Chapter 9
FOURIER Transform Method For PDEs

The general idea of the Fourier method for real-world problems is as follows. First, we transform the given differential equation from x-domain to ω-domain, do all calculations in ω-domain, and finally use the inverse transform to represent the solution in x-domain. As we will see in the sequel, the work in ω-domain is much simpler than working in x-domain. Fig.9.1 shows that process schematically.

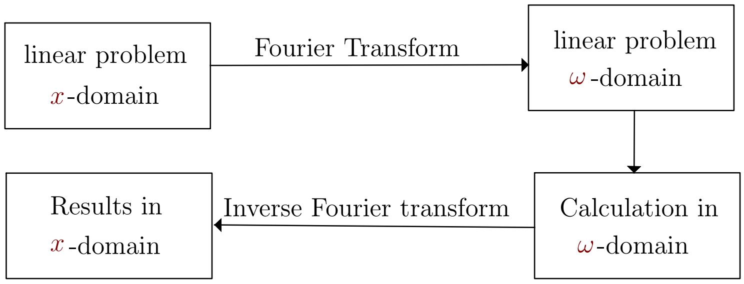

Figure 9.1.

9.1 1D problems

9.1.1 Heat problem and heat kernel

Consider the following homogeneous heat problem

$$\begin{cases} \partial_t u = \partial_{xx} u, -\infty < x < \infty, t > 0 \\ u(0,x) = u_0(x) \end{cases}. \tag{9.1}$$

Note that there is no boundary condition for the problem. However, from the physical point of view, we pose the *boundedness condition* so that a true solution must be bounded at $|x| \to \infty$ for all $t > 0$. Let us first assume that the initial condition $u_0(x)$ has the FOURIER transform $\hat{u}_0(\omega)$. By taking the FOURIER transform of the problem with respect to x, we reach the following equation

$$\begin{cases} \partial_t \hat{u}(t,\omega) = -\omega^2 \hat{u}(t,\omega) \\ \hat{u}(0,\omega) = \hat{u}_0(\omega) \end{cases}, \tag{9.2}$$

where $\hat{u}(t,\omega)$ is the transform of $u(t,x)$ *with respect to* x. Note that the obtained problem is an ordinary equation in ω-domain which is solved for

$$\hat{u}(t,\omega) = \hat{u}_0(\omega)\, e^{-\omega^2 t}. \tag{9.3}$$

Now, the solution $u(t,x)$ is written by the aid of inverse transform as

$$u(t,x)=\mathcal{F}^{-1}\{\hat{u}_0(\omega)\,e^{-\omega^2 t}\}=\frac{1}{2\pi}\int_{\mathbb{R}}\hat{u}_0(\omega)\,e^{-\omega^2 t}\,e^{i\omega x}\,d\omega. \tag{9.4}$$

Example 9.1. Consider the following problem

$$\begin{cases}\partial_t u=\partial_{xx}u, -\infty<x<\infty, t>0\\ u(0,x)=\delta(x)\end{cases},$$

where $\delta(x)$ is the DIRAC delta function. The problem in ω domain reads

$$\begin{cases}\partial_t\hat{u}=-\omega^2\,\hat{u}\\ \hat{u}_0(\omega)=1\end{cases},$$

that is solved for $\hat{u}(t,\omega)=e^{-\omega^2 t}$. Therefore, the solution $u(t,x)$ is derived by the inverse transform

$$u(t,x)=\mathcal{F}^{-1}\{e^{-\omega^2 t}\}=\frac{1}{\sqrt{4\pi t}}\,e^{-\frac{x^2}{4t}}.$$

This is the *fundamental solution* of a simple heat equation.

Example 9.2. Let us solve the following heat problem

$$\begin{cases}\partial_t u=k\partial_{xx}u, -\infty<x<\infty, t>0\\ u(0,x)=e^{-|x|}\end{cases},$$

where $k>0$ is a constant. Taking transform of the above equation, we reach

$$\begin{cases}\partial_t\hat{u}=-k\omega^2\,\hat{u}\\ \hat{u}(0,\omega)=\frac{2}{1+\omega^2}\end{cases},$$

and thus

$$\hat{u}(t,\omega)=\frac{2}{1+\omega^2}e^{-k\omega^2 t}.$$

Note that $\hat{u}$ is even, and therefore

$$u(t,x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{-k\omega^2 t}}{1+\omega^2}\cos(\omega x)\,d\omega.$$

The graph of the solution is shown in Fig.9.2 for $k=1$ and for $t=0, 0.1, 0.5$ and $t=2$.

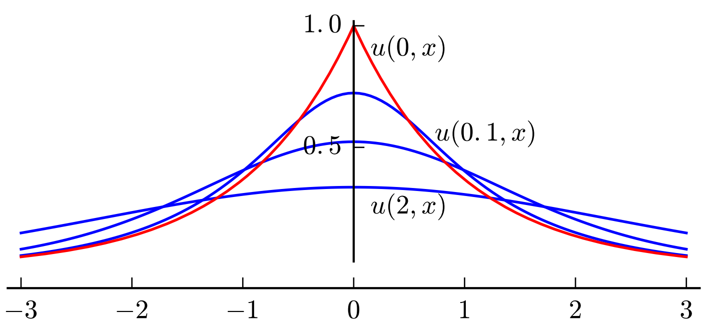

Figure 9.2.

Remark 9.1. Observe that the solution spreads out in time and that the sharp point in the initial heat profile (at $x=0$) immediately disappear. In fact, the solution is averaged for $t>0$. This averaging process conserves the total thermal energy of the system, and makes the solution smooth.

