Steady states

Steady states [equilibria, fixed points] for the differential equation of the form

$$x'(t) = f(x)$$

are those values of x that satisfy $f(x) = 0$.

Question of interest: what is the stability of such steady states? If x is perturbed from its steady state value x^*, does it return to x^* or move away from x^*?

Stability analysis

For equations of the form $x'(t) = f(x)$, there are two approaches to determine the stability of fixed points:

- Graphical stability analysis
- Linear stability analysis

Graphical stability analysis for the cooling problem with equation $x'(t) = k(21 - x)$

Boardwork . . .

Graphical stability analysis for the general problem with equation $x'(t) = f(x)$

Boardwork . . .

Graphical stability analysis: observations

We note the following relationship between the stability of a fixed point x^* of the differential equation $x'(t) = f(x)$ and the slope of $f(x)$:

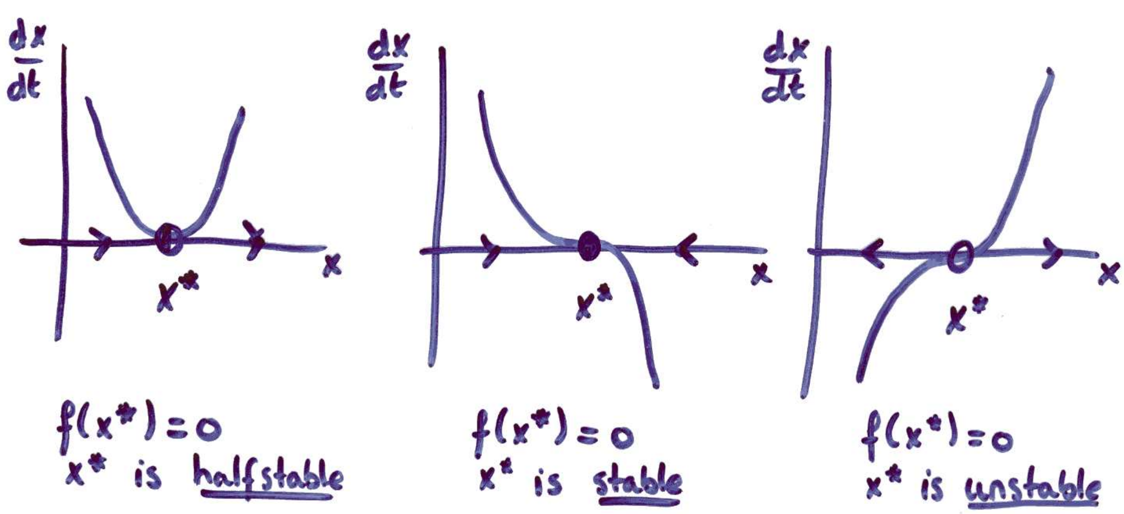
- If $f'(x^*) > 0$, then x^* is unstable
- If $f'(x^*) < 0$, then x^* is stable

Theorem

Let x^* be a fixed point of $x'(t) = f(x)$.

- If $f'(x^*) > 0$, then x^* is unstable.
- If $f'(x^*) < 0$, then x^* is stable.
- If $f'(x^*) = 0$, then no conclusions can be drawn without further work.

Anything can happen when $f'(x^*) = 0$



Linear stability analysis

We are considering

$$\frac{dx}{dt} = f(x) \tag{1}$$

with steady state x^*, that is, $f(x^*) = 0$.

Introduce a small perturbation y from x^*, that is, let

$$x = x^* + y \tag{2}$$

Substitute (2) into (1), and expand the right-hand-side with a Taylor series to get:

$$\begin{aligned} \frac{d(x^* + y)}{dt} &= f(x^* + y) \\ \frac{dy}{dt} &= f(x^*) + f'(x^*)y + \mathcal{O}(y^2) \end{aligned}$$

Since x^* is a fixed point, we can replace $f(x^*)$ on the right hand side by 0. If, in addition, we can safely neglect all the terms in the Taylor series that have been collected in the term $\mathcal{O}(y^2)$, then we are left with the following equation for the perturbation:

$$\frac{dy}{dt} = f'(x^*)y.$$

We recognize that $f'(x^*)$ is some constant, λ say. The equation for the perturbation thus is the linear equation

$$\frac{dy}{dt} = \lambda y,$$

which we studied previously (the world's simplest differential equation).

The solution for this last differential equation is

$$y(t) = y_0 e^{\lambda t}.$$

- If $\lambda < 0$, then $y(t) \to 0$ as $t \to \infty$.
- If $\lambda > 0$, then $y(t) \to \pm\infty$ as $t \to \infty$.

That is, the perturbation dies out if $\lambda = f'(x^*) < 0$, and grows if $\lambda = f'(x^*) > 0$. In the special case that $\lambda = f'(x^*) = 0$, the terms collected in the term $\mathcal{O}(y^2)$ become important, and other techniques of analysis are required. The theorem presented earlier follows.

Example

The rate of formation of a certain chemical in a reaction is known to be governed by the equation

$$x'(t) = (a - x)(b - x)$$

where $x(t)$ is the amount (mass) of the chemical of interest at time t, and a and b are amounts of other chemicals present when $t = 0$, with $0 < a < b$.

Find the steady states of the reaction and determine their stability (with a graphical analysis as well as a linear stability analysis).

What does the model predict? Sketch representative solutions $x(t)$ as a function of t.