{P}^1(\mathbb{F}_q)$ and $\mathcal{F} := \mathbb{F}_q((s))$ the completion of F at the origin. Write $\mathbf{G}$ for the central extension of the loop group of a finite dimensional group $\mathbf{G}_o$ and denote by $\mathbf{G}^e$ the full affine Kac–Moody group. We shall let G_o, G, and G^e be the corresponding set of $\mathcal{F}$-points. The first aim of this note is to establish a decomposition for a certain sub*set* $G^+ \subset G^e$ singled by Garland in [15] in the course of his study of p-adic Cartan decompositions; this subset (and close variants of it) continue to play a central role in the study of p-adic loop groups. Letting K and K_∞ denote subgroups of G roughly [(1)] of the form equal to $\mathbf{G}(\mathbb{F}_q[[s]])$ and $\mathbf{G}(\mathbb{F}_q[s^{-1}])$ respectively, we show

Theorem. — *(Local Birkhoff Decomposition, see Theorem 5.1) Each (K, K_∞) double coset of G^+ has a representative in the torus corresponding to a dominant coweight.*

Note this is *not* the decomposition of G corresponding to a pair of opposed Borel (or parabolic) subgroups, a result sometimes called the Birkhoff factorization (see [36, Ch. 8]). Whereas the factorization of *loc. cit.* depends only on algebraic structure of loop groups over an arbitrary field, the factorization we establish here depends both on this algebraic structure as well as that of the local field $\mathscr{F}$. For this reason, we call the factorization studied here a *local Birkhoff decomposition* to distinguish it from the purely (algebraic) Birkhoff decomposition studied earlier. We might remark that although local Birkhoff decompositions for $\mathbf{G}_o(\mathcal{F})$ can be seen as algebraic Birkhoff decompositions for $\mathbf{G}(\mathbb{F}_q)$ (for a pair of opposed, maximal parabolics), we do not know of an analogous statement for the local Birkhoff decomposition of $G := \mathbf{G}(\mathcal{F})$.

The second result of this note is the following finiteness result.

Theorem. — *(Finiteness theorem, see Theorem 5.2) For $\check{\lambda}$ dominant and $\check{\mu}$ any coweight, writing $\pi^{\check{\lambda}}$ and $\pi^{\check{\mu}}$ for the corresponding elements in the torus of G, the quotient*

$$K \setminus K\pi^{\check{\lambda}}K \cap K\pi^{\check{\mu}}K_\infty^1 \tag{1.1}$$

is a finite set and nonempty only if $\check{\mu} \leq \check{\lambda}$. Here $K_\infty^1 \subset K_\infty$ consists of elements which are equal to the identity 'modulo s^{-1}' (see §4.2.1).

The sets studied in the above theorem are conjectured to have cardinality which is polynomial in q. We do not investigate the polynomiality here, but note that our proof of finiteness can be extended to prove an Iwahori variant (see §5.2.1).

To recall the motivation behind our study of the above finiteness result, recall that $\mathbf{G}$ is replaced by $\mathbf{G}_o$, a quotient as in the previous theorem has finite cardinality given by a polynomial in q that is equal to the (parabolic or spherical) affine R-polynomials of Kazhdan and Lusztig, see [30, §2, Appendix A] and [14]. Recently D. Muthiah (partly with D. Orr) [31–33] has proposed a double-affine analogue of Kazhdan–Lusztig theory for the Iwahori–Hecke type algebras of [2, 13] that were attached to G^+ in which the cardinalities of the sets (1.1) are to play the role of spherical R-polynomials. However, prior to this work, these sets were not known to be finite.

On the other hand, Muthiah (for certain affine types, see [32]) and N. Bardy-Panse, A.Hébert, and G.Rousseau (for more general Kac–Moody types, see [5]) have developed a substantial theory to import such questions into the study of *masures*, a combinatorial/geometric object introduced by S. Gaussent and G. Rousseau ([22, 37]) that generalizes Bruhat–Tits buildings to the Kac–Moody context. In both the works of Muthiah and Bardy-Panse *et al.*, a theory is built conditional on the existence of certain local Birkhoff factorizations (more general than the ones considered here it seems, see the conjectures in [5, §4.1.1]). Building upon this and proving subtle combinatorial results about the "affine" Bruhat order on the Tits–Weyl semi-group, *i.e.*, the affine Weyl group-type object governing p-adic Kac–Moody groups, (see [25, 33, 35, 41]), A. Hébert and P. Philippe [26] have now proven the polynomiality of Iwahori versions of (4.2.1). These last results are valid in a general Kac–Moody setting, though still conditional on the existence of certain local Birkhoff decomposition. We hope some ideas from this paper can be used to eventually lift this restriction.

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(1) In the main body of the paper, we adopt slightly different, but *possibly* equivalent, definitions for these subgroups, see Remark 4.2.1

In place of masures, our approach rests on a global-to-local principle which we learnt some time ago from Howard Garland: local Birkhoff decompositions are restatements of Grothendieck's theorem on the splitting of bundles over the projective line and hence follow from reduction theory for groups defined over the function field of the projective line. Moreover, Garland has also explained to the author how this principle can be adapted to loop groups. A version of his arguments were recorded in [34, Prop. 17.2.2], where they were used to give an alternative treatment of a result due to M. Kapranov about Grothendieck's theorem for loop group valued bundles [29, §7.1]. We also note that the idea of treating local question on G in terms of global questions on $\mathbb{P}^1$ appears in [10, 12]. As such, one might hope that variants of our methods may have applications to transverse slices in the double affine Grassmannian and associated affine Coloumb branches (see [8], [11]). In this direction, we also refer to earlier work on calorons by H. Garland and M. Murray [16].

As for the finiteness result, we combine global techniques with results in representation theory of affine Lie algebras (such results are similar to those appearing in [1, 7]). We also need to use the finiteness results established earlier in [7, 13]. Such results also hold for general Kac-Moody groups (see [23, 24] for an approach using masures, or [1] using representation theory). We also use a convexity result here that follows from the formula for Satake transform established in the affine case by [13] or in the general Kac–Moody case in [4]. We expect that with slightly different arguments, the methods of this paper can be made independent of this formula. The reduction theory of Garland, on which our proofs rest, is only known in affine type, and this poses an obstacle to extending our works to general Kac–Moody type. Along these lines, it may be worth emphasizing that our proofs require working in the complete affine Kac–Moody group of [18] whose dependency on a defining representation is exploited a few times. This is in contrast to the minimal groups appearing in [6, 32] or even [13].

Although motivated by them, we have not touched upon the connections to the Iwahori–Hecke algebras of [13] and [3] in this note. For example, it would be interesting to understand if the spherical R-polynomials defined here are consistent with affine versions of the Kato–Lusztig or 'geometric' Casselman–Shalika formulas, both of which should follow from existing techniques (though not yet appearing in precisely the form we need, the work of Viswanath [39] is close). Muthiah has undertaken a similar study in his context, see Remark 5.2 and the references therein.

Acknowledgments: This project started as a joint one with Dinakar Muthiah who explained to us the constructions in [32] and proposed a number of intriguing questions. We thank him for the many discussions we have had about these topics and for sharing his thoughts on related matters. This note would not exist but for them.

We would also like to thank Howard Garland for explaining to us his proof of a local Birkhoff decomposition in finite dimensions a number of years ago– it is essentially just reproduced here with 'standard loop group' modifications (that he also explained to us on different occasions). We thank him for generously sharing his ideas with us over the years.

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2. Basic notation on fields and adeles

As in the introduction, we let $\mathbb{F}_q$ denote the finite field of order q.

2.0.1. Local fields. — Let $\mathcal{F}$ be a non-archimedean local field (in this paper, we always focus on the case that $\mathcal{F}$ has positive characteristic). Write $\mathcal{O} \subset \mathcal{F}$ be the ring of integers of $\mathcal{F}$ and we reserve the notation π for a choice of uniformizing element. Denote by κ the residue field and by q its cardinality. Write $\mathrm{val}_{\mathcal{F}} : \mathcal{F} \to \mathbb{Z} \cup \{\infty\}$ be the valuation on $\mathcal{F}$ and write $|\cdot|_{\mathcal{F}} := |\cdot|$ for the corresponding norm, $|x| = q^{-\mathrm{val}_{\mathcal{F}}(x)}$.

2.0.2. Global Fields. — Let $F = \mathbb{F}_q(s)$ be a function field of transcendence degree 1 over $\mathbb{F}_q$. Let $|F|$ denote the set of places of F and for each place $\nu \in |F|$ denote by F_ν the corresponding completion with valuation ring $\mathcal{O}_\nu \subset F_\nu$, and residue field $\kappa(\nu)$, a finite field of size $q_\nu := q^{[\kappa(\nu):\mathbb{F}_q]}$. Denote by $\mathrm{val}_\nu : F_\nu \to \mathbb{Z} \cup \{\infty\}$ the valuation, and choose $s_\nu \in \mathcal{O}_\nu$ a uniformizing element. For each write the local norm as $|\cdot|_\nu$.

2.0.3. Places of F. — The places of $F = \mathbb{F}_q(s)$ can be described explicitly as follows, see [40, §3.1, Thm. 2] for more details. Two special places will be used in this paper.

- The place ν_0: if $x \in F^*$, there exists a unique integer n such that $x = s^n \frac{\beta}{\alpha}$ where $\beta, \alpha \in k[s]$ are relatively prime to each other and to s. Set $\mathrm{val}_{\nu_0}(x) = n$ and $|x|_\nu = q^{-n}$. In this case, we have $\mathcal{O}_{\nu_0} = \mathbb{F}_q[[s]]$ and $F_{\nu_0} = \mathbb{F}_q((s))$. We also pick π_{ν_0} to be the uniformizer corresponding to s and note that in this case $q_{\nu_0} = q$,
- The place ν_∞ is the unique place ν such that $|s|_\nu > 1$. We have that $\mathcal{O}_{\nu_\infty}$ and F_{ν_∞} are isomorphic to $\mathbb{F}_q[[s^{-1}]]$ and $\mathbb{F}_q((s^{-1}))$ respectively. Again, in this case we have $q_{\nu_\infty} = q$ and we shall reserve the term π_{ν_∞} for the uniformizer corresponding to s^{-1}.

Lemma. — *Let $x \in F$ be such that $|x|_\nu \leq 1$ for all $\nu \in |F| \setminus \{\nu_\infty\}$, then $x \in k[s]$. Similarly for $x \in F$ such that $|x|_\nu \leq 1$ for all $\nu \in |F| \setminus \{\nu_0\}$, we have $x \in k[s^{-1}]$.*

2.0.4. Adeles and ideles. — Letting $\mathbb{A}$ and $\mathbb{I}$ denote the ring of adeles and the group of ideles of F, one has an embeddings $\iota : F \hookrightarrow \mathbb{A}$ obtained from the completion maps $\iota_\nu : F \hookrightarrow F_\nu$. We continue to denote the restriction of ι to F^* as $\iota : F^* \hookrightarrow \mathbb{I}$. For each $x \in F$, we often just write $x_\nu := \iota_\nu(x)$ and so that $\iota(x) = (x_\nu)_{\nu \in |F|}$.

The global norm on $\mathbb{I}$ is defined as the product, $|x|_{\mathbb{I}} = \prod_{v \in |F|} |x_v|_v$ for $x = (x_v) \in \mathbb{I}$. Let us record here also the *product formula*: for $a \in F^*$, we have $|a|_{\mathbb{I}} = 1$. By a *divisor* of F we shall mean a formal sum $\mathfrak{a} = \sum_{\nu \in |F|} a(\nu)[\nu]$ with $a(\nu) \in \mathbb{Z}$ and $a(\nu) = 0$ for almost all $v \in |F|$. For any divisor $\mathfrak{a}$, its degree is defined to be $\deg(\mathfrak{a}) := \sum_{v \in |F|} a(v)[\kappa(\nu) : \mathbb{F}_q]$. Given any divisor $\mathfrak{a}$, we define

$$\Omega(\mathfrak{a}) = \prod_{\nu \in |F|} \pi_\nu^{-a(\nu)} \mathcal{O}_\nu \subset \mathbb{A}; \tag{2.1}$$

i.e., an adele x lies in $\Omega(\mathfrak{a})$ if $\mathrm{val}_v(x) \geq -a(v)$ for all $v \in |F|$, and so for such an $x \in \Omega(\mathfrak{a})$

$$|x|_{\mathbb{I}} = \prod_v |x|_v = \prod_v q_v^{-\mathrm{val}_v(x)} \leq \prod_v q_v^{a(\nu)} = q^{\deg(\mathfrak{a})} \tag{2.2}$$

Proposition. — [40, Corollary 3, p. 101] *If $\mathfrak{a}$ is any divisor with* $\deg(\mathfrak{a}) \geq -1$ *then* $\mathbb{A}_F = F + \Omega(\mathfrak{a})$

Remarks. — For any function field L, there exists an integer $g \geq 0$ (the *genus* of the function field) so that a similar statement holds when $\deg(\mathfrak{a}) > 2g - 2$ (the genus of F is known to be 0).

2.0.5. Idelic approximation. — For any subset $U \subsetneq |F|$ with finite complement, we also define $\mathbb{A}(U), \mathbb{I}(U)$ and $\mathbb{O}(U)$ as restricted direct product over the places in U, *i.e.,* $\mathbb{I}(U) = \prod'_{v \in U} F_v^*$ and $\mathbb{O}(U) := \prod_{v \in U} \mathscr{O}_v^*$. Then one has the following approximation theorem (*cf.* [40, Chap VI])

$$\mathbb{I}(U) = \mathbb{O}(U) F^*, \tag{2.3}$$

where again $F^* \hookrightarrow \mathbb{I}(U)$ via ι.