Example 9.3. Let us solve the following problem

$$\begin{cases} \partial_t u = \partial_{xx} u + \frac{\delta(t)}{1+x^2} \\ u(0,x) = 0 \end{cases}.$$

By the FOURIER transform of the equation, we reach the following equation

$$\begin{cases} \partial_t \hat{u} = -\omega^2 \hat{u} + \pi e^{-|\omega|} \delta(t) \\ \hat{u}(0,\omega) = 0 \end{cases}.$$

The above differential equation can be solved, for example, by the LAPLACE transform method. The solution is

$$(s+\omega^2)\hat{U}(s,\omega) = \pi e^{-|\omega|},$$

and therefore $\hat{u}(t,\omega) = \pi e^{-|\omega|} e^{-\omega^2 t}$. Thus, the solution u is

$$u(t,x) = \int_0^\infty e^{-\omega} e^{-\omega^2 t} \cos(\omega x) \mathrm{d}\omega.$$

It is simply verified that the obtained solution is the same as the solution to the following problem

$$\begin{cases} \partial_t u = \partial_{xx} u \\ u(0,x) = \frac{1}{1+x^2} \end{cases}.$$

9.1.2 Convolution representation of the solution

Back to formula (9.3), we can use the convolution formula and write the solution to a heat equation as follows

$$\mathcal{F}^{-1}\{\hat{u}_0(\omega)\, e^{-\omega^2 t}\} = u_0(x) * \mathcal{F}^{-1}\{e^{-\omega^2 t}\} = u_0(x) * \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}.$$

The function $\Phi(t,\, x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$ is called the fundamental solution of the heat equation or the *heat kernel*l. Therefore,

$$u(t,x) = \int_{\mathbb{R}} u_0(y) \Phi(t, x-y)\, dy = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} u_0(y)\, e^{-\frac{(x-y)^2}{4t}}\, dy. \tag{9.5}$$

Formulation presented in (9.5) has an advantage over (9.4). In fact, one can use (9.4) only when $u_0(x)$ has a FOURIER transform.

Example 9.4. Let us solve the following heat problem

$$\begin{cases} \partial_t u = \partial_{xx} u, -\infty < x < \infty, t > 0 \\ u(0, x) = \left\{ \begin{smallmatrix} 1 & x < 0 \\ 2 & x > 0 \end{smallmatrix} \right. \end{cases}$$

The relation (9.5) gives the solution

$$u(t, x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4t}} dy + \frac{2}{\sqrt{4\pi t}} \int_{0}^{\infty} e^{-\frac{(x-y)^2}{4t}} dy.$$

Notice that

$$\frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{y^2}{4t}} dy = 1,$$

and thus, at $x = 0$ we obtain $u(t, 0) = \frac{3}{2}$. Fig.9.3 shows the solution for $t = 0.01, 0.05, 0.2$. As it is observed from the figure, the solution becomes smooth immediately for $t > 0$.

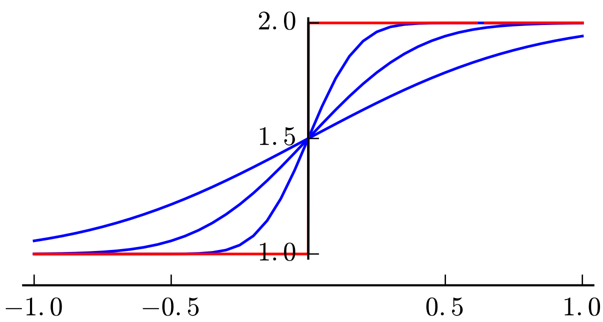

Figure 9.3.

9.1.3 Heat problems in semi-infinite domain

Consider a heat problem defined on the semi-infinite domain $x \in (0, \infty)$

$$\begin{cases} \partial_t u = \partial_{xx} u, 0 < x < \infty, t > 0 \\ u(t, 0) = 0 \\ u(0, x) = f(x) \end{cases}. \tag{9.6}$$

Because the domain of x is defined in $(0, \infty)$, we should use the sine or cosine FOURIER integrals. According to the given boundary condition, we choose the sine FOURIER transform for the problem and write

$$\begin{cases} \partial_t \hat{u}_s(\omega) = -\omega^2 \hat{u}_s(\omega) \\ \hat{u}_s(0, \omega) = \hat{f}_s(\omega) \end{cases}.$$

THe solution $\hat{u}_s(t, \omega)$ is

$$\hat{u}_s(t, \omega) = \hat{f}_s(\omega) e^{-\omega^2 t},$$

and thus

$$u(t, x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \hat{f}_s(\omega) e^{-\omega^2 t} \sin(\omega x) \, d\omega.$$

Note that $u(t,0)=0$ for all $t\geq 0$.

Example 9.5. Let us solve the following heat problem

$$\begin{cases} \partial_t u = \partial_{xx} u + t f(x), 0 < x < \infty, t > 0 \\ u(t,0) = 0 \\ u(0,x) = 0 \end{cases},$$

where $f(x)=1$ for $0<x<1$ and zero otherwise. Let us take the sine FOURIER transform of the problem and rewrite the equation as follows

$$\begin{cases} \partial_t \hat{u}_s(t,\omega) = -\omega^2 \hat{u}_s(t,\omega) + t\,\frac{1-\cos\omega}{\omega} \\ \hat{u}_s(0,\omega) = 0 \end{cases}.$$

The above problem is solved for

$$\hat{u}_s(t,\omega) = \frac{1-\cos\omega}{\omega} e^{-\omega^2 t} \int_0^t s\, e^{\omega^2 s}\, ds = \frac{1-\cos\omega}{\omega} \left\{ \frac{t}{\omega^2} - \frac{1-e^{-\omega^2 t}}{\omega^4} \right\}.$$

The solution $u(t,x)$ is derived then by the following formula

$$u(t,x) = \frac{2}{\pi} \int_0^\infty \hat{u}_s(t,\omega) \sin(\omega x)\, d\omega.$$

Fig.9.4 shows the solution for $t=0.5$.

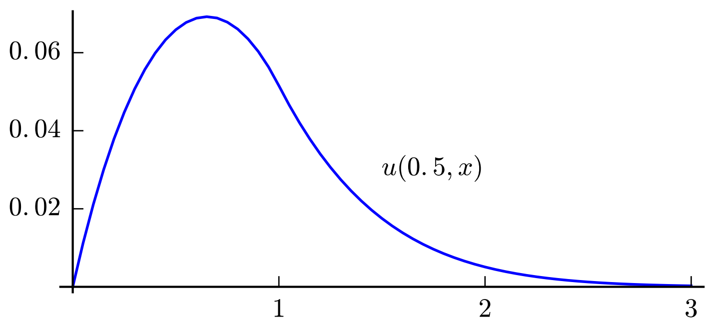

Figure 9.4.

