Chapter 7

Fourier Transform Method

In this chapter, we delve into the Fourier transform and its application in solving linear second-order partial differential equations on unbounded domains. Up until now, we have focused on studying these equations solely on bounded domains with specific boundary conditions. However, with the Fourier transform, we are able to extend our problem-solving capabilities to unbounded domains. The Fourier transform provides us with a remarkable tool to analyze functions defined on unbounded domains, offering insights into their frequency components and facilitating the solution of differential equations, particularly partial differential equations.

This powerful technique is closely associated with the contributions of Joseph Fourier, a renowned French physicist and mathematician whose pioneering work paved the way for tackling linear PDEs in new and innovative ways.

7.1 Introduction

We will delve into the development of the Fourier transform from its roots in Fourier series and its application in solving partial differential equations using the eigenfunction expansion method.

7.1.1 From Fourier series to Fourier transform

In the previous sections, we explored the Fourier series representation of a piecewise continuously differentiable function $f(x)$ defined on $x \in [-L, L]$. We demonstrated that such a function can be elegantly expressed using trigonometric functions, namely $\{\cos(n\omega x), \sin(n\omega x)\}$, where $\omega = \frac{\pi}{L}$. By utilizing the Euler formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta),$$

we can equivalently represent the trigonometric Fourier series in its complex form as follows:

$$f(x) = \sum_{n=-\infty}^{\infty} F_n e^{i\omega_n x}, \tag{7.1}$$

where the coefficients F_n are given by:

$$F_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-in\omega x}\, dx. \tag{7.2}$$

These formulas are derived from the orthogonality property of the complex exponential functions

$$\langle\!\langle e^{in\omega x}, e^{im\omega x}\rangle\!\rangle := \int_{-L}^{L} e^{i\frac{n\pi}{L}x}\, e^{-i\frac{m\pi}{L}x}\, dx = \begin{cases} \frac{1}{2L} & n = m \\ 0 & n \neq m \end{cases}.$$

In this context, we employ the complex version of the inner product $\langle\!\langle , \rangle\!\rangle$, defined as

$$\langle\!\langle f, g\rangle\!\rangle = \int_a^b f(x)\, \bar{g}(x)\, dx,$$

where $\bar{g}\,(x)$ represents the complex conjugate of $g(x)$, and the integration is over the interval $[a, b]$.

The concept of the Fourier series representation can be extended to functions defined on an unbounded domain by introducing the Fourier transform. Starting from the complex Fourier series representation of a function $f(x)$ in $[-L, L]$, and substituting (7.2) into (7.1), we can write

$$f(x) = \frac{1}{2\pi}\int_{-L}^{L} f(z)\left(\sum_{n=-\infty}^{\infty} e^{i(x-z)n\omega_L}\, \omega_L\right) dz,$$

where $\omega_L = \frac{\pi}{L}$. As L approaches infinity, and ω_L approaches zero, we interpret the summation as an integral:

$$\lim_{L\to\infty} \sum_{n=-\infty}^{\infty} e^{i(x-z)n\omega_L}\, \omega_L = \lim_{L\to\infty}\int_{-L}^{L} e^{i(x-z)\omega}\, d\omega,$$

resulting in:

$$f(x) = \frac{1}{2\pi}\lim_{L\to\infty}\int_{-L}^{L}\left(\int_{-L}^{L} f(z)\, e^{-i\omega z}\, dz\right) e^{i\omega x}\, d\omega.$$

The inner integral in this expression is recognized as the Fourier transform of the function f, denoted as $\mathcal{F}\{f\}$, and defined as:

$$\mathcal{F}\{f\} = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx.$$

Note that the integral is taken over x, resulting in a function of ω. For convenience, we denote the transformed function as $\hat{f}(\omega)$. Thus, we obtain the final result:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega x}\, d\omega.$$

To ensure convergence of the integral for $\mathcal{F}\{f\}$, we establish the integrability condition for the function $f(x)$.

Theorem 7.1. *Assume that a function $f(x)$ is integrable over $(-\infty,\infty)$, i.e.,*

$$\int_{-\infty}^{\infty} |f(x)|\,dx < \infty.$$

Then, $\hat{f}(\omega)$ exists, and it is continuous with respect to ω. Furthermore, the supremum of $|\hat{f}(\omega)|$ is finite, i.e.,

$$\sup_{\omega} |\hat{f}(\omega)| < \infty.$$

Proof. The existence of $\hat{f}$ is verified by the inequality:

$$\left|\int_{-\infty}^{\infty} f(x)\,e^{-i\omega x}\,dx\right| \le \int_{-\infty}^{\infty} |f(x)|\,dx < \infty,$$

ensuring that $\hat{f}(\omega)$ converges for any ω. To demonstrate continuity, let ω_0 be fixed, and as ω approaches ω_0:

$$\lim_{\omega\to\omega_0} \int_{-\infty}^{\infty} f(x)\,e^{-i\omega x}\,dx = \int_{-\infty}^{\infty} \lim_{\omega\to\omega_0} f(x)\,e^{-i\omega x}\,dx = \int_{-\infty}^{\infty} f(x)\,e^{-i\omega_0 x}\,dx.$$

The passage of the limit inside the integral is allowed by the dominant convergence theorem (see the appendix of this book). Hence,

$$\lim_{\omega\to\omega_0} \hat{f}(\omega) = \hat{f}(\omega_0).$$

The final claim is proved as follows:

$$\sup_{\omega} \int_{-\infty}^{\infty} f(x)\,e^{-i\omega x}\,dx \le \sup_{\omega}\left|\int_{-\infty}^{\infty} f(x)\,e^{-i\omega x}\,dx\right| \le \sup_{\omega} \int_{-\infty}^{\infty} |f(x)|\,dx < \infty,$$

confirming that the supremum of $|\hat{f}(\omega)|$ is finite, and this completes the proof. $\square$

The following theorem, known as the Fourier theorem states the convergence of $\mathcal{F}^{-1}$ defined as:

$$\mathcal{F}^{-1}\{\hat{f}\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\,e^{-i\omega x}\,d\omega.$$

Theorem 7.2. *Let $f(x)$ be an integrable function defined on $(-\infty,\infty)$ and piecewise continuously differentiable, with $f'(x)$ being a piecewise continuous function. Then, the inverse Fourier transform, denoted as $\mathcal{F}^{-1}\{\hat{f}\}$, is equal to $f(x)$ at continuity points of f. However, at discontinuity points of f, the inverse Fourier transform is given by:*

$$\mathcal{F}^{-1}\{\hat{f}\} = \frac{f(x^+) + f(x^-)}{2}$$

where $f(x^+)$ and $f(x^-)$ represent the right and left limits of f at x, respectively.

Example 7.1. Consider the function $f(x)$ given by:

$$f(x) = \begin{cases} \frac{1}{\sqrt{|x|}} & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

This function is integrable:

$$\int_{-1}^{1} \frac{1}{\sqrt{|x|}}\, dx = 4 < \infty.$$

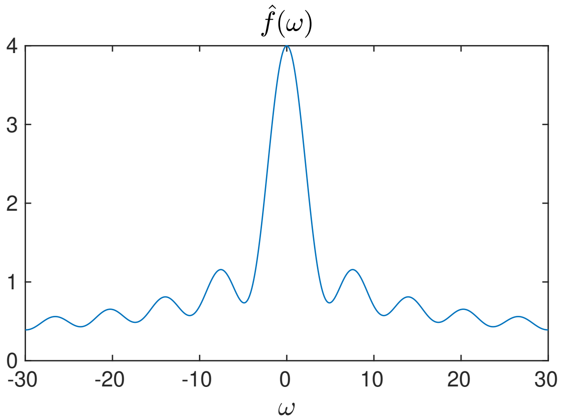
By the formula, we have

$$\mathcal{F}\left\{\frac{1}{\sqrt{|x|}}\right\} = \lim_{L\to\infty} \int_{-L}^{L} \frac{1}{\sqrt{|x|}} e^{-i\omega x}\, dx = \lim_{L\to\infty} \left\{\int_{-L}^{L} \frac{\cos(\omega x)}{\sqrt{|x|}}\, dx + \int_{-L}^{L} \frac{i\sin(\omega x)}{\sqrt{|x|}} dx\right\}.$$

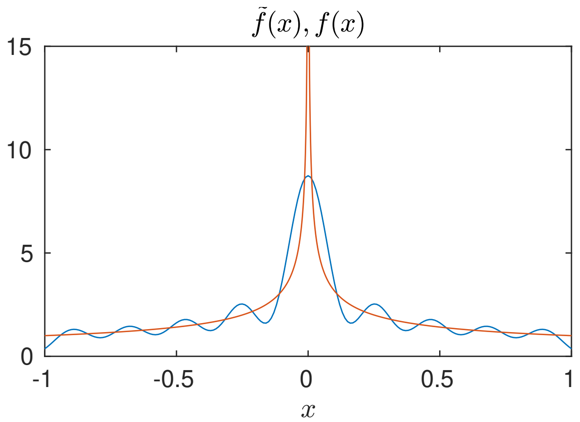
Since the function $\frac{1}{\sqrt{|x|}}$ is even, the second integral at the right-hand side is zero, and we can write:

$$\mathcal{F}\left\{\frac{1}{\sqrt{|x|}}\right\} = \lim_{L\to\infty} \int_{-L}^{L} \frac{\cos(\omega x)}{\sqrt{|x|}}\, dx.$$

The figure below depicts the Fourier transform of the given function in $\omega \in (-30, 30)$



Even though, the function is not piecewise continuous, its inverse Fourier transform $\mathcal{F}^{-1}\{\hat{f}\}$ converges outside of the singular point $x = 0$ as shown below:



Exercise 7.1. Show that if $f(x)$ is an even function, $\mathcal{F}\{f\}$ is an even function in ω and real. If $f(x)$ is an odd function, $\mathcal{F}\{f\}$ is an odd function in ω and pure imaginary.

7.1.2 Fourier transform as frequency distribution

When we view a function $f(x)$ as an electrical signal or a wave in time or space, its Fourier transform $\hat{f}(\omega) = \mathcal{F}\{f(x)\}$ reveals the frequency components embedded in the signal. For a function $f(x)$ defined on $x \in [-L, L]$, the frequency distribution is discrete, characterized by the terms $F_n e^{in\omega_L x}$, where $\omega_L = \frac{\pi}{L}$, and the magnitude F_n is given by:

$$F_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-in\omega_L x}\, dx.$$

As we move to functions defined on the entire real line $(-\infty, \infty)$, this frequency spectrum evolves into a continuous distribution over the ω-domain.

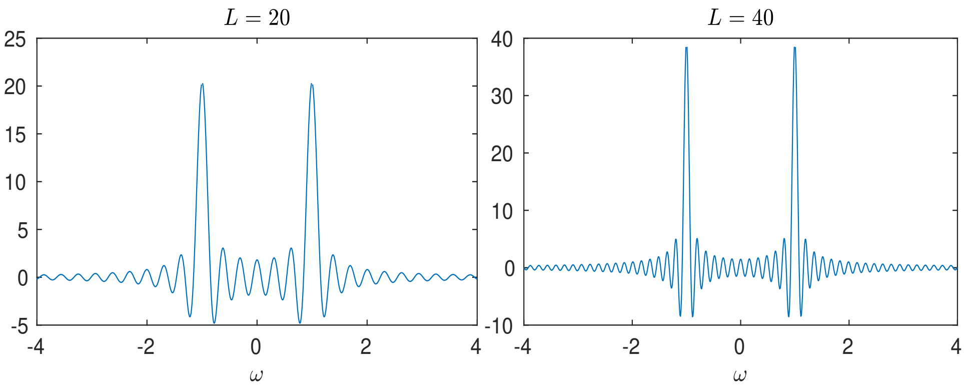
To illustrate this, let's consider the function $f(x) = \cos(\omega_0 x)$. This function is periodic with a period $T_0 = \frac{2\pi}{\omega_0}$ and the angular frequency $\omega = \omega_0$. Thus, the signal has a single periodic frequency component, which is ω_0. We expect that $\mathcal{F}\{\cos(\omega_0 x)\}$ will exhibit a spike at ω_0:

$$\mathcal{F}\{\cos(\omega_0 x)\} = \lim_{L\to\infty}\int_{-L}^{L} \cos(\omega_0 x)\, e^{-i\omega x} dx.$$

Using Euler's formula $\cos(\omega_0 x) = \frac{e^{i\omega_0 x} + e^{-i\omega_0 x}}{2}$, we obtain:

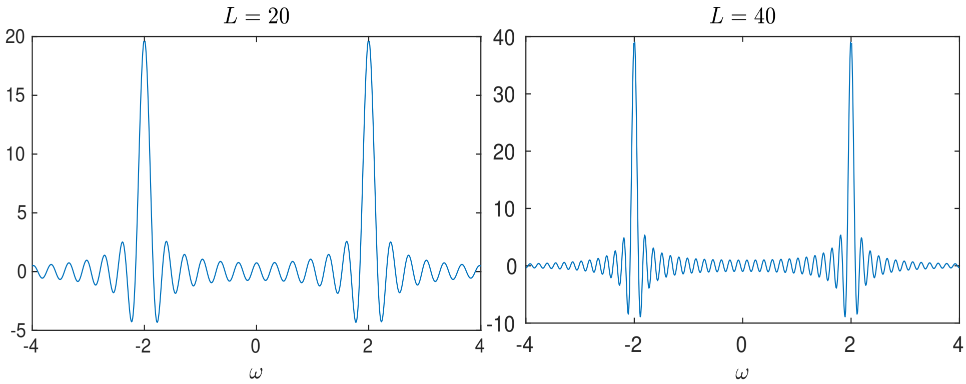
$$\mathcal{F}\{\cos(\omega_0 x)\} = \lim_{L\to\infty}\left\{\frac{\sin[(\omega-\omega_0)L]}{\omega-\omega_0} + \frac{\sin[(\omega+\omega_0)L]}{\omega+\omega 0}\right\}.$$

The figure below depicts the graph of this Fourier transform for $\omega_0 = 1$ for $L = 20$ and $L = 40$.



As observed, the spectrum or frequency distribution of the function $\cos(x)$ exhibits a spike at the physical frequency $\omega = 1$ and a non-physical (or purely mathematical) frequency at $\omega = -1$, which becomes more pronounced as L approaches infinity. Therefore, $\hat{f}(\omega) = 0$ for $\omega \neq \pm 1$.

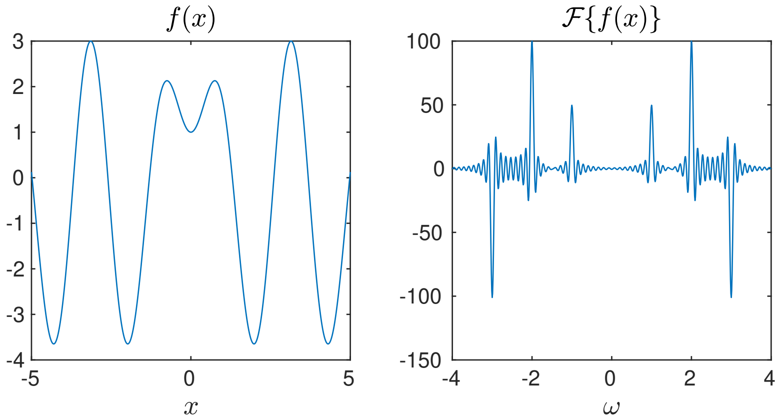
Now, let's compare this with the function $f(x) = \cos(2x)$. This function rotates twice faster than $\cos(x)$. The function $\mathcal{F}\{\cos(2x)\}$ exhibits spikes at $\omega = 2$, and $\omega = -2$.



Now, consider an electrical or mechanical signal composed of various sub-signals with different angular frequencies:

$$f(x) = c_1 \cos(x) + c_2 \operatorname{co}(2x) + \cdots + c_n \cos(nx).$$

The function $\mathcal{F}\{f(x)\}$ has spikes at physical frequencies $\omega = 1, 2, \ldots, n$ weighted by the values $c_1, c_2, \ldots, c_n$. The figure below depicts the function $f(x) = \cos(x) + 2\cos(2x) - 2\cos(3x)$, and its Fourier transform. It is observed that $\mathcal{F}\{f(x)\}$ has spikes at $\omega = 1, 2, 3$ with weights $1, 2, -2$:



In general, a function $f(x)$ can have a continuous distribution of angular frequencies ω, which is commonly represented as its frequency spectrum $\hat{f}(\omega)$ using the Fourier transform $\mathcal{F}$. The inverse Fourier transform allows us to retrieve $f(x)$ from its frequency spectrum through the formula

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega x}\, d\omega.$$

In physics and engineering contexts, it is often convenient to use the notation $\omega = 2\pi f$ and express the inverse Fourier transform as:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i2\pi f x}\, df.$$

However, this notation may lead to confusion, as we typically denote our function by $f(x)$.

Exercise 7.2. Let's compare the frequency spectrum of two functions $f_1(x) = e^{-0.2|x|}\cos(x)$ and $f_2(x) = e^{-0.2|x|}\cos(2x)$. The function f_2 changes twice faster than f_1. Draw these two functions in x-domain. Now draw the functions $\hat{f}_1(\omega)$ and $\hat{f}_2(\omega)$ and explain what you observe. You can use the following code:

```
x=-10:0.01:10;
f1=@(x) exp(-abs(x)/5).*cos(x);
f2=@(x) exp(-abs(x)/5).*cos(2*x);
subplot(1,2,1)
plot(x,f1(x))
subplot(1,2,2)
plot(x,f2(x))
figure()
w=-2:0.01:2;
fw1=2*integral(@(x) f1(x).*cos(w.*x),0,40, ...
 'ArrayValued',true);
subplot(1,2,1)
plot(w,fw1);
w=-3:0.01:3;
fw2=2*integral(@(x) f2(x).*cos(w.*x),0,40, ...
 'ArrayValued',true);
subplot(1,2,2)
plot(w,fw2)
```

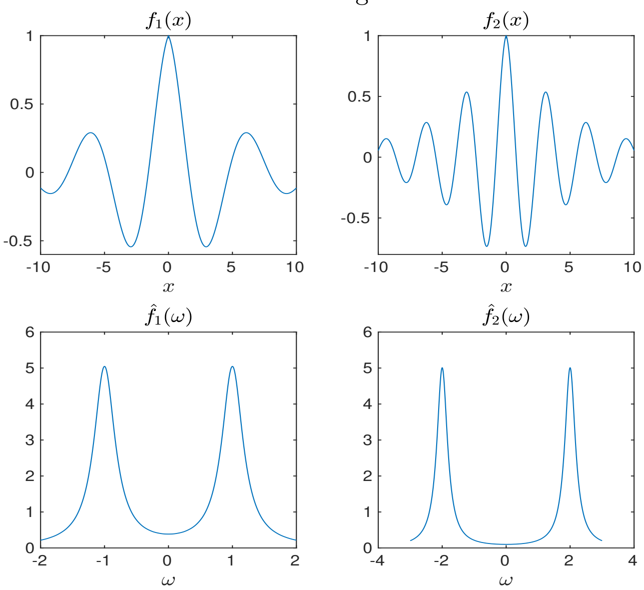
Explain why we used in the above code the integral

$$2\int_0^{40} f_1(x)\cos(\omega x)\,dx,$$

instead of the integral

$$\int_{-40}^{40} f_1(x)\,e^{-i\omega x}\,dx.$$

Your figures should be like the following ones:



Exercise 7.3. The function $f(x) = 1$ is not integrable. Its Fourier transform is

$$\mathcal{F}\{1\} = \lim_{L\to\infty}\int_{-L}^{L} e^{-i\omega x}\,dx = \lim_{L\to\infty}\frac{2\sin(\omega L)}{\omega}.$$

Plot this function for different values of L. What do you observe at $\omega = 0$? The function $f(x) = 1$ is the limiting function of $\cos(\omega_0 x)$ when ω_0 approaches 0. Compare this result to the Fourier transform of $\cos(\omega_0 x)$ when $\omega_0 \to 0$.

