Differential Equations

A **differential equation** is an equation containing derivatives of an unknown function.

Finding a **solution** to the differential equation means finding an expression for the unknown function that **satisfies the differential equation**.

Example: The World's Simplest Differential Equation

$$\frac{dx}{dt} = kx \quad \text{OR} \quad x'(t) = kx(t)$$

This equation says that the rate of change of substance x at time t is proportional to the value of x at time t.

We can find the solution to this differential equation by inspection. We look for a function $x(t)$ that satisfies the equation. That is, we look for a function $x(t)$ so that when we differentiate it, we obtain $kx(t)$.

Guess:

$$x(t) = e^{kt}$$

Check:

$$x'(t) = ke^{kt} = kx(t)$$

But the following also works:

$$\begin{aligned} x(t) &= 2e^{kt} \\ x'(t) &= 2ke^{kt} = k\left(2e^{kt}\right) = kx(t) \end{aligned}$$

We say that

$$x(t) = Ce^{kt}$$

is the general solution to the differential equation, where C is an arbitrary constant.

So far, we have that

$$x(t) = Ce^{kt}$$

is a **general solution** to the differential equation

$$x'(t) = kx(t)$$

To determine C, we need some additional information, for example an initial condition:

$$x(0) = x_0$$

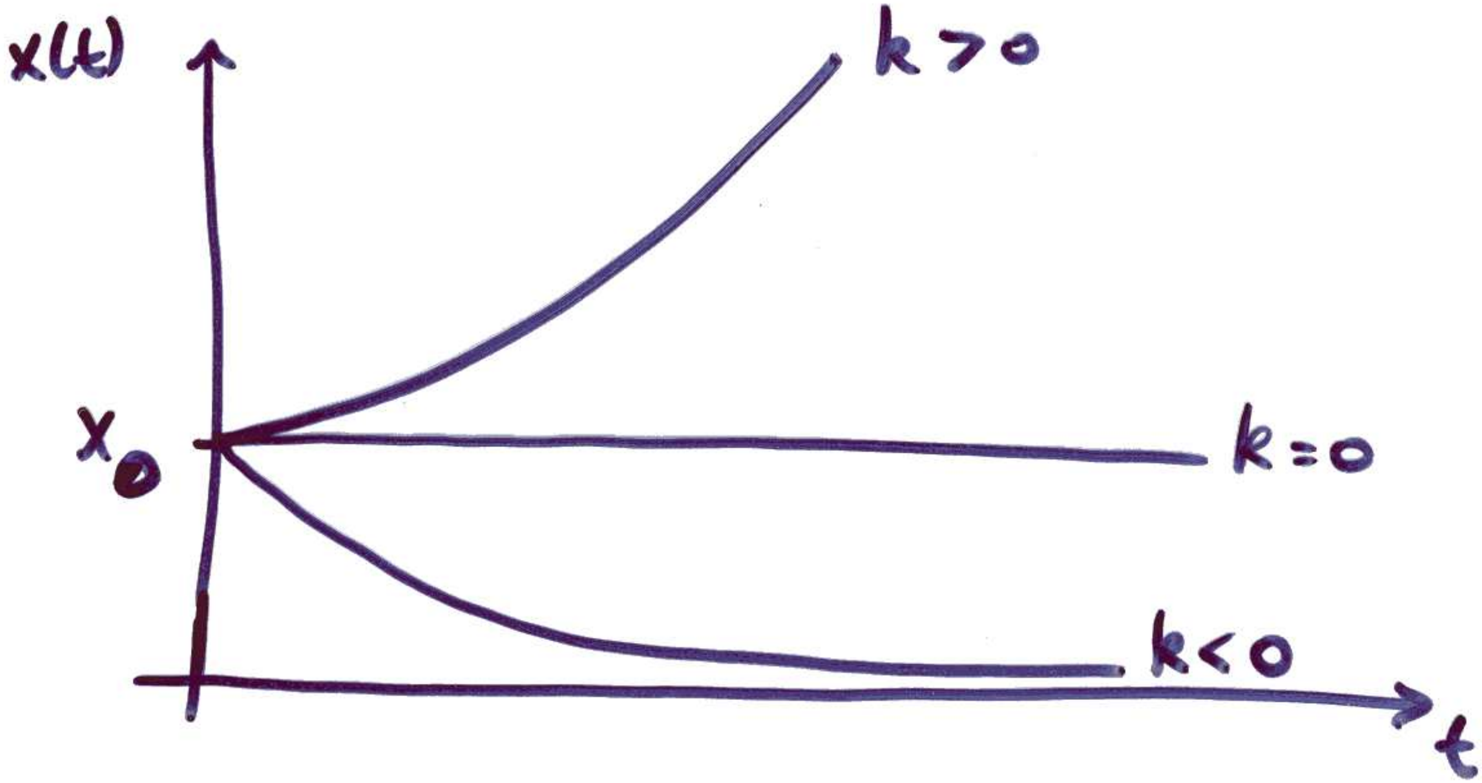
Then:

$$x(0) = Ce^0 = C = x_0$$

We say that

$$x(t) = x_0 e^{kt}$$

is the **specific solution** to the differential equation.



$$x(t) = x_0 e^{kt}$$

Modelling with Differential Equations

Construction of many dynamic models with differential equations is based on one of the following:

1. Translation of a physical law into a differential equation
   - Newton's law of motion
   - Newton's law of cooling
   - . . .

2. Using the word equation

$$\text{Rate of change of a `substance'} = \text{rate in} - \text{rate out}$$

   - rate of change of volume = inflow - outflow
   - rate of change of population = birth rate - death rate
   - . . .