Remark 9.2. If the boundary condition is $\partial_x u(t,\,0)=0$, we choose the cosine FOURIER transform for the problem.

9.1.4 Wave problem and d'Alembert formula

The method to solve a wave equation is completely similar to solve a heat problem. Consider the following equation

$$\begin{cases} \partial_{tt} u = c^2\, \partial_{xx} u, -\infty < x < \infty \\ u(0,x) = f(x) \\ \partial_t u(0,x) = g(x) \end{cases}.$$

Let us write the solution as follows

$$u(t,x)=\frac{1}{2\pi}\int_{\mathbb{R}}\hat{u}(t,\omega)\,e^{i\omega x}\,d\omega,$$

for an undetermined function $\hat{u}(t,\omega)$. To determine $\hat{u}$, we substitute the integral into the equation and obtain

$$\begin{cases}\partial_{tt}\,\hat{u}(t,\omega)=-c^2\omega^2\,\hat{u}(t,\omega)\\ \hat{u}(0,\omega)=\hat{f}(\omega),\partial_t\hat{u}(0,\omega)=\hat{g}(\omega)\end{cases}.$$

The general solution to the above initial value problem is

$$\hat{u}(t,\omega)=\hat{f}(\omega)\cos(c\omega t)+\frac{1}{c\omega}\hat{g}(\omega)\sin(c\omega t),$$

and thus

$$u(t,x)=\mathcal{F}^{-1}\{\hat{f}(\omega)\cos(c\omega t)\}+\frac{1}{c}\mathcal{F}^{-1}\left\{\frac{1}{\omega}\,\hat{g}(\omega)\sin(c\omega t)\right\}.$$

According to the convolution formula, one can write

$$\mathcal{F}^{-1}\{\hat{f}(\omega)\cos(c\omega t)\}=f(x)*\frac{1}{2}[\delta(x-ct)+\delta(x+ct)]=\frac{1}{2}[f(x-ct)+f(x+ct)].$$

For the second term, we have

$$\frac{1}{c}\mathcal{F}^{-1}\left\{\frac{1}{\omega}\,\hat{g}(\omega)\sin(c\omega t)\right\}=\frac{-1}{2c}\left(\int_0^x g(s)ds\right)*[\delta(x-ct)-\delta(x+ct)]=$$
$$=\frac{1}{2c}\int_{x-ct}^{x+ct}g(s)ds.$$

Therefore, the solution can be written as

$$u(t,x)=\frac{1}{2}[f(x-ct)+f(x+ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}g(s)ds. \tag{9.7}$$

Example 9.6. Let us solve the following damped wave equation

$$\begin{cases}\partial_{tt}u+2\xi\partial_t u=c^2\partial_{xx}u\\ u(0,x)=f(x)\\ \partial_t u(0,x)=g(x)\end{cases}.$$

By the FOURIER transform, the problem reduces to

$$\begin{cases}\partial_{tt}\hat{u}+2\xi\partial_t\hat{u}=-c^2\omega^2\hat{u}\\ \hat{u}(0,\omega)=\hat{f}(\omega),\partial_t\hat{u}(0,\omega)=\hat{g}(\omega)\end{cases}.$$

For example, if $g=0$ then the solution is

$$\hat{u}(t,\omega)=e^{-\xi t}\hat{f}(\omega)\left[\cos(\mu t)+\frac{\xi}{\mu}\sin(\mu t)\right],$$

where $\mu = \sqrt{\omega^2 - \xi^2}$. The figure (9.5) shows the solution $u(t, x)$ when $c = 1$ and $\xi = 0.5$ if $f(x) = \begin{cases} (1-x^2)/4 & -1 \leq x \leq 1 \\ 0 & \text{othersise} \end{cases}$. Observe that two branches are not separated completely due to the damping factor ξ.

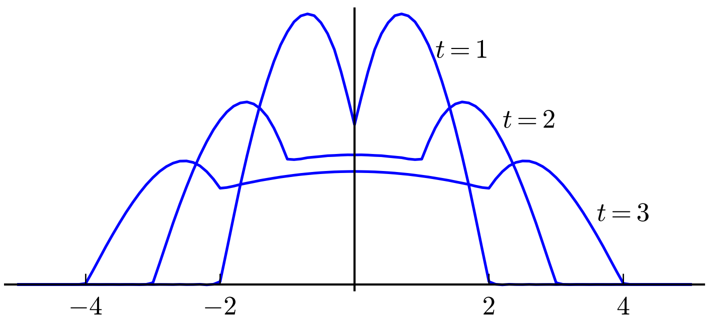

Figure 9.5.

9.1.5 Heat kernel

Consider the following heat problem

$$\begin{cases} \partial_t u = \partial_{xx} u, -\infty < x < \infty, t > 0 \\ u(0, x) = \delta(x - x_0) \end{cases},$$

where x_0 is an arbitrary point in $(-\infty, \infty)$, and δ is the DIRAC delta function. According to relation (9.5), the solution can be written as

$$u(t, x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \delta(y - x_0)\, e^{-\frac{(x-y)^2}{4t}}\, dy = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-x_0)^2}{4t}},$$

and thus the fundamental solution $\Phi(t, x)$ is called sometime the *impulse response* of the heat equation. Accordingly, the response to an arbitrary initial heat profile can be written as

$$u(t, x) = u_0(x) * \Phi(t, x).$$

As it is observe from Fig.9.6, the solution shrinks at x_0 for $t \to 0$.