Exercise 7.4. If $f(x)$ is an odd function, show that $\mathcal{F}\{f\}$ is an odd and pure imaginary function in ω-domain. Let $f(x) = \sin(\omega_0 x)$. This function is not integrable. Plot the imaginary part of $\mathcal{F}\{\sin(\omega_0 x)\}$. What do you observe at ω_0 and $-\omega_0$? Use the following code in Matlab to draw $\mathcal{F}\{\sin(\omega_0 x)\}$.

```
w0=1;%you can change  this value
w=-4:0.01:4; %you can change this interval
xinf=50% you can change this value
fw=integral(@(x) sin(w0*x).*sin(w.*x),-xinf,xinf,'ArrayValued',true);
plot(w,fw)
```

7.1.3 Eigenfunction expansion method

The Eigenfunction expansion method for problems on unbounded domain can be illustrated by considering the heat equation defined on $x \in (-\infty, \infty)$:

$$\begin{cases} u_t = k u_{xx} \\ u(x,0) = f(x) \end{cases}.$$

While there are no boundary conditions specified in this problem, we assume that the solution $u(x,t)$ is integrable to guarantee the existence of the Fourier transform $\hat{u}(\omega,t)$. We then write the solution $u(x,t)$ as the following integral:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(\omega,t)\, e^{i\omega x}\, d\omega, \tag{7.3}$$

where $\hat{u}(\omega,t)$ is an undetermined function. This representation is analogous to how we expressed solutions using eigenfunctions $\{e^{in\omega_L x}\}$ or equivalently, trigonometric functions $\left\{\cos\left(\frac{n\pi}{L}x\right), \sin\left(\frac{n\pi}{L}x\right)\right\}$ for boundary value problems defined on a bounded domain $(-L, L)$.

To determine $\hat{u}(\omega,\ t)$, we substitute the integral (7.3) into the heat equation $u_t = k u_{xx}$ and obtain the ordinary differential equation:

$$\hat{u}_t(\omega,t) = -k\omega^2\, \hat{u}(\omega,t).$$

This differential equation is with respect to time t, and it can be solved as:

$$\hat{u}(\omega,t) = C(\omega)\, e^{-k\omega^2 t},$$

where $C(\omega)$ is an arbitrary function with respect to ω. The specific form of $C(\omega)$ will be determined by the initial condition $u(x,0) = f(x)$ through the Fourier transform: $\hat{u}(\omega,\ 0) = \hat{f}(\omega)$. Thus, we obtain $\hat{u}(\omega,\ t) = \hat{f}(\omega)\ e^{-k\omega^2 t}$. Consequently, the integral solution for $u(x,t)$ is retrieved by the inverse Fourier transform:

$$u(x,t) = \mathcal{F}^{-1}\{\hat{f}(\omega)\, e^{-k\omega^2 t}\}. \tag{7.4}$$

This solution can be represented in integral form as:

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{-k\omega^2 t}\, e^{i\omega x}\, d\omega. \tag{7.5}$$

Now, let's compare this integral solution to the series solution of the same heat problem defined on the interval $[-L, L]$:

$$\begin{cases} u_t = k u_{xx} & -L < x < L \\ u(-L,t) = u(L,t) \\ u_x(-L,t) = u_x(L,t) \\ u(x,0) = f(x) \end{cases}.$$

The solution to this problem in complex form and in terms of the eigenfunctions $\{e^{in\omega x}\}$ for $\omega = \frac{\pi}{L}$ is given as:

$$u(x,t) = \sum_{n=-\infty}^{\infty} F_n \, e^{-kn^2\omega_L^2 t} \, e^{in\omega_L x}.$$

It turns out that when $L \to \infty$, this solution converges to the integral solution (7.5).

Problem 7.1. Consider the following heat problem

$$\begin{cases} u_t = u_{xx} + h(x) \\ u(x,0) = 0 \end{cases}.$$

Assume that h is integrable. Find an integral solution to this problem.

7.2 Fourier transform of important functions

In this section, we will review the Fourier transform of some important functions that will be useful in solving heat and wave equations on unbounded domains in the future examples. We will examine the Fourier transform of commonly encountered functions, which will aid us in finding integral solutions and understanding the behavior of these functions under Fourier transformation.

7.2.1 Pulse or rectangle function

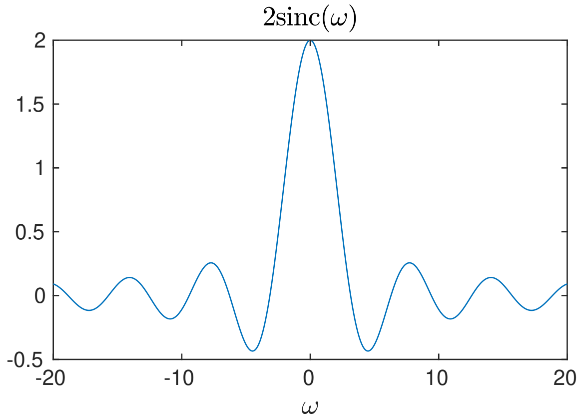
Consider the following function:

$$f(x) = \begin{cases} 1 & -a < x < a \\ 0 & \text{otherwise} \end{cases},$$

where $a > 0$ is a constant. By definition of the Fourier transform, we have

$$\mathcal{F}\{f\} = \int_{-a}^{a} e^{-i\omega x} \, dx = \frac{2\sin(a\omega)}{\omega}.$$

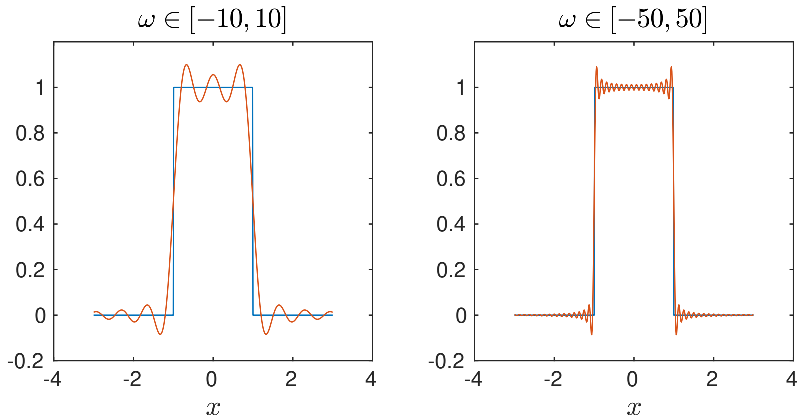
In particular, when $a = 1$, the transform function is $\hat{f}(\omega) = 2\,\mathrm{sinc}(\omega)$ for the sinc function: $\mathrm{sinc}(z) = \frac{\sin(z)}{z}$.



Now, let's investigate the inverse Fourier transform of $\frac{2\sin(a\omega)}{\omega}$. The expression for the inverse transform is given as:

$$\mathcal{F}^{-1}\left\{\frac{2\sin(a\omega)}{\omega}\right\} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(a\omega)}{\omega}\cos(\omega x)\,d\omega.$$

The figure below depicts the inverse transform with integration over intervals $\omega \in (-01, 10)$ and $\omega \in (-50, 50)$:



If a approaches ∞, the function $f(x)$ approaches the constant function $f \equiv 1$. Even though the constant function $f \equiv 1$ is not integrable, we can write

$$\mathcal{F}\{1_x\} = \lim_{a\to\infty}\frac{2\sin(a\omega)}{\omega}.$$

It can be shown that the limiting function at the right hand side is a the Dirac delta function.

Problem 7.2. Show first the equality

$$\int_{-\infty}^{\infty}\frac{\sin(a\,z)}{z}\,dz = \pi.$$

Now, let f be a continuous function and integrable. Show the relation

$$\lim_{a\to\infty}\int_{-\infty}^{\infty}\frac{\sin(a\,z)}{\pi\,z}\,f(z)\,dz = f(0).$$

Now, with these results, we can symbolically write:

$$\mathcal{F}\{1_x\} = 2\pi\delta(\omega).$$

Note also that

$$\mathcal{F}^{-1}\{2\pi\delta(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega)\, e^{i\omega x}\, d\omega = 1_x.$$

Using the inverse transform formula, we can find the transform of the function $f(x) = \frac{\sin(ax)}{x}$. We have

$$\int_{-\infty}^{\infty} \frac{\sin(a\omega)}{\omega} e^{i\omega x}\, d\omega = \begin{cases} \pi & -a < x < a \\ 0 & \text{otherwise} \end{cases}$$

Changing $\omega \to x$ and $x \to -\omega$, we obtain

$$\int_{-\infty}^{\infty} \frac{\sin(ax)}{x} e^{-i\omega x}\, dx = \begin{cases} \pi & -a < \omega < a \\ 0 & \text{otherwise} \end{cases},$$

and thus

$$\mathcal{F}\left\{\frac{\sin(ax)}{x}\right\} = \begin{cases} \pi & -a < \omega < a \\ 0 & \text{otherwise} \end{cases}.$$

The function at the right-hand side represents a low-pass filter in the frequency domain. This filter allows low frequencies to pass through and filters out high frequencies in a given function. Such filters are commonly used in signal processing and communication systems to eliminate unwanted noise and interference while preserving essential information.

Exercise 7.5. Consider the function

$$f_a(x) = \begin{cases} \frac{1}{2a} & -a < x < a \\ 0 & \text{otherwise} \end{cases},$$

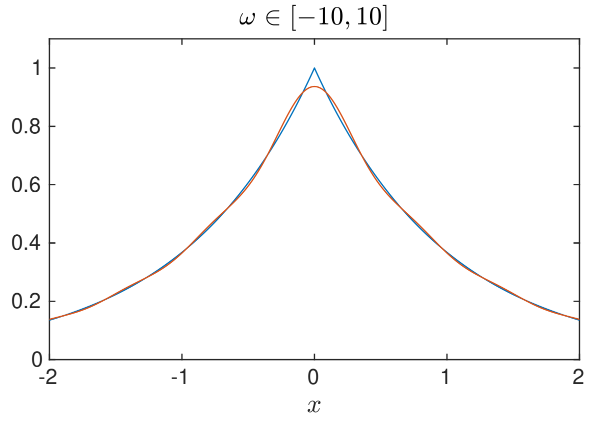
for $a > 0$. Let a approaches 0, find $\mathcal{F}\{f_a(x)\}$. What is limiting function $f_a(x)$ when $a \to 0$?

7.2.2 Exponential functions

Let f be given by: $f_a(x) = e^{-a|x|}, a > 0$. This function is integrable and continuously differentiable everywhere except $x = 0$. By definition of the Fourier transform, we have

$$\mathcal{F}\{f_a(x)\} = \int_{-\infty}^{\infty} e^{-a|x|} e^{-i\omega x}\, dx = \int_{-\infty}^{0} e^{(a-i\omega)x} + \int_{0}^{\infty} e^{-(a+i\omega)x} = \frac{2a}{a^2+\omega^2}.$$

The figure below illustrates the reconstruction of $f_a(x)$ for $a = 1$ by its Fourier transform:



When a approaches zero, $f_a(x)$ approaches the constant function $f\equiv 1$. Then, we expect that $\mathcal{F}\{f_a(x)\}$ approaches the Dirac delta function $2\pi\delta(\omega)$.

Problem 7.3. Show first that

$$\int_{-\infty}^{\infty}\frac{a}{a^2+x^2}\,dx=\pi,$$

for any $a>0$. Let $h(x)$ be a continuous function which is zero outside an interval $(-R,R)$ for some $R>0$. Show the following relation

$$\lim_{a\to 0}\int_{-\infty}^{\infty}\frac{a\,h(x)}{a^2+x^2}\,dx=\pi\,h(0).$$

Exercise 7.6. Use the inverse Fourier transform and show the following relation

$$\int_0^{\infty}\frac{\cos(ax)}{1+x^2}\,dx=\frac{\pi}{2}\,e^{-|a|}.$$

Exercise 7.7. Use the inverse Fourier transform and show the relation

$$\mathcal{F}\left\{\frac{1}{a^2+x^2}\right\}=\frac{\pi}{a}\,e^{-a|\omega|}.$$

Exercise 7.8. The function given by

$$f(x)=\begin{cases}1 & x>0\\ -1 & x<0\end{cases}$$

is not integrable. The function $f(x)$ is the limiting function of $f_\varepsilon(x)=\begin{cases}e^{-\varepsilon x} & x>0\\ -e^{\varepsilon x} & x<0\end{cases}$. Find $\mathcal{F}\{f_\varepsilon\}$ and let ε approaches zero.

Problem 7.4. Let $g_a(x)=\frac{1}{2}a\,f_a(x)$. Show that $g_a(x)$ is a Dirac delta sequence function, i.e.,

$$\lim_{a\to\infty}\int_{-a}^{a}g_a(x)\,h(x)\,dx=h(0),$$

for any continuous function $h(x)$ which vanishes outside an interval $(-R,R)$ for some $R>0$.

7.2.3 Gaussian function

The Gaussian function shows itself in several fields of pure and applied sciences. One of its beauty is that the Fourier transform of a Gaussian function is a Gaussian. Let $f(x)$ be given by: $f(x)=e^{-ax^2}$. Its Fourier transform is:

$$\mathcal{F}\{e^{-ax^2}\}=\int_{-\infty}^{\infty}e^{-ax^2}\,e^{-i\omega x}\,dx=e^{-\frac{\omega^2}{4a}}\int_{-\infty}^{\infty}e^{-a\left(x-\frac{i\omega}{2}\right)^2}\,dx.$$

The last integral in the above equality is

$$\int_{-\infty}^{\infty}e^{-a\left(x-\frac{i\omega}{2}\right)^2}\,dx=\sqrt{\frac{\pi}{a}},$$

and thus

$$\mathcal{F}\{e^{-ax^2}\}=\sqrt{\frac{\pi}{a}}\,e^{-\frac{\omega^2}{4a}}.$$

In particular, if $a=\frac{1}{2}$, we obtain

$$\mathcal{F}\left\{e^{-\frac{x^2}{2}}\right\}=\sqrt{2\pi}\,e^{-\frac{\omega^2}{2}}.$$

Exercise 7.9. Show the relation

$$\mathcal{F}^{-1}\{e^{-t\omega^2}\} = \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{x^2}{4t}},$$

for $t > 0$. Show also the equality

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{x^2}{4t}}\, dx = 1.$$

Furthermore, show that the function $f_t(x) = \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{x^2}{4t}}$ is a Dirac delta sequence function for $t \to 0$, i.e., for any continuous function $h(x)$ vanishing outside an interval $(-R, R)$ for some $R > 0$, it satisfies the relation

$$\lim_{t \to 0} \int_{-\infty}^{\infty} f_t(x)\, h(x)\, dx = h(0).$$

7.2.4 Dirac delta function

Recall that the Dirac delta function $\delta(x)$ is defined by the relation:

$$\int_{-\infty}^{\infty} h(x)\, \delta(x)\, dx = h(0),$$

for any continuous and bounded function h. The Dirac delta function plays a fundamental role in the theory of Fourier transform and is essential for solving linear partial differential equations using Fourier transform. It acts as a distribution and has many important properties, including the shifting property:

$$\int_{-\infty}^{\infty} h(x)\, \delta(x - x_0)\, dx = h(x_0).$$

Additionally, we can define the Dirac delta function sequence as follows: A sequence of functions $\delta_n(x)$ is called a Dirac delta sequence if for any continuous and bounded function h, we have:

$$\lim_{n \to \infty} \int_{-a}^{a} h(x)\, \delta_n(x)\, dx = h(0).$$

The Dirac delta function sequence is used to approximate the Dirac delta function $\delta(x)$ for a continuous function h. By considering the limit of the Dirac delta sequence as n approaches infinity, we can define the Dirac delta function as:

$$\delta(x) = \lim_{n \to \infty} \delta_n(x).$$

Exercise 7.10. Prove that the following functions are Dirac delta sequences:

a) The sequence

$$\delta_n(x) = \begin{cases} \frac{n}{2} & -\frac{1}{n} < x < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}.$$

b) The sequence

$$\delta_n(x) = \begin{cases} n^2 x + n & -\frac{1}{n} < x < 0 \\ -n^2 x + n & 0 < x < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

The Fourier transform of the function $f(x) = \delta(x - x_0)$ is defined as

$$\mathcal{F}\{\delta(x - x_0)\} = \int_{-\infty}^{\infty} \delta(x - x_0)\, e^{-i\omega x}\, dx = e^{-i\omega x_0} \int_{-\infty}^{\infty} \delta(x - x_0)\, dx = e^{-i\omega x_0}.$$

In particular, if $x_0 = 0$, then $\mathcal{F}\{\delta(x)\} = 1_\omega$. On the other hand, by the definition of the inverse Fourier transform, we have

$$\mathcal{F}^{-1}\{1_\omega\} = \frac{1}{2\pi} \lim_{L\to\infty} \int_{-L}^{L} e^{i\omega x}\, d\omega = \frac{1}{\pi} \lim_{L\to\infty} \frac{\sin(xL)}{x}.$$

The sequence $\frac{1}{\pi}\frac{\sin(nx)}{x}$ is a Dirac delta sequence for $L \to \infty$, and thus:

$$\mathcal{F}^{-1}\{1_\omega\} = \delta(x).$$

Exercise 7.11. Show that the sequence of functions $\frac{1}{\pi}\frac{\sin(nx)}{x}$ is a Dirac delta sequence.

