THE RANDOM WALK PROBLEM

Imagine standing at the origin of a number line. On the toss of a fair coin, you take one step to the right (positive direction) if the coin shows a head, and you take one step to the left (negative direction) if the coin shows a tail. You repeat the process until you reach an exit at either the (integer) point $N < 0$ or the (integer) point $M > 0$, at which point the walk ends.

Let p_k denote the probability that the walk will end at N if you are standing at the point k.

If you move to the left at the next step, the probability that you reach the left exit is p_{k-1}, and if you move to the right at the next step, the probability that you reach the left exit is p_{k+1}. Since these moves have the same probability, the sequence of probabilities satisfies the difference equation

$$p_k = \frac{1}{2}\left(p_{k-1} + p_{k+1}\right)$$

for $k = N+1, \ldots, M-1$.

(a) What are the boundary conditions, p_N and p_M?

(b) Solve the difference equation subject to the boundary conditions.

(c) What is the probability that if you are standing at the origin, you will reach the left exit at N? Check the solution in the special case that $N = -M$. Is it consistent with your intuition?

Solution:

(a) The boundary conditions are $p_N = 1$ and $p_M = 0$.

(b) We use the trial solution $p_k = q^k$. Substituting this trial solution into the difference equation gives

$$q^k = \frac{1}{2}\left(q^{k-1} + q^{k+1}\right).$$

Dividing by q^{k-1} gives

$$q = \frac{1}{2} + \frac{1}{2}q^2$$

or

$$q^2 - 2q + 1 = (q-1)^2 = 0.$$

So $q = 1$ is a double root. Two linearly independent solutions then are

$$\begin{aligned} p_k &= 1^k = 1, \\ p_l &= k \cdot 1^k = k. \end{aligned}$$

Any linear combination of these two solutions also is a solution. Thus, the general solution for our equation is

$$p_k = A \cdot 1 + B \cdot k = A + Bk,$$

where A and B are determined by the boundary conditions.

Applying the boundary conditions $p_N = 1$ and $P_M = 0$ gives two equations for the two unknowns A and B, as follows:

$$\begin{aligned} p_N &= A + bN = 1, \\ p_M &= A + bM = 0. \end{aligned}$$

Solving for A and B, we obtain

$$\begin{aligned} A &= -\frac{M}{N-M}, \\ B &= \frac{1}{N-M}, \end{aligned}$$

so that the solution is

$$p_k = -\frac{M}{N-M} + \frac{k}{N-M} = \frac{M-k}{M-N}.$$

(c) We are interested in p_0, which is

$$p_0 = \frac{M}{M-N}.$$

If $N = -M$, we get

$$p_0 = \frac{M}{M+M} = \frac{1}{2},$$

as expected from intuition.