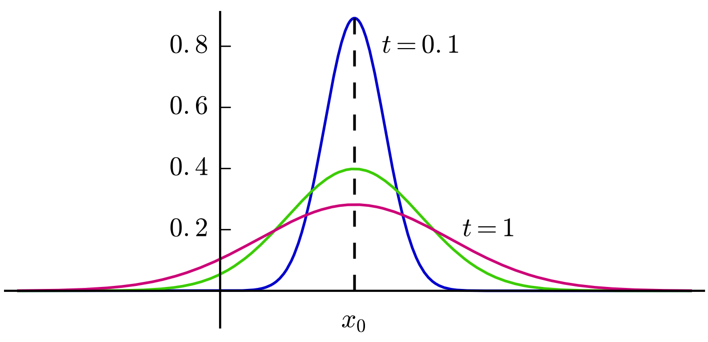

Figure 9.6.

The following proposition states that the initial thermal energy (if it is finite) is conserved for $t>0$.

Proposition 9.1. *Assume that the initial heat profile $u_0(x)$ in (9.1) is integrable. Then the solution decay in long term*

$$\lim_{t\to\infty} u(t,x)=0, \tag{9.8}$$

and furthermore

$$\int_{\mathbb{R}} u(t,x)\,\mathrm{d}x=\int_{\mathbb{R}} u_0(x)\,\mathrm{d}x. \tag{9.9}$$

Proof. We have

$$\lim_{t\to\infty} u(t,x)=\lim_{t\to\infty}\int_{\mathbb{R}} u_0(y)\,\Phi(t,x-y)\,dy=\int_{\mathbb{R}} u_0(y)\lim_{t\to\infty}\Phi(t,x-y)\,dy.$$

The limit can be passed inside the integral according to the dominant convergence theorem. Note that

$$|u_0(y)\Phi(t,x-y)|<|u_0(y)|,$$

and $u_0(x)$ is integrable. On the other hand, since we have

$$\lim_{t\to\infty}\Phi(t,x-y)=0,$$

we conclude (9.8). To show (9.9), we take $x-y=z$ and write

$$\int_{\mathbb{R}} u(t,x)\,\mathrm{d}x=\int_{\mathbb{R}}\int_{\mathbb{R}} u_0(y)\Phi(t,x-y)\mathrm{d}y=\int_{\mathbb{R}}\Phi(t,z)\left(\int_{\mathbb{R}} u_0(x-z)\,dx\right)\mathrm{d}z.$$

Since u_0 is integrable, we have

$$\int_{\mathbb{R}} u_0(x-z)\,dx=\int_{\mathbb{R}} u_0(x)\,\mathrm{d}x,$$

and therefore

$$\int_{\mathbb{R}} u(t,x)\,\mathrm{d}x=\left(\int_{\mathbb{R}} u_0(x)\,dx\right)\left(\int_{\mathbb{R}}\Phi(t,z)dz\right)$$

It is simply verified that

$$\int_{\mathbb{R}}\Phi(t,x)dx=1,$$

and hence (9.9). □

Remark 9.3. Note that in general we can pass the limit inside the integrals and write

$$\lim_{t\to\infty}\int_{\mathbb{R}} u(t,x)\,dx=\int_{\mathbb{R}}\lim_{t\to\infty} u(t,x)\,dx.$$

In fact, if the relation holds, by property $\lim_{t\to\infty} u(t,x)=0$, we should have

$$\lim_{t\to\infty}\int_{\mathbb{R}} u(t,x)\,dx=0,$$

which is evidently wrong. Another example is Φ itself as

$$\lim_{t\to\infty}\int_{\mathbb{R}}\Phi(t,x)\,\mathrm{d}x=\lim_{t\to\infty}1=1,$$

and

$$\int_{\mathbb{R}}\lim_{t\to\infty}\Phi(t,x)\,\mathrm{d}x=0.$$

Proposition 9.2. *Assume that $f(x)$ is a bounded continuous function in $\mathbb{R}$, then*

$$\lim_{t\to 0}\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}}f(y)\,e^{-\frac{(x-y)^2}{4t}}\,dy=f(x).$$

Proof. Let $y=2\sqrt{t}\,z+x$ and then

$$\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}}f(y)\,e^{-\frac{(x-y)^2}{4t}}\,dy=\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}f(2\sqrt{t}\,z+x)\,e^{-z^2}\,dz.$$

According to the dominant convergence theorem, we can write

$$\lim_{t\to 0}\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}}f(y)\,e^{-\frac{(x-y)^2}{4t}}\,dy=\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}\lim_{t\to 0}f(2\sqrt{t}y+x)\,e^{-z^2}\,dz$$
$$=\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}f(x)\,e^{-\frac{z^2}{4t}}\,dz=\frac{f(x)}{\sqrt{\pi}}\int_{\mathbb{R}}e^{-\frac{z^2}{4t}}\,dz=f(x),$$

and this completes the proof. □

Problem 9.1. Assume u_0 is an admissible functions, and $u(t,x)$ is the solution to the following problem

$$\begin{cases}\partial_t u=\partial_{xx}u\\ u(0,x)=u_0(x)\end{cases}.$$

Show the following relation

$$\lim_{t\to 0}u(t,x)=u_0(x)$$

Now, we use the relation (9.5) to show that $u(t,x)$ is smooth even if the initial data is discontinuous. The proof of the following theorem is given in the appendix to this chapter.

Theorem 9.1. *Assume that the initial data $u_0(x)$ is admissible, then $u(t,x)$ given in (9.5) is smooth for $t>0$.*

Proof. We show that u is continuous. Fix (t,x), $t>0$ and let (t_n,x_n), $t_n>0$ be an arbitrary sequence converging (t,x). We have

$$\lim_{n\to\infty}u(t_n,x_n)=\lim_{n\to\infty}\int_{\mathbb{R}}u_0(y)\Phi(t_n,x_n-y)dy.$$

Since $t_n>0$, $t_n\to t$, and $t>0$, without loss of generality, we can assume that $t_n>\varepsilon$ for some $\varepsilon>0$ and for all n. On the other hand, for any $\varepsilon>0$ there is $M>0$ such that $\Phi(t_n,x-y)<M$ regardless of x,y. In fact, we have

$$\Phi(t_n,x-y)=\frac{1}{\sqrt{4\pi t_n}}e^{-\frac{(x-y)^2}{4t_n}}\leq\frac{1}{\sqrt{4\pi t_n}}<\frac{1}{\sqrt{4\pi\varepsilon}}=M.$$