3. Preliminary material on Loop Groups

3.1. Affine Kac–Moody algebras. —

3.1.1. Generalized Cartan Matrices. — Fix a natural number ℓ and write $I = \{1, \dots, \ell+1\}$ and $I_o = \{1, \dots, \ell\}$. Choose $\mathsf{A} = (a_{ij})_{i,j \in I}$ an indecomposable generalized Cartan matrix (GCM) of *untwisted, affine* type [27, Ch.6]. As A is of affine type, its null space is one dimensional and we write $\delta = (d_1, \dots, d_{\ell+1})$ for the unique vector with integral, relatively prime, and positive entries in this space (cf. [27, Theorem 4.8 (b)]). Similarly, the transpose ${}^t\mathsf{A}$ is again an indecomposable GCM of affine type and we define an analogous vector $\delta^\vee = (\check{d}_1, \dots, \check{d}_{\ell+1})$ in its null space. Note that one always has $\check{d}_{\ell+1} = 1$. We have $d_{\ell+1} = 1$ as A was assumed to be of *untwisted* affine type.

3.1.2. The Lie algebras $\mathfrak{g}$ and $\mathfrak{g}_o$. — Let $\mathfrak{g}(\mathsf{A})$ be the (complex) affine Kac–Moody algebra attached to A. As A is automatically *symmetrizable* (see [27, Lemma 4.6]), $\mathfrak{g}(\mathsf{A})$ admits the following description (see [27, §2.1]): it has generators $e_i, f_i, \check{a}_i \, (i \in I)$ subject to the relations:

$$\begin{array}{lllll} [\check{a}_i, \check{a}_j] = 0, & [e_i, f_j] = \delta_{ij} \check{a}_i, & [\check{a}_i, e_j] = a_{ij} e_j, & [\check{a}_i, f_j] = -a_{ij} f_j & \text{for } i, j \in I \\ (\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0, & (\operatorname{ad} f_i)^{1-a_{ij}} f_j = 0 & & & \text{for } i, j \in I, \, i \neq j. \end{array} \tag{3.1}$$

Let $\mathfrak{h}(\mathsf{A})$ be the linear span of $\check{a}_i \, (i \in I)$; it is an abelian Lie subalgebra of $\mathfrak{g}(\mathsf{A})$ and $\mathfrak{n}_+(\mathsf{A})$ (resp. $\mathfrak{n}_-(\mathsf{A})$) be the Lie algebras generated by $e_i, i \in I$ (resp. $f_i, i \in I$). One has a triangular decomposition $\mathfrak{g}(\mathsf{A}) = \mathfrak{n}_-(\mathsf{A}) \oplus \mathfrak{h}(\mathsf{A}) \oplus \mathfrak{n}_+(\mathsf{A})$. Often we drop the A from the notation and just write $\mathfrak{g}$, $\mathfrak{h}$, *etc.* We define $\mathfrak{g}_o$ to be the Lie algebra generated by $e_i, f_i, \check{a}_i$ for $i \in I_o$, and we often regard it as a subalgebra of $\mathfrak{g}$. We also let $\mathfrak{h}_o$ for the span of $\check{a}_1, \dots, \check{a}_\ell$. Next, we introduce the elements

$$\check{\vartheta} := \check{d}_1 \check{a}_1 + \cdots + \check{d}_\ell \check{a}_\ell \in \mathfrak{h}_o \quad \text{and} \quad \delta = d_1 a_1 + \cdots + d_{\ell+1} a_{\ell+1} \tag{3.2}$$

where $d_i, \check{d}_i$ were introduced above in §3.1.1. Then one verifies that $\mathbf{c} := \check{a}_{\ell+1} + \vartheta^\vee \in \mathfrak{h}$ is central in $\mathfrak{g}$. Write $\langle \cdot, \cdot \rangle : \mathfrak{h} \times \mathfrak{h}^* \to \mathbb{C}$ for the dual pairing and define $a_i \in \mathfrak{h}^* \, (i \in I)$ by specifying $\langle h, a_i \rangle e_i = [h, e_i]$ for $h \in \mathfrak{h}$. Thus we have $a_{ij} = \langle \check{a}_i, a_j \rangle$ for $i, j \in I$. If we define, with a slight abuse of notation, $\delta = d_1 a_1 + \cdots + d_{\ell+1} a_{\ell+1}$, we have $\langle h, \delta \rangle = 0$ for all $h \in \mathfrak{h}$.

3.1.3. Loop presentation and the extended algebra $\mathfrak{g}^e$. — Sometimes it is useful to describe $\mathfrak{g}$ in terms of its "loop presentation," *i.e.* one has an isomorphism [18, Thm. 3.7] of Lie algebras

$$\mathfrak{g} \cong \mathfrak{g}_o \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{c}, \tag{3.3}$$

where the bracket on the right hand side is specified as in [27, 7.2.2] in terms of the Lie bracket on $\mathfrak{g}_o$ as well as normalized Killing form on $\mathfrak{g}_o$. The image of $e_i, f_i, \check{a}_i \in \mathfrak{g}$ for $i \in I_o$ will be equal to the corresponding elements in $\mathfrak{g}_o$. On the other hand, the elements $e_{\ell+1}$ and $f_{\ell+1}$ will be polynomials of degree 1 and -1 in t as specified in (3.6) of *op. cit.* and $\check{a}_{\ell+1}$ is sent to $-\check{\vartheta} + \mathbf{c}$. Let $\mathbf{D}$ be the derivation on $\mathfrak{g}_o \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{c}$ which acts as $t \frac{d}{dt}$ on $\mathfrak{g}_o \otimes \mathbb{C}[t, t^{-1}]$ and 0 on $\mathbb{C}\mathbf{c}$. One can then form the extended affine Lie algebra as the semi-direct product (in the sense of Lie algebras) on the space

$$\mathfrak{g}^e := \left(\mathfrak{g}_o \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{c} \right) \rtimes \mathbb{C}\mathbf{D}. \tag{3.4}$$

This *extended affine Kac–Moody algebra* has a 'Cartan subalgebra' of the form

$$\mathfrak{h}^e = \mathfrak{h}^e(\mathsf{A}) := \mathfrak{h} \oplus \mathbb{C}\mathbf{D} = \mathbb{C}\mathbf{c} \oplus \mathfrak{h}_o \oplus \mathbb{C}\mathbf{D}, \tag{3.5}$$

and denote by $\langle \cdot,\cdot \rangle : \mathfrak{h}^e \times (\mathfrak{h}^e)^* \to \mathbb{C}$ the dual pairing extending the one defined above. Extend the elements a_i above to elements of $(\mathfrak{h}^e)^*$ by requiring $\langle \mathbf{D}, a_i \rangle = 0$ for $i \in I_o$ and $\langle \mathbf{D}, a_{\ell+1} \rangle = 1$.

3.1.4. Invariant bilinear form. — Setting $\varepsilon_i := d_i (\check{d_i})^{-1}$ for $i \in I$, we define a symmetric bilinear form $(\cdot,\cdot)$ on $\mathfrak{h}^e$ by requiring $(\check{a}_i, h) = \langle h, a_i \rangle \varepsilon_i$ for $i \in I$, $h \in \mathfrak{h}$ and $(\mathbf{D},\mathbf{D}) = 0$. Then $(\cdot,\cdot)$ is non-degenerate (see [27, Lemma 2.1b]) and induces an isomorphism $\nu : \mathfrak{h}^e \to (\mathfrak{h}^e)^*$. Write also $(\cdot,\cdot)$ for the corresponding form on $(\mathfrak{h}^e)^*$. We have that $(\delta,\delta) = 0$.

3.1.5. Roots. — The roots of $\mathcal{R}$ of $\mathfrak{g}^e$ will be set of all non-zero $\varphi \in (\mathfrak{h}^e)^*$ such that

$$\mathfrak{g}^\varphi := \{ x \in \mathfrak{g} \mid [h,x] = \langle h, \varphi \rangle x \text{ for all } h \in \mathfrak{h}^e \} \neq 0. \tag{3.6}$$

Each such φ is a unique integral linear combination of the simple roots $\Pi := \{a_1, \ldots, a_{\ell+1}\}$ with all non-negative or all non-positive coefficients. This allows us to define the positive and negative roots as $\mathcal{R}_+$ and $\mathcal{R}_- := -\mathcal{R}_+$, and we also define the *root lattice* to be $\mathcal{Q} := \mathbb{Z}a_1 + \cdots + \mathbb{Z}a_{\ell+1}$.

Let $\mathcal{R}_o$ denote the set of roots φ such that $\mathfrak{g}^\varphi \subset \mathfrak{g}_o$. The set $\mathcal{R}$ can partitioned into two sets $\mathcal{R} = \mathcal{R}_{re} \sqcup \mathcal{R}_{im}$ where $\mathcal{R}_{re}$ denotes the real roots and $\mathcal{R}_{im}$ denotes the imaginary roots. Recalling the definition of δ from (3.2), we have

$$\mathcal{R}_{im} = \{ n\delta \mid n \in \mathbb{Z} \setminus \{0\} \} \quad \text{and} \quad \mathcal{R}_{re} = \{ \alpha + n\delta \mid \alpha \in \mathcal{R}_o, n \in \mathbb{Z} \}, \ . \tag{3.7}$$

The positive imaginary roots $\mathcal{R}_{im,+} = \mathcal{R}_{im} \cap \mathcal{R}_+$ are characterized by the condition $n > 0$, and the positive real roots $\mathcal{R}_{re,+} := \mathcal{R}_+ \cap \mathcal{R}_{re}$ are further specified by the condition that $\alpha > 0$ and $n \geq 0$ or $\alpha < 0$ and $n > 0$. For $a \in \mathcal{R}$, define the root multiplicity $\mathrm{m}(a) := \dim_{\mathbb{C}} \mathfrak{g}^a$. If $a \in \mathcal{R}_{re}$, we have $\mathrm{m}(a) = 1$ and if $a \in \mathcal{R}_{im}$, we have $\mathrm{m}(a) = \ell$.

3.1.6. (Real) coroots. — For $a_i \in \Pi$ we have defined the coroot $\check{a}_i \in \mathfrak{h}$. We extend this and associate a coroot $\check{a} \in \mathfrak{h}$ to $a \in \mathcal{R}$ as follows. Define $x_{a_i} := \varepsilon_i^{-1} \check{a}_i$ for $i \in I$; and for $\varphi := \sum_{i \in I} c_i a_i \in \mathcal{R}_{re}$, set $x_\varphi := \sum_{i \in I} c_i x_{a_i}$. Then let $\check{a} := \frac{2}{(a,a)} x_a$ for $a \in \mathcal{R}_{re}$; if $a = a_i$, the two meanings of $\check{a}_i$ coincide.

3.1.7. Chevalley basis. — In [17, §4], a basis Ψ of $\mathfrak{g}$ that extends the usual notion of a Chevalley basis from finite-dimensional Lie theory was introduced. It is again called a Chevalley basis (for $\mathfrak{g}$) and can be described as follows starting from a certain Chevalley basis Ψ_o for $\mathfrak{g}_o$ constructed in *op. cit.* and containing the elements $h_i := \check{a}_i, i \in I_o$ together with elements $E_\alpha \in \mathfrak{g}_o^\alpha$ for each $\alpha \in \mathcal{R}_o$. Now for $a \in \mathcal{R}_{re}$ of the form $a = \alpha + n\delta$ with $\alpha \in \mathcal{R}_o$, we let $\xi_a := E_\alpha \otimes t^n$; and, for each $a = n\delta \in \mathcal{R}_{im}$ we let $\xi_i(n) := h_i \otimes t^n$ for $i \in I_o$. The basis Ψ of $\mathfrak{g}$ then consists of the elements

$$\Psi := \{ h_i := \check{a}_i, i \in I \} \ \cup \{ \xi_i(n), i \in I_o, n \in \mathbb{Z}_{\neq 0} \} \ \cup \ \{ \xi_a, a \in \mathcal{R}_{\mathrm{Re}} \}. \tag{3.8}$$

3.1.8. Weights. — An element $\lambda \in (\mathfrak{h}^e)^*$ is called an integral weight if $\langle \check{a}_i, \lambda \rangle \in \mathbb{Z}$ for $i \in I$ and if $\langle \mathbf{D}, \lambda \rangle \in \mathbb{Z}$. Denote by Λ the set of all such elements. We define the set of dominant, integral weights $\Lambda_+ \subset \Lambda$ by the condition $\langle \check{a}_i, \lambda \rangle \geq 0$ for $i \in I$. If $\lambda \in \Lambda$ and there exists some $i \in I$ such that $\langle \check{a}_i, \lambda \rangle \neq 0$, we say that λ is *normal*.

For each $j \in I$, define the fundamental weights $\lambda_j \in (\mathfrak{h}^e)^*$ for $j \in I$ as

$$\langle \check{a}_i, \lambda_j \rangle = \delta_{ij} \text{ and } \langle \mathbf{D}, \lambda_j \rangle = 0, \tag{3.9}$$

we find that the elements $\delta, \lambda_1, \ldots, \lambda_{\ell+1}$ form a basis of Λ. Let us also set

$$\rho = \sum_{i \in I} \lambda_i. \tag{3.10}$$

It has the important property that $\langle \check{a}_i, \rho \rangle = 1$ for $i \in I$ and $\langle \mathbf{D}, \rho \rangle = 0$.