Problem 7.5. A function $f(x)$ is called self-dual if $\hat{f}(\omega) = f(\omega)$. For this reason, and to make a balance between $\mathcal{F}$ and $\mathcal{F}^{-1}$, some textbooks change the definitions as follows:

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx,$$

and

$$\mathcal{F}^{-1}\{\hat{f}(\omega)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega x}\, d\omega.$$

It is simply seen that $f(x) = \sqrt{2\pi}\, e^{-\frac{x^2}{2}}$ is a self-dual function with the above definitions as:

$$\mathcal{F}\left\{\sqrt{2\pi}\, e^{-\frac{x^2}{2}}\right\} = \sqrt{2\pi}\, e^{-\frac{\omega^2}{2}} = f(\omega).$$

Consider the function defined as:

$$f(x) = \sqrt{2\pi} \sum_{n=-\infty}^{\infty} \delta(x - 2n\pi).$$

Show that its Fourier transform with respect to the balanced form of $\mathcal{F}$ is:

$$\hat{f}(\omega) = \lim_{n\to\infty} \frac{\sin[(2n+1)\pi\omega]}{\sin(\pi\omega)}.$$

Draw this function for some values of n and explain what you observe at $n \in \mathbb{Z}$. We can show that the function at these point behave like a Dirac delta function. Fix $n = n_0$ and assume that h is function continuous in the interval $(n_0 - \varepsilon, n_0 + \varepsilon)$ for small $\varepsilon > 0$ and zero outside of this interval. Show the relation

$$\lim_{n\to\infty} \int_{n_0-\varepsilon}^{n_0+\varepsilon} \frac{\sin[(2n+1)\pi\omega]}{\sin(\pi\omega)} h(x) = h(n_0).$$

This implies that $\hat{f}(\omega)$ behaves like a Dirac delta sequence function at $\omega = n \in \mathbb{Z}$. The function $\sum_{n=-\infty}^{\infty} \delta(x - 2n\pi)$ is self-dual with respect to the unbalanced form of $\mathcal{F}$.

7.2.5 Trigonometric sine and cosine functions

Consider the function $f(x) = \cos(\omega_0 x)$. As we saw above, its Fourier transform is

$$\mathcal{F}\{\cos(\omega_0 x)\} = \lim_{L\to\infty} \left\{ \frac{\sin[(\omega - \omega_0)L]}{\omega - \omega_0} + \frac{\sin[(\omega + \omega_0)L]}{\omega + \omega 0} \right\}.$$

On the other hand, the sequences $\frac{\sin[(\omega-\omega_0)n]}{\pi(\omega-\omega_0)}$ and $\frac{\sin[(\omega+\omega_0)n]}{\pi(\omega+\omega_0)}$ are Dirac delta sequence functions and approach $\delta(\omega-\omega_0)$ and $\delta(\omega+\omega_0)$ respectively. Therefore, we obtain:

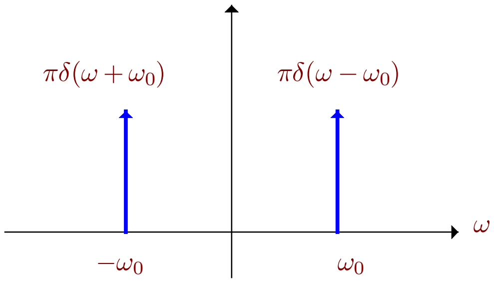
$$\mathcal{F}\{\cos(\omega_0 x)\} = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0).$$

These are two spikes at physical frequencies ω_0 and $-\omega_0$ that we discussed in the previous section.

Exercise 7.12. Calculate the inverses $\mathcal{F}^{-1}\{\delta(\omega-\omega_0)\}$, and $\mathcal{F}^{-1}\{\delta(\omega+\omega_0)\}$ and conclude

$$\mathcal{F}\{\cos(\omega_0 x)\} = \frac{1}{2}\{\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\}.$$

This is depicted in the figure below:



Find the Fourier transform $\mathcal{F}\{\sin(\omega_0 x)\}$ and draw its imaginary part in ω-domain.

Exercise 7.13. Show the relation

$$\mathcal{F}\{\sin(\omega_0 x)\} = i\pi\delta(\omega-\omega_0) - i\pi\delta(\omega-\omega_0).$$

7.2.6 Fourier transform of semi-bounded functions

For functions defined on $[0,\infty)$ instead of $(-\infty,\infty)$, we can either use the odd or even extension of f for $\mathcal{F}\{f(x)\}$. This in particular is important to solve linear partial differential equations defined on semi-bounded domains.

Let $f(x)$ be a function defined on $[0,\infty)$. Defining the odd extension:

$$f_o = \begin{cases} f(x) & x>0 \\ -f(-x) & x<0 \end{cases},$$

we can write $\mathcal{F}\{f_o\}$ as:

$$\mathcal{F}\{f_o\} = -i\int_{-\infty}^{\infty} f_o(x)\sin(\omega x)\,dx = -2i\int_0^{\infty} f(x)\sin(\omega x)\,dx.$$

This is called the Fourier sine transform while its imaginary part is called the Fourier sine integral

$$\mathcal{F}_s\{f(x)\} := \hat{f}_s(\omega) = \int_0^{\infty} f(x)\sin(\omega x)\,dx.$$

Note that $\hat{f}_s(\omega)$ is an odd function and $\hat{f}_s(0)=0$. By the inverse Fourier transform, we have

$$f_o(x) = \mathcal{F}^{-1}\{-2i\,\hat{f}_s(\omega)\} = \frac{1}{\pi}\int_{-\infty}^{\infty}\hat{f}_s(\omega)\sin(\omega x)\,d\omega.$$

Therefore, we can express the function $f(x)$ defined on $x \in [0,\infty)$ as the integral:

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \hat{f}_s(\omega)\sin(\omega x)\,d\omega,$$

where $\hat{f}_s$ is the Fourier sine integral of f.

The Fourier cosine transform is defined similarly. The even extension of a function $f(x)$, $x \in [0,\infty)$ is defined as:

$$f_e(x) = \begin{cases} f(x) & x > 0 \\ f(-x) & x < 0 \end{cases},$$

and thus:

$$\mathcal{F}\{f_e(x)\} := 2\int_0^{\infty} f(x)\cos(\omega x)\,dx.$$

The Fourier cosine integral is defined as:

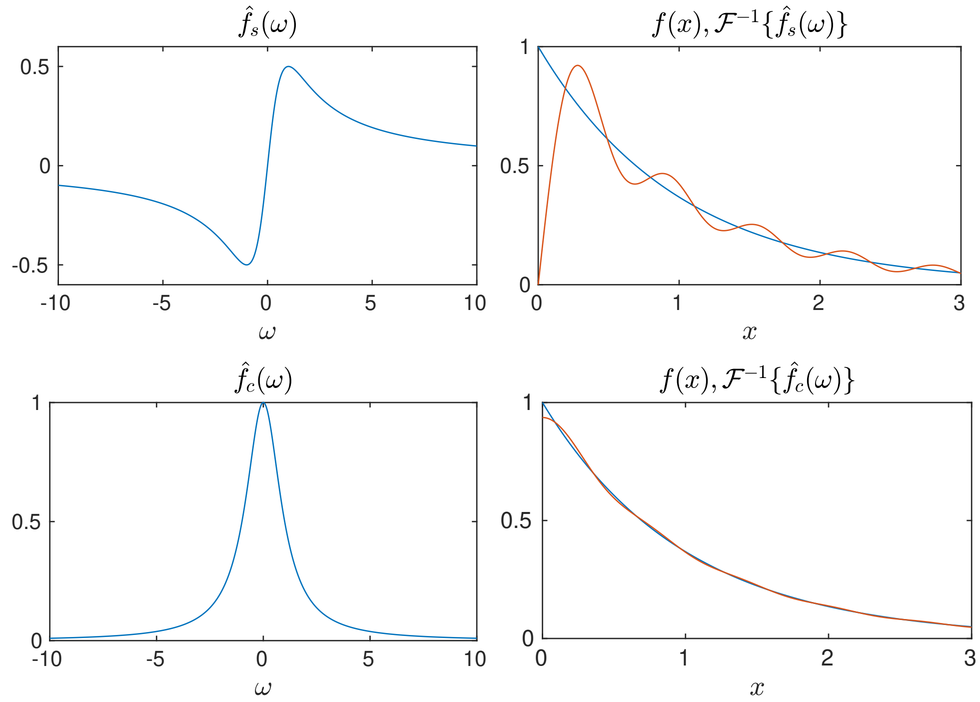
$$\mathcal{F}_c\{f(x)\} = \hat{f}_c(\omega) = \int_0^{\infty} f(x)\cos(\omega x)\,dx.$$

Note that $\hat{f}_c(\omega)$ is an even function in ω and $f_c'(0) = 0$. Therefore, a function $f(x)$ which is defined on the domain $[0,\infty)$ can be expressed as:

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \hat{f}_c(\omega)\cos(\omega x)\,d\omega,$$

where $\hat{f}_c(\omega)$ is the Fourier cosine of $f(x)$.

Example 7.2. Consider the function $f(x) = e^{-x}$ for $x \in [0,\infty)$. The following figure shows its Fourier sine and cosine integrals integrated over $\omega \in [-10, 10]$:



The Fourier sine integral is simply derived as:

$$\hat{f}_s(\omega) = \int_0^\infty e^{-x} \sin(\omega x)\, dx = \frac{\omega}{1+\omega^2},$$

and the Fourier cosine integral as:

$$\hat{f}_c(\omega) = \int_0^\infty e^{-x} \cos(\omega x)\, dx = \frac{1}{1+\omega^2}.$$

Exercise 7.14. Use the above results and determine the value of following integrals:

$$\int_0^\infty \frac{\cos(x)}{1+x^2}\, dx, \int_0^\infty \frac{x\sin(x)}{1+x^2}\, dx.$$

Exercise 7.15. Draw the Fourier sine and cosine integrals of the function $f(x) = e^{-x^2}$ define din $x \in [0, \infty)$. You can use the following code in Matlab:

```
w=-10:0.01:10;% The range of plot of $\hat{f}_s$
X=40;%Interval for integration over x
fw=integral(@(x) exp(-x.^2).*sin(w.*x),0,X, ...
 'ArrayValued',true);
plot(w,fw)
```

7.3 Properties of Fourier transform

7.3.1 Linearity

The Fourier transform is a continuous linear integral transformation that allows us to analyze the frequency components embedded in a given function. In particular, for any integrable functions f_1 and f_2, and any constants c_1 and c_2, the Fourier transform satisfies the linearity property:

$$\mathcal{F}\{c_1 f_1 + c_2 f_2\} = c_1 \hat{f}_1(\omega) + c_2 \hat{f}_2(\omega).$$

This means that the Fourier transform of the linear combination of two functions is equal to the corresponding linear combination of their individual Fourier transforms. In other words, the frequency components present in the sum $c_1 f_1 + c_2 f_2$ are equal to the sum of the frequency components in f_1 and f_2, weighted by their respective constants c_1 and c_2. This property makes the Fourier transform a powerful tool for analyzing the linear systems.

Problem 7.6. Prove that if f_1 and f_2 are integrable, then $c_1 f_1 + c_2 f_2$ is for any constant c_1, c_2.

Exercise 7.16. Consider the function

$$u(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}.$$

Write $u(x)$ as $u(x) = f(x) + \frac{1}{2}$, where $f(x)$ is the function

$$f(x) = \begin{cases} \frac{1}{2} & x > 0 \\ -\frac{1}{2} & x < 0 \end{cases}.$$

Show that

$$\mathcal{F}\{f(x)\} = \frac{1}{i\omega}, \mathcal{F}\left\{\frac{1}{2}\right\} = \pi\delta(\omega),$$

and thus

$$\mathcal{F}\{u(x)\} = \frac{1}{i\omega} + \pi\delta(\omega).$$

7.3.2 Shift in frequency

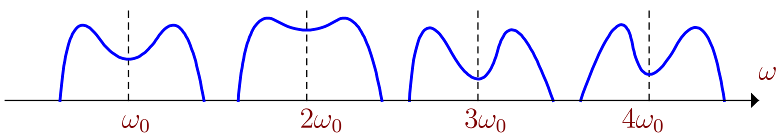
Let $f(x)$ be a function with frequency distribution $\hat{f}(\omega)$. This frequency distribution is shifted around the point ω_0 if f is multiplied by $\cos(\omega_0 x)$. This is simply verified by the following calculation:

$$\mathcal{F}\{f(x)\cos(\omega_0 x)\} = \frac{1}{2}\int_{\mathbb{R}} f(x)e^{i\omega_0 x}\,e^{-i\omega x} + \frac{1}{2}\int_{\mathbb{R}} f(x)e^{-i\omega_0 x}\,e^{-i\omega x}.$$

The first integral is just $\hat{f}(\omega-\omega_0)$ and the second one is $\hat{f}(\omega+\omega_0)$, and finally

$$\mathcal{F}\{f(x)\cos(\omega_0 x)\} = \frac{1}{2}\hat{f}(\omega-\omega_0) + \frac{1}{2}\hat{f}(\omega+\omega_0).$$

Frequency modulation is one of the most important applications of this property. It allows us to efficiently transmit data from different users along the same cable by allocating specific frequency bands to each user's data. By multiplying each data stream $f_k(t)$ by $\cos(k\omega_0 t)$, the frequency band of each user's data is shifted to $\omega = k\omega_0$, as shown in the figure below:



This modulation technique ensures that each user's data occupies a unique frequency band, preventing interference and enabling simultaneous transmission of multiple data streams through the same cable. It is widely used in various communication technologies, including wireless communication, radio broadcasting, and digital signal processing.

Exercise 7.17. Show the following relation

$$\mathcal{F}\left\{\frac{\cos(\omega_0 x)}{1+x^2}\right\} = \frac{\pi}{2}\{e^{-|\omega-\omega_0|} + e^{-|\omega+\omega_0|}\},$$

and conclude

$$\int_0^\infty \frac{\cos(x)}{1+x^2}\,dx = \frac{\pi}{2}e^{-1}.$$

Exercise 7.18. A band pass filter is constructed by the function

$$f(x) = \frac{2\sin(ax)}{x}\cos(\omega_0 x).$$

for $a=1$, find ω_0 such that the filter passes all frequencies in the range $[99, 101]$.

7.3.3 Shift in spatial variable x

For a function $f(x)$ with the frequency distribution $\hat{f}(\omega)$, the Fourier transform of $f(x-x_0)$ is

$$\mathcal{F}\{f(x-x_0)\} = e^{-i\omega x_0}\,\hat{f}(\omega).$$

In particular, if $\mathcal{F}\{\delta(x)\} = 1$, then $\mathcal{F}\{\delta(x - x_0)\} = e^{-i\omega x_0}$. This property is the dual of the previous one. Shift in x causes a phase in ω, and phase in x causes a shift in ω.

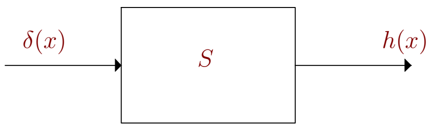
7.3.4 Convolution

Convolution is a fundamental operation in the analysis of linear systems, and it arises naturally in the context of several fields of applied mathematics, engineering and physics. The convolution between two functions $f(x)$ and $g(x)$ defined on the real line $(-\infty, \infty)$ is defined as:

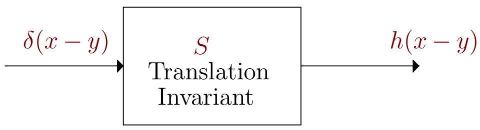
$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)\, g(x - y)\, dy.$$

This operation combines two functions into a new function, and it can be thought of as a "weighted average" of the two functions, where the weighting is determined by the function $g(x - y)$ as y ranges over the entire real line. The convolution operation is commutative, meaning that $f * g = g * f$. It is also associative and distributive over addition.

This special integral operation naturally arises from the analysis of linear systems. Consider a linear system S with the response $h(x)$ to the impulse $\delta(x)$. This response $h(x)$ is called the impulse response of the system. It describes how the system reacts to an impulse input at time $x = 0$.



The system is called translation-invariant if its response to the shifted input $\delta(x - y)$ is $h(x - y)$. This property means that the behavior of the system does not change with respect to time shifts in the input signal. In other words, a delay or shift in the input signal results in a corresponding delay or shift in the system's response, represented by the function $h(x - y)$:



Now, let's consider the input to the system, denoted as $f(x)$. The function $f(x)$ can be expressed in the integral form as:

$$f(x) = \int_{-\infty}^{\infty} f(y)\, \delta(x - y)\, dy,$$

or equivalently as a Riemann sum as:

$$f(x) = \lim_{n \to \infty} \sum_{k=-n}^{\infty} f(y_k)\, \delta_k(x - y_k)\, \Delta y_k.$$

In this form, we can see that the input $f(x)$ is represented as a summation of shifted impulses $\delta(x - y_k)$ multiplied by weights $f(y_k)\, \Delta y_k$. A system is linear if its response to the summation is a summation as:

Therefore, the response of a linear translation-invariant system with impulse response $h(x)$ to an input $f(x)$ is given by the convolution integral:

$$(f*h)(x)=\int_{-\infty}^{\infty} f(y)\,h(x-y)\,dy.$$

This integral can be challenging to compute directly, but the Fourier transform provides an elegant method to solve this convolution integral. The property of the Fourier transform $\mathcal{F}$ states that the transform of a convolution of two functions is the product of their individual transforms:

$$\mathcal{F}\{f*g\}=\mathcal{F}\{f\}\,\mathcal{F}\{g\},$$

for any two functions f, g with the Fourier transforms $\hat{f}$, $\hat{g}$ respectively. This property allows us to simplify the convolution integral and express the response of the system to an input as the product of their individual Fourier transforms. This property is particularly useful in simplifying calculations and solving differential equations through Fourier transforms.

Proof. Let f and g be two functions with Fourier transforms $\mathcal{F}\{f\}$ and $\mathcal{F}\{g\}$. We have

$$\mathcal{F}\{f*g\}=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(y)\,g(x-y)\,dy\right)e^{-i\omega x}\,dx.$$

Using the Fubini theorem, and changing the order of integrals, we obtain

$$\mathcal{F}\{f*g\}=\int_{-\infty}^{\infty} f(y)\left(\int_{-\infty}^{\infty} g(x-y)\,e^{-i\omega x}\,dx\right)dy.$$

The internal integral can be written by the substitution $z=x-y$ as:

$$\int_{-\infty}^{\infty} g(x-y)\,e^{-i\omega x}\,dx=e^{-i\omega y}\int_{-\infty}^{\infty} g(z)\,e^{-i\omega z}\,dz,$$

and thus

$$\mathcal{F}\{f*g\}=\left(\int_{-\infty}^{\infty} f(y)\,e^{-i\omega y}\,dy\right)\left(\int_{-\infty}^{\infty} g(z)\,e^{-i\omega z}\,dz\right)=\mathcal{F}\{f\}\,\mathcal{F}\{g\},$$

and this completes the proof. □

Exercise 7.19. Use the convolution property and determine the inverse Fourier transform:

$$\mathcal{F}^{-1}\left\{\frac{e^{-t\omega^2}}{1+\omega^2}\right\},$$

where $t>0$ is a constant.