Therefore

$$|u_0(y)\Phi(t_n, x_n - y)| < M|u_0(y)|.$$

Since f in integrable, we use the dominant convergence theorem and write

$$\lim_{n\to\infty} u(t_n, x_n) = \int_{\mathbb{R}} u_0(y) \lim_{n\to\infty} \Phi(t_n, x_n - y)\mathrm{d}y = \int_{\mathbb{R}} u_0(y)\Phi(t, x - y)dy = u(t, x).$$

Hence, the continuity is proved. Let us show that $u(t, x)$ is continuously differentiable with respect to t for $t > 0$. Fix t and let h be small enough such that $t + h > 0$. We have

$$\frac{u(t+h, x) - u(t, x)}{h} = \int_{\mathbb{R}} u_0(y) \frac{\Phi(t+h, x-y) - \Phi(t, x-y)}{h} dy.$$

$\Phi(t, x)$ is smooth for $t > 0$, and thus there is $M = M(t)$ such that

$$|\partial_t \Phi(t, x)| < M.$$

Therefore, we have

$$\lim_{h\to 0} \frac{u(t+h, x) - u(t, x)}{h} = \int_{\mathbb{R}} u_0(y) \lim_{h\to 0} \frac{\Phi(t+h, x-y) - \Phi(t, x-y)}{h} \mathrm{dy} = \int_{\mathbb{R}} u_0(y)\partial_t \Phi(t, x-y)dy.$$

The differentiability of higher orders with respect to t and x is proved by a similar argument. □

Problems

Problem 9.2. Solve the following heat problem in $(-\infty, \infty)$ and draw the temperature at $x = 1$ with respect to time

$$\begin{cases} \partial_t u = \partial_{xx} u, \\ u(0, x) = e^{-x^2} \end{cases}.$$

Problem 9.3. Solve the following damped wave equation on $-\infty < x < \infty$

$$\begin{cases} \partial_{tt} u + 2\xi \partial_t u = \partial_{xx} u \\ u(0, x) = 0, \partial_t u(0, x) = e^{-|x|} \end{cases},$$

where $\xi > 0$ is a constant.

Problem 9.4. Solve the following heat problem on $0 < x < \infty$

$$\begin{cases} \partial_t u = k\partial_{xx} u \\ u(t, 0) = 0 \\ u(0, x) = \frac{1}{1+x^2} \end{cases},$$

where $k > 0$ is a constant.

Problem 9.5. Solve the following heat problem

$$\begin{cases} \partial_t u = \partial_{xx} u + e^{-|x|} \\ u(0, x) = 0 \end{cases},$$

and draw the solution for $t = 1$.

Problem 9.6. Solve the following heat problem

$$\begin{cases} \partial_t u = \partial_{xx} u + \frac{e^{-|x|}}{1+x^2} \\ u(0,x) = 0 \end{cases},$$

and draw the solution for $t = 1$.

Problem 9.7. If $f(x)$ is admissible function, show that following problems have same solution

$$\begin{cases} \partial_t u = \partial_{xx} u + \delta(t)\, f(x) \\ u(0,x) = 0 \end{cases}, \begin{cases} \partial_t u = \partial_{xx} u \\ u(0,x) = f(x) \end{cases}.$$

Problem 9.8. Solve the following heat problem in the domain $(-\infty, \infty)$

$$\begin{cases} \partial_t u = \partial_{xx} u \\ u(0,x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \end{cases}$$

Problem 9.9. Solve the following heat equation on the domain $(0, \infty)$

$$\begin{cases} \partial_t u = \partial_{xx} u \\ \partial_x u(t,0) = 0 \\ u(0,x) = e^{-x} \end{cases}.$$

Problem 9.10. Solve the following wave equation in the domain $(0, \infty)$

$$\begin{cases} \partial_{tt} u = 4\partial_{xx} u \\ u(t,0) = 0 \\ u(0,x) = 0, \partial_t u(0,x) = e^{-\frac{x^2}{2}} \end{cases}.$$

Problem 9.11. Assume that the initial data $f(x)$ in (9.1) is admissible.

a) Directly differentiate $\Phi(t,x)$ and verify

$$\partial_t \Phi(t,x) = \partial_{xx} \Phi(t,x).$$

b) Us the result in part (a) to show that the integral solution (9.5) is a true solution to the problem (9.1).

Problem 9.12. Draw the solution of the problem

$$\begin{cases} \partial_{tt} u = \partial_{xx} u \\ u(0,x) = \begin{cases} \frac{(1-x^2)}{4} & -1 \le x \le 1 \\ 0 & \text{othersise} \end{cases} \\ \partial_t u(0,x) = 0 \end{cases},$$

at times $t = 0.5, 1, 1.5, 2$.

Problem 9.13. Solve the following wave problem and draw the solution for different instance of time

$$\begin{cases} \partial_{tt} u = \partial_{xx} u \\ u(0,x) = \delta(t-1)e^{-x^2/2} \\ \partial_t u(0,x) = 0 \end{cases}.$$

Problem 9.14. Show that the solution to the problem

$$\begin{cases} \partial_t u = \partial_{xx} u & x \in (0,\infty) \\ \partial_x u(t,0) = 0 \\ u(0,x) = f(x) & x \in (0,\infty) \end{cases},$$

is

$$u(t,x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\hat{f}_c(\omega)\,e^{\omega^2 t}\cos(\omega x)\,d\omega,$$

where $F_c(\omega)$ is the cosine FOURIER transform of $f(x)$.

Problem 9.15. Solve the following problem

$$\begin{cases}\partial_t u=\partial_{xx}u & x\in(0,\infty)\\ u(t,0)=1 & \\ u(0,x)=\frac{1}{1+x^2} & x\in(0,\infty)\end{cases}.$$

Problem 9.16. Even though we have the formula (9.9), use the PLANCHEREL theorem and (9.3) to show

$$\lim_{t\to\infty}\int_{-\infty}^{\infty}|u(t,x)|^2dx=0.$$

9.2 Higher dimensional problems

9.2.1 Laplace equation

We solve the LAPLACE equation $\Delta u=0$ on semi-bounded domains.

