The multi-marginal optimal transportation problem*

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Optimal transportation is an active and exciting area of research; for background and an extensive list of references, see the books by Villani [3, 4]. However, most of the progress made in this field to date has been restricted to problems with two marginals; problems with three or more marginals have thus far received relatively little attention. This abstract briefly summarizes recent progress made by the author on these multi-marginal problems; a more detailed exposition can be found in [1] and [2].

The multi-marginal transportation problem asks how to couple several distributions of mass with maximal efficiency, as measured by a prescribed surplus function. More precisely, for $i = 1, 2, ..., m$, let M_i be a compact smooth manifold of dimension n_i, endowed with a Borel probability measure μ_i and let $s : M_1 \times M_2 \times ... \times M_m \to \mathbb{R}$ be a C^2 smooth function, which we will call the surplus function. The optimal transportation problem then has two formulations. In the *Monge* formulation, the goal is to maximize

$$S(G_2, G_3, ..., G_m) := \int_{M_1} s(x_1, G_2(x_1), G_3(x_1), ..., G_m(x_1))d\mu_1 \qquad \textbf{(M)}$$

among all $(m-1)$-tuples of measurable maps $(G_2, G_3, ..., G_m)$, where $G_i : M_1 \to M_i$ pushes μ_1 forward to μ_i for all $i = 2, 3, ..., m$.

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In the *Kantorovich*, or relaxed, formulation of the problem one maximizes

$$S(\mu) := \int_{M_1 \times M_2 \times ... \times M_m} s(x_1, x_2, x_3, ..., x_m) d\mu \tag{\textbf{K}}$$

among all positive Borel measures μ on $M_1 \times M_2 \times ... \times M_m$ such that the canonical projection

$$\pi_i : M_1 \times M_2 \times ... \times M_m \to M_i$$

pushes μ forward to μ_i for all i. Heuristically, in the Monge formulation, mass at almost every point $x_1 \in M_1$ must be coupled with mass at unique points $x_i \in M_i$ for $i = 2, 3, ..., m$, whereas in the Kantorovich formulation a coupling may *split* a piece of mass at x_1 among two or more destination points in M_i for $i = 2, 3, ..., m$.

It is straightforward to show that a solution μ to the Kantorovich problem exists. When $m = 2$ and $n_1 = n_2$, it is possible, under weak conditions on s and μ_1, to show that the solution is concentrated on the graph of a function over x_1. This function then solves the Monge problem and in this case it is not hard to show that the solutions to both the Monge and Kantorovich problems are unique. Gangbo and Święch, Heinich, and Carlier have extended these results to the multi-marginal setting for certain special surplus functions; a complete list of references may be found in [1].

In [1], I develop a geometric framework to study the multi-marginal optimal transportation problem. I define a convex family G of semi-Riemannian metrics on $M_1 \times M_2 \times ... \times M_m$, derived from the mixed, second order partial derivatives of s. For any semi-metric g in this family, I then prove that near a point $\vec{x} := (x_1, x_2, ..., x_m) \in M_1 \times M_2 \times ... \times M_m$, the support of the optimal measure μ is contained in a Lipschitz submanifold of $M_1 \times M_2 \times ... \times M_m$, whose dimension is the number of non-timelike directions in the signature of g at $\vec{x}$. This generalizes a similar result in the two marginal setting, due to McCann, Warren and myself, which asserts that when $m = 2$ and $n_1 = n_2 := n$, μ is supported on an n-dimensional Lipschitz submanifold, under a weak local condition on s. In that case, the family G contains only one semi-metric, whose signature is (n, n); this is exactly the semi-metr! ic used by Kim and McCann to study the regularity of solutions to the Monge problem. When $m \geq 3$, the dimension of the support of μ depends on an entire family of semi-metrics whose signatures may vary; generically, each g will have between n_{max} and $N - n_{max}$ non-timelike directions, where $n_{max} := \max_i n_i$ and $N := \sum_{i=1}^{m} n_i$.

In certain cases, all the elements of G have many non-timelike directions, and we demonstrate by example that in such cases the support of the solution may actually concentrate on a high dimensional submanifold. In particular, it will not be concentrated on the graph of a function over the first marginal; moreover, I show that in some of these cases the solution is non-unique. This stands in stark contrast to the two marginal case; in addition, it suggests that stronger conditions must be assumed in order to prove the existence and uniqueness of solutions to the Monge problem as well as uniqueness of solutions to the Kantorovich problem.

These questions are resolved in [2]. The conditions I impose on s are much stronger than conditions required to prove analogous results for two marginal problems, as one would expect given the preceding discussion. Nonetheless, they apply to the surplus functions considered by Gangbo and Święch and Heinich as well as several other interesting examples, which are outlined in [2].

References

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