3.1.9. Coweights. — Dually, we define the coweights $\check{\Lambda} \subset \mathfrak{h}^e$ as those elements $\check{\lambda} \in \mathfrak{h}^e$ such that $\langle \check{\lambda}, a_i \rangle \in \mathbb{Z}$ for $i \in I$ and $\langle \check{\lambda}, \lambda_{\ell+1} \rangle \in \mathbb{Z}$ as well. Similarly to what we did above, we define a notion of dominant coweights $\check{\Lambda}_+$ and introduce a partial order on $\check{\Lambda}$ by specifying $\check{\lambda} \leq \check{\mu}$ when $\check{\mu} - \check{\lambda}$ is a non-negative (integral) sum of $\check{a}_i, i \in I$. It is easy to see that for fixed $\check{\lambda}, \check{\mu} \in \check{\Lambda}$, the set $\{\check{\xi} \in \check{\Lambda} \mid \check{\mu} \leq \check{\xi} \leq \check{\lambda}\}$ is always finite.

3.1.10. Weyl group and Tits cone. — Let $W := W(\mathsf{A})$ be the Weyl group associated to A. It is a Coxeter group whose generators we shall write as $\{s_i\}_{i \in I}$. Denote by $\ell : W \to \mathbb{N}$ the length function. The set of elements with length bounded above is finite.

Define the *Tits cone* $\overline{X} \subset \check{\Lambda}$ as

$$\overline{X} := \bigcup_{w \in W} w(\check{\Lambda}_+). \tag{3.11}$$

From [28, Proposition 1.9(a)], we have $\overline{X} = \{\check{\lambda} \in \Lambda^\vee | \langle \check{\lambda}, \delta \rangle > 0\} \sqcup \mathbb{Z}\mathbf{c}$. In terms of the decomposition of (3.5), the elements of the Tits cone can be written as

$$\check{\lambda} = m\mathbf{c}, m \in \mathbb{Z} \quad \text{or} \quad \check{\lambda} = m\mathbf{c} + \check{\lambda}_o + r\mathbf{D} \text{ with } m, r \in \mathbb{Z}, \check{\lambda}_o \in \mathfrak{h}_o, \text{ and } r > 0 \ . \tag{3.12}$$

The collection of elements of the second type will be called the *interior of the Tits* cone and denoted by X from now on. Note that X is stable under the action of W and each $\check{\lambda} \in X$ has finite stabilizer.

3.2. Representations and Chevalley Forms. —

3.2.1. Enveloping algebras and filtrations. — For a Lie algebra $\mathfrak{s}$, let $\mathcal{U}(\mathfrak{s})$ denotes its universal enveloping algebra. Note that $\mathcal{U}(\mathfrak{n})$ is isomorphic to the tensor algebra of $\mathfrak{n}$. Denoting by $F^d\mathcal{U}(\mathfrak{n})$ the subspace of linear combination of elements of the form

$$e_{i_1}^{n_1} \cdots e_{i_k}^{n_k} \text{ such that } n_1 + \cdots + n_k = d, \tag{3.13}$$

we obtain a natural filtration $F^\bullet\mathcal{U}(\mathfrak{n})$.

Generalizing the $\mathbb{Z}$-form of Chevalley–Kostant from finite-dimensions (see [38, Ch. 2]), Garland has introduced a $\mathbb{Z}$-form $\mathcal{U}_\mathbb{Z}$ of $\mathcal{U} := \mathcal{U}(\mathfrak{g})$ in [17, §5] which is generated as an algebra by the divided powers of elements from the Chevalley basis from §3.1.7, namely $\xi_a^{(n)} := \frac{\xi_a^n}{n!}$ for $a \in \mathcal{R}_{re}$ and $n \geq 0$. He has also produced a natural $\mathbb{Z}$-basis of this algebra in terms of monomials (see [17, Thm 5.8]).

3.2.2. The modules V^λ. — Given any *normal* $\lambda \in \Lambda_+$ (cf. §3.1.8), there exists an *irreducible* highest weight module for $\mathfrak{g}$ (and actually $\mathfrak{g}^e$) with highest weight λ which we denote by V^λ.

1. [17, Lemma 10.4] and [18, Lemma 7.12] For any $v \in V^\lambda$ and $a \in \mathcal{R}_{re}$ there exists a positive integer r (depending on v) such that $\xi_a^r.v = 0$.
2. [17, Theorem 11.3] There exists an admissible basis [(2)] Ω of V^λ such that if $V^\lambda_\mathbb{Z}$ denotes the $\mathbb{Z}$-span of Ω, then $V^\lambda_\mathbb{Z}$ is invariant under the integral form $\mathcal{U}_\mathbb{Z}$. Such a basis is called an *integral, admissible basis*.

We note that $\Omega \cap V^\lambda_\lambda$ consists of a single element, and that such an element is a *primitive* element of $V_\mathbb{Z}$ (here we mean primitive in the sense of lattices, *i.e.* it is not a positive integral multiple element of another element in $V_\mathbb{Z}$). Such a highest weight vector is often denoted as $\mathbf{v}$ or $\mathbf{v}_\lambda$ and is called a *primitive, highest weight vector.*

Using the admissible basis, for any field k we can construct the module $V^\lambda_k := V^\lambda_\mathbb{Z} \otimes_\mathbb{Z} k$ for the Lie algebra $\mathfrak{g}_k := \mathfrak{g}_\mathbb{Z} \otimes_\mathbb{Z} k$ (or even $\mathfrak{g}^e_k := \mathfrak{g}^e_k \rtimes k\mathbf{D}$.)

(2) A basis Ω of V^λ is called admissible if $\Omega = \cup_{\nu \in \Xi_\lambda} (\Omega \cap V^\lambda_\nu)$

3.2.3. Weights of a representation. — For each representation $V := V^{\lambda}$ as in the previous paragraph, we denote the set of weights of the representation as

$$\Xi(V) = \{\mu \in (\mathfrak{h}^e)^* \mid V_\mu := \{v \in V \mid hv = \mu(h)v \text{ for all } h \in \mathfrak{h}^e\} \neq 0\}. \tag{3.14}$$

We have $\Xi(V) \subset \Lambda$. Each $\mu \in \Xi(V)$ satisfies $\mu \leq \lambda$ and we define the depth of $\mu \in \Xi(V)$ as

$$\text{depth}(\mu) = \langle \rho, \lambda - \mu \rangle \geq 0. \tag{3.15}$$

We also remark here that the admissible basis $\Omega = \{\mathbf{v}_0 := \mathbf{v}, \mathbf{v}_1, \dots\}$ constructed in the previous paragraph can be chosen so that $\mathbf{v}_i \in V_{\mu_i}$ and so that if $i \geq j$, then $\text{depth}(\mu_i) \geq \text{depth}(\mu_j)$ (the latter condition is sometimes referred to as being *coherently ordered*).

Write $P(V) \subset \Lambda$ for the lattice generated by $\Xi(V)$, we have

1. [18, Lemma 15.2] $\mathcal{Q} \subset P(V) \subset \Lambda$ where $\mathcal{Q}$ is the root lattice
2. [18, Lemma 15.7] If λ, μ are two normal, dominant weights, then there exists a positive integer m such that $P(V^{m\mu}) \subset P(V^{\lambda})$.

3.2.4. Demazure modules. — Fix V^{λ} as above with weight lattice $\Xi := \Xi^{\lambda}$, integral, admissible basis Ω, and primitive highest weight vector $\mathbf{v} \in \Omega$. For each $w \in W$, let $\Xi(w)$ be the set of $\mu \in \Xi^{\lambda}$ such that $\mu \geq w\lambda$. The weight space $V^{\lambda}_{w\lambda}$ are known to be one-dimensional and contain a unique primitive vector $\mathbf{v}_{w\lambda} \in \Omega \cap V_{w\lambda}$. The Demazure module is the $\mathfrak{g}$-submodule

$$V^{\lambda}(w) := V(w) := \mathcal{U}(\mathfrak{n})\mathbf{v}_{w\lambda}, \tag{3.16}$$

and it has weights in $\Xi(w)$, and hence is finite-dimensional. Write $V^{\lambda}_{\mathbb{Z}}(w)$ for the $\mathbb{Z}$-module generated by the elements from Ω which lie in $V^{\lambda}(w)$. Using it, we can then base change $V^{\lambda}(w)$ to any field L in the usual way and write the corresponding object as $V^{\lambda}(w)_L$.

3.2.5. A result of Joseph. — Suppose we look at the representation with highest weight ρ introduced in (3.10). Denote by $\mathbf{v}_\rho \in V_\rho$ any highest weight vector. For each $w \in W$, pick $\mathbf{v}_{w\rho}$ as in the previous paragraph. Then one has the following result due to A. Joseph

Lemma. — [9, Lemma 18.2] *If $\mathbf{v}_\rho \in F^d\,\mathcal{U}(\mathfrak{n})\mathbf{v}_{w\rho}$ then we must have $d \geq \ell(w)/2$.*

3.3. Loop groups over a general field. — Let $\mathfrak{g}^e$ be extended affine Kac-Moody algebra constructed in §3.1.2, 3.1.3, equipped with Chevalley basis Ψ as in §3.1.7. Fix $\lambda \in \Lambda_+$ a dominant, *normal*, and integral weight (cf. §3.1.8) and denote by $V := V^{\lambda}$ be the irreducible, highest weight representation as in §3.2.2. Choose an integral, admissible, coherently ordered basis Ω with highest weight vector denoted by $\mathbf{v}$ as in §3.2.2. The groups we construct below depend on λ, but whenever λ is to be fixed implicitly, we drop it from our notation. We shall also write Ξ for $\Xi(V^{\lambda})$, i.e., the set of weights. Occasionally, we do have to compare the groups for different λ, and this is facilitated by the arguments in [18, §15] as we review below.

Let k be an arbitrary field.

3.3.1. Defining $\mathbf{G}$ and $\mathbf{G}^e$. — Let $a \in \mathcal{R}_{re}$ and $s \in k$. Then we define

$$\chi^{\lambda}_a(s) := \chi_a(s) := \exp(s\xi_a) = 1 + s\xi_a + s^2\xi_a^{(2)} + \cdots \tag{3.17}$$

Using point (1) from §3.2.2, we may regard $\chi_a(s)$ are defining an (invertible) operator on V^{λ}_k. The subgroup of $\text{Aut}(V_k)$ these elements define will be denoted as $\mathbf{U}_a(k)$. These are the *one parameter subgroups* corresponding to real roots. Next, for $\alpha \in \mathcal{R}_o$ and $\sigma = \sum_{i \geq i_0} c_i t^i \in k((t))$ the expression

$$\chi^{\lambda}_{\alpha}(\sigma) := \chi_{\alpha}(\sigma) := \Pi_{i \geq i_0}\chi_{\alpha + i\delta}(c_i) \tag{3.18}$$

also defines an element of $\text{Aut}(V^{\lambda}_k)$ using [18, Lemma 7.16].

Define the $\mathbf{G}^{\lambda}(k) := \mathbf{G}(k) \subset \mathrm{Aut}(V_k)$ to be the subgroup generated by $\chi_\alpha(\sigma)$ for $\alpha \in \mathcal{R}_o, \sigma \in k((t))$. We also set $\mathbf{G}^{\lambda,e}(k) := \mathbf{G}^e(k)$ to be the group generated by $\mathbf{G}^{\lambda}(k)$ together with the automorphisms $\eta(s) \in \mathrm{Aut}(V_k)$ for $s \in k^*$ defined as follows:

$$\eta(s).v = s^{\langle \mathbf{D}, \mu \rangle} v \quad \text{for} \quad v \in V_{k,\mu}, \mu \in \Xi. \tag{3.19}$$

3.3.2. The torus $\mathbf{H}$ *of* $\mathbf{G}$. — For each $a \in \mathcal{R}_{re}$ and $s \in k^*$, we define

$$w_a(s) := \chi_a(s)\chi_{-a}(-s^{-1})\chi_a(s), \quad \dot{w} := w_a(-1), \quad \text{and} \quad h_a(s) = w_a(s)w_a(-1)\ . \tag{3.20}$$

If $s \in k^*$, then $h_a(s)$ acts diagonally on V_k [18, Lemma 11.2], *i.e.*

$$h_a(s).v = s^{\langle \check{a}, \mu \rangle} v \quad \text{for} \quad v \in V_{k,\mu}, \mu \in \Xi. \tag{3.21}$$

where $\check{a}$ is the coroot attached to $a \in \mathcal{R}_{re}$ defined in §3.1.6. We shall sometimes write $h_{a_i}(s)^{\mu}$ for $s^{\langle \check{a}, \mu \rangle}$. One can also check that if $a, b \in \mathcal{R}_{re}$ we have

$$h_a(s)\chi_b(t)h_a(s)^{-1} = \chi_b(s^{\langle \check{a}, b \rangle} t) \text{ for } s \in k^*, t \in k. \tag{3.22}$$

Using this relation, one can verify the element

$$h_{\mathbf{c}}(s) := h_{a_{\ell+1}}(s)\, h_{a_1}(s)^{\check{d}_1} \cdots h_{a_\ell}(s)^{\check{d}_\ell} \text{ for } s \in k^*, \check{d}_i \text{ as in (3.2)}, \tag{3.23}$$

is central in $\mathbf{G}^e(k)$. Define abelian subgroups

$$\mathbf{H}(k) = \langle h_{a_i}(s) \mid i \in I, s \in k^* \rangle \quad \text{and} \quad \mathbf{H}^e(k) = \mathbf{H}(k) \rtimes_\eta k^* = \{ h\eta(s) \mid h \in \mathbf{H}(k), s \in k^* \}\ . \tag{3.24}$$

One can show that every element in $\mathbf{H}^e(k)$ can be written (not necessarily uniquely) in the form

$$h\eta(s) \text{ where } h := h_{a_1}(s_1) \cdots h_{a_{\ell+1}}(s_{\ell+1}) \text{ with } s, s_i \in k^*, i \in I. \tag{3.25}$$

As such, if we define for each $\mu \in \Xi$ and $h\eta(s) \in \mathbf{H}^e(k)$ (as above) the number $(h\eta(s))^{\mu} \in k^*$ by the condition that $(h\eta(s))v = (h\eta(s))^{\mu} v$ for any $v \in V^{\lambda}_{\mu}$, we may compute

$$(h\eta(s))^{\mu} = s_1^{\langle \check{a}_1, \mu \rangle} \cdots s_{\ell+1}^{\langle \check{a}_{\ell+1}, \mu \rangle} s^{\langle \mathbf{D}, \mu \rangle}, \tag{3.26}$$

and note that this quantity is independent of the expression (3.25).