Exercise 7.20. Show the relation:

$$\mathcal{F}^{-1}\{\cos(\omega c t)\}=\frac{1}{2}\{\delta(x-ct)+\delta(x+ct)\},$$

and conclude:

$$\mathcal{F}^{-1}\{\hat{f}(\omega)\cos(c\omega t)\} = \frac{1}{2}\{f(x-ct) + f(x+ct)\}.$$

Problem 7.7. Show the relation:

$$\mathcal{F}\{f(x)\, g(x)\} = \frac{1}{2\pi}\,\hat{f}(\omega) * \hat{g}(\omega).$$

This equality justifies the fact that multiplication in x-domain result sin the convolution in ω-domain, and convolution in x-domain results in the multiplication in ω-domain.

Problem 7.8. If f and g are integrable functions, prove that the integral $f * g$ converges.

7.3.5 Differentiation and integration

Let $f(x)$ be an integrable function and differentiable with the derivative $f'(x)$. The Fourier transform $\mathcal{F}\{f'(x)\}$ is given by

$$\mathcal{F}\{f'\} = i\omega\,\hat{f}(\omega). \tag{7.6}$$

This is simply seen by the direct calculation:

$$\mathcal{F}\{f'(x)\} = \int_{-\infty}^{\infty} f'(x)\, e^{-i\omega x}\, dx = f(x)\, e^{-i\omega x}\Big|_{-\infty}^{\infty} + i\omega\int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx.$$

Since f is integrable, it vanishes at $x \to \pm\infty$, and thus:

$$\mathcal{F}\{f'(x)\} = i\omega\mathcal{F}\{f(x)\}.$$

In a similar manner, we obtain the formula:

$$\mathcal{F}\{f''(x)\} = i\omega\mathcal{F}\{f'(x)\} = -\omega^2\hat{f}(\omega),$$

as long as f is a twice differentiable function and $\mathcal{F}\{f(x)\}$ exists.

This property is specially useful to solve linear second-order differential equations. For example, consider the following heat problem:

$$\begin{cases} u_t = u_{xx} \\ u(x,0) = f(x) \end{cases}.$$

By taking Fourier transform of the problem with respect to the spatial variable x, we arrive at the following ordinary differential equation:

$$\begin{cases} \hat{u}_t(\omega, t) = -\omega^2\,\hat{u}(\omega, t) \\ \hat{u}(\omega, 0) = \hat{f}(\omega) \end{cases},$$

as long as $\mathcal{F}\{f(x)\}$ exists. This first-order ODE can be solved for $\hat{u}(\omega, t)$, and thus $u(x,t)$ can be retrieved by the inverse Fourier $\mathcal{F}^{-1}$.

Now, let $F(x)$ be an anti-derivative of $f(x)$:

$$F(x) = \int_{-\infty}^{x} f(t)\, dt.$$

If $f(x)$ be an integrable function, then we have

$$\mathcal{F}\{F'(x)\} = \mathcal{F}\{f(x)\} = \hat{f}(\omega).$$

On the other hand, we have

$$\mathcal{F}\{F'(x)\} = i\omega\mathcal{F}\{F(x)\},$$

and thus

$$\mathcal{F}\left\{\int_{-\infty}^{x} f(t)dt\right\} = \frac{1}{i\omega}\mathcal{F}\{f(x)\}. \tag{7.7}$$

Problem 7.9. In one of previous exercises, we obtained the formula

$$\mathcal{F}\{u(x)\} = \frac{1}{i\omega} + \pi\delta(\omega).,$$

where $u(x) = \left\{\begin{smallmatrix} 1 & x>0 \\ 0 & x<0 \end{smallmatrix}\right.$. Show the relation

$$u(x) = \int_{-\infty}^{x} \delta(t)\, dt.$$

On the other hand, if we use the formula (7.7), we have

$$\mathcal{F}\{u(x)\} = \frac{1}{i\omega}\mathcal{F}\{\delta(x)\} = \frac{1}{i\omega}.$$

What is wrong here? Show that if we use the inverse Fourier transform, the relation $\mathcal{F}\{u(x)\} = \frac{1}{i\omega}$ can not be true as we should have:

$$\mathcal{F}^{-1}\{\hat{u}(\omega)\} = \frac{u(0^+) + u(0^-)}{2} = \frac{1}{2}$$

7.3.6 Multiplication by x

If the function $g(x) = x\, f(x)$ is integrable, and if $\hat{f}(\omega)$ is differentiable, we have:

$$\mathcal{F}\{x\, f(x)\} = i\,\frac{d\,\hat{f}(\omega)}{d\omega}:$$

This demonstrates a duality between f' and $\mathcal{F}\{f'\}$ on one side and $x\, f$ and $\mathcal{F}\{x\, f\}$ on the other side.

The following theorem provides us with a sufficient condition for the differentiability of $\hat{f}(\omega)$. The proof is given in the appendix to this chapter.

Theorem 7.3. *Assume that $f(x)$ is an integrable, and piecewise continuously differentiable function. Moreover, assume that there is $R > 0$ such that the function $f(x)$ decay sub-exponentially outside $x \in (-R, R)$, i.e., there are some $\alpha, \beta > 0$ such that $|f(x)| < \alpha e^{-\beta|x|}$, for $|x| > R$. Then $\hat{f}(\omega)$ is continuously differentiable.*

Exercise 7.21. Show the following relation

$$\mathcal{F}\left\{\frac{x}{1+x^2}\right\} = -i\pi e^{-|\omega|}\,\mathrm{sign}(\omega).$$

7.3.7 Expansion and shrinking

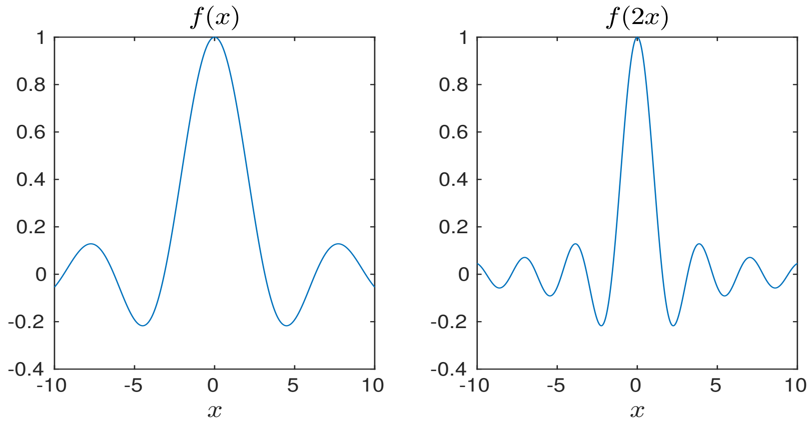
Consider an integrable function $f(x)$, and let $a > 0$ be a constant. If $0 < a < 1$, the function $f(ax)$ stretches in the x-domain, and if $a > 1$, the function shrinks in the x-domain. This behavior is reflected in the Fourier transform of $f(ax)$ as follows:

$$\mathcal{F}\{f(ax)\} = \int_{-\infty}^{\infty} f(ax)\, e^{-i\omega x}\, dx = \frac{1}{a}\int_{-\infty}^{\infty} f(y)\, e^{-i\omega y/a}\, dy = \frac{1}{a}\hat{f}\left(\frac{\omega}{a}\right).$$

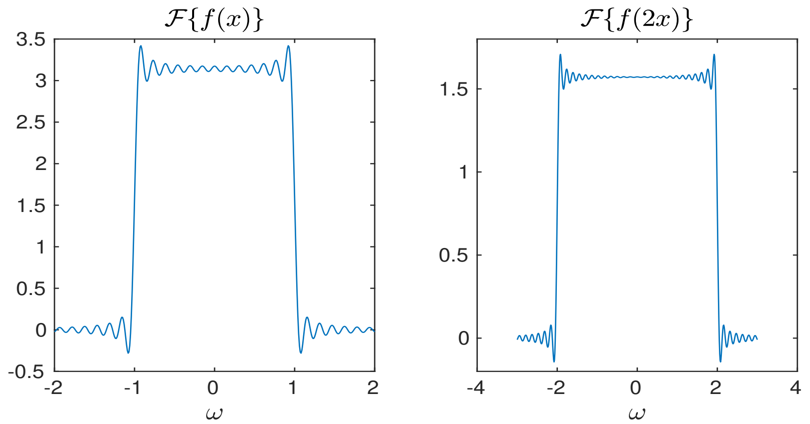
Hence, if $a>1$, the Fourier transform $\mathcal{F}\{f(ax)\}$ stretches in the ω-domain, while if $0<a<1$, $\mathcal{F}\{f(ax)\}$ shrinks in the ω-domain.

The interpretation of this behavior lies in the frequency spectrum of $\mathcal{F}\{f(ax)\}$, which represents the angular frequency components present in the function $f(ax)$. The frequency spectrum indicates how fast or slow a function changes periodically. When a function shrinks in the x-domain ($a>1$), it undergoes faster changes, leading to a wider frequency spectrum. In contrast, for $0<a<1$, the function changes more slowly, resulting in a narrower frequency spectrum that predominantly covers low frequencies.

For instance, the figure below illustrates the graphs of functions $f(x)=\frac{\sin(x)}{x}$ and $f(2x)$. We can observe how $f(2x)$ is contracted around $x=0$:



The figure below depicts the Fourier transforms $\mathcal{F}\{f(x)\}$ and $\mathcal{F}\{f(2x)\}$, and as can be seen, $\mathcal{F}\{f(2x)\}$ is wider than $\mathcal{F}\{f(x)\}$:



Additionally, this behavior is related to a fundamental concept known as the uncertainty principle. As we saw, the shrinking in the x-domain results in the expansion in the ω-domain and vice versa. This expansion in ω-domain introduces uncertainty in measurement. Thus, when there is certainty in measuring a function $f(x)$, corresponding to the shrinking in the x-domain, there will be uncertainty in measuring $\hat{f}(\omega)$, which is a consequence of the expansion of $\hat{f}(\omega)$. Similarly, the shrinking of $\hat{f}(\omega)$ results in uncertainty in measuring $f(x)$. This uncertainty principle is a fundamental characteristic of the Fourier transform and has far-reaching implications in various fields, including signal processing, quantum mechanics, and more. In the problem set, we prove an important result of this principle.

7.3.8 Uncertainty principle

The shrinking and expansion property of the Fourier transform is deeply connected to a fundamental concept known as the uncertainty principle. As we have observed, when a function shrinks in the x-domain, it leads to an expansion in the ω-domain and vice versa. This expansion in the ω-domain introduces uncertainty in the measurement process. Hence, when there is high certainty in measuring a function $f(x)$, corresponding to the shrinking in the x-domain, there will be greater uncertainty in measuring $\hat{f}(\omega)$, a result of the expansion of $\hat{f}(\omega)$ in the frequency domain. Conversely, the shrinking of $\hat{f}(\omega)$ results in uncertainty in measuring $f(x)$.

This uncertainty principle is a fundamental characteristic of the Fourier transform and has significant implications in various fields, including signal processing, quantum mechanics, and more. It fundamentally describes the trade-off between the certainty in the time or spatial domain and the certainty in the frequency domain.

To quantify the dispersion or spread of a function f around $x=0$, we define the dispersion of f as:

$$D(f) = \frac{1}{\sqrt{E(f)}} \left(\int_{-\infty}^{\infty} |x f(x)|^2 \right)^{1/2},$$

provided that both integrals exist. A similar definition holds for $D(\hat{f})$, the dispersion of $\hat{f}(\omega)$ around $\omega = 0$. These definitions closely resemble the definition of standard variation in probability theory.

Theorem 7.4. *As long as $D(f)$ and $D(\hat{f})$ are bounded, the uncertainty principle establishes the following inequality:*

$$D(f)\, D(\hat{f}) \geq \frac{1}{2}.$$

This principle clearly indicates that the more certainty we have about the localization of a function in the time or spatial domain (i.e., higher concentration around $x=0$), the less certainty we will have about its localization in the frequency domain (i.e., $\hat{f}(\omega)$ will be more spread out around $\omega=0$), and vice versa. This inherent trade-off is why the inequality is termed the uncertainty principle. For a simple proof of the theorem, please refer to the problem set.

7.3.9 Energy and Plancherel theorem

In physics and engineering, the energy of a function $f(x)$ is defined as the integral:

$$E(f) = \int_{-\infty}^{\infty} |f(x)|^2\, dx,$$

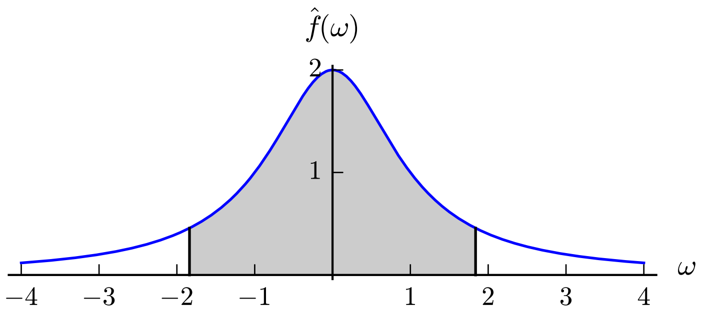
as long as the integral is bounded. The Plancherel theorem establishes a fundamental relationship between the energy of $f(x)$ and the energy of its Fourier transform, $\hat{f}(\omega)$.

Theorem 7.5. (Plancherel) *Assume that $f(x)$ is square integrable function. Then the following relation holds*

$$E(\hat{f}) = 2\pi E(f). \tag{7.8}$$

This powerful theorem states that the energy of the Fourier transform, $\hat{f}(\omega)$, is equal to 2π times the energy of the original function, $f(x)$. Consequently, any filtration of $\hat{f}(\omega)$ will directly impact the original function $f(x)$.

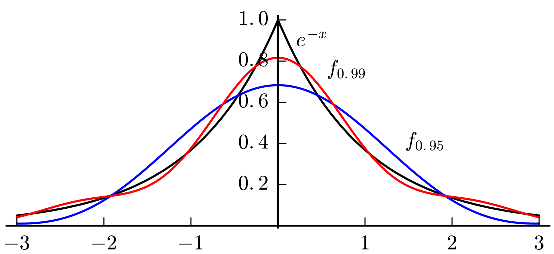
To illustrate this theorem, let's consider the function $f(x)=e^{-|x|}$. By applying the Plancherel theorem, we find that $E(f)=1$. Furthermore, after calculating the Fourier transform of $f(x)$, which is $\hat{f}(\omega)=\frac{2}{1+\omega^2}$, we determine that $E\left(\frac{2}{1+\omega^2}\right)=2\pi$. In the graph of $\hat{f}(\omega)$, we observe the interval $[-1.838, 1.838]$, which contains 95 percent of the energy of $\hat{f}(\omega)$. Engineers often use this cutoff value to reconstruct an approximate function $\tilde{f}(x)$ from the original function $f(x)$.



The reconstruction $\tilde{f}(x)$ can be defined as:

$$\tilde{f}(x)=\frac{1}{2\pi}\int_{-1.838}^{1.838}\hat{f}(\omega)\,e^{i\omega x}\,d\omega.$$

Similarly, a higher frequency band for $\hat{f}(\omega)$ will result in a more accurate reconstruction of $f(x)$. This technique plays a crucial role in various applications, including digital communication and signal processing, where obtaining an accurate representation of the original signal is essential. The graph depicts the results of reconstructions $f_{0.95}$ and $f_{0.99}$ using the ω-interval $[-3.373, 3.373]$ for the latter. As evident, the broader frequency band results in a more precise reconstruction.



Exercise 7.22. Use Plancherel theorem and prove

$$\int_0^\infty \frac{\sin(ax)\sin(bx)}{x^2} = \begin{cases} \frac{a\pi}{2} & a<b \\ \frac{b\pi}{2} & b<a \end{cases}.$$

Hint: use the relation

$$\mathcal{F}\left\{\frac{\sin(ax)}{x}\right\} = \begin{cases} \pi & -a<\omega<a \\ 0 & \text{otherwise} \end{cases},$$

for any $a>0$.

Problem 7.10. Suppose f, g are real admissible functions.

a) Use the relation:

$$\mathcal{F}\{f(x)g(x)\} = \frac{1}{2\pi}\hat{f}(\omega) * \hat{g}(\omega).$$

and prove the following relation

$$\int_{-\infty}^{\infty} f(x)g(x)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\,\overline{\hat{g}(\omega)}\,d\omega.$$

This proves the Plancherel theorem.

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|\,d\omega.$$

7.3.10 Nyquist-Shannon rate

The Nyquist-Shannon theorem plays a crucial role in digital signal processing, particularly when transmitting electrical signals over transmission lines. In practical applications, we often transmit a sampled signal (digital signal) rather than the continuous function $f(t)$ itself. To reconstruct the original signal at the destination without any loss of information, the sampling rate should satisfy the Nyquist-Shannon rate.

Assuming that the signal $f(t)$ has a bounded frequency band, i.e., $-\omega_0 \le \omega \le \omega_0$, Nyquist and later Shannon showed that the original function can be accurately reconstructed from the sampled signal if the sampling rate is at least $\frac{\omega_0}{\pi}$. This result is fundamental in digital signal processing, as sampling a signal with the Nyquist rate ensures the preservation of all information in the signal.

