Linear Stability Analysis for Systems of Ordinary Differential Equations

Consider the following two-dimensional system:

$$\begin{aligned}\dot{x} &= f(x,y),\\ \dot{y} &= g(x,y),\end{aligned}$$

and suppose that $(\bar{x},\bar{y})$ is a steady state, that is, $f(\bar{x},\bar{y})=0$ and $g(\bar{x},\bar{y})=0$.

The question of interest is whether the steady state is stable or unstable.

Consider a small perturbation from the steady state by letting

$$\begin{aligned}x &= \bar{x}+u,\\ y &= \bar{y}+v,\end{aligned}$$

where both u and v are understood to be small. The question of interest translates into the following: will u and v grow (so that x and y move away from the steady state), or will they decay to zero (so that x and y move towards the steady state)?

In the former case, we say that the steady state is unstable, in the latter it is stable.

To see whether the perturbation grows or decays, we need to derive differential equations for u and v. We do so as follows:

$$
\begin{aligned}
\dot{u} &= \dot{x} \quad \text{(since } \bar{x} \text{ is constant)} \\
&= f(x,y) \quad \text{(by definition)} \\
&= f(\bar{x}+u, \bar{y}+v) \quad \text{(by substitution)} \\
&= f(\bar{x},\bar{y}) + \frac{\partial f}{\partial x}(\bar{x},\bar{y})u + \frac{\partial f}{\partial y}(\bar{x},\bar{y})v + \ldots \\
&\quad \text{(Taylor series expansion)} \\
&= \frac{\partial f}{\partial x}(\bar{x},\bar{y})u + \frac{\partial f}{\partial y}(\bar{x},\bar{y})v + \ldots \\
&\quad \text{(since } f(\bar{x},\bar{y}) = 0\text{)}
\end{aligned}
$$

Similarly,

$$
\dot{v} = \frac{\partial g}{\partial x}(\bar{x},\bar{y})u + \frac{\partial g}{\partial y}(\bar{x},\bar{y})v + \ldots
$$

The ... denote higher order terms, involving u^2, v^2, uv, etc. Since u and v are assumed to be small, these higher order terms are extremely small.

If we can safely neglect the higher order terms, we obtain the following *linear* system of equations governing the evolution of the perturbations u and v:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x}(\bar{x},\bar{y}) & \frac{\partial f}{\partial y}(\bar{x},\bar{y}) \\ \frac{\partial g}{\partial x}(\bar{x},\bar{y}) & \frac{\partial g}{\partial y}(\bar{x},\bar{y}) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

We refer to the matrix as the *Jacobian matrix* of the original system at the steady state $(\bar{x},\bar{y})$.

The above linear system for u and v has the trivial steady state $(u,v) = (0,0)$, and the stability of this trivial steady state is determined by the *eigenvalues* of the matrix, as follows:

- If the eigenvalues of the Jacobian matrix all have real parts less than zero, then the steady state is stable.

- If at least one of the eigenvalues of the Jacobian matrix has real part greater than zero, then the steady state is unstable.

- Otherwise there is no conclusion (then we have a borderline case between stability and instability; such cases require an investigation of the higher order terms we neglected, and this requires more sophisticated mathematical machinery discussed in advanced courses on ordinary differential equations).

Last but not least, there is a theorem (the Hartman-Grobman Theorem) that guarantees that the stability of the steady state $(\bar{x}, \bar{y})$ of the original system is the same as the stability of the trivial steady state $(0, 0)$ of the linearized system.

Thus, the procedure to determine stability of $(\bar{x}, \bar{y})$ is as follows:

1. Compute all partial derivatives of the right-hand-side of the original system of differential equations, and construct the Jacobian matrix.

2. Evaluate the Jacobian matrix at the steady state.

3. Compute eigenvalues.

4. Conclude stability or instability based on the real parts of the eigenvalues.

Note: The theorem and procedure apply to N-dimensional systems.