Remarks. — Let $\mathbf{H}_o$ be the subgroup generated by $h_{a_i}(s)$ for $i \in I_o$ and $s \in k^*$. One can also show that every $a \in \mathbf{H}^e(k)$ has a decomposition $a = h_{\mathbf{c}}(t)\, h_o \eta(s)$ for $t, s \in k^*$ and $h_o \in \mathbf{H}_o$. In fact , in such an expression, t, h_o, and s are uniquely defined.

Using (3.26) and the fact that $P(V)$ is the lattice generated by $\Xi(V)$, we define an isomorphism

$$\mathbf{H}^e(k) \xrightarrow{\cong} \mathrm{Hom}(P(V), k^*), h\eta(s) \mapsto (\mu \mapsto (h\eta(s))^{\mu}. \tag{3.27}$$

The $s \in k^*$ appearing in (3.25) is well-defined, and in the case that each of the fundamental weights $\lambda_i \in P(V)$, the numbers s_i are as well. In other words, if $P(V) \supset \mathrm{Span}_{\mathbb{Z}}(\lambda_i, i \in I)$, then the expression (3.25) is well-defined (this is called the simply-connected case).

3.3.3. — For each $\alpha \in \mathcal{R}_o$ and a non-zero element $\sigma \in k((t))^*$, we define elements of $\mathbf{G}(k)$

$$w_\alpha(\sigma) = \chi_\alpha(\sigma)\chi_{-\alpha}(-\sigma^{-1})\chi_\alpha(\sigma) \quad \text{and} \quad h_\alpha(\sigma) = w_\alpha(\sigma)w_\alpha(1)^{-1}. \tag{3.28}$$

Note that the elements $h_\alpha(\sigma)$ no longer act diagonally on V for a general σ. If $\sigma \in (k[[t]])^*$ is such that $\sigma \equiv 1 \bmod t$ (*i.e.* $\sigma = 1 + \sum_{j \geq 1} \sigma_j t^j, \sigma_j \in k$) we can factorize it as

$$\sigma = \prod_{j=1}^{\infty} (1 - b_j t^j) \tag{3.29}$$

for uniquely defined $b_j \in k$ ([20, Lemma 8.1]). It follows that

$$h_\alpha(\sigma) = \prod_{j\geq 1} h_\alpha(1 - b_j t^j). \tag{3.30}$$

Lemma. — *Let $j \in \mathbb{Z}_{>0}$ and fix $\alpha \in \Pi_o, x \in k$. For any $a \in \mathbf{H}^e(k)$,*

$$a h_\alpha(1 - xt^j) a^{-1} = h_\alpha(1 - a^{j\delta} x t^j). \tag{3.31}$$

Proof. — Following Remark (3.3.2), we write $a \in \mathbf{H}^e(k)$ as $a = h_{\mathbf{c}}(z) a_o \eta(s)$ for $a_o \in \mathbf{H}_o(k)$ and $s, z \in k^*$. Hence, we have $a^{j\delta} = \eta(s)^{j\delta} = s^j$. It suffices to show (3.31) for the following three cases: (i) $a = h_c(z)$ for $z \in k^*$; (ii) $a \in \mathbf{H}_o(k)$; and (iii) $a = \eta(s)$ for $s \in k^*$. Note that in case (i) and (iii), the assertion is that a commutes with $h_\alpha(1 - xt^j)$.

Now, case (i) is clear since $h_c(z)$ is central in $\mathbf{G}(k)$. As for case (ii), if $a = a_o$, we may argue as in [20, p. 249-250]. First one verifies (3.31) holds modulo the central extension (i.e. modulo the subgroup $A_{\mathbf{c}} := \langle h_{\mathbf{c}}(t), t \in k^* \rangle$). This quotient group is some Chevalley group (roughly) of the form $\mathbf{G}_o(k((t)))$ where the elements a_o and $h_\alpha(1 - xt^j)$ both lie in the torus, and hence commute. Hence,

$$a_o h_\alpha(1 - xt^j) a_o^{-1} = \zeta h_\alpha(1 - xt^j) \tag{3.32}$$

where $\zeta \in A_{\mathbf{c}}$. Now we conclude that $\zeta = 1$ since the left hand side of (3.32) must fix the highest weight vector $\mathbf{v}$. Finally, consider case (iii) so that $a = \eta(s), s \in k^*$. Here, one may argue using an explicit formula for the action of $h_\alpha(1 - xt^j)$ on a highest weight module from [7, Proposition 4.8]. Alternatively, one may go back to the Steinberg relations, i.e. (3.28) and verifying the fact that

$$\eta(s) \chi_\alpha(\sigma(t)) \eta(s)^{-1} = \chi_\alpha(\sigma(st)) \text{ for } \alpha \in \mathcal{R}_o, \ \sigma(t) \in k((t)). \tag{3.33}$$

□

3.3.4. — Define the subgroup $\mathbf{B}(k) \subset \mathbf{G}(k)$ generated by elements of the form

$$(1) \chi_\alpha(\sigma) \text{ for } \alpha \in \mathcal{R}_{o,+}, \sigma \in k[[t]]; \ \ (2) \ \chi_\alpha(\sigma) \text{ for } \alpha \in \mathcal{R}_{o,-}, \sigma \in tk[[t]]; \tag{3.34}$$

$$(3) \ h_\alpha(\sigma) \text{ for } \alpha \in \mathcal{R}_{o,+}, \sigma \in k[[t]]^*; \quad \text{and } (4) \ h_{a_{l+1}}(s) \text{ for } s \in k^*. \tag{3.35}$$

Let $\mathbf{U}(k) \subset \mathbf{B}(k)$ be the subgroup generated by elements of the form (1), (2), and (3) above where in (3) we require also that $\sigma \in k[[t]]^*$ satisfies $\sigma \equiv 1 \bmod t$, we have

$$\mathbf{B}(k) = \mathbf{H}(k) \ltimes \mathbf{U}(k). \tag{3.36}$$

We also define $\mathbf{U}^-(k)$ as the subgroup generated by $\mathbf{U}_{-a}(k)$ for $a \in \mathcal{R}_{re}$.

3.3.5. Iwahori–Matsumoto Coordinates on $\mathbf{U}$. — The 'complete group' $\mathbf{U}(k)$ possess a set of coordinates which allow one to carry out certain inductive constructions. Introduce the elements

$$u_+ = \prod_{\alpha \in \mathcal{R}_{o,+}} \chi_\alpha(\sigma_\alpha) \text{ where } \sigma_\alpha = \sum_{j\geq 0} \sigma_{\alpha,j} t^j \in k[[t]], \tag{3.37}$$

$$u_0 = \prod_{i \in I_o} h_{\alpha_i}(\sigma_i) \text{ where } \sigma_i = 1 + \sum_{j\geq 1} \sigma_{i,j} t^j \in (k[[t]])^*, \text{ and} \tag{3.38}$$

$$u_- = \prod_{\alpha \in \mathcal{R}_{o,+}} \chi_{-\alpha}(\sigma'_\alpha) \text{ where } \sigma'_\alpha = \sum_{j\geq 1} \sigma'_{\alpha,j} t^j \in tk[[t]], \tag{3.39}$$

where the products in the first and third line are with respect to some fixed order on $\mathcal{R}_{o,+}$. Given a family of coordinates $\Sigma := (\sigma_{\alpha,j}, \sigma_{i,j}, \sigma'_{\alpha,j})$ as in the above expressions, we write $u(\Sigma)$ for the product of the elements $u_+ u_0 u_-$ defined as above

Proposition. — *Every $u \in \mathbf{U}(k)$ can be written as $u = u(\Sigma)$ for a unique family $\Sigma := (\sigma_{\alpha,j}, \sigma_{i,j}, \sigma'_{\alpha,j})$.*

3.3.6. Variant. — Using (3.30) we may also describe u_0 above as

$$u_0 = \prod_{i \in I_o} \prod_{j \geq 0} h_{\alpha_i}(1 - b_{i,j} t^j) \text{ for } b_{i,j} \in k. \tag{3.40}$$

So, we could also take $\Sigma := (\sigma_{\alpha,j}, b_{i,j}, \sigma'_{\alpha,j})$ and we continue to write $u(\Sigma)$ for the corresponding element in $\mathbf{U}(k)$, replacing u_0 from (3.37) with (3.40). The corresponding version of Proposition 3.3.5 then holds.

3.3.7. Weyl group, inversion sets, unipotent subgroups. — For each $w \in W$, if we write a reduced decomposition $w = s_{i_1} \cdots s_{i_r}$ with each $i_j \in I$, then we define the element

$$\dot{w} := \dot{w}_{a_{i_1}} \cdots \dot{w}_{a_{i_r}} \in \mathbf{G}(k), \tag{3.41}$$

where $\dot{w}_a$ was defined in (3.20). This element does not depend on the choice of reduced decomposition. For each $w \in W$, if $\mathbf{v}$ denotes the (unique) primitive highest weight vector in the Chevalley lattice $V_{\mathbb{Z}}$, then from [18, Lemma 11.2 (i)] and an argument as in one can verify that $\dot{w}\mathbf{v} = \pm \mathbf{v}_{w\lambda}$ where $\mathbf{v}_{w\lambda} \in V^{\lambda}_{w\lambda} \cap \Omega$ was the vector introduced in §3.2.4.

For each $w \in W$, we define the inversion sets

$$S_w := \{a \in \mathcal{R}_{re} \mid w^{-1} a < 0\}. \tag{3.42}$$

Then we may factor $\mathbf{U}(k)$ [19, Cor. 6.5] as

$$\mathbf{U}(k) = \mathbf{U}^w(k) \ltimes \mathbf{U}_w(k) \quad \text{where } \mathbf{U}^w(k) = \mathbf{U}(k) \cap \dot{w}\mathbf{U}(k)\dot{w}^{-1} \quad \text{and} \quad \mathbf{U}_w(k) := \mathbf{U}(k) \cap \dot{w}\mathbf{U}^-(k)\dot{w}^{-1} \tag{3.43}$$

Furthermore, for any fixed order on S_w, every element of $\mathbf{U}_w(k)$ can be uniquely written as

$$\chi_{b_1}(s_1) \cdots \chi_{b_r}(s_r) \text{ with } S_w = \{b_1, \ldots, b_r\}. \tag{3.44}$$

We sometimes write $\mathbf{U}^a(k)$ for $\mathbf{U}^w(k)$ with $w = w_a$ and a simple. Finally, we mention that the (refined) Bruhat decomposition for $\mathbf{G}(k)$ asserts that

$$\mathbf{G}(k) = \sqcup_{w \in W} \mathbf{U}_w(k)\, \dot{w} \mathbf{H}(F) \mathbf{U}(F). \tag{3.45}$$

3.3.8. Varying the defining weight. — Let λ, μ be two normal, dominant, integral weights of $\mathfrak{g}^e$, and define the groups $\mathbf{G}^{\lambda}$ and $\mathbf{G}^{\mu}$ with respect to the representations V^{λ} and V^{μ} respectively. Then Garland has shown [18, Thm. 15.9] that if we have an inclusion at the level of lattices of weights $P(V^{\mu}) \subset P(V^{\lambda})$, then there exists a unique homomorphism $\mathbf{G}^{\lambda}(k) \to \mathbf{G}^{\mu}(k)$ which sends each generator $\chi^{\lambda}_{\alpha}(\sigma)$ of $\mathbf{G}^{\lambda}(k)$ (as defined in (3.18)) to the corresponding generator $\chi^{\mu}_{\alpha}(\sigma(t))$ of $\mathbf{G}^{\mu}(k)$. Combined with the last observation from §3.2.3, for any λ and $i \in I$, there exists a map $\mathbf{G}^{\lambda}(k) \to \mathbf{G}^{m\lambda_i}(k)$ for some positive integer m (here λ_i was a fundamental weight).