Since $\hat{f}(\omega)$ is bounded with spectrum band $[-\omega_0, \omega_0]$, we can express the Fourier transform $\hat{f}(\omega)$ using the complex form of the Fourier series as follows:

$$\hat{f}(\omega) = \sum_{n=-\infty}^{\infty} c_n\, e^{-i\frac{n\pi}{\omega_0}\omega},$$

where c_n is obtained through the inner product:

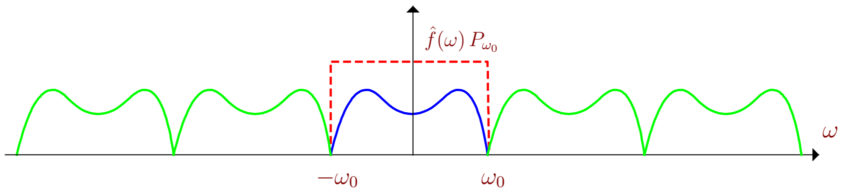
$$c_n = \frac{1}{2\omega_0}\int_{-\omega_0}^{\omega_0} \hat{f}(\omega)\, e^{i\frac{n\pi}{\omega_0}\omega}\, d\omega = \frac{\pi}{\omega_0}\left\{\frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} \hat{f}(\omega)\, e^{i\frac{n\pi}{\omega_0}\omega}\, d\omega\right\} = \frac{\pi}{\omega_0} f\!\left(\frac{n\pi}{\omega_0}\right).$$

Let's denote $\frac{n\pi}{\omega_0}$ by t_n which can be considered as the sampling times of the function $f(t)$. This means that the original function $f(t)$ is sampled at a rate of $T_0 = \frac{\pi}{\omega_0}$ or equivalently, at a rate of $\frac{\omega_0}{\pi}$. If we replace the angular frequency ω_0 by $2\pi f_0$, we obtain $T_0 = \frac{1}{2 f_0}$, and the sampling rate $f_N = 2 f_0$. Thus, we can write:

$$\hat{f}(\omega) = \sum_{n=-\infty}^{\infty} \frac{\pi}{\omega_0} f(t_n)\, e^{-i t_n \omega}. \tag{7.9}$$

This implies that to reconstruct $\hat{f}(\omega)$, we only need the data set $\{f(t_n)\}_{n=-\infty}^{\infty}$. Since we can reconstruct $f(t)$ from its Fourier transform $\hat{f}(\omega)$ by the inverse Fourier transform, we only need to sample $f(t)$ at the rate $f_N = 2 f_0$. This rate is known as the Nyquist rate.

It's important to note that the Fourier series of $\hat{f}(\omega)$ given in formula (7.9) is periodic with a period of $2\omega_0$.



To filter out the periodic copies of the principal part shown with the blue line in the figure, we need to multiply this Fourier series with a low-pass filter:

$$\hat{p}(\omega) = \begin{cases} 1 & \omega \in (-\omega_0, \omega_0) \\ 0 & \text{otherwise} \end{cases}.$$

Therefore, the original function $f(t)$ can be reconstructed as:

$$f(t) = \mathcal{F}^{-1}\{\hat{f}(\omega)\,\hat{p}(\omega)\} = \sum_{n=-\infty}^{\infty} \frac{1}{2\omega_0} f(t_n) \int_{-\omega_0}^{\omega_0} e^{-it_n\omega}\, e^{i\omega t}\, d\omega =$$
$$\sum_{n=-\infty}^{\infty} f(t_n) \frac{\sin[\omega_0(t-t_n)]}{\omega_0(t-t_n)}.$$

The last expression represents the discrete convolution of $f(t)$ and $\frac{\sin[\omega_0 t)]}{\omega_0 t}$ at instance $t_n = \frac{n\pi}{\omega_0}$. This demonstrates how the Nyquist rate and the use of low-pass filters ensure accurate reconstruction of the original signal from the sampled one.

Example 7.3. Consider the following function

$$f(t) = \frac{2(\sin(t) - t\cos(t))}{\pi t^3}.$$

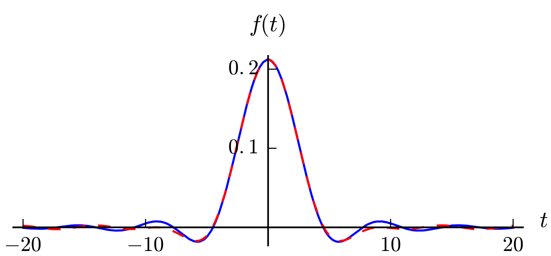
The FOURIER transform of f is

$$\hat{f}(\omega) = \begin{cases} 1-\omega^2 & -1 \le \omega \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

For $\omega_0 = 1$, sequence $\{f(t_n)\}_{n=-\infty}^{\infty}$ captures all information in $f(t)$. In fact, the reconstruction is done by the following series

$$f(t) = \sum_{n=-\infty}^{\infty} f(n\pi) \frac{\sin(t-n\pi)}{t-n\pi},$$

which converges fast to the original function. The figure below illustrates the original function and three terms of the summations:



7.4 Higher dimensional transform

The Fourier transform, as described above, can indeed be extended to functions of two or more independent variables. This extension is particularly essential when dealing with partial differential equations, such as the heat or wave equations, defined on a plane or in space. For a function $f:\mathbb{R}^2\to\mathbb{R}$ that satisfies the condition

$$\iint_{\mathbb{R}^2} |f(x,y)|\,dx\,dy<\infty,$$

its Fourier transform with frequency parameters ω_1 and ω_2 associated with x and y, respectively, is defined as follows:

$$\hat{f}(\omega_1,\omega_2)=\int_{\mathbb{R}^2} f(x,y)\,e^{-i\omega_1 x}\,e^{-i\omega_2 y}\,dx\,dy.$$

Similarly, the inverse Fourier transform is given by the following formula:

$$f(x,y)=\mathcal{F}^{-1}\{\hat{f}\}=\frac{1}{4\pi^2}\int_{\mathbb{R}^2}\hat{f}(\omega_1,\omega_2)\,e^{i\omega_1 x}\,e^{i\omega_2 y}\,d\omega_1\,d\omega_2,$$

provided that the integral converges.

As an example, consider the Gaussian function $f(x,y)=e^{-\frac{x^2+y^2}{2}}$. The Fourier transform of f is calculated as:

$$\hat{f}(\omega_1,\omega_2)=\int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}}\,e^{-i\omega_1 x}\,e^{-i\omega_2 y}\,dx\,dy=\left(\int_{\mathbb{R}} e^{-\frac{x^2}{2}}\,e^{-i\omega_1 x}\,dx\right)\left(\int_{\mathbb{R}} e^{-\frac{y^2}{2}}\,e^{-i\omega_2 y}\,dy\right).$$

Each integral on the right-hand side represents a 1D Fourier transform, which results in:

$$\hat{f}(\omega_1,\omega_2)=2\pi e^{-\frac{\omega_1^2+\omega_2^2}{2}}.$$

This extension of the Fourier transform to multiple variables allows us to analyze and solve complex problems in physics, engineering, and other fields where functions are defined in two or more dimensions. It is a powerful tool for understanding the frequency content and characteristics of multidimensional signals and functions.

Exercise 7.23. Find the inverse Fourier transform: $\mathcal{F}^{-1}\{e^{-t(\omega_1^2+\omega_2^2)}\}$.

Exercise 7.24. Find the Fourier transform: $\mathcal{F}\{e^{-t|x|-s|y|}\}$ for positive constants t,s.

Exercise 7.25. If $f(x,y)$ is an odd function with respect to x, show

$$\hat{f}(-\omega_1,\omega_2)=-\hat{f}(\omega_1,\omega_2).$$

The same relation holds for the y variable. If f is even function with respect to x, then

$$\hat{f}(-\omega_1,\omega_2)=\hat{f}(\omega_1,\omega_2).$$

The extension of the Fourier transform to functions of n independent variables is indeed defined in a similar manner. For an integrable function $f:\mathbb{R}^n\to\mathbb{R}$, the n-dimensional Fourier transform is given by:

$$\mathcal{F}\{f\}=\hat{f}(\omega_1,\ldots,\omega_n)=\int_{\mathbb{R}^n} f(x_1,\ldots,x_n)\,e^{-i\omega_1 x_1}\cdots e^{-i\omega_n x_n}\,dV,$$

where dV is the volume element in $\mathbb{R}^n$, and ω_1, ..., ω_n are frequency parameters associated with coordinates $x_1, \ldots, x_n$. The inverse Fourier transform, allowing us to reconstruct the original function, is defined as:

$$\mathcal{F}^{-1}\{\hat{f}\} = f(x_1, \ldots, x_n) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\omega_1, \ldots, \omega_n)\, e^{i\omega_1 x_1} \cdots e^{i\omega_n x_n}\, d\omega.$$

Just like the properties of the 1D Fourier transform, the higher-dimensional Fourier transform also exhibits similar properties. For instance, if f is smooth enough with respect to its arguments x and y, then we have the following relationships:

$$\mathcal{F}\{\partial_x f\} = i\omega_1 \hat{f}(\omega_1, \omega_2), \quad \mathcal{F}\{\partial_{xy} f\} = -\omega_1 \omega_2 \hat{f}(\omega_1, \omega_2).$$

The n-dimensional Fourier transform plays a crucial role in various fields, especially when dealing with functions defined in multiple dimensions, such as in solving partial differential equations in physics and engineering, image processing, and many other applications.

Proposition 7.1. *Assume that $u(x,t)$ is continuously differentiable function with respect to t, and u_t and u are integrable functions with respect to x for any t, then we have*

$$\hat{u}_t(\omega, t) = \int_{-\infty}^{\infty} u_t(x,t)\, e^{-i\omega x}\, dx.$$

Proof. Since $u(x,t)$ is integrable, for any t in x, we can define the Fourier transform of $u(x,t)$ as

$$\hat{u}(\omega, t) = \int_{-\infty}^{\infty} u(x,t)\, e^{-i\omega x}\, dx.$$

For $\varepsilon \neq 0$, we have:

$$\frac{\hat{u}(\omega, t+\varepsilon) - \hat{u}(\omega, \varepsilon)}{\varepsilon} = \int_{-\infty}^{\infty} \frac{u(x, t+\varepsilon) - u(x,t)}{\varepsilon}\, e^{-i\omega x}\, dx.$$

By the mean value theorem, we can write

$$\frac{u(x, t+\varepsilon) - u(x,t)}{\varepsilon} = u_t(x, \tau_\varepsilon),$$

for some τ_ε in the interval t and $t+\varepsilon$ (if $\varepsilon > 0$). Therefore, we have

$$\frac{\hat{u}(\omega, t+\varepsilon) - \hat{u}(\omega, t)}{\varepsilon} = \int_{-\infty}^{\infty} u_t(x, \tau_\varepsilon)\, e^{-i\omega x}\, dx.$$

Now, let ε approaches zero:

$$\lim_{\omega \to 0} \frac{\hat{u}(\omega, t+\varepsilon) - \hat{u}(\omega, t)}{\varepsilon} = \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} u_t(x, \tau_\varepsilon)\, e^{-i\omega x}\, dx.$$

Since u_t is integrable, we can pass the limit inside the integral, and write

$$\lim_{\varepsilon \to 0} \frac{\hat{u}(\omega, t+\varepsilon) - \hat{u}(\omega, t)}{\varepsilon} = \int_{-\infty}^{\infty} \lim_{\varepsilon \to 0} u_t(x, \tau_\varepsilon)\, e^{-i\omega x}\, dx.$$

Since u_t is continuous is continuous, we obtain

$$\hat{u}_t(\omega,t) := \int_{-\infty}^{\infty} u_t(x,t)\, e^{i\omega x}\, dx.$$

The continuity of $\hat{u}_t(\omega,\, t)$ with respect to t follows from the continuity and the integrability of $u_t(x,t)$ and the dominant convergence theorem. □

Problems

Problem 7.11. Find the Fourier transform of following functions

a) $f(x) = \frac{1}{a^2+x^2}$

b) $f(x) = e^{i\omega_0 x}$ for $|x|<1$ and $f(x)=0$ in $|x|>1$

c) $f(x) = \begin{cases} e^{-x} & x>0 \\ 0 & x<0 \end{cases}$

d) $f(x) = \begin{cases} x\,e^{-x} & x>0 \\ 0 & x<0 \end{cases}$

e) $f(x) = \begin{cases} 1 & a<x<b \\ 0 & \text{otherwise} \end{cases}$

Problem 7.12. Find the inverse Fourier transform of the following functions

a) $\hat{f}(\omega) = \frac{1}{\omega^2-2\omega+2}$. Hint: a shift in ω.

b) $\hat{f}(\omega) = e^{-|\omega|-i\omega}$

Problem 7.13. Find the Fourier cosine integral of the following functions

a) $f(x) = e^{-x}$

b) $f(x) = x\,e^{-x}$

Problem 7.14. Use the definition of $\mathcal{F}^{-1}$ and show the relation:

$$\mathcal{F}^{-1}\left\{\frac{1}{i\omega} + \pi\delta(\omega)\right\}(x) = \begin{cases} 1 & x>0 \\ \frac{1}{2} & x=0 \\ 0 & x<0 \end{cases}.$$

Hint: note that

$$\int_{-\infty}^{\infty} \frac{\sin(\omega x)}{\omega} d\omega = \begin{cases} \pi & x>0 \\ 0 & x=0 \\ -\pi & x<0 \end{cases}.$$

Problem 7.15. Show the following relation

$$\mathcal{F}\{\cos(x^2)\} = \sqrt{\pi}\cos\left(\frac{\omega^2-\pi}{4}\right).$$

Find a relation for $\mathcal{F}\{\sin(x^2)\}$. Hint: May you wish to use the formula

$$\int_{-\infty}^{\infty} e^{ix^2}\, dx = \sqrt{i\pi}$$

Problem 7.16.

a) Show the following relation for $t>0$

$$\mathcal{F}\left\{\frac{e^{-t|x|}}{\sqrt{|x|}}\right\} = \sqrt{2\pi}\left(\frac{t+\sqrt{t^2+\omega^2}}{t^2+\omega^2}\right)^{1/2}.$$

Hint: you may wish to use the formula

$$\int_{-\infty}^{\infty} e^{-\alpha z^2}\, dz = \sqrt{\frac{\pi}{\alpha}},$$

for $\text{Re}\{\alpha\} > 0$.

b) Conclude the following relation

$$\mathcal{F}\left\{\frac{1}{\sqrt{|x|}}\right\} = \sqrt{\frac{2\pi}{|\omega|}}.$$

In particular, the function $f(x) = \frac{1}{\sqrt{|x|}}$ is self-dual with respect to the balanced version of the Fourier transform.

c) Find the following integral

$$\int_0^\infty \frac{\cos(x)}{\sqrt{x}}\,dx.$$

Problem 7.17. Prove the relation:

$$\lim_{s\to\infty}\int_{-\infty}^{\infty}\frac{\cos(sx)}{1+x^2}\,dx = 0.$$

Hint: use the relation for $\mathcal{F}\{\cos(\omega_0 x)\, f(x)\}$.

Problem 7.18. Consider the function $g(x) = \sqrt{2}\, e^{-2|x|}$.

a) Find the energy of $g(x)$ and verify the Plancherel identity.

b) Find the frequency band of the 95 percent cut off energy of $g(x)$.

c) Draw $g(x)$ and the reconstructed function based on the band you found in the part (b).

Problem 7.19. We give a proof for the uncertainty principle.

$$\frac{1}{\int_R |f(x)|^2 \int_{\mathbb{R}} |\hat{f}(\omega)|^2}\int_{\mathbb{R}} x^2|f(x)|^2\int_{\mathbb{R}}\omega^2|\hat{f}(\omega)|^2 \geq \frac{1}{4}.$$

a) Without loss of generality, assume $\|f\| = 1$ (why is this plausible?) Then, according to the PLANCHEREL identity, we have $\|\hat{f}\| = \sqrt{2\pi}$. On the other hand, by the relation $\mathcal{F}\{f'\} = i\omega\,\hat{f}(\omega)$, we obtain

$$\int_{\mathbb{R}}\omega^2\hat{f}(\omega) = \int_{\mathbb{R}}|f'(x)|^2.$$

We show first the following inequality

$$\left(\int_{\mathbb{R}} x^2|f(x)|^2\right)^{1/2}\left(\int_{\mathbb{R}}|f'(x)|^2\right)^{1/2} \geq \frac{\sqrt{2\pi}}{2}.$$

b) Use the CAUCHY-SCHWARZ inequality and show the following inequality

$$\left(\int_{\mathbb{R}} x^2|f(x)|^2\right)^{1/2}\left(\int_{\mathbb{R}}|f'(x)|^2\right)^{1/2} \geq \int_{\mathbb{R}} x\,f(x)\,f'(x).$$

Show also that the equality holds only if $x f(x) = \lambda f'(x)$ for some $\lambda\in\mathbb{R}$.

c) Show that

$$\int_{\mathbb{R}} x\,f(x)\,f'(x)\,dx \geq \frac{1}{2}$$

and conclude the uncertainty principle. Show that the equality holds only if $f(x) = ce^{ax^2}$ where c, a are some constants.

Problem 7.20.

a) Verify that the tent function

$$f(x) = \begin{cases} x+1 & -1\leq x\leq 0\\ 1-x & 0\leq x\leq 1\\ 0 & \text{otherwise}\end{cases}$$

is the convolution of the following pulse function with itself:

$$p(x) = \begin{cases} 1 & -1/2\leq x\leq 1/2\\ 0 & \text{otherwise}\end{cases}.$$

b) Use the property of the Fourier transform of convolutions to determine $\mathcal{F}\{f(x)\}$.

Problem 7.21. Assume that $f(x)$ is a smooth function with the Fourier transform $\hat{f}(w)$, and furthermore

$$\sum_{n=-\infty}^{\infty} f(n) < \infty.$$

a) Prove the following relation which is known the POISSON summation

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(2n\pi).$$

Hint: note that

$$f(n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{in\omega}\, d\omega,$$

and that for any $m \in \mathbb{Z}$, we have

$$\hat{f}(2m\pi) = \sum_{n=-\infty}^{\infty} f_{n,2m},$$

where

$$f_{n,2m} = \frac{1}{2\pi}\int_{(2m-1)\pi}^{(2m+1)\pi} \hat{f}(\omega)\, e^{-in\omega}\, d\omega.$$

b) Use the result in part a) to prove the following identity:

$$\sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} = \pi \coth\pi.$$

Problem 7.22. Find the Fourier transform of the following two-variable functions:

a) $f(x,y) = e^{-|x|-|y|}$

b) $f(x,y) = e^{-t|x|-s|y|}$

c) $f(x,y) = e^{-x^2-y^2}$

d) $f(x,y) = \frac{1}{(a^2+x^2)(b^2+y^2)}$

Problem 7.23. Assume that $f(x,y)$ is a smooth and integrable function. Furthermore, assume that f is of order 2, i.e.,

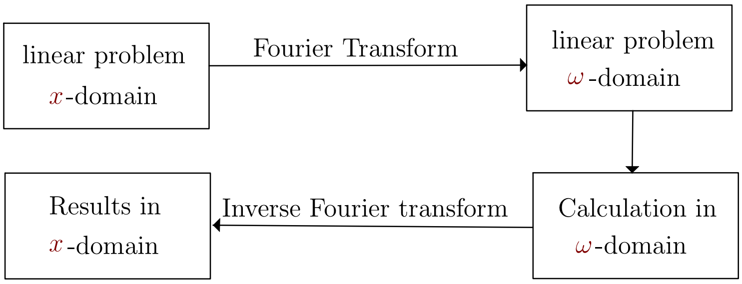
$$f(\lambda x, \lambda y) = \lambda^2 f(x,y),$$

for any $\lambda \neq 0$. Find the formula of the Fourier transform $\mathcal{F}\{\Delta f\}$.