LAPLACE equation on a half-plane

Consider the following problem defined on $\Omega=\{(x,y),y\geq 0\}$

$$\begin{cases}\Delta u=0\\ u(x,0)=f(x)\end{cases}.$$

We look for solutions that remain bounded in Ω, that is,

$$\sup_{(x,y)\in\Omega}|u(x,y)|<\infty. \tag{9.10}$$

The above problem is equivalent to the following one in ω-domain

$$\begin{cases}\partial_{yy}\hat{u}(\omega,y)-\omega^2\,\hat{u}(\omega,y)=0\\ \hat{u}(\omega,0)=\hat{f}(\omega)\end{cases}.$$

Therefore, $\hat{u}(\omega,y)$ is

$$\hat{u}(\omega,y)=A(\omega)\,e^{\omega y}+B(\omega)\,e^{-\omega y},$$

for some undetermined coefficients functions $A(\omega),B(\omega)$. To determine these coefficients, we apply the boundedness condition. The coefficient $A(\omega)$ must be zero for $\omega>0$, otherwise the solution goes unbounded for $y\to\infty$. Similarly, $B(\omega)$ must be zero for $\omega<0$. Hence, $\hat{u}$ has the following form

$$\hat{u}(\omega,y)=\begin{cases}B(\omega)\,e^{-\omega y} & \omega>0\\ A(\omega)\,e^{\omega y} & \omega<0\end{cases}.$$

This is equivalent to $\hat{u}(\omega, y) = C(\omega) e^{-y|\omega|}$ for some function $C(\omega)$. It is simply seen that $C(\omega) = \hat{f}(\omega)$ and therefore

$$u(x, y) = \mathcal{F}^{-1}\{\hat{u}\} = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\omega)\, e^{-\omega|y|}\, e^{i\omega x}\, d\omega. \tag{9.11}$$

Problem 9.17. Show that the above solution can be written in the following form

$$u(x, y) = \frac{y}{\pi} \int_{\mathbb{R}} \frac{f(\xi)}{y^2 + (x - \xi)^2}\, d\xi. \tag{9.12}$$

LAPLACE equation in quadrant

Now let us solve the LAPLACE equation in quadrant $\Omega = \{(x, y); x > 0, y > 0\}$ with boundary condition $u(0, y) = 0$ and $u(x, 0) = f(x)$ for $x > 0$. To solve the problem, we extend the problem on the half-plane $\{(x, y), y > 0\}$ by extending $f(x)$ as an odd function $f_{\text{odd}}(x)$ for $-\infty < x < \infty$. Now, formula (9.11) implies

$$u(x, y) = \frac{2}{\pi} \int_0^{\infty} \hat{f}_s(\omega)\, e^{-\omega|y|} \sin(\omega x)\, d\omega,$$

where $\hat{f}_s(\omega)$ is the sine FOURIER transform of $f(x)$.

LAPLACE equation in a strip

Consider the following problem on strip $\Omega = \{(x, y); a < y < b\}$

$$\begin{cases} \Delta u = 0 \\ u(x, a) = f(x), u(x, b) = g(x) \end{cases},$$

Since the domain is infinite in x-direction, we take the FOURIER transform with respect to x and obtain the following problem

$$\begin{cases} \partial_{yy}\hat{u} - \omega^2 \hat{u} = 0 \\ \hat{u}(\omega, a) = \hat{f}(\omega) \\ \hat{u}(\omega, b) = \hat{g}(\omega) \end{cases}.$$

The solution of the above equation is

$$\hat{u}(\omega, y) = A(\omega)\, e^{\omega y} + B(\omega)\, e^{-\omega y},$$

for some undetermined functions $A(\omega), B(\omega)$ which are determined by the aid of the given boundary conditions $y = a$ and $y = b$ as follows

$$\begin{cases} \hat{f}(\omega) = A(\omega)\, e^{\omega a} + B(\omega)\, e^{-\omega a} \\ \hat{g}(\omega) = A(\omega)\, e^{\omega b} + B(\omega)\, e^{-\omega b} \end{cases}.$$

Thus,

$$A(\omega) = \frac{\hat{f}(\omega)\, e^{-\omega b} - \hat{g}(\omega)\, e^{-\omega a}}{2\sinh(\omega(a - b))}, \qquad B(\omega) = -\frac{\hat{f}(\omega)\, e^{\omega b} - \hat{g}(\omega)\, e^{\omega a}}{2\sinh(\omega(a - b))},$$

and finally

$$u(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}} [A(\omega)\, e^{\omega y} + B(\omega)\, e^{-\omega y}]\, e^{i\omega x}\, d\omega.$$

9.2.2 Heat and wave problems

We solve simple heat and wave problem in 2D. In particular, we solve problems $\partial_t u = k\,\Delta u$, and $\partial_{tt} u = c^2 \Delta u$ on semi-bounded domains.

Solution to heat equation

We first find the form of heat kernel in $\mathbb{R}^2$. For higher dimension, the form of the solution is completely similar. Consider the following problem

$$\begin{cases} \partial_t u = \Delta u, (x,y) \in \mathbb{R}^2 \\ u(0,x,y) = u_0(x,y) \end{cases}. \tag{9.13}$$

If the solution $u(t,x,y)$ has a transform $\hat{u}$, then it satisfies the following equation

$$\begin{cases} \partial_t \hat{u}(t,\omega_x,\omega_y) = -\omega^2\, \hat{u}(t,\omega_x,\omega_y) \\ \hat{u}(0,\omega_x,\omega_y) = \hat{u}_0(\omega_x,\omega_y) \end{cases},$$

where $\omega = \sqrt{\omega_x^2 + \omega_y^2}$. The above equation is a linear ordinary equation with respect to t and thus

$$\hat{u}(t,\omega_x,\omega_y) = \hat{u}_0(\omega_x,\omega_y)\, e^{-\omega^2 t}. \tag{9.14}$$