3.4. Loop groups over local fields. — We now specialize the constructions of the previous section to the case when $k = \mathcal{F}$ is a local field as in §2.0.1. Let us write with roman letters the $\mathcal{F}$-valued points of the corresponding bold faced object, *i.e.* $G := \mathbf{G}(\mathcal{F}), B := \mathbf{B}(\mathcal{F}), H^e := \mathbf{H}^e(k)$, *etc.*

3.4.1. On the lattice $V_{\mathcal{O}}$ and the group K.— Recall the defining representation $V := V^{\lambda}$ for our group $\mathbf{G}$, and the lattice $V_{\mathbb{Z}} \subset V$ from §3.2.2 with primitive highest weight vector $\mathbf{v}$. By abuse of notation, we often just write V for $V_{\mathcal{F}}$, and also define $V_{\mathcal{O}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}$. Construct a norm $|\cdot|_{\mathcal{F}} := |\cdot|$ on V as follows:

$$|v| = q^n \text{ where } n \text{ is the smallest integer such that } \pi^n v \in V_{\mathcal{O}}. \tag{3.46}$$

We often continue to denote by $\mathbf{v}$ the element in $V_{\mathcal{O}}$ corresponding to the highest weight vector from Ω; note that it satisfies $|\mathbf{v}| = 1$. Let $K \subset G$ denote the subgroup of elements $k \in G$ such that $|kv| = |v|$

for all $v \in V$, *i.e.,*

$$K = \{g \in G \mid gV_{\mathscr{O}} \subset V_{\mathscr{O}}\}. \tag{3.47}$$

One can show (in many instances) that K is also equal to the subgroup $\mathbf{G}(\mathscr{O})$ generated by $\chi_\alpha(\sigma)$ with $\alpha \in \mathcal{R}_o$ and $\sigma \in \mathscr{O}((t))$ (see [34, Prop. 7.4.1] or the techniques from [13, Appendix A]), but we shall not need these fact here. Let us also note here that if $w \in W$, then $\dot{w} \in K$ and one can verify that $||\dot{w}\mathbf{v}|| = ||\mathbf{v}||$, and that, as noted earlier in §3.3.7,we have $\dot{w}\mathbf{v} = \pm \mathbf{v}_{w\lambda}$.

3.4.2. On the torus H. — As is our convention, we write $H := \mathbf{H}(\mathcal{F})$, $H^e := \mathbf{H}^e(\mathcal{F})$, and also set $H_{\mathscr{O}} := \mathbf{H}(\mathcal{F}) \cap K$. For each $a = h\eta(s) \in H^e$ with $a \in H, s \in \mathcal{F}^*$, define an element in

$$\check{\lambda}_a \in P(V)^\vee := \mathrm{Hom}_{\mathbb{Z}}(P(V), \mathbb{Z}) \subset \check{\Lambda} \tag{3.48}$$

as follows: $\langle \mu, \check{\lambda}_a \rangle = n$ where $(h\eta(s))^\mu \in \mathcal{F}^*$ has a factorization as $\pi^n u$ with $n \in \mathbb{Z}, u \in \mathscr{O}^*$.

For each $\check{\lambda} \in P(V)^\vee$, let $\pi^{\check{\lambda}}$ be the unique element in H^e which is sent to $\mu \mapsto \pi^{\langle \mu, \check{\lambda} \rangle}$ under the isomorphism (3.27). One may check that each $a = h\eta(s) \in H^e$ has a factorization

$$a = \pi^{\check{\lambda}_a} \eta(s_{\mathscr{O}}) h_{\mathscr{O}} \text{ where } h_{\mathscr{O}} \in H_{\mathscr{O}} \text{ and } s = s_{\mathscr{O}} \pi^r \text{ with } s_{\mathscr{O}} \in \mathscr{O}^*, r \in \mathbb{Z}. \tag{3.49}$$

3.4.3. Iwasawa and Cartan decomposition. — The Iwasawa decomposition states that every x in G^e can be written

$$x = k\pi^{\check{\lambda}} u \quad \text{with} \quad k \in K, \check{\lambda} \in P(\check{V}), u \in U \tag{3.50}$$

with $\check{\lambda}$ uniquely determined by x and sometimes written as $\check{\lambda}_x$. While this decomposition holds for all of G^e, Garland [15] has argued that a Cartan decomposition only holds over the certain subset

$$G^+ := \{x \in G^e \mid \check{\lambda}(x) \in X\}, \tag{3.51}$$

where the (interior of the) Tits cone was defined in §3.1.10: each $x \in G^+$ may be written as

$$x = k_1 \pi^{\check{\lambda}} k_2 \text{ with } k_1, k_2 \in K \tag{3.52}$$

and $\check{\lambda} \in P(V)^\vee_+$, the set of dominant elements in $P(V)^\vee$, is uniquely defined from x.

Remarks. — Note that this definition of G^+ is not quite the semi-group studied in [12,13]. For one, we are working with complete groups as opposed to the minimal group. Perhaps more importantly though, here we do not allow the 'boundary' points of the Tits cone to be in G^+, *i.e.* the elements $\pi^{n\mathbf{c}}$ are not allowed in our G^+ whereas they do lie in the semi-group considered in these other papers.

3.4.4. — The following results were proven in [7] and [13] for a slightly different definition of the group G. Let us call the group from *op. cit.* informally as the 'minimal' group and distinguish it from the 'complete' group considered here.

Proposition. —

1. [7, Thm. 1.9(1)] *For $\check{\lambda}, \check{\mu} \in P(V)^\vee$, if $K\pi^{\check{\lambda}}U \cap K\pi^{\check{\mu}}U^- \neq \emptyset$, then $\check{\lambda} \leq \check{\mu}$. Moreover, when this set is non-empty, it is finite modulo K.*
2. [7, Thm. 1.9(2)] *If $\check{\lambda} \in P(V)^\vee_+$ and $\check{\mu} \in P(V)^\vee$ the following set is finite*

$$M(\check{\lambda}, \check{\mu}) := K \setminus K\pi^{\check{\lambda}}K \cap K\pi^{\check{\mu}}U. \tag{3.53}$$

3. *If the set $M(\check{\lambda}, \check{\mu})$ as in the previous part is non-empty, then so is $M(\check{\lambda}, w\check{\mu})$ for any $w \in W$.*

Remarks. — As for the proofs, the first assertion of (1) is quite simple, and follows from some elementary representation theory which holds regardless of whether one works with the complete group of the minimal group. The second assertion of (2) is the so-called Gindikin–Karpelevich finiteness which is the main subject of [7] in affine type and of [1] and [24] for general Kac–Moody type. As U^- is the same for both the minimal group of [7] and the group G we study here, the arguments in *op. cit.* go through here as well.

Part (2) is sometimes called the spherical finiteness, and was first established in [12] using some algebro–geometrical techniques and then again in [7] (affine type) and [1] (general Kac–Moody type) using more elementary representation theoretic techniques. It was also extended to general Kac–Moody types using (and developing) the theory of masures in [23]. The argument in [7, §6.2] for the spherical finiteness uses the Gindikin–Karpelevich finiteness and so again holds for complete groups. Moreover, one can deduce that the intersection (and quotient by K) is the same as in the minimal group.

Part (3) follows in the minimal group by using the formula for the Satake transform established in [13] for affine type (and in general Kac–Moody type in [4]). By the remarks at the end of the previous paragraph, it also holds in the complete group.

4. Reduction theory of Garland and the Birkhoff decomposition

Let $F = \mathbb{F}_q(s)$ and recall the notations from §2.0.2, which we maintain here. Fix also $V := V^\lambda$ a highest weight representation with λ normal, dominant, and integral and construct the group $\mathbf{G}$ (or $\mathbf{G}^e$) with respect to this representation. In particular, we have a choice of an admissible basis $\Omega \subset V_{\mathbb{Z}}$ as well a primitive highest weight vector $\mathbf{v}$ as in §3.2.3.

4.1. Adelic Loop Groups and Reduction Theory. —

4.1.1. — For each $\nu \in |F|$, let $G_\nu := \mathbf{G}(F_\nu)$ and let K_ν be the subgroup as in (3.47). Define $G_{\mathbb{A}} := \prod'_{\nu \in |F|} G_\nu$ where the product is restricted in the sense that, for almost all places ν, the component lies in K_ν. For any $x = (x_\nu) \in G_{\mathbb{A}}$ we define the finite set $\mathrm{Supp}(x) := \{\nu \in |F| \text{ such that } x_\nu \notin K_\nu\}$. Write $K_{\mathbb{A}} := \prod_{\nu \in |F|} K_\nu$, and let us also introduce in the natural way the groups $H_{\mathbb{A}}, U_{\mathbb{A}}$, etc. If $U \subset |F|$ is a set with finite complement, we shall also write $G_{\mathbb{A}(U)}, H_{\mathbb{A}(U)}$, *etc.* for the restricted direct product taken over the places U instead of all of $|F|$. Generalizing (2.3) we have

$$H_{\mathbb{A}(U)} = H_{\mathbb{O}(U)} H_F. \tag{4.1}$$

4.1.2. — We can also consider the group $G^e_{\mathbb{A}} := G_{\mathbb{A}} \rtimes \mathbb{I}$ and $H^e_{\mathbb{A}} := H_{\mathbb{A}} \rtimes \mathbb{I}$. We shall usually restrict ourselves to the subset $G^\tau_{\mathbb{A}} \subset G^e_{\mathbb{A}}$ whose projection onto $\mathbb{I}$ is equal to a fixed element $\tau \in \mathbb{I}$ that satisfies the following condition:

$$\textbf{SPos:} \quad \tau = (\tau_\nu) \in \mathbb{I} \quad \text{such that} \quad |\tau_\nu| \leq 1 \text{ for all } \nu \in |F| \ . \tag{4.2}$$

As a matter of notation, we shall often refer to elements in $G^\tau_{\mathbb{A}}$ as $x = g\eta(\tau)$ with $g \in G_{\mathbb{A}}$ and $\tau \in \mathbb{I}$. So for example, each $a \in H^\tau_{\mathbb{A}}$ may be written as

$$a = h_{a_1}(s_1) \cdots h_{a_{\ell+1}}(s_{\ell+1}) \eta(\tau), \text{ where } s_i \in \mathbb{I}, i \in I. \tag{4.3}$$

4.1.3. — For each $\nu \in |F|$, we may use the maps $\iota_\nu : F \hookrightarrow F_\nu$ from §2.0.4 to obtain a map that we continue to denote as $\iota_\nu : \mathbf{G}(F) \to \mathbf{G}(F_\nu)$. We can use these to define a map $\iota : \mathbf{G}(F) \to \prod_{\nu \in |F|} \mathbf{G}(F_\nu)$, $g \mapsto \iota_\nu(g)$, but the image does not in general lie in $G_{\mathbb{A}}$. We then define

$$\Gamma := \{\gamma \in \mathbf{G}(F) \mid \iota(\gamma) \in G_{\mathbb{A}}\} \tag{4.4}$$

and note that we still have $\Gamma/\Gamma \cap \mathbf{B}(F) = \mathbf{G}(F)/\mathbf{B}(F)$. For each finite set $S \subset |F|$ we also define

$$\Gamma_S := \mathbf{G}(F_S) := \{\gamma \in \mathbf{G}(F) \mid \gamma_\nu := \iota_\nu(\gamma) \in K_\nu \text{ for all } \nu \notin S\}. \tag{4.5}$$

Note that the subgroup $\mathbf{G}(F_S)$ is contained in Γ. When $S = \{\xi\}$ is a singleton, we have

$$\Gamma_\xi := \{\gamma \in \Gamma \mid \gamma_\nu \in K_\nu \text{ unless } \nu = \xi\}. \tag{4.6}$$

4.1.4. A global norm. — Recall that $\mathbf{v}$ denoted the primitive, highest weight for V, which we assume lies in $V_{\mathbb{Z}}$. For each place $\nu \in |F|$, we continue to write $\mathbf{v}$ for the image of $\mathbf{v}$ under base change in the vector space V_ν. Recall that for each ν, we have $|\mathbf{v}|_\nu = 1$. With these conventions, for $x := (x_\nu) \in G^e_{\mathbb{A}}$ we define

$$||x.\mathbf{v}|| := \prod_{\nu \in |F|} |x_\nu.\mathbf{v}|_\nu. \tag{4.7}$$

For almost all ν, we have $x_\nu \in K_\nu$ and $|\mathbf{v}|_\nu = 1$ for all ν, so (4.7) makes sense. In fact

$$||x.\mathbf{v}|| = \prod_{\nu \in \mathrm{Supp}(x)} |x_\nu.\mathbf{v}|_\nu \tag{4.8}$$

4.1.5. — Let $\mu \in P(V)$ and $a \in H^\tau_{\mathbb{A}}$. Writing a as in (3.25), we then set

$$|a^\mu| = |s_1|^{\langle \mu, a_1^\vee \rangle} \cdots |s_{\ell+1}|^{\langle \mu, a_{\ell+1}^\vee \rangle} |\tau|^{\langle \mu, \mathbf{D} \rangle}. \tag{4.9}$$

For each positive constant $c > 0$, we define

$$H^\tau_{\mathbb{A}}(c) := \{a = h\eta(\tau) \in H^\tau_{\mathbb{A}} \mid |(h\eta(\tau))^{a_i}| < c \text{ for } i = 1, \ldots, \ell+1\}. \tag{4.10}$$

4.1.6. The sets $U_{\Omega(\mathfrak{a})}$. — For each divisor $\mathfrak{a}$, we have defined the subset $\Omega(\mathfrak{a}) \subset \mathbb{A}$ in (2.1). Recall also the coordinates we have put on $\mathbf{U}$ from §3.3.5, especially Proposition 3.3.5 and the remark following it. We then define the subset $U_{\Omega(\mathfrak{a})} \subset U_{\mathbb{A}}$ by requiring all of the coordinates $\Sigma := (\sigma_{\alpha,j}, b_{i,j}, \sigma'_{\alpha,j})$ as in (3.40) to lie in $\Omega(\mathfrak{a})$.

4.1.7. — The reduction theory of H. Garland [18, §19] (and also see [19, §8]) adapted to the case of the function field $F = \mathbb{F}_q(s)$ yields

Theorem. — *Fix* $\mathbf{G} := \mathbf{G}^\rho$ *for* ρ *as in* (3.10)*, and let* $\tau = (\tau_\nu) \in \mathbb{I}$ *with* $\tau_\nu = 1$ *for all* $\nu \neq \nu_0$ *and* $|\tau_{\nu_0}| < 1$ *(hence* τ *satisfies* **SPos***). Then for any* $x \in G^\tau_{\mathbb{A}} := G_{\mathbb{A}}\eta(\tau)$*, there exists* $\gamma \in \Gamma$ *such that* $x\gamma \in K_{\mathbb{A}} H^\tau_{\mathbb{A}}(1)$.