7.5 Fourier transform and PDEs

The Fourier method is a powerful technique for solving real-world problems involving differential equations. The general idea of this method is as follows:

1. Start with the linear problem described in the x-domain.
2. Apply the Fourier transform to convert the problem into the ω-domain. This step involves transforming the differential equation and any initial or boundary conditions.
3. Perform calculations and manipulations in the ω-domain, where the problem becomes simpler due to the properties of the Fourier transform.
4. Use the inverse Fourier transform to obtain the solution back in the x-domain. This step gives us the final solution to the original problem.



Working with the Fourier method in the frequency domain often allows us to tackle complex problems more efficiently, as it unveils the frequency components and their behavior, which can be advantageous for various applications in physics, engineering, signal processing, and many other fields.

7.5.1 1D heat equations

Let Ω be a conductive rod extending from $-\infty$ to ∞ (in fact, it models a very long and thin homogeneous conductive rod). Consider the following heat problem on Ω for $t>0$:

$$\begin{cases} u_t = u_{xx}, \\ u(x,t) = f(x) \end{cases}, \tag{7.10}$$

where $f(x)$ is an integrable function. Note that there is no boundary condition for this problem; however, the boundedness condition should be satisfied by the solution as x approaches $\pm\infty$. By utilizing the Fourier transform in x, the problem reduces to the following "ordinary" differential equation in t:

$$\begin{cases} \hat{u}_t(\omega,t) = -\omega^2 \hat{u}(\omega,t) \\ \hat{u}(\omega,0) = \hat{f}(\omega) \end{cases}, \tag{7.11}$$

where $\hat{u}(\omega,t)$ represents the Fourier transform of $u(x,t)$ denoted by $\mathcal{F}\{u(x,t)\}$. This ordinary differential equation is then solved to find the solution in the frequency domain:

$$\hat{u}(\omega,t) = \hat{f}(\omega)\, e^{-\omega^2 t}. \tag{7.12}$$

Now, we can return to the x-domain by performing the inverse Fourier transform. We have two choices for the inverse Fourier transform in this case:

1. Using the definition of the inverse Fourier transform and representing the solution u as:

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{-\omega^2 t}\, e^{i\omega x}\, d\omega.$$

   This is an integral formula over the frequency spectrum ω. This integral formula is very similar to the series solutions to the heat problems on bounded domains.

2. Using the convolution as to express u as

$$u(x,t) = \mathcal{F}^{-1}\{\hat{f}(\omega)\, e^{-\omega^2 t}\} = f(x) * \mathcal{F}^{-1}\{e^{-\omega^2 t}\}.$$

As we saw in the previous section, we have

$$\mathcal{F}^{-1}\{e^{-\omega^2 t}\} = \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{x^2}{4t}}.$$

This function is called the heat kernel and is denoted by $\Phi(x, t)$ for $t > 0$. Therefore, we can write u as the convolution

$$u(x,t) = \int_{-\infty}^{\infty} f(z)\,\Phi(x-z,t)\,dz = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} f(z)\, e^{-\frac{(x-z)^2}{4t}}\,dz.$$

Both representations of $u(x,t)$ are valid solutions to the given heat problem.

Example 7.4. Let $f(x)$ be the rectangle function

$$f(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases},$$

and let us solve the following heat problem on $x \in (-\infty, \infty)$

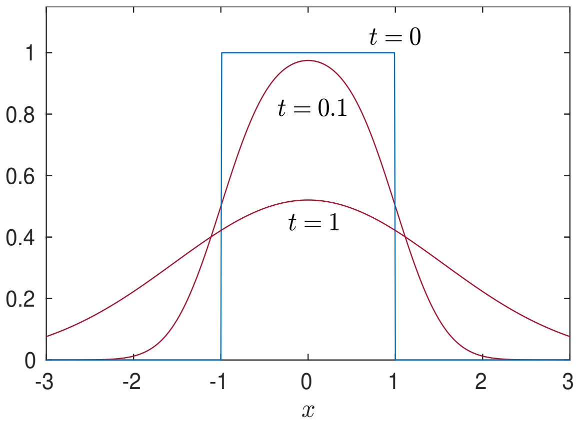
$$\begin{cases} u_t = u_{xx} \\ u(x,0) = f(x) \end{cases}.$$

The first form of the inverse Fourier transform results in

$$u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(\omega)}{\omega}\, e^{-\omega^2 t}\, e^{i\omega x}\,d\omega.$$

The integral at the right-hand side at $t = 0$ is equal to $\mathcal{F}^{-1}\left\{\frac{2\sin(\omega)}{\omega}\right\}$ which is the rectangle function $f(x)$. The figure below depicts the solution u by numerical integration in the ω-band $[-20, 20]$. For a faster calculation, we used the following equivalent formula:

$$u(x,t) = \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin(\omega)}{\omega}\, e^{-\omega^2 t}\cos(\omega x)\,d\omega.$$



Notice that for any $t > 0$, the solution $u(x,t)$ becomes smooth, even if the initial condition $u(x,0)$ is discontinuous. This smoothing effect is a characteristic behavior of the heat equation, which helps in smoothing out the initial conditions. This fact becomes more apparent when we use the second representation of $u(x,t)$, namely the convolution form:

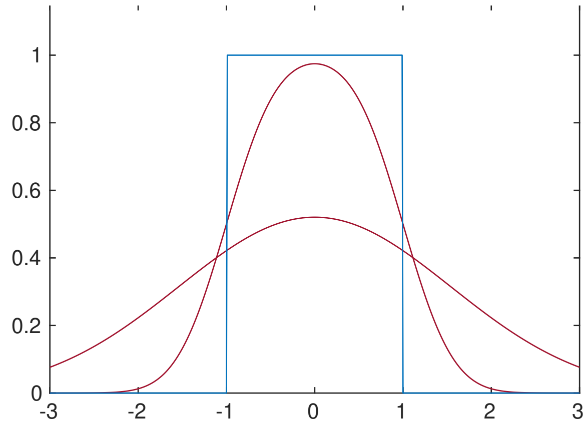
$$u(x,t) = f(x) * \Phi(x,t).$$

The function $\Phi(x, t)$ is smooth for all $t > 0$, and when it is convoluted with any piecewise continuous function, it effectively removes discontinuities and imparts smoothness to the function.

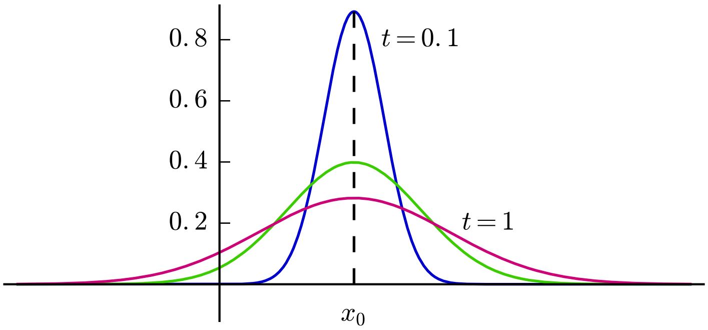
For the given $f(x)$, we have

$$u(x,t) = f(x) * \Phi(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-1}^{1} e^{-\frac{(x-z)^2}{4t}}\, dz.$$

The figure below depicts the solution using the convolution:



Remark 7.1. (Heat kernel) As previously discussed, the heat kernel $\Phi(x,t)$ plays a fundamental role in solving the heat equation with a point source $\delta(x)$ as the initial condition. For $t > 0$, $\Phi(x, t)$ represents the temperature distribution throughout space as time evolves from the point source:



As depicted in the graph, the behavior of $\Phi(x, t)$ for various instances of time shows how heat diffuses and spreads over space. Interestingly, as time approaches zero, the heat kernel approaches a Dirac delta function, which is a mathematical representation of an instantaneous point source. Additionally, the convolution of $\Phi(x,t)$ with any piecewise continuous function results in a smooth function, reflecting its smoothing effect. The heat kernel's remarkable properties make it a valuable tool in understanding heat conduction, diffusion processes, and solving heat-related problems in various scientific and engineering applications.

Problem 7.24. Show that if $f(x)$ is a piecewise continuous and integrable function, the convolution $f(x) * \Phi(x,t)$ for any $t > 0$ is a smooth function. Show that

$$\int_{-\infty}^{\infty} \Phi(x,t)\, dx = 1,$$

for any $t>0$, and that:

$$\lim_{t\to 0}\int_{-\infty}^{\infty} f(x)\,\Phi(x,t)\,dx = f(0),$$

for any integrable and continuous function $f(x)$. Show that the following relation holds for any integrable function $f(x)$ and if x is a continuity point of f.:

$$\lim_{t\to 0} f(x)*\Phi(x,t) = f(x).$$

Problem 7.25. Let $u(x,t)$ be the solution of the equation

$$\begin{cases} u_t = u_{xx} \\ u(x,0) = f(x) \end{cases}.$$

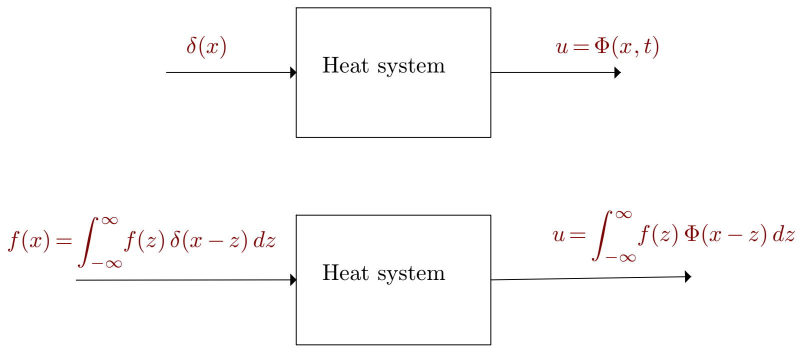
Use the convolution form of the solution and show the relation:

$$\int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty} f(x)\,dx.$$

Exercise 7.26. Show that $\Phi(x,t)$ is the solution of the following heat problem

$$\begin{cases} u_t = u_{xx} \\ u(x,0) = \delta(x) \end{cases},$$

where $\delta(x)$ is the Dirac delta function. Thus, $\Phi(x,t)$ is the impulse response of the heat system to the initial condition $\delta(t)$. Due to the relation $f(x)=f(x)*\delta(x)$, the response of the heat system to the input $f(x)$ is $u=f(x)*\Phi(x,t)$ as shown in the figure below:



Exercise 7.27. The convolution form of the representing solution is specially useful if the Fourier transform of the initial condition is not easy to calculate. Consider the following heat problem

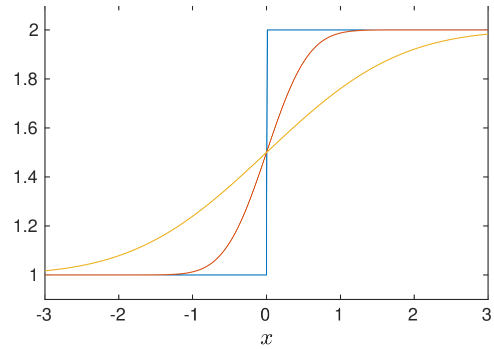
$$\begin{cases} u_t = u_{xx} \\ u(x,0) = \begin{cases} 1 & x<0 \\ 2 & x>0 \end{cases} \end{cases}.$$

Write down the convolution solution and show that $u(0,t)=\frac{3}{2}$ for all $t>0$. Sketch the graph of $u(x,t)$ at time $t=0.1,1$.

You may use the following code in Matlab:

```
x=-3:0.01:3;
f=@(x) 1*(x<=0)+2*(x>0);
t=0.1;
ux1=integral(@(z) f(z).*exp(-(x-z).^2/(4*t)),-20,20, ...
 'ArrayValued',true)/sqrt(4*pi*t);
t=1;
ux2=integral(@(z) f(z).*exp(-(x-z).^2/(4*t)),-20,20, ...
 'ArrayValued',true)/sqrt(4*pi*t);
plot(x,f(x),x, ux1,x,ux2)
```

You will obtain a graph similar to below:



Exercise 7.28. Find and integral solution for the following problem defined on $x\in(-\infty,\infty)$:

$$\begin{cases} u_t = k u_{xx} \\ u(x,t) = e^{-|x|} \end{cases},$$

and draw the solution $u(0,t)$ for $t\geq 0$.

The solutions to non-homogeneous equations follow a similar approach. Let's consider the heat problem on the domain $(-\infty,\infty)$:

$$\begin{cases} u_t = k u_{xx} + h(x,t) \\ u(x,0) = 0 \end{cases}.$$

Taking the Fourier transform of the equation leads to the following ordinary differential equation:

$$\hat{u}_t + k\omega^2 \hat{u} = \hat{h}(\omega,t),$$

where $\hat{h}$ is equal to $\mathcal{F}\{h(x,t)\}$. This ordinary differential equation can be solved using the methods discussed in the textbook on ordinary differential equations (ODEs).

Example 7.5. Let's solve the following problem:

$$\begin{cases} u_t = u_{xx} + \frac{\delta(t-1)}{1+x^2} \\ u(x,0) = 0 \end{cases}.$$

The ordinary differential equation for $\hat{u}(\omega,t)$ is:

$$\begin{cases} \hat{u}_t(\omega,t) = -\omega^2 \hat{u}(\omega,t) + \pi e^{-|\omega|}\delta(t-1) \\ \hat{u}(\omega,0) = 0 \end{cases}.$$

To solve this ordinary differential equation, we will use the Laplace transform method. The unilateral Laplace transform for a function $f(t)$, for $t>0$, is defined as:

$$\mathcal{L}\{f\} = \int_0^\infty f(t)\, e^{-st}\, dt.$$

In particular, we have:

$$\mathcal{L}\{\delta(t-1)\} = e^{-s}.$$

For a detailed discussion on this subject, the reader can refer to the textbook on ordinary differential equations (ODEs). Utilizing the Laplace transform for the derived first-order ODE, we obtain:

$$(s+\omega^2)\hat{U}(s,\omega)=\pi e^{-|\omega|}\,e^{-s},$$

where $\hat{U}(\omega,s)$ is equal to $\mathcal{L}\{\hat{u}(\omega,t)\}$. By inverse Laplace transform $\mathcal{L}^{-1}$, we obtain $\hat{u}(\omega,t)$ as

$$\hat{u}(\omega,t)=\mathcal{L}^{-1}\{\hat{U}(\omega,s)\}=\pi e^{-|\omega|}\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s+\omega^2}\right\}=\pi e^{-|\omega|}\,e^{-\omega^2(t-1)}\,u(t-1),$$

where $u(t-1)$ is the Heaviside function $u(t)$ shifted to $t-1$. The solution u in the x-domain can be obtained by taking the inverse Fourier transform as:

$$u(x,t)=\mathcal{F}^{-1}\{\hat{u}(\omega,t)\}=\mathcal{F}^{-1}\{\pi e^{-|\omega|}\,e^{-\omega^2(t-1)}\}\,u(t-1).$$

The convolution form of u is given by:

$$u(x,t)=\frac{1}{1+x^2}*\Phi(x,t-1)\,u(t-1).$$

Problem 7.26. Let $f(x)$ be an integrable function. Show that the solution of the problem:

$$\begin{cases} u_t=u_{xx}+\delta(t)\,f(x) \\ u(x,0)=0 \end{cases},$$

is equal to the solution to the following homogeneous equation:

$$\begin{cases} u_t=u_{xx} \\ u(x,0)=f(x) \end{cases}$$

Exercise 7.29. Solve the following heat problem

$$\begin{cases} u_t=u_{xx}+\delta(t)\,e^{-x^2} \\ u(x,0)=e^{-|x|} \end{cases}.$$

Problem 7.27. Find an integral solution for the following partial differential equation

$$\begin{cases} u_t=u_{xx}+u_x \\ u(x,0)=\frac{\cos(x)}{1+x^2} \end{cases}.$$

Problem 7.28. Solve the following heat problem

$$\begin{cases} u_t=u_{xx}+\frac{e^{-|x|}}{1+x^2} \\ u(x,0)=0 \end{cases},$$

and sketch the graph of the solution at $t=1$.

7.5.2 1D Wave problems

The method to solve a wave equation is completely similar to solve a heat problem. Consider the following problem defined on $x \in (-\infty, \infty)$:

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,x) = f(x) \\ \partial_t u(0,x) = g(x) \end{cases}.$$

Taking Fourier transform of the problem results in the following second-order ordinary differential equation:

$$\begin{cases} \hat{u}_{tt} = -c^2\omega^2 \hat{u} \\ \hat{u}(\omega,0) = \hat{f}(\omega) \\ \hat{u}_t(\omega,0) = \hat{g}(\omega) \end{cases}.$$

This differential equation is solved for:

$$\hat{u}(t,\omega) = \hat{f}(\omega)\cos(c\omega t) + \frac{1}{c\omega}\hat{g}(\omega)\sin(c\omega t),$$

and thus:

$$u(t,x) = \mathcal{F}^{-1}\{\hat{f}(\omega)\cos(c\omega t)\} + \frac{1}{c}\mathcal{F}^{-1}\left\{\frac{1}{\omega}\hat{g}(\omega)\sin(c\omega t)\right\}.$$

By the definition of inverse Fourier transform, we have

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)\cos(c\omega t)\, e^{i\omega x}\, d\omega + \frac{1}{2\pi c}\int_{-\infty}^{\infty}\frac{\hat{g}(\omega)}{\omega}\sin(c\omega t)\, e^{i\omega x}\, d\omega.$$

Here again we have two choices to represent the inverse Fourier transform $\mathcal{F}^{-1}$:

1. To use the definition of the inverse Fourier transform and represent the solution u as:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)\cos(c\omega t)\, e^{i\omega x}\, d\omega + \frac{1}{2\pi c}\int_{-\infty}^{\infty}\frac{\hat{g}(\omega)}{\omega}\sin(c\omega t)\, e^{i\omega x}\, d\omega. \quad (7.13)$$

2. Use the convolution and write u as:

$$u(x,t) = f(x) * \mathcal{F}^{-1}\{\cos(c\omega t)\} + \frac{1}{c}\mathcal{F}^{-1}\left\{\frac{\hat{g}(\omega)}{\omega}\right\} * \mathcal{F}^{-1}\{\sin(c\omega t)\}. \quad (7.14)$$

Recall the relation

$$\mathcal{F}\left\{\int_0^x g(\tau)\, d\tau\right\} = \frac{1}{i\omega}\hat{g}(\omega),$$

and hence:

$$\mathcal{F}^{-1}\left\{\frac{\hat{g}(\omega)}{\omega}\right\} = i\int_0^x g(\tau)\, d\tau$$

Problem 7.29. Show the relations

$$\mathcal{F}^{-1}\{\cos(c\omega t)\} = \frac{1}{2}[\delta(x-ct) + \delta(x+ct)],$$

and

$$\mathcal{F}^{-1}\{\sin(c\omega t)\} = \frac{1}{2i}[\delta(x-ct)-\delta(x+ct)].$$

The result of the above problem leads to the following formula for u:

$$u(x,t) = \frac{1}{2}[f(x-ct)+f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau.$$

This formula is called D'Alembert formula for the 1D wave problem.