This gives the solution u as follows

$$u(t,x,y) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \hat{u}_0(\omega_x,\omega_y)\, e^{-\omega^2 t}\, e^{i\omega_x x}\, e^{i\omega_y y}\, d\omega_x\, d\omega_y. \tag{9.15}$$

By the convolution, we can write

$$u(t,x,y) = \mathcal{F}^{-1}\{\hat{u}_0(\omega_x,\omega_y)\, e^{-\omega^2 t}\} = u_0(x,y) * \mathcal{F}^{-1}\{e^{-(\omega_x^2+\omega_y^2)t}\}.$$

On the other hand, we have

$$\mathcal{F}^{-1}\{e^{-(\omega_x^2+\omega_y^2)t}\} = \frac{1}{4\pi t}\, e^{-\frac{(x^2+y^2)}{4t}}. \tag{9.16}$$

The functions

$$\Phi(t,x,y) = \frac{1}{4\pi t}\, e^{-\frac{(x^2+y^2)}{4t}}$$

is called the *fundamental solution* or the *heat kernel* of 2D heat problem. Therefore, the solution can be put in the following convolution form

$$u(t,x,y) = \frac{1}{4\pi t} \int_{\mathbb{R}^2} u_0(z_1,z_2)\, e^{-\frac{(x-z_1)^2+(y-z_2)^2}{4t}}\, dz_1\, dz_2. \tag{9.17}$$

Heat problems on semi-infinite domains

Consider the following heat problem on the half-plane $y > 0$:

$$\begin{cases} \partial_t u = \Delta u \\ u(t,x,0) = 0 \\ u(0,x,y) = u_0(x,y) \end{cases}. \tag{9.18}$$

If we extend $u_0(x,y)$ to $\mathbb{R}^2$ as an odd function with respect to y, then we reach a problem on $\mathbb{R}^2$. Let $u_{\text{odd}}(x,y)$ be the odd extension of u_0 with respect to y in $\mathbb{R}^2$. We have

$$u_{\text{odd}}(t,x,y) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \hat{u}_{\text{odd}}(\omega_x,\omega_y)\, e^{-\omega^2 t}\, e^{i\omega_x x}\, e^{i\omega_y y}\, d\omega_x\, d\omega_y.$$

We claim $u_{\text{odd}}(t,x,0)=0$. In fact, $\hat{u}_{\text{odd}}(\omega_x,-\omega_y) = -\hat{u}_{\text{odd}}(\omega_x,\omega_y)$ and thus

$$\int_{\mathbb{R}^2} \hat{u}_{\text{odd}}(\omega_x,\omega_y)\, e^{-\omega^2 t}\, e^{i\omega_x x}\, e^{i\omega_y y} = i\int_{\mathbb{R}^2} \hat{u}_{\text{odd}}(\omega_x,\omega_y)\, e^{-\omega^2 t}\, e^{i\omega_x x}\sin(\omega_y y),$$

that verifies the claim. Therefore, we can write

$$u_{\text{odd}}(t,x,y) = \frac{i}{4\pi^2}\int_{\mathbb{R}^2} \hat{u}_{\text{odd}}(\omega_x,\omega_y)\, e^{-\omega^2 t}\, e^{i\omega_x x}\sin(\omega_y y)\, d\omega_x\, d\omega_y.$$

The solution $u(t,x,y)$ then can be written as follows in the convolution form

$$u(t,x,y) = u_{\text{odd}}(x,y) * \Phi(t,x,y).$$

Non-homogeneous heat equations

Consider the following non-homogeneous equation

$$\begin{cases} \partial_t u = \Delta u \\ u(t,x,0) = f(x) \\ u(0,x,y) = u_0(x,y) \end{cases}.$$

In order to solve the problem, we use the *superposition principle* and split up the problem into two sub-problems

$$\begin{cases} \Delta v = 0 \\ v(x,0) = f(x) \end{cases} + \begin{cases} \partial_t w = \Delta w \\ w(t,x,0) = 0 \\ w(0,x,y) = u_0(x,y) - v(x,y) \end{cases}.$$

It is simply seen that the solution of the original problem is as follows

$$u(t,x,y) = v(x,y) + w(t,x,y).$$

The function $v_e(x,y)$ is called the *steady state* solution contributed by the boundary term. We have the following property

$$\lim_{t\to\infty} u(t,x,y) = v(x,y) + \lim_{t\to\infty} w(t,x,y) = v(x,y).$$

Solution to wave equations

Now, we consider the wave equation $\partial_{tt} u = c^2 \Delta u$ in $\mathbb{R}^2$. For the sake of simplicity, we solve the following problem

$$\begin{cases} \partial_{tt} u = c^2 \Delta u \\ u(0,x,y) = f(x,y) \\ \partial_t u(0,x,y) = 0 \end{cases}.$$

The ordinary differential equation for $\hat{u}(t,\omega_x,\omega_y)$ is

$$\begin{cases} \partial_{tt}\hat{u} + c^2\omega^2\hat{u} = 0 \\ \hat{u}(0,\omega_x,\omega_y) = \hat{f}(\omega_x,\omega_y) \\ \partial_t\hat{u}(0,\omega_x,\omega_y) = 0 \end{cases},$$

where $\omega = \sqrt{\omega_x^2+\omega_y^2}$ as before. The solution of the above ordinary initial value problem is

$$\hat{u}(t,\omega_x,\omega_y) = \hat{f}(\omega_x,\omega_y)\cos(ct\omega). \tag{9.19}$$

Therefore, the solution $u(t,x,y)$ can be written in the following convolution form

$$u(t,x,y) = f(x,y) * \mathcal{F}^{-1}\{\cos(ct\omega)\}. \tag{9.20}$$

Let us try to find the inverse FOURIER transform in the above formula. To do that, we write ω_x,ω_y as $\omega_x = \omega\cos\gamma$, $\omega_y = \omega\sin\gamma$ for $\gamma \in [-\pi,\pi]$. Therefore, we can write

$$\mathcal{F}^{-1}\{\cos(ct\omega)\} = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_0^\infty \cos(ct\omega)e^{-i\omega(x\cos\gamma+y\sin\gamma)}\,\omega\,d\omega\,d\gamma.$$