As the proof is very similar to arguments in [18] and [19], we just sketch the main ideas. Let us fix the following notation throughout: $S \subset |F|$ is a finite set such that $\mathrm{Supp}(x) \subset S$ in the notation from §4.1.1. Note that if $\gamma \in \Gamma_S$, then $\mathrm{Supp}(x\gamma) \subset S$ as well.

4.1.8. Proof of Theorem 4.1.7, step 1: existence of minima. — For $x \in G^\tau_{\mathbb{A}}$ with τ satisfying **SPos**, we argue that there exists $\gamma \in \Gamma_S$ such then $||x\gamma.\mathbf{v}_\rho||$ achieves a minimum value. As $\mathrm{Supp}(x\gamma) \subset S$,

$$||x\gamma.\mathbf{v}_\rho|| = \prod_{\nu \in S} |x_\nu\, \iota_\nu(\gamma).\mathbf{v}_\rho|_\nu. \tag{4.11}$$

Using the Bruhat decomposition over F (see (3.45)), write $\gamma = u_{w,F}\, \dot{w} h_F u_F$ for $u_{w,F} \in \mathbf{U}_w(F), h_F \in \mathbf{H}(F), u \in \mathbf{U}(F)$ for some $w \in W$, then

$$||x\gamma.\mathbf{v}_\rho|| = ||x u_{w,F} \dot{w} \mathbf{v}_\rho|| \tag{4.12}$$

by using the fact that u_F fixes $\mathbf{v}_\rho$ and the product formula for h^ρ_F discussed in 2.0.4. Now writing $x = k_{\mathbb{A}} h_{\mathbb{A}} \eta(s) u_{\mathbb{A}}$, we have

$$||x\gamma.\mathbf{v}_\rho|| = ||h_{\mathbb{A}}\eta(s) u_{\mathbb{A}} u_{w,F} \dot{w}\, \mathbf{v}_\rho|| \geq ||h_{\mathbb{A}}\eta(s) \mathbf{v}_{w\rho}||, \tag{4.13}$$

where the last equality follows from the fact that elements in $U_{\mathbb{A}}$ are unipotent 'upper triangular' (with respect to our coherently ordered basis) operators. Using this, we can verify

Lemma. — *(cf.* [21, p.731-733] *) Fix $M > 0$. Then for almost all $w \in W$, if $\gamma \in \mathbf{B}(F)\dot{w}\mathbf{B}(F)$ we have $||x\gamma.\mathbf{v}_\rho|| > M$*

So we are reduced to arguing that, for a fixed $w \in W$, the minimum exists over all $\Gamma_S \cap \mathbf{B}(F)\,\dot{w}\,\mathbf{B}(F)$, *i.e.* we need to verify that, for a fixed w,

$$\min_{\gamma\in\Gamma_S\cap\mathbf{B}(F)\,\dot{w}\,\mathbf{B}(F)} ||x\gamma.\mathbf{v}_\rho|| \tag{4.14}$$

exists. As $V_F^\rho(w)$ is a finite-dimensional F-vector space, it is discrete in its adelization which we denote as $V_{\mathbb{A}}^\rho(w)$ (constructed as in [40, pp. 60-61]). As $\gamma\mathbf{v}_\rho \in V_F^\rho(w)$, the set of values in (4.14) is also discrete and non-zero, hence the minimum is achieved. As $\gamma.\mathbf{v}_\rho \in V_F^\rho(w)$, the set of values in (4.14) is also discrete and non-zero, hence the minimum is achieved.

4.1.9. Proof of Theorem 4.1.7, step 2: choosing a minimizing γ appropriately.— Although the functional $\gamma \mapsto ||x\gamma.\mathbf{v}||$ achieves a minimum, the γ which produces the minimum is not unique. We shall exploit this freedom below. For example, if γ achieves the minimum, so does $\gamma \cdot h_F n_{\mathbb{A}}$ for any $h_F \in H_F$ and $n_{\mathbb{A}} \in U_{\mathbb{A}}$. Indeed, $n_{\mathbb{A}}$ fixes $\mathbf{v}$ and $h_F.\mathbf{v} = \alpha\mathbf{v}$ for $\alpha \in F^*$, so that

$$||x\gamma h_F n_{\mathbb{A}}.\mathbf{v}|| = ||x\gamma\alpha.\mathbf{v}|| = |\alpha|_{\mathbb{I}}\,||x\gamma.\mathbf{v}|| = ||x\gamma.\mathbf{v}|| \tag{4.15}$$

using the product formula for ideles discussed in §2.0.4.

Suppose now $\gamma \in \Gamma$ is any element minimizing the functional $\gamma \mapsto ||x\gamma.\mathbf{v}||$. Write

$$x\gamma = k_{\mathbb{A}} h_{\mathbb{A}} \eta(\tau) u_{\mathbb{A}}, \text{ with } u_{\mathbb{A}} \in U_{\mathbb{A}}, k_{\mathbb{A}} \in K_{\mathbb{A}}, h_{\mathbb{A}} \in H_{\mathbb{A}}. \tag{4.16}$$

Let $v_0 \in |F|$ be a fixed place. We would first like to argue that γ can be chosen to minimize the functional and such that we also have

1. $\mathrm{Supp}(h_{\mathbb{A}}) = \{v_0\}$
2. $u_{\mathbb{A}} \in U_{\Omega(\mathfrak{a})}$ for $\mathfrak{a} = [v_0]$ with $U_{\Omega(\mathfrak{a})}$ defined as in (4.1.6).

As for the requirement (1), note that for a given $h_{\mathbb{A}} \in H_{\mathbb{A}}$ translating by an element h_F we can arrange that $h_{\mathbb{A}} h_F$ has support at v_o by (4.1). By what was written at the start of this step, γh_F also minimizes the functional in question, and we compute

$$x\gamma h_F = k_{\mathbb{A}}(h_{\mathbb{A}} h_F)\eta(\tau)\tilde{u}_{\mathbb{A}} \tag{4.17}$$

where $\tilde{u}_{\mathbb{A}} \in U_{\mathbb{A}}$ is obtained from $u_{\mathbb{A}}$ by conjugation by h_F. In this way we can achieve condition (1).

As for condition (2), $\tilde{u}_{\mathbb{A}}$ need not lie in $U_{\Omega(\mathfrak{a})}$ but we can arrange it to lie in this set after multiplying by an element from $n_{\mathbb{A}} \in U_{\mathbb{A}}$. In fact we will show that we can choose $n_{\mathbb{A}} \in U_F \cap U_{\mathbb{A}}$. The argument below is not sensitive to $\mathfrak{a}$ being chosen to be v_0 and works for any $\mathfrak{a}$ with $\deg(\mathfrak{a}) \geq -1$. Note that this is where we have used the genus zero assumption of $F = k(s)$. Indeed, in this case, writing S for the support of $\mathfrak{a}$, we find from (2.0.4) that

$$\mathbb{A} = \Omega(\mathfrak{a}) + F. \tag{4.18}$$

Hence if $x \in \mathbb{A}$ has support S, we can find $\gamma \in F_S$ so that $x = \omega + \gamma$ with $\omega \in \Omega(\mathfrak{a})$. Next, we proceed by induction (*cf.* [18, Lemma 18.2]) using the Iwahori–Matsumoto coordinates from §3.3.5 to find $n_F \in \Gamma_S \cap U_{\mathbb{A}} := U_{F_S}$ so that $\tilde{u}_{\mathbb{A}} n_F \in U_{\Omega(\mathfrak{a})}$.

4.1.10. Proof of Theorem 4.1.7, step 3: deriving the condition on $H_{\mathbb{A}}^e(1)$. — In this step, we emphasize that we are using the fact that the defining representation of $\mathbf{G}$ is ρ. Consider γ which satisfies both conditions of the previous step and write $z := x\gamma$. Now consider zw_a for w_a the reflection associated to a simple root a. As above, write an Iwasawa decomposition $z = k_{\mathbb{A}} h_{\mathbb{A}} \eta(\tau) u_{\mathbb{A}}$

which satisfies conditions (1) and (2) from previous step 2. Since $\dot{w}_a \in \Gamma$, by the minimum property of z, we must have

$$|(h_{\mathbb{A}}\eta(\tau))^{\rho}| = ||z.\mathbf{v}_{\rho}|| \leq ||z\dot{w}_a.\mathbf{v}_{\rho}||. \tag{4.19}$$

Decomposing $u_{\mathbb{A}} \in U_{\mathbb{A}}$ as $u = u_a(\sigma)u^a$ with $u_a(\sigma) \in U_a(\mathbb{A}), \sigma \in \Omega(\mathfrak{a})$ and $u^a \in \dot{w}_a U_{\mathbb{A}} \dot{w}_a^{-1} \cap U_{\mathbb{A}}$ one has (see [18, p.103-104])

$$||z\dot{w}_a.\mathbf{v}_{\rho}|| = ||h_{\mathbb{A}}\eta(\tau)u_a(\sigma)\mathbf{v}_{\rho-a}|| = ||h_{\mathbb{A}}\eta(\tau)\mathbf{v}_{\rho-a} + h_{\mathbb{A}}\eta(\tau)\sigma\mathbf{v}_{\rho}||. \tag{4.20}$$

Thus we may conclude that

$$|(h_{\mathbb{A}}\eta(\tau))^{\rho}| \leq \sup\{|(h_{\mathbb{A}}\eta(\tau))^{\rho-a}|, |\sigma||(h_{\mathbb{A}}\eta(\tau))^{\rho}|\}. \tag{4.21}$$

As $\sigma \in \Omega(\mathfrak{a})$ with $\deg(\mathfrak{a}) = \deg(\nu_0) \geq 1$. Hence, using (2.2) we have $|\sigma| \leq q^{-\deg\mathfrak{a}} < 1$. Thus (4.20) implies that $|(h_{\mathbb{A}}\eta(\tau))^{\rho}| \leq |(h_{\mathbb{A}}\eta(\tau))^{\rho-a}|$ or that $|(h_{\mathbb{A}}\eta(\tau))^{a}| \leq 1$ for any simple root a.

Note: This step does not depend on the genus $g = 0$ condition.

4.1.11. Proof of Theorem 4.1.7, step 4: conclusion.— We emphasize here that we need both the genus 0 assumption and that we are working in V^{ρ}. Suppose $\mathfrak{a} = [\nu_0]$ and combine the conclusion of Step 2 and 3 to find γ such that (1) and (2) from Step 2 are satisfied, and such that $x\gamma = k_{\mathbb{A}}h_{\mathbb{A}}\eta(\tau)u_{\mathbb{A}}$ with $(h_{\mathbb{A}}\eta(\tau)) \in H_{\mathbb{A}}^{\tau}(1)$.

Since $\mathrm{Supp}(h_{\mathbb{A}}\eta(\tau)) = \{\nu_0\}$, we have $h_{\nu} \in H_{\nu} \cap K_{\nu}$ and $\tau_{\nu} \in \mathscr{O}_{\nu}^*$ for all $\nu \neq \nu_0$. We also have

$$h_{\nu}\eta(\tau_{\nu})\mathbf{v}_{\mu} = (h_{\nu}\eta(\tau_{\nu}))^{\mu}\,\mathbf{v}_{\mu} \text{ for any } \mu \in \Xi(V). \tag{4.22}$$

Hence for any root $\beta \in \mathcal{R}$, we have $(h_{\nu}\eta(\tau_{\nu})^{\beta} \in \mathscr{O}_{\nu}^*$, $\nu \neq \nu_0$.

As for $\nu = \nu_0$, for any *positive* root $\beta \in \mathcal{R}_+$, we have $|\tau_{\nu_0}| < 1$ and $(h_{\nu_0}\eta(\tau_{\nu_0}))^{\beta} \in \mathscr{O}_{\nu_0}$ since $|(h_{\mathbb{A}}\eta(\tau))^{\beta}|_{\mathbb{I}} = |h_{\nu_0}\eta(\tau_{\nu_0})|_{\nu_0} \leq 1$.

Let us now argue that $(h_{\mathbb{A}}\eta(\tau))u_{\mathbb{A}}(h_{\mathbb{A}}\eta(\tau))^{-1} \in K_{\mathbb{A}}$, where $u_{\mathbb{A}} := u(\Sigma) = (u_{\nu}) \in U_{\Omega(\mathfrak{a})}$. Note that the last condition implies, in particular that all of the Iwahori–Matsumoto coordinates $\sigma_{\alpha,j}, \sigma'_{-\alpha,j}$ and $b_{i,j}$ from (3.40) lie in $\mathscr{O}_{\nu}$ for all ν (this fact fails for higher genus function fields, where $\deg(\mathfrak{a})$ must always be strictly positive). Now, since

$$(h_{\mathbb{A}}\eta(\tau))\chi_{\beta}(u)(h_{\mathbb{A}}\eta(\tau))^{-1} = \chi_{\beta}((h_{\mathbb{A}}\eta(\tau))^{\beta}u), \tag{4.23}$$

we conclude from the last paragraph that for any positive real root β, the above expression is in $K_{\mathbb{A}}$.

Now using (3.3.3), we also have that

$$(h_{\mathbb{A}}\eta(\tau))h_{\alpha_i}(1 - b_{i,j}t^j)(h_{\mathbb{A}}\eta(\tau))^{-1} = h_{\alpha_i}(1 - \tilde{b}_{i,j}t^j) \tag{4.24}$$

with $\tilde{b}_{i,j} \in \mathbb{O}$. Since a general element in $U_{\Omega(\mathfrak{a})}$ is a (possibly infinite) product of $\chi_{\beta}(u_{\beta})$ and $h_{\alpha_i}(1 - b_{i,j}t^j)$ with u_{β} and $b_{i,j} \in \Omega(\mathfrak{a})$ the desired conjugation result follows.