Exercise 7.30. With the formula (7.13), use the Euler formula for $\cos(c\omega t)$ and $\sin(c\omega t)$ and conclude the convolution formula (7.14). Hence, two formula for $u(x,t)$ are equivalent.

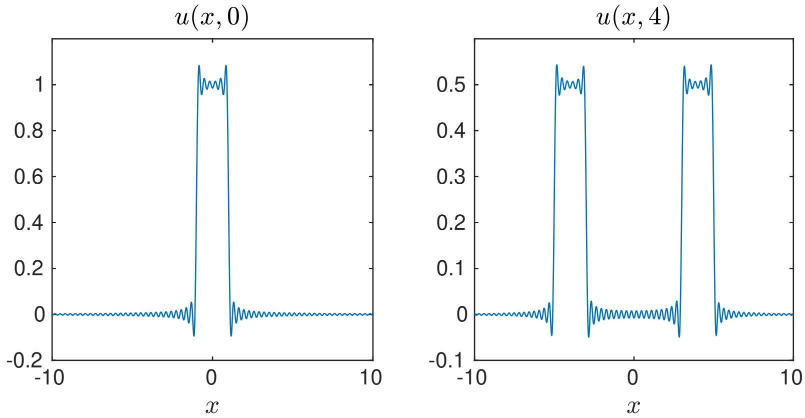
Example 7.6. Let's solve the following wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = f(x) \\ u_t(x,0) = 0 \end{cases},$$

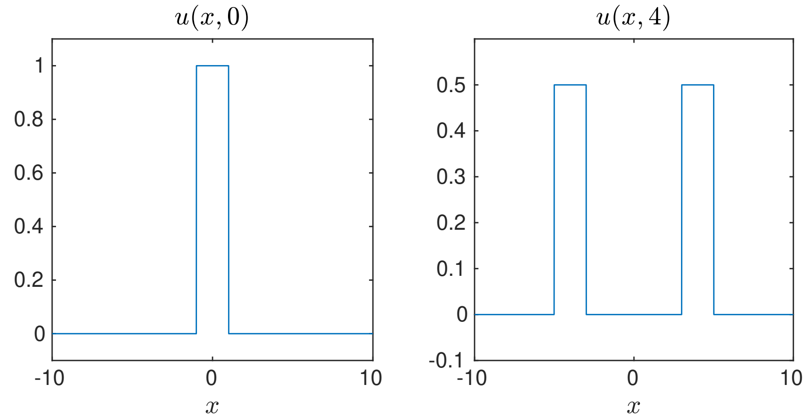
where $f(x)$ is the rectangle function $f(x) = \begin{cases} 1 & -1<x<1 \\ 0 & \text{otherwise} \end{cases}$. The inverse Fourier transform can be written as

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{\sin(\omega)}{\omega}\cos(c\omega t)\cos(\omega x)\,dx.$$

The figure below uses the numerical integration on the frequency band $[-20,20]$ for $c=1$ to represent the solution at times $t=0$and $t=4$:



The convolution solution does not need the numerical integration and yields the exact solution:



Example 7.7. Let us solve the following damped wave equation

$$\begin{cases} u_{tt} + 2\xi u_t = c^2 u_{xx} \\ u(x,0) = f(x) \\ u_t(x,0) = 0 \end{cases},$$

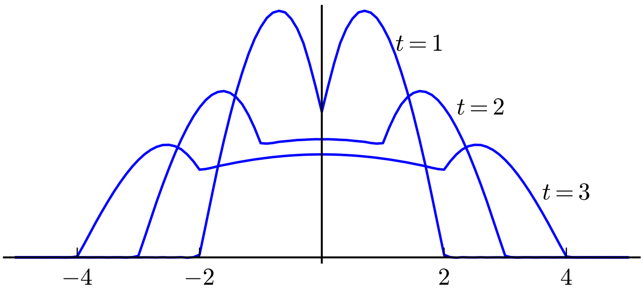
where $f(x)$ is the function $f(x) = \begin{cases} (1-x^2)/4 & -1 \le x \le 1 \\ 0 & \text{othersise} \end{cases}$. Taking the Fourier transform, the given problem reduces to the following ordinary differential equation:

$$\begin{cases} \hat{u}_{tt} + 2\xi \hat{u}_t = -c^2\omega^2 \hat{u} \\ \hat{u}(\omega,0) = \hat{f}(\omega), \hat{u}_t(\omega,0) = 0 \end{cases},$$

which is solved for the function:

$$\hat{u}(t,\omega) = e^{-\xi t} \hat{f}(\omega) \left[\cos(\mu t) + \frac{\xi}{\mu} \sin(\mu t) \right],$$

where $\mu = \sqrt{\omega^2 - \xi^2}$. The figure below shows the solution $u(x,t)$ for $c = 1$ and $\xi = 0.5$. Observe that two branches are not separated completely due to the damping factor ξ.



Exercise 7.31. Find the solution of the following wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = \delta(x) \\ u_t(x,0) = 0 \end{cases}.$$

This solution is the response of the wave system to the impulse displacement at time $t = 0$. What will be the solution if the initial conditions changes to the following one:

$$\begin{cases} u(x,0) = 0 \\ u_t(x,0) = \delta(x) \end{cases}.$$

Find the response of the damped wave system

$$u_{tt} + 2\xi u_t = u_{xx},$$

to the impulse displacement exercised at $t = 0$.

Exercise 7.32. Consider the following wave equation on the domain $x \in (-\infty, \infty)$:

$$\begin{cases} u_{tt} = u_{xx} + \delta(t-1)\, f(x) \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases},$$

where $f(x)$ is the rectangle function

$$f(x)=\begin{cases} 1 & -1<x<1 \\ 0 & \text{otherwise} \end{cases}.$$

a) Use the Fourier transform and find the function $\hat{u}(\omega,t)$.

b) Show that $u(x,t)$ can be determined as the convolution

$$u(x,t)=\frac{1}{2}f(x)*g_t(x).$$

What is $g_t(x)$?

Exercise 7.33. Determine the integral solution of the wave equation

$$\begin{cases} u_{tt}+0.6u_t=u_{xx} \\ u(x,0)=f(x) \\ u_t(x,0)=0 \end{cases},$$

where $f(x)$ is the function

$$f(x)=\begin{cases} \frac{\sin(x)}{x} & -\pi<x<\pi \\ 0 & \text{otherwise} \end{cases}$$

and draw the solution at $t=4$. You can use the following code in Matlab to draw the solution:

```
x=-5:0.01:5;
xi=0.3;t=4;
f=@(x) sin(x).*(x>-pi & x<pi)./x;
fw=@(w) 2*integral(@(x) f(x).*cos(w.*x),0,pi);
u=exp(-xi*t)*integral(@(w) cos(w.*x).*fw(w).*(cos(sqrt(w.^2-
xi^2)*t)+xi*sin(sqrt(w.^2-xi^2)*t)./sqrt(w.^2-xi^2)),0,40, ...
 'ArrayValued',true)/pi;
plot(x,u)
```

7.5.3 1D problems in semi-unbounded domains

The differential equations defined on semi-unbounded domains such as $(0,\infty)$ can be effectively solved using the Fourier sine or cosine integrals. As introduced before, the Fourier sine transform of a function $f(x)$ defined on $[0,\infty)$ is defined as:

$$\mathcal{F}_s\{f(x)\}:=\hat{f}_s(\omega)=\int_0^\infty f(x)\sin(\omega x)\,dx,$$

with the inverse

$$\mathcal{F}^{-1}\{\hat{f}_s(\omega)\}=\frac{1}{\pi}\int_{-\infty}^{\infty}\hat{f}_s(\omega)\sin(\omega x)\,d\omega.$$

It is important to note that $\mathcal{F}^{-1}\{\hat{f}_s(\omega)\}|_{x=0}=0$, meaning that the inverse Fourier transform converges to zero at $x=0$. This transform is particularly useful in solving Dirichlet problems, where boundary conditions are specified at one end of the semi-unbounded domain.

Example 7.8. Let's consider the heat problem defined on the domain $x\in(0,\infty)$:

$$\begin{cases} u_t=u_{xx}, & 0<x<\infty, t>0 \\ u(0,t)=0, & t>0 \\ u(x,0)=f(x) & \end{cases}. \tag{7.15}$$

Since the boundary condition at $x=0$ is of the Dirichlet type, we will use the Fourier sine series to solve this problem:

$$\begin{cases} \frac{\partial}{\partial t}\hat{u}_s(\omega,t) = -\omega^2\,\hat{u}_s(\omega,t) \\ \hat{u}_s(\omega,0) = \hat{f}_s(\omega) \end{cases},$$

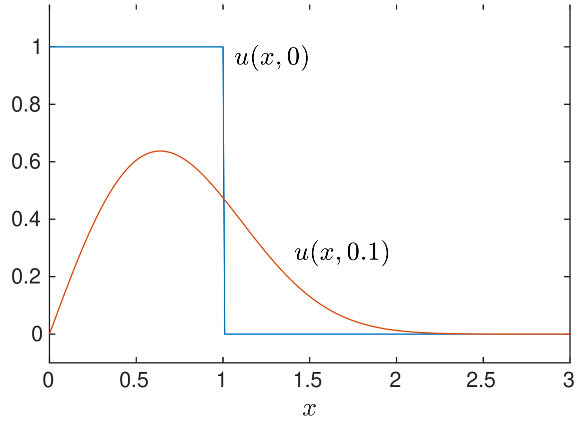
where $\hat{u}_s$ is equal to the Fourier sine transform of u. This ordinary differential equation is then solved for the function:

$$\hat{u}_s(\omega,t) = \hat{f}_s(\omega)\,e^{-\omega^2 t}.$$

By performing the inverse Fourier transform, we obtain the expression for u as:

$$u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \hat{f}_s(\omega)\,e^{-\omega^2 t}\sin(\omega x)\,d\omega.$$

It is important to note that $u(0,\,t)=0$ for all $t>0$, which satisfies the Dirichlet condition at $x=0$. The figure below illustrates the solution $u(x,\,t)$ for the initial condition $f(x)=e^{-x}$.



If the boundary condition at $x=0$ is Neumann, the solution can be expressed in terms of the Fourier cosine transform:

$$\mathcal{F}_c\{f(x)\} := \hat{f}_c(\omega) = \int_0^{\infty} f(x)\cos(\omega x)\,dx,$$

with its inverse given by:

$$\mathcal{F}^{-1}\{\hat{f}_c(\omega)\} = \frac{1}{\pi}\int_{-\infty}^{\infty} \hat{f}_c(\omega)\cos(\omega x)\,d\omega.$$

If we modify the boundary condition for the above example to $u_x(0,\,t)=0$, the integral solution changes to:

$$u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \hat{f}_c(\omega)\,e^{-\omega^2 t}\cos(\omega x)\,d\omega.$$

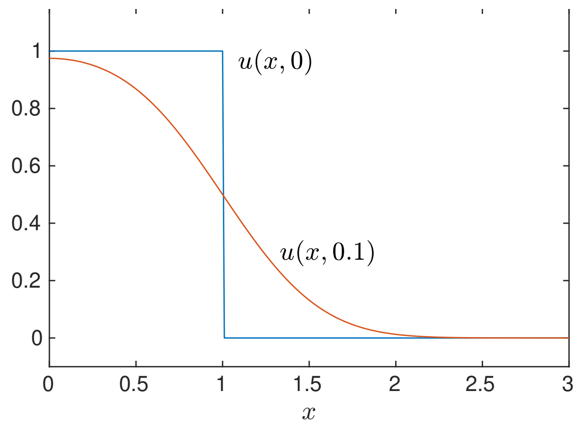
For the given piecewise function $f(x)$ defined as: $f(x) = \begin{cases} 1 & 0<x<1 \\ 0 & \end{cases}$, we can calculate its Fourier cosine transform as:

$$\hat{f}_c(\omega) = \int_0^1 \cos(\omega x)dx = \frac{\sin(\omega)}{\omega}.$$

Consequently, the solution to the heat problem with the Neumann boundary condition at time $t>0$ is:

$$u(x,t)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(\omega)}{\omega}\,e^{-\omega^2 t}\cos(\omega x)\,d\omega.$$

The figure below illustrates the solution at time $t=0.1$. By comparing this solution to the solution of the previous example (Dirichlet boundary condition), we can observe that the Neumann boundary condition enforces the derivative of the solution at $x=0$ to be zero for all $t>0$:



Remark 7.2. An alternative approach to solve Dirichlet problems is to extend the functions involved as odd functions over the entire domain $(-\infty,\infty)$. For example, the odd extension of the problem (7.15) is formulated as:

$$\begin{cases} u_t=u_{xx}, & -\infty<x<\infty, t>0 \\ u(x,0)=f_o(x) & \end{cases},$$

where f_o is the odd extension of f, defined as:

$$f_o(x)=\begin{cases} f(x) & x>0 \\ -f(-x) & x<0 \end{cases}.$$

Taking the Fourier transform of this extended problem results in the solution:

$$\hat{u}_t(\omega,t)=\hat{f}_o(\omega)\,e^{-\omega^2 t},$$

and finally $u(x,t)$ is given by the convolution as:

$$u(x,t)=f_o(x)*\frac{1}{4\pi t}\,e^{-\frac{x^2}{4t}}=\int_{-\infty}^{\infty}f_{o(z)}\,e^{-\frac{(x-z)^2}{4t}}\,dz.$$

It can be observed that $u(0,t)$ is equal to zero as:

$$u(0,t)=\int_{-\infty}^{\infty}f_{o(z)}\,e^{-\frac{z^2}{4t}}\,dz=0.$$

This approach is particularly useful to represent the solutions in convolution form. Both methods, either extending the functions as odd over the entire domain or using the Fourier sine integral, yield the same results but in different expressions. The choice of method depends on the convenience of the problem and the desired form of the solution.

Exercise 7.34. Find and integral solution for the following heat problem defined on $[0,\infty)$:

$$\begin{cases} u_t = u_{xx} + t f(x), & 0 < x < \infty, t > 0 \\ u(0,t) = 0 & t > 0 \\ u(x,0) = 0 & \end{cases},$$

where $f(x) = \left\{ \begin{smallmatrix} 1 & 0 < x < 1 \\ 0 & \end{smallmatrix} \right.$.

Exercise 7.35. Consider the following wave equation defined on $[0,\infty)$:

$$\begin{cases} u_{tt} = u_{xx} + \delta(t-1) e^{-x} \\ u_x(0,t) = 0 \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$$

Use even extension of the function involved and then employ the Fourier transform to solve the problem. Verify that your solution satisfies the boundary condition $u_x(0,t) = 0$.

Exercise 7.36. Let Ω be the interval $[1,\infty)$. Solve the following problem on Ω

$$\begin{cases} u_t = u_{xx} + \delta(t-1) e^{-x} \\ u_x(1,t) = 0 \\ u(x,0) = 0 \end{cases}$$

Exercise 7.37. Consider the following heat problem on a semi-infinite domain

$$\begin{cases} u_t = u_{xx} \\ u(0,x) = 1 \\ u(x,0) = f(x) \end{cases}.$$

The boundary condition at $x = 0$ is non-homogeneous. To solve this problem, we assume the solution $u(x,\, t)$ can be written as the sum of a steady-state solution $V(x)$ and a transient solution $w(x,t)$, i.e., $u(x,t) = V(x) + w(x,t)$. The steady-state solution $V(x)$ satisfies the non-homogeneous boundary condition $u(0,x) = 1$. Determine this function, and then find an integral solution for w.

7.6 Higher dimensional problems

n higher dimensional problems, we extend the Fourier transform method to solve various partial differential equations on unbounded domains in $\mathbb{R}^2$. Specifically, we focus on solving the heat equation, wave equation, Poisson equation, and Laplace equation.

These problems are considered over the entire space $\mathbb{R}^2$, as well as on specific regions such as strips in the plane and half-planes. The Fourier transform provides an efficient and powerful tool for transforming these partial differential equations into simpler ordinary differential equations in the frequency domain.

7.6.1 Equations defined on $\mathbb{R}^2$

Consider the heat equation $u_t = \Delta u$ with the initial condition $u(x, y, 0) = f(x, y)$. To solve this partial differential equation in two dimensions, we employ the two-variable form of the Fourier transform. This results in the following ordinary differential equation:

$$\hat{u}_t(\omega_1, \omega_2, t) = -(\omega_1^2 + \omega_2^2)\, \hat{u}(\omega_1, \omega_2, t),$$

where ω_1 and ω_2 are the frequency variables associated with the x and y directions, respectively. The Fourier transform of the function $u(x, y, t)$ is defined as:

$$\hat{u}(\omega_1, \omega_2, t) = \iint_{\mathbb{R}^2} u(x, y, t)\, e^{-i\omega_1 x}\, e^{-i\omega_2 y}\, dx\, dy.$$

Using the definition of the Laplacian operator Δ, we can express the Fourier transform Δ as:

$$\mathcal{F}\{\Delta u\} = \mathcal{F}\{u_{xx} + u_{yy}\} = -(\omega_1^2 + \omega_2^2)\, \hat{u}(\omega_1, \omega_2, t).$$

The resulting ordinary differential equation, along with the given initial condition, is solved to obtain the solution:

$$\hat{u}(\omega_1, \omega_2, t) = \hat{f}(\omega_1, \omega_2)\, e^{-(\omega_1^2 + \omega_2^2)t}.$$

Consequently, the solution u in the xy-domain can be obtained by taking the inverse Fourier transform:

$$u(x, y, t) = \mathcal{F}^{-1}\{\hat{f}(\omega_1, \omega_2)\, e^{-(\omega_1^2 + \omega_2^2)t}\}.$$

This inverse transform can be expressed either as a double integral:

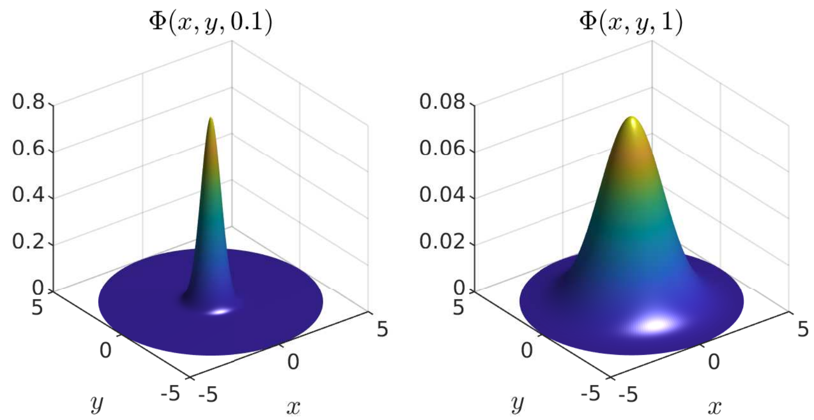
$$\mathcal{F}^{-1}\{\hat{f}(\omega_1, \omega_2)\, e^{-(\omega_1^2 + \omega_2^2)t}\} = \frac{1}{4\pi^2} \iint_{\mathbb{R}^2} \hat{f}(\omega_1, \omega_2)\, e^{-(\omega_1^2 + \omega_2^2)t}\, e^{i\omega_1 x}\, e^{i\omega_2 y}\, d\omega_1\, d\omega_2,$$

or using the convolution:

$$\mathcal{F}^{-1}\{\hat{f}(\omega_1, \omega_2)\, e^{-(\omega_1^2 + \omega_2^2)t}\} = f(x, y) * \mathcal{F}^{-1}\{e^{-(\omega_1^2 + \omega_2^2)t}\}.$$

The Fourier transform method provides an effective approach to solve the heat equation in two dimensions, enabling us to study the evolution of temperature distributions over time in complex spatial domains.