If we write $x = \rho\cos\theta$, $y = \rho\sin\theta$, we reach

$$\mathcal{F}^{-1}\{\cos(ct\omega)\} = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_0^\infty \cos(ct\omega)\cos(\rho\omega\cos(\gamma-\theta))\omega\,d\omega\,d\gamma.$$

It turns out that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(s\cos\gamma)\,d\gamma = J_0(s),$$

where J_0 is the BESSEL function of the first kind. Therefore, we have

$$\mathcal{F}^{-1}\{\cos(ct\omega)\} = \frac{1}{2\pi}\int_0^\infty J_0(\rho\omega)\cos(ct\omega)\omega\,d\omega.$$

We can write the right hand side of the above formula as follows

$$\frac{1}{2\pi}\int_0^\infty J_0(\rho\omega)\cos(ct\omega)\omega\,d\omega = \frac{1}{2\pi c}\frac{d}{dt}\frac{1}{\rho}\int_0^\infty J_0(\omega)\sin\left(\frac{ct}{\rho}\omega\right)d\omega.$$

We use the following formula

$$\int_0^\infty J_0(\omega)\sin\left(\frac{ct}{\rho}\omega\right)d\omega = \frac{1}{\sqrt{\frac{c^2t^2}{\rho^2}-1}},\ \rho < ct$$

and obtain

$$\mathcal{F}^{-1}\{\cos(ct\omega)\} = \frac{1}{2\pi c}\frac{d}{dt}\frac{1}{\sqrt{c^2t^2-\rho^2}},\ \rho < ct.$$

Finally, the solution $u(t,x,y)$ in formula (9.20) can be written as follows

$$u(t,x,y) = \frac{1}{2\pi c}\frac{d}{dt}\int_{B_{ct}(x,y)}\frac{f(z_1,z_2)}{\sqrt{c^2t^2-(x-z_1)^2-(y-z_2)^2}}\,dz_1dz_2,$$

where $B_{ct}(x,y)$ is the disk centered at (x,y) with radius ct. By a similar calculation, it is shown that if $\partial_t u(0,x,y) = g(x)$ then the solution is

$$u(t,x,y) = \frac{1}{2\pi c}\frac{d}{dt}\int_{B_{ct}(x,y)} \frac{f(z_1,z_2)}{\sqrt{c^2t^2-(x-z_1)^2-(y-z_2)^2}}\,dz_1dz_2 + \frac{1}{2\pi c}\int_{B_{ct}(x,y)} \frac{g(z_1,z_2)}{\sqrt{c^2t^2-(x-z_1)^2-(y-z_2)^2}}\,dz_1dz_2.$$

Problems

Problem 9.18. Solve the following heat equation on $\mathbb{R}^2$

$$\begin{cases} \partial_t u = 4\Delta u \\ u(0,x,y) = e^{-x^2-y^2} \end{cases}.$$

Problem 9.19. Consider the following heat problem in the upper half plane $y > 0$

$$\begin{cases} \partial_t u = \Delta u \\ u(t,x,0) = e^{-|x|} \\ u(0,x,y) = e^{-x^2-y^2} \end{cases}.$$

What method you suggest to solve this problem? Try it and write the solution in the integral form.

Problem 9.20. Solve the following heat equation on the upper half plane $y > 0$

$$\begin{cases} \partial_t u = \Delta u \\ \partial_y u(t,x,0) = 0 \\ u(0,x,y) = e^{-|x|}e^{-y} \end{cases}.$$

Problem 9.21. Consider the LAPLACE equation

$$\begin{cases} \Delta u = 0 \\ u(x,0) = f(x) \end{cases}.$$

Show the the solution can be written as

$$u(x,y) = \frac{y}{\pi}\int_{\mathbb{R}} \frac{f(\xi)}{y^2+(x-\xi)^2}\,d\xi.$$

At first glance, you may think $\lim_{y\to 0} u(x,y) = 0$. Plot the solution $u(x,y)$ for $y = 0.1, 0.05, 0.01$ if $f(x) = e^{-|x|}$ and observe it converges to $f(x)$.

Problem 9.22. Solve the LAPLACE equation on the domain $\Omega = \{(x,y), x > 0, y > 0\}$ with the boundary conditions $u(0,y) = u(x,1) = 0$ and $u(x,0) = e^{-x}$.

Problem 9.23. Write the solution of the following heat problem defined in the quadrant $x > 0$, $y > 0$ in the integral form

$$\begin{cases} \partial_t u = \Delta u \\ u(t,0,y) = f(x) \\ u(t,x,0) = 0 \\ u(0,x,y) = u_0(x,y) \end{cases}.$$

Problem 9.24. Solve the LAPLACE equation $\Delta u = 0$ on the domain $\Omega = \{(x,y), 0 < x < 1\}$ with the boundary conditions $u(0,y) = 0$, $u(1,y) = e^{-y^2}$

Problem 9.25. Solve the LAPLACE equation on the domain $\Omega = \{(x,y), 0<y<1, x>0\}$ with the boundary conditions $u(0,y)=0$, $u(x,1)=0$ and $u(x,0)=e^{-x^2}$

Problem 9.26. Let Ω denote the strip $\{(x,y); 0<y<1\}$

a) Show that the solution to the LAPLACE equation

$$\begin{cases} \Delta u = 0 \\ u(x,0)=0, u(x,1)=\delta(x+1)+\delta(x-1) \end{cases}$$

is

$$u(x,y)=\frac{2}{\pi}\int_0^\infty \frac{\sinh(\omega y)}{\sinh(\omega)}\cos(\omega)\cos(\omega x)\,d\omega.$$

b) Find a solution if the boundary data changes to $u(x,1)=\delta(x+1)-\delta(x-1)$.

Problem 9.27. Solve the following heat problem in the strip $\Omega=\{(x,y); 0<y<1\}$

$$\begin{cases} \partial_t u = \Delta u \\ u(t,x,0)=u(t,x,1)=1 \\ u(0,x,y)=\begin{cases} 1 & -1<x<1 \\ 0 & \text{otherwise} \end{cases} \end{cases}.$$