4.2. On the groups K_{∞} and K_{∞}^1. — Recall that $F_{\nu_0} = k((s))$ is the completion of F at ν_0, a local field which we now denote as $\mathcal{F}$. We also adopt the conventions from §2.0.1 and so write π for the uniformizer s, $\mathscr{O} := k[[s]] \subset \mathcal{F}$ for ring of integers, and $q = \kappa(\nu_0)$ for the cardinality of the residue field. Write $G := \mathbf{G}(\mathcal{F})$. We shall mostly work within the semi-group $G^+ \subset \mathbf{G}^e(\mathcal{F})$ introduced in (3.51), but also consider the place ν_{∞} corresponding to prime s^{-1} described in §2.0.3 (in fact, playing off the two is what makes our argument work).

In this section, we introduce subgroups $K_{\infty}^1 \subset K_{\infty} \subset G$ using the global techniques we have discussed in §4.1 and then study how the (algebraic) Bruhat decomposition coming from $\mathbf{G}(F)$ interacts with the p-adic Iwasawa decomposition on $\mathbf{G}(\mathcal{F})$. Throughout, we shall let

$$K := K_{\nu_0} \subset G. \tag{4.25}$$

4.2.1. On K_∞ and K^1_∞. — Using the notation from (4.6), we can now make the following.

Definition. — Recalling that

$$\Gamma_{v_0} := \{\gamma \in \mathbf{G}(F) \mid \gamma_v \in K_v, v \neq v_0\}, \tag{4.26}$$

we write $K_\infty \subset G$ for the image of Γ_{v_0} under the map $\mathbf{G}(F) \to \mathbf{G}(F_0) := G$.

Note that the elements $\dot{w} \in \mathbf{G}(F)$ introduced in (3.41) will naturally live in K_∞. In fact, suppose $\sigma \in F$ is such that $\sigma_v \in \mathscr{O}_v$ for all $v \neq v_0$. Then from Lemma 2.0.3, we have $\sigma \in \mathbb{F}_q[s^{-1}]$ and so we identify σ with σ_{v_0} in this case and note that $\chi_a(\sigma) \in K_\infty$ for each $a \in \mathcal{R}_{re}$. Similarly, one can show that if $\xi(t) \in F((t))$ is of the form $\xi(t) = \sum_n \xi_n t^n$ with each $\xi_n \in \mathbb{F}_q[s^{-1}]$, then $\chi_\alpha(\xi(t)) \in K_\infty$ for each $\alpha \in \mathcal{R}_o$.

Remarks. — Let $R_\infty \subset G$ be the subgroup generated by elements just mentioned. One might try to use the arguments from [34, §7.4] (these arguments were due to H. Garland) to compare R_∞ with K_∞, but we do not pursue this point here. Muthiah has pointed out the issue may be more subtle, and has brought to our attention [6, Remark 6.8 (1)].

Definition. — Define the subgroup

$$\Gamma^1_{v_0} := \{\gamma \in \mathbf{G}(F) \mid \gamma_v \in K_v \text{ for all } v \neq v_0, \gamma_{v_\infty} \equiv 1 \bmod \pi_{v_\infty}\}, \tag{4.27}$$

and let K^1_∞ be the image of elements from $\Gamma^1_{v_0}$ under the natural map $\mathbf{G}(F) \to G$.

Note that the elements $\dot{w}$ do not lie in K^1_∞.

4.2.2. — Let $\gamma \in \mathbf{G}(F)$ lie in the Bruhat cell $\mathbf{B}(F)\dot{w}\mathbf{B}(F)$. Write

$$\gamma = u_w \dot{w}_\gamma h_\gamma n \tag{4.28}$$

with $u_w \in \mathbf{U}_w(F), n \in \mathbf{U}(F), w_\gamma \in W$, $\dot{w}_\gamma$ its lift to G as in §3.3.7, and $h_\gamma \in \mathbf{H}(F)$. Note that w_γ and h_γ are uniquely determined by γ. Write γ_v, $u_{w,v}, h_{\gamma,v}$, and n_v for the images in the completions, one has a Bruhat decomposition (3.45) in $\mathbf{G}(F_v)$

$$\gamma_v = u_{w,v} \dot{w}_\gamma h_{\gamma,v} n_v, \tag{4.29}$$

where we have used the fact that $\dot{w}_\gamma$ can be identified with its image in the completion.

Lemma. — *If $\gamma \in \Gamma_{v_0}$, then we have*

$$|h^\lambda_{\gamma,v}| \leq 1 \text{ for } v \neq 0 \quad \text{and} \quad |h^\lambda_{\gamma,v_0}| \geq 1 \tag{4.30}$$

for λ the defining representation of $\mathbf{G}$.

Proof. — This is a simple consequence of the product formula. For h_γ as in (4.28), if $v \neq 0$, we know that $\gamma_v.\mathbf{v}_\rho \in V_{\mathscr{O}_v}$. Hence using (4.29),

$$\gamma_v.\mathbf{v}_\lambda = h^\lambda_{\gamma,v} \mathbf{v}_{w_\gamma \lambda} + \text{ terms of higher depth} \tag{4.31}$$

and so $h^\lambda_{\gamma,v} \in \mathscr{O}_v$ for all $v \neq v_0$, *i.e.* $|h^\lambda_{\gamma,v}|_v \leq 1$ for $v \neq v_0$. As $h^\lambda_\gamma \in F^*$, by the product formula,

$$|h^\lambda_\gamma|_{\mathbb{I}} = \prod_{v \in |F|} |h^\lambda_{\gamma,v}|_v = 1, \tag{4.32}$$

and the result follows. □

Remarks. — Decomposing $h_{\gamma,v} = \pi_v^{\check{\zeta}_v} h'$ with $\check{\zeta}_v \in P(V^\lambda)$ and $h' \in H_{\mathscr{O}_v}$ as in (3.49), we may conclude from the fact that $h^\lambda_{\gamma,v} \in \mathscr{O}_v$ that $\langle \check{\zeta}_v, \lambda \rangle \geq 0$ for $v \neq v_0$ and similarly that $\langle \check{\zeta}_{v_0}, \lambda \rangle \leq 0$. Now even though the group $\mathbf{G}$ was defined with respect to the weight λ, as we have seen in §3.3.8 for each $i \in I$, there is a positive integer m_i and a map $\mathbf{G}^\lambda \to \mathbf{G}^{m\lambda_i}$. Acting on the highest weight vector $\mathbf{v}_{m\lambda_i}$ allows us to conclude that $\langle \check{\zeta}_{v_0}, m_i \lambda_i \rangle \leq 0$ for each $i \in I$ and hence

$$\check{\zeta}_{v_0} \leq 0. \tag{4.33}$$

4.2.3. Joseph estimate. — In this section, we need to assume that the defining representation for the group $\mathbf{G}$ is ρ, the weight defined in (3.10). We again adopt the same notation as in the previous paragraph, but also now recall the place v_∞ from §2.0.3.

Lemma. — *Let $\gamma \in \Gamma^1_{v_0}$. Then in the notation of* (4.29),

$$|h^\rho_{\gamma, v_\infty}|_{v_\infty} \leq q^{-\ell(w_\gamma)/2} \tag{4.34}$$

Proof. — By definition, we have $\gamma_{v_\infty} \mathbf{v}_\rho \in V_{\mathscr{O}_{v_\infty}}$. As such it makes sense to speak of $\gamma_{v_\infty} \mathbf{v}_\rho \bmod \pi_{v_\infty}$ using the map $V_{\mathscr{O}_{v_\infty}} \to V_{\kappa(v_\infty)}$. As $\gamma_{v_\infty} \in \Gamma^1_{v_0}$, it follows that $\gamma_{v_\infty} \mathbf{v}_\rho = \mathbf{v}_\rho \bmod \pi_{v_\infty}$. In other words, the coefficient of $\mathbf{v}_\rho$ in $\gamma_{v_\infty} \mathbf{v}_\rho$ has valuation 0.

We can now compute $\gamma_{v_\infty} \mathbf{v}_\rho$ in a different way by using (4.28) and (4.29) to write $\gamma = u_w \dot{w}_\gamma h_\gamma n$ and $\gamma_{v_\infty} = u_{w,v_\infty} \dot{w}_\gamma h_{\gamma,v_\infty} n_{\gamma,v_\infty}$. Then applying to $\mathbf{v}_\rho$

$$\gamma_{v_\infty} \mathbf{v}_\rho = u_{w,v_\infty} h^\rho_{\gamma,v_\infty} \mathbf{v}_{w_\gamma \rho}. \tag{4.35}$$

Let us now assume the following Claim, whose proof will be given in §4.2.4.

Claim. — *Any element $u_{w,v_\infty} \in \mathbf{U}_w(F_{v_\infty})$ can be written as*

$$u_{w,v_\infty} = k\, p \tag{4.36}$$

where $k \in \mathbf{U}_w(\mathscr{O}_{v_\infty})$ and $p \in \mathbf{U}_w(F_{v_\infty})$ consists of a product over root subgroups where each coefficients has π_{v_∞}-valuation strictly negative, i.e.

$$p = \textstyle\prod_{\beta \in S_w} \chi_\beta(d_\beta) \quad \textit{with each} \quad |d_\beta|_{v_\infty} > 1 \tag{4.37}$$

Keeping the notation from the above claim and using (4.36), we find (4.35) that

$$p\, h^\rho_{\gamma,\infty}.\mathbf{v}_{w_\gamma \rho} = k^{-1} \gamma_{v_\infty} \mathbf{v}_\rho. \tag{4.38}$$

From the first paragraph of this proof, the coefficient of $\mathbf{v}_\rho$ in the right hand side of (4.38) is of the form $c\,\mathbf{v}_\rho$ modulo π_{v_∞} with $c \in \mathscr{O}^*_{v_\infty}$. Hence the coefficient of $\mathbf{v}_\rho$ in the left hand side of (4.38) must also have norm 1 with respect to $|\cdot|_{v_\infty}$.

By Lemma 3.2.5, one needs at least $\ell(w_\gamma)/2$ operators to move from $\mathbf{v}_{w_\gamma \rho}$ to $\mathbf{v}_\rho$ and each of these increase the norm by at least $q := q_{v_\infty}$. Hence the coefficient, say z, of $\mathbf{v}_\rho$ in $\mathbf{v}_{w_\gamma \rho}$ satisfies $|z|_{v_\infty} \geq q^{\ell(w_\gamma)/2}$. As $|z h^\rho_{\gamma,v_\infty}|_{v_\infty} = 1$, we have

$$|h^\rho_{\gamma,\infty}|_{v_\infty} = |z|^{-1}_{v_\infty} \leq q^{-\ell(w_\gamma)/2}. \tag{4.39}$$

□

4.2.4. Proof of Claim. — For the purposes of this argument, let us write $\pi := \pi_{v_\infty}$, so that $k((\pi_{v_\infty})) = k((\pi))$, $\mathscr{O}_{v_\infty} = k[[\pi]]$, *etc.* Moreover, the elements in $\pi^{-1}k[\pi^{-1}]$ have strictly negative valuation. The claim follows by a 'Gindikin-Karpelevich' type induction. If $w := w_a$ for a simple root a, the assertion follows from the fact that $\chi_a(x)\chi_a(y) = \chi_a(x+y)$ and the decomposition $k((\pi)) = k[[\pi]] + \pi^{-1}k[\pi^{-1}]$. Assume that the claim has been proven for w of length at most r and pick $w \in W$ so that $w = w'w_a$ with a simple and $\ell(w') = r$. By (3.43), we have

$$U_w = \widetilde{U} \cdot U_a \quad \text{where} \quad \widetilde{U} := U_w \cap U^a = w_a U_{w'} w_a. \tag{4.40}$$

Note that $\widetilde{U} \subset U^a$ is normalized by U_a. Indeed, if $u_a \in U_a$ and $\widetilde{u} \in \widetilde{U}$, we know that $u_a \widetilde{u} u_a^{-1} \in U^a$. But it must also be in U_w, and hence it must lie in $\widetilde{U}$. Moreover, we note that even though the claim does not apply to $\widetilde{U}$, it does apply to $\dot{w}_a \widetilde{U} \dot{w}_a^{-1} = U_{w'}$.

To start the induction, assume $u_w := (w_a \cdot u_{w'} w_a)\, u_a$ with $u_a \in U_a$ and $u_{w'} \in U_{w'}$, and where we shall write $\tilde{u} := w_a \cdot u_{w'} w_a \in \widetilde{U}$. We may write $u_a = k_a p_a$ with $k_a \in U_a(\mathscr{O}_\infty)$ and $p_a = \chi_a(z)$ with $z \in \pi^{-1}k[\pi^{-1}]$. As U_a normalizes $\widetilde{U}$, we may conclude that

$$u_w = k_a \widetilde{u}_1 p_a \tag{4.41}$$

for some $\widetilde{u}_1 \in \widetilde{U}$. Applying the inductive claim to $U_{w'}$ we find that $\widetilde{u}_1 = \dot{w}_a k_{w'} p_{w'} \dot{w}_a^{-1}$ with $k_{w'} \in U_{w'}$ integral and $p_{w'} \in U_{w'}$ a product of elements with negative valuation. Since conjugation by $\dot{w}_a$ does not alter the valuation, the claim follows.

4.2.5. — The following is the key to proving the finiteness result Theorem 5.2 in the next section.

Lemma. — *Let $\check{\lambda} \in P(V)_+$ and $\check{\mu}, \check{\xi} \in P(V)$.*

1. *Suppose there exists $\gamma \in \Gamma_{v_0}$ so that $k_\infty := \gamma_{v_0}^{-1} \in K_\infty$ satisfies the condition that $\pi^{\check{\mu}} k_\infty \in K\pi^{\check{\xi}}U$, then $w_\gamma \check{\xi} \geq \check{\mu}$ with w_γ defined as in* (4.28).
2. *There are only finitely many $\check{\xi}$ such that there exists some $\gamma \in \Gamma^1_{v_0}$ so that $k^1_\infty := \gamma_{v_0} \in K^1_\infty$ satisfies the condition that $\pi^{\check{\mu}} k_\infty \in K\pi^{\check{\xi}}U \cap K\pi^{\check{\lambda}}K$.*

Proof. — As for (1), suppose we have $x := \pi^{\check{\mu}} k_\infty = k\pi^{\check{\xi}} u$ with $k \in K, u \in U$. Regard x adelically living at the place v_0, *i.e.,* write $\underline{x} = (x_v)$ with $x_v = 1$ for $v \neq v_0$ and $x_{v_0} = x$. Then by definition of γ, $x_v \gamma_v \in K_v$ for $v \neq v_0$ and $x\gamma_{v_0} \in K\pi^{\check{\mu}}$. Decomposing $\gamma \in \mathbf{G}(F)$ as in (4.28) and using Lemma 4.2.2 and the remark after it (and keeping the same notation as was introduced there), we have $h_{\gamma,v_0} = \pi^{\check{\zeta}_{v_0}} h_{\mathscr{O}}$ with $h_{\mathscr{O}} \in H_{\mathscr{O}_{v_0}}$ and $\check{\zeta}_{v_0} \leq 0$. In sum, we have

$$\pi^{\check{\xi}} u\, u_{w,v_0} \dot{w}_\gamma \pi^{\check{\zeta}_{v_0}} h_{\mathscr{O}} n_{v_0} \in K\pi^{\check{\mu}}. \tag{4.42}$$

From here, using Proposition 3.4.4(1), we conclude that

$$w_\gamma \check{\xi} \geq \check{\mu} - \check{\zeta}_{v_0} \geq \check{\mu}. \tag{4.43}$$

As for part (2), keeping the same notation as above (but now with $\gamma \in \Gamma^1_{v_0}$) we recall from Lemma 4.2.2 that $|h^{\rho}_{\gamma,v_0}|_{v_0} \geq 1$ but that $|h^{\rho}_{\gamma,v}|_v \leq 1$ for all other v. Hence, from the product formula

$$|h^{\rho}_{\gamma,v_\infty}|_{v_\infty} = |h^{\rho}_{\gamma,v_0}|_0^{-1} \prod_{v \neq v_0, v_\infty} |h^{\rho}_{\gamma,v}|_v^{-1} \geq |h^{\rho}_{\gamma,v_0}|_{v_0}^{-1}. \tag{4.44}$$

From part (1), we have $w_\gamma \check{\xi} \geq \check{\mu}$. On the other hand, if $K\pi^{\check{\lambda}}K \cap K\pi^{\check{\xi}}U$ then $\check{\lambda} \geq \check{\xi}$ and, in fact, $\check{\lambda} \geq w\check{\xi}$ for any $w \in W$ from Proposition 3.4.4(3), so that in particular $\check{\lambda} \geq w_\gamma \check{\xi}$. However there are only finitely many coweights in the set $\{H \mid \check{\lambda} \geq H \geq \check{\mu}\}$. Hence $w_\gamma \check{\xi}$ assumes a finite set of values, and from (4.43) $\check{\zeta}_{v_0}$ also assumes a finite set of values. In particular, $\langle \rho, \check{\zeta}_{v_0} \rangle \leq 0$ is bounded from below, *i.e.* $|h^{\rho}_{\gamma,v_0}|_{v_0} = q^{-\langle\rho,\check{\zeta}_{v_0}\rangle}$ is bounded above, by some constant depending on $\check{\lambda}$ and $\check{\mu}$. Therefore $|h^{\rho}_{\gamma,v_0}|^{-1}_{v_0}$ is bounded below and so combining (4.44) and Lemma 4.2.3 we obtain

$$q^{-\ell(w_\gamma)/2} \geq |h^{\rho}_{\gamma,v_\infty}|_{v_\infty} \geq c(\check{\lambda},\check{\mu}) \tag{4.45}$$

for some constant $c(\check{\lambda},\check{\mu})$. Hence $\ell(w_\gamma)$ is bounded above, and so there are only finitely many such elements w_γ. Since we have already seen that $\{w_\gamma\check{\xi}\}$ ranges over a finite set (depending on $\check{\lambda},\check{\mu}$), part (2) follows. □

5. Birkhoff Decomposition and a Finiteness Theorem

We are now ready to prove the main results of this paper. We assume in this section that the defining representation for the group **G** is $V := V^{\rho}$ with ρ as in (3.10). We also maintain the same notation as in the previous section.

5.1. A Birkhoff decomposition. —

Theorem. — *(Birkhoff) Each $x \in G^+$ may be written as $x = k\pi^{\check{\lambda}}k_\infty$ where $k \in K$, $k_\infty \in K_\infty$ and $\check{\lambda} \in P(V)_+$. The element $\check{\lambda}$ is uniquely determined by x.*

Proof. — Any $x \in G^+$ may be factored as $x = k'h\eta(\tau)u$ with $k' \in K, h \in H_{\mathcal{F}}, u \in \mathbf{U}(\mathcal{F})$ and $\tau \in \mathcal{F}^*$ such that $|\tau| < 1$. Hence, if we define as in the proof of Lemma 4.2.5,

$$\underline{x} := (\underbrace{x}_{v=v_0},\underbrace{1,1,\cdots}_{v\neq v_0}) \quad \text{and} \quad \underline{\tau} := (\tau,1,1,\cdots) \quad , \tag{5.1}$$

the element $\underline{x} \in G^{\underline{\tau}}_{\mathbb{A}}$ satisfies the hypothesis of Theorem 4.1.7. As Supp(x) and Supp(τ) are $\{v_0\}$, there exists $\gamma \in \Gamma_{v_0}$ such that $\underline{x}\gamma = k_{\mathbb{A}}h_{\mathbb{A}}\eta(\tau)$ where $k_{\mathbb{A}} \in K_{\mathbb{A}}$ and $h_{\mathbb{A}} \in H_{\mathbb{A}}$. Writing k and k_∞ for the component of $k_{\mathbb{A}}$ and γ at v_0 respectively, the desired factorization follows. Next, we note that since both K and K_∞ contain $\dot{w}$ for each $w \in W$, we may write $x = k\pi^{\check{\lambda}}k_\infty$ with $\check{\lambda} \in P(V)_+$.

Let us now turn to the uniqueness assertion: suppose

$$x = k\pi^{\check{\lambda}}k_\infty x = k'\pi^{\check{\mu}}k'_\infty \quad \text{with} \quad k,k' \in K, \check{\lambda},\check{\mu} \in P(V)_+, \quad \text{and} \quad k_\infty, k'_\infty \in K_\infty. \tag{5.2}$$

It then follows that there exists $r \in K, r_\infty \in K_\infty$ so that $r\pi^{\check{\lambda}}r_\infty = \pi^{\check{\mu}}$. As $r_\infty \in K_\infty$, there exists $\gamma \in \Gamma_{v_0}$ such that $\gamma_{v_0} = r_\infty$. Using Lemma 4.2.2 and the remark after it, if we write $\gamma = u_w\dot{w}_\gamma h_\gamma n$ as in (4.28), have $h_{\gamma,v_0} = \pi^{\check{\zeta}_{v_0}}h_{\mathcal{O}}$ with $\check{\zeta}_{v_0} \leq 0$. Hence, we find that

$$r\pi^{\check{\lambda}}u_{w,v_0}\dot{w}_\gamma\pi^{\check{\zeta}_{v_0}}h_{\mathcal{O}}n_{v_0} = \pi^{\check{\mu}} \tag{5.3}$$

from which we can conclude using Proposition 3.4.4(1) that

$$\check{\lambda} \geq w_\gamma\check{\lambda} \geq \check{\mu} - \check{\zeta}_{v_0} \geq \check{\mu}, \tag{5.4}$$

where the first inequality uses the fact that $\check{\lambda}$ was dominant and the third that $\check{\zeta}_{v_0} \leq 0$. So we have concluded that $\check{\lambda} \geq \check{\mu}$, but reversing the roles of $\check{\lambda}$ and $\check{\mu}$, we obtain the opposite inequality and hence $\check{\lambda} = \check{\mu}$. □

5.2. Finiteness Theorem. — Recall the group K_∞^1 introduced earlier in §4.2.1.

Theorem. — *For $\check{\lambda} \in P(V)_+$ and $\check{\mu} \in P(V)$,*

$$K \backslash K\pi^{\check{\lambda}} K \cap K\pi^{\check{\mu}} K_\infty^1 \tag{5.5}$$

is finite and non-empty only when $\check{\mu} \leq \check{\lambda}$.

Proof. — To prove the finiteness, in light of Proposition 3.4.4(2), it suffices to show that the set of $\check{\xi}$ such that $K\pi^{\check{\xi}} U \cap K\pi^{\check{\lambda}} K \cap K\pi^{\check{\mu}} K_\infty^1 \neq \emptyset$ is finite. This however follows from Lemma 4.2.5, part (2). As we already remarked in the proof of *loc. cit.*, $\check{\lambda} \geq w\check{\xi}$ for each such $\check{\lambda}$ and any $w \in W$. Using part (1) of the same Lemma (4.2.5), we can then conclude that $\check{\lambda} \geq \check{\mu}$ as well.

□

Remarks. — One could ask how the cardinality of the above set, say for fixed $\check{\lambda}$ and $\check{\mu}$, varies with q, where we recall that $F = \mathbb{F}_q(s)$. Using the point of view of masures, Muthiah [32] has argued in his context (and with slightly differently defined groups) these are polynomial in q and has also explicitly computed these polynomials in certain cases, see *op. cit.* §5.5. Comparing his answers to the Coulomb branch perspective (a candidate for the double affine geometric Satake), he found that the polynomials from the masure theoretic point of view seem to be of the wrong degree. We have not yet considered such polynomiality results in our context. In a related vein, A. Hébert and Paul Philippe [26] have now established the polynomiality (in the Iwahori case as well, see below §5.2.1) for general Kac–Moody type.

5.2.1. On an Iwahori variant.— For $\kappa = \mathscr{O}/\pi\mathscr{O}$ the residue field of $\mathcal{F}$, we have a natural map $\omega : V_{\mathscr{O}} \to V_\kappa$ defined by 'working modulo π'. Then each $k \in K$ defines a map that we denote as $k \operatorname{mod} \pi : V_\kappa \to V_\kappa$. Define the Iwahori subgroup

$$I := \{k \in K \mid k \operatorname{mod} \pi = \mathbf{1}_{V_\kappa}\} \tag{5.6}$$

where $\mathbf{1}_{V_\kappa}$ is the identity map on V_κ. Moreover, one also has that K is a disjoint union, over W, of double cosets $I\dot{w}I$.

Define now the "affine" or Tits–Weyl group, as $W_X := W \rtimes X$, where the action of W on X is the standard one. Note that W_X is not a Coxeter group, though it does admit a Bruhat-type order and a length function [31, 33]. For each $x = (w, \check{\lambda}) \in W_X$ with $w \in W$ and $\check{\lambda} \in X$, we define an element $\dot{x} := \dot{w}\pi^{\check{\lambda}} \in G^+$ and note that as in [13, Prop 3.4.2], each $x \in G^+$ can be written as $x = i_1 \dot{z} i_2$ for some $z \in W_X$.

An Iwahori variant of Theorem 5.2 states: for $x, y \in W_X$, the set $I \backslash I\dot{x}K_\infty^1 \cap I\dot{y}I$ is finite. Presumably the non-emptiness of this intersection would also imply a relation between x and y in the Bruhat order studied by Muthiah. As for finiteness, here is an approach which uses Theorem 5.2: following [13, Lemma 5.2.5] we can reduce to showing that

$$K_{x,y} = \{k \in K \mid k(I\dot{x}K_\infty^1 \cap I\dot{y}I) \subset (I\dot{x}K_\infty^1 \cap I\dot{y}I)\}, \tag{5.7}$$

which is a left I-invariant set, satisfies $|I \backslash K_{x,y}| < \infty$. To address the latter claim, it suffices to show that for a fixed $y \in W_X$, the number of w' such that $I\dot{w}'I\dot{y}I \cap I\dot{y}I \neq \emptyset$ is finite. To show this question, one may work in the Iwahori–Hecke algebra of [13][3] and switch to the Bernstein basis introduced

[3] We are eliding over the distinction between formal and minimal groups here

there (see op. cit. §5-6). Adopting the notation of *op. cit.* we may write $T_y = \sum_{\sigma,\check{\mu}} c_{\sigma,\check{\mu}} T_\sigma \theta_{\check{\mu}}$ (the sum is finite), and then consider the product in the Iwahori–Hecke algebra

$$T_{w'} \cdot T_y = \sum_{\sigma,\check{\mu}} c_{\sigma,\check{\mu}} T_{w'} T_\sigma \theta_{\check{\mu}}. \tag{5.8}$$

If $\ell(w')$ is large compared to $\ell(\sigma)$, then $T_{w'} T_\sigma$, is also a sum of T_τ's with $\tau \in W$ of large length. This bounds the length of possible w'.

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