It turns out that $\mathcal{F}^{-1}\{e^{-(\omega_1^2 + \omega_2^2)t}\}$ is given by $\frac{1}{4\pi t}\, e^{-\frac{x^2 + y^2}{4t}}$, which is known as the 2D heat kernel and is denoted by $\Phi(x,\ y,\ t)$. This heat kernel shares similar properties with the one-dimensional heat kernel we encountered in problems defined on unbounded one-dimensional conductive rods. Just like its one-dimensional counterpart, the 2D heat kernel has the property of smoothing out initial conditions. As time progresses $(t > 0)$, the heat kernel acts as a convolution operator with the initial condition function $f(x, y)$. It diffuses the initial information and spreads it over the entire domain, leading to a smooth solution at each time instance. Additionally, for $t \to 0$, the heat kernel approaches the Dirac delta function centered at the initial point $(x, y) = (0, 0)$. The figure below illustrates $\Phi(x, y, t)$ for $t = 0.1$ and $t = 1$:



Problem 7.30. Show the relation

$$\mathcal{F}^{-1}\{e^{-(\omega_1^2+\omega_2^2)t}\} = \frac{1}{4\pi t}\, e^{-\frac{x^2+y^2}{4t}}.$$

This is called the 2D heat kernel:

$$\Phi(x,y,t) = \frac{1}{4\pi t}\, e^{-\frac{x^2+y^2}{4t}}.$$

Show the relation

$$\iint_{\mathbb{R}^2} \Phi(x,y,t)\, dx\, dy = 1.$$

Furthermore, show that the following relation holds:

$$\lim_{t\to 0} \iint_{\mathbb{R}^2} f(x,y)\, \Phi(x,y,t)\, dx\, dy = f(0,0),$$

for any integrable and continuous function $f(x,y)$.

Exercise 7.38. Consider the heat problem

$$\begin{cases} u_t = \Delta u \\ u(x,y,0) = f(x) \end{cases}.$$

Show that the solution of the problem is

$$u(x,y,t) = f(x) * \Phi(x,t).$$

Exercise 7.39. Write an integral solution for the following heat problem

$$\begin{cases} u_t = \Delta u \\ u(x,y,0) = e^{-|x|-|y|} \end{cases}.$$

Exercise 7.40. Solve the following heat equation on $\mathbb{R}^2$

$$\begin{cases} u_t = 4\Delta u \\ u(x,y,0) = e^{-x^2-y^2} \end{cases}.$$

Exercise 7.41. Solve the following heat problem on $\mathbb{R}^2$

$$\begin{cases} u_t = \Delta u + r^{-x^2-y^2} \\ u(x,y,0) = 0 \end{cases}.$$

Exercise 7.42. Solve the following heat problem on $\mathbb{R}^2$

$$\begin{cases} u_t = \Delta u + \frac{\delta(t-1)}{(1+x^2)(1+y^2)} \\ u(x,y,0) = 0 \end{cases}.$$

Exercise 7.43. Solve the following heat problem on $\mathbb{R}^2$

$$\begin{cases} u_t = \Delta u + \delta(t-1)\,\delta(x)\,\delta(y) \\ u(x,y,0) = 0 \end{cases}.$$

Now, let's consider wave equation $u_{tt} = c^2 \Delta u$ in $\mathbb{R}^2$. For simplicity, we consider the following problem:

$$\begin{cases} u_{tt} = c^2 \Delta u \\ u(x,y,0) = f(x,y) \\ u(x,y,0) = 0 \end{cases}.$$

Indeed, solving the wave equation in two dimensions using Fourier transform leads us to the ordinary differential equation:

$$\hat{u}_{tt} + c^2 \omega^2 \hat{u} = 0$$

where $\omega = \sqrt{\omega_1^2 + \omega_2^2}$. With the initial conditions:

$$\begin{cases} \hat{u}(\omega_1, \omega_2, 0) = \hat{f}(\omega_1, \omega_2) \\ \hat{u}(\omega_1, \omega_2, 0) = 0 \end{cases},$$

we obtain the solution:

$$\hat{u}(\omega_1, \omega_2, t) = \hat{f}(\omega_1, \omega_2) \cos(ct\omega). \tag{7.16}$$

To express the solution $u(x,\ y,\ t)$ in the xy-domain, we use the inverse Fourier transform. The solution can be written as an integral over the frequency domain:

$$u(x,y,t) = \frac{1}{4\pi^2} \iint_{\mathbb{R}^2} \hat{f}(\omega_1, \omega_2) \cos(ct\omega)\, e^{i\omega_1 x}\, e^{i\omega_2 y}\, d\omega_1\, d\omega_2.$$

Alternatively, the solution can also be expressed in the convolution form:

$$u(x,y,t) = f(x,y) * \mathcal{F}^{-1}\{\cos(ct\omega)\}. \tag{7.17}$$

The inverse Fourier transform of the function $\cos\left(c\sqrt{\omega_1^2 + \omega_2^2}\, t\right)$ is not straightforward and requires intricate calculations. Let us try to find this inverse transform. To do that, we rewrite ω_1, ω_2 as $\omega_1 = \omega \cos\gamma$, $\omega_2 = \omega \sin\gamma$ for $\gamma \in [-\pi, \pi]$. Therefore, we can write

$$\mathcal{F}^{-1}\{\cos(ct\omega)\} = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_0^{\infty} \cos(ct\omega) e^{-i\omega(x\cos\gamma + y\sin\gamma)}\, \omega\, d\omega\, d\gamma.$$

If we use the polar coordinate formula for x and y: $x = r\cos\theta$, $y = r\sin\theta$, we reach

$$\mathcal{F}^{-1}\{\cos(ct\omega)\} = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_0^{\infty}\cos(ct\omega)\cos(r\omega\cos(\gamma-\theta))\omega\, d\omega\, d\gamma.$$

Using the formula:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(s\cos\gamma)\, d\gamma = J_0(s),$$

where J_0 is the Bessel function of the first kind, we obtain:

$$\mathcal{F}^{-1}\{\cos(ct\omega)\} = \frac{1}{2\pi}\int_0^{\infty} J_0(r\omega)\cos(ct\omega)\omega\, d\omega.$$

The integral at the right-hand side can be rewritten as:

$$\frac{1}{2\pi}\int_0^{\infty} J_0(\rho\omega)\cos(ct\omega)\omega\, d\omega = \frac{1}{2\pi c}\frac{d}{dt}\frac{1}{\rho}\int_0^{\infty} J_0(\omega)\sin\left(\frac{ct}{\rho}\omega\right)d\omega.$$

Using the formula:

$$\int_0^{\infty} J_0(\omega)\sin\left(\frac{ct}{\rho}\omega\right)d\omega = \frac{1}{\sqrt{\frac{c^2t^2}{\rho^2}-1}},\ \rho < ct,$$

we finally obtain:

$$\mathcal{F}^{-1}\{\cos(ct\omega)\} = \frac{1}{2\pi c}\frac{d}{dt}\frac{1}{\sqrt{c^2t^2-\rho^2}},\ \rho < ct.$$

Finally, the solution $u(t,x,y)$ can be written as follows:

$$u(t,x,y) = \frac{1}{2\pi c}\frac{d}{dt}\int_{B_{ct}(x,y)}\frac{f(z_1,z_2)}{\sqrt{c^2t^2-(x-z_1)^2-(y-z_2)^2}}dz_1dz_2,$$

where $B_{ct}(x,y)$ is the disk centered at (x,y) of radius ct. By a similar calculation, it is shown that if $u_t(0,x,y) = g(x)$, the solution is:

$$u(t,x,y) = \frac{1}{2\pi c}\frac{d}{dt}\int_{B_{ct}(x,y)}\frac{f(z_1,z_2)}{\sqrt{c^2t^2-(x-z_1)^2-(y-z_2)^2}}dz_1dz_2 + \frac{1}{2\pi c}\int_{B_{ct}(x,y)}\frac{g(z_1,z_2)}{\sqrt{c^2t^2-(x-z_1)^2-(y-z_2)^2}}dz_1dz_2.$$

7.6.2 Equations on half-plane

Consider the domain $\Omega = \{(x,\ y);\ y > 0\}$. We are interested in solving the heat equation $u_t = \Delta u$ on Ω, subject to the boundary condition $u(x,\ 0,\ t) = 0$ and the initial condition $u(x,y,0) = f(x,y)$. To tackle this problem, we can make use of the odd extension of $f(x,y)$ with respect to y, which can be defined as follows:

$$f_o(x,y) = \begin{cases} f(x,y) & y>0 \\ -f(x,-y) & y<0 \end{cases}.$$

By performing the odd extension, we obtain the heat equation on the entire domain $\mathbb{R}^2$ as:

$$\begin{cases} u_t = \Delta u \\ u(x, y, 0) = f_o(x, y) \end{cases}.$$

Next, we take the Fourier transform to derive the following ordinary differential equation:

$$\begin{cases} \hat{u}_t(\omega_1, \omega_2, t) = -(\omega_1^2 + \omega_2^2)\, \hat{u}(\omega_1, \omega_2, t) \\ \hat{u}(\omega_1, \omega_2, 0) = \hat{f}_o(\omega_1, \omega_2) \end{cases}.$$

Solving this ordinary differential equation yields the function:

$$\hat{u}(\omega_1, \omega_2, t) = \hat{f}_o(\omega_1, \omega_2)\, e^{-(\omega_1^2 + \omega_2^2)t}.$$

To obtain the solution $u(x, y, t)$, we apply the inverse Fourier transform and express it in the convolution form:

$$u(x, y, t) = f_o(x, y) * \frac{1}{4\pi t}\, e^{-\frac{x^2 + y^2}{4t}} = \frac{1}{4\pi t} \iint_{-\infty}^{\infty} f_o(z_1, z_2)\, e^{-\frac{(x - z_1)^2 + (y - z_2)^2}{4t}}\, dz_1\, dz_2.$$

It is important to note that the solution satisfies the boundary condition $u(x, 0, t) = 0$ due to the symmetry argument:

$$u(x, 0, t) = \frac{1}{4\pi t} \int_{-\infty}^{\infty} e^{-\frac{(x - z_1)^2}{4t}} \left\{ \int_{-\infty}^{\infty} f_o(z_1, z_2)\, e^{-\frac{z_2^2}{4t}}\, dz_2 \right\} dz_1 = 0.$$

For problems with Neumann boundary conditions, the even extension of the functions involved should be utilized.

Remark 7.3. An alternative approach to solve these problems is by using Fourier sine or cosine integrals, depending on the type of boundary condition involved.

Exercise 7.44. Consider the following problem

$$\begin{cases} u_t = \Delta u + e^{-y} e^{-|x|} \\ u_y(x, 0, t) = 0 \\ u(x, y, 0) = 0 \end{cases},$$

on the half-plane $y \geq 0$. Express the solution in the convolution form and in term of the integral over ω-domain. Verify that both expressions satisfy the prescribed boundary condition.

Exercise 7.45. Solve the following heat problem on the half-plane $x \geq 0$:

$$\begin{cases} u_t = \Delta u \\ u(0, y, t) = 0 \\ u(x, y, 0) = \frac{e^{-x}}{1 + y^2} \end{cases}.$$

We are now addressing the Laplace equations on half-planes, which arise when dealing with non-homogeneous boundary conditions for heat or wave equations. Consider the following heat problem on the domain $\Omega = \{(x, y); y \geq 0\}$:

$$\begin{cases} u_t = \Delta u \\ u(x, 0, t) = f(x) \\ u(x, y, 0) = g(x) \end{cases}.$$

To solve this problem, we take the solution as $u(x,y,t) = V(x,y) + w(x,y,t)$, where V satisfies the Laplace equation with the boundary condition:

$$\begin{cases} \Delta V = 0 \\ V(x,0) = f(x) \end{cases}.$$

Since x is unbounded in this domain, we take the Fourier transform of the equation with respect to x:

$$\begin{cases} \hat{V}_{yy}(\omega, y) - \omega^2 \hat{V}(\omega, y) = 0 \\ \hat{V}(\omega, 0) = \hat{f}(\omega) \end{cases}.$$

The general solution of this equation is:

$$\hat{V}(\omega, y) = A(\omega)\, e^{-\omega y} + B(\omega)\, e^{\omega y}.$$

However, to ensure the solution remains bounded as y approaches infinity, we assume that $B(\omega) = 0$ for $\omega > 0$ and $A(\omega) = 0$ for $\omega < 0$. This implies that the solution $\hat{V}(\omega, y)$ can be represented as:

$$\hat{V}(\omega, y) = C(\omega)\, e^{-y|\omega|},$$

where $C(\omega)$ is an undetermined function. The initial condition $\hat{V}(\omega, 0)$ enforces $C(\omega)$ to be equal to $\hat{f}(\omega)$, and therefore:

$$\hat{V}(\omega, y) = \hat{f}(\omega)\, e^{-y|\omega|}.$$

The function $V(x,y)$ can be retrieved using the inverse Fourier transform $\mathcal{F}^{-1}$, which can be expressed in convolution form as:

$$V(x,y) = f(x) * \frac{1}{\pi} \frac{y}{y^2 + x^2} = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(z)}{y^2 + (x-z)^2}\, dz.$$

An alternative representation using the definition of $\mathcal{F}^{-1}$ is:

$$V(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{-y|\omega|}\, e^{i\omega x}\, d\omega.$$

Exercise 7.46. Let Ω be the domain $\Omega = \{(x,y), x > 0, y > 0\}$. Solve the following Laplace equation

$$\begin{cases} \Delta u = 0 & \text{on } \Omega \\ u(x,0) = f(x) & \\ u(0,y) = 0 & \end{cases}.$$

Express the solution in convolution form.

Exercise 7.47. Solve the following heat problem

$$\begin{cases} u_t = \Delta u & \text{on } \Omega \\ u(x,0,t) = \frac{1}{1+x^2} & \\ u(x,y,0) = 0 & \end{cases},$$

where Ω is the half-plane $y > 0$.

Problem 7.31. Let Ω be the domain $\Omega = \{(x, y); y > 0\}$. Use the Fourier transform method and show that the bounded solution to the Laplace equation $\Delta u = 0$ on Ω subject to the boundary condition $u(x, 0) = \frac{1}{1 + x^2}$ is:

$$u(x, y) = \frac{y + 1}{(y + 1)^2 + x^2}.$$

Problem 7.32. Let Ω be the domain $\Omega = \{(x, y), y > 0\}$. Use the Fourier transform method and show that the bounded solution to the Laplace equation $\Delta u = 0$ on Ω subject to the boundary condition $u(x, 0) = \left\{\begin{smallmatrix} 1 & -1 < x < 1 \\ 0 & \text{otherwise} \end{smallmatrix}\right.$ satisfies the relation:

$$u(0, y) = \frac{2}{\pi} \tan^{-1}\left(\frac{1}{y}\right).$$

What is $u(x, y)$?

Problem 7.33. Show that the solution to the Laplace equation $\Delta u = 0$ in the half-plane $y > 0$ subject to the boundary condition

$$u(x, 0) = \begin{cases} 1 & x > 0 \\ \frac{1}{2} & x = 0 \\ 0 & x < 0 \end{cases},$$

is

$$u(x, y) = \frac{1}{\pi} \tan^{-1}\left(\frac{x}{y}\right) + \frac{1}{2}.$$

7.6.3 Equations on strips

Consider the heat equation $u_t = \Delta u$ on the strip $\Omega = (-\infty, \infty) \times (0, L)$ in the xy-plane, subject to the boundary conditions:

$$\begin{cases} u(x, 0, t) = 0 \\ u(x, L, t) = 0 \end{cases}.$$

Since the domain is unbounded in x, we take the Fourier transform of the equation with respect to x and reach:

$$\begin{cases} \hat{u}_t(\omega, y, t) = \hat{u}_{yy}(\omega, y, t) - \omega^2 \hat{u}(\omega, y, t) \\ \hat{u}(\omega, 0, t) = 0 \\ \hat{u}(\omega, L, t) = 0 \end{cases}.$$

This leads to a heat equation in the (y, t) space, for $0 < y < L$, which can be solved using the method we introduced for problems in bounded domains. The associated eigenvalue problem is:

$$\begin{cases} \phi'' - \omega^2 \phi = -\lambda \phi \\ \phi(0) = \phi(L) = 0 \end{cases}.$$

Solving for the eigenfunctions and eigenvalues yields:

$$\phi_n(y) = \sin\left(\frac{n\pi}{L} y\right), \lambda_n = \omega^2 + \frac{n^2 \pi^2}{L^2}.$$

Hence, the solution to the resulting heat equation in (y, t) is given by:

$$\hat{u}(\omega, y, t) = \sum_{n=1}^{\infty} C(\omega)\, e^{-\omega^2 t}\, e^{-\frac{n^2 \pi^2}{L^2} t} \sin\left(\frac{n\pi}{L} y\right).$$

The coefficient functions $C(\omega)$ can be determined by the initial condition of the problem if provided. Let the initial condition be given by: $u(x,y,0)=f(x,y)$. By taking the Fourier transform, we obtain $\hat{u}(\omega,y,0)=\hat{f}(\omega,y)$, an thus

$$\hat{f}(\omega,y)=\sum_{n=1}^{\infty} C(\omega)\,\sin\Big(\frac{n\pi}{L}y\Big),$$

and by performing the inner product, we obtain:

$$C(\omega)=\frac{2}{L}\int_0^L \hat{f}(\omega,y)\sin\Big(\frac{n\pi}{L}y\Big)\,dy.$$

The solution $u(x,y,t)$ can be retrieved by performing the inverse Fourier transform as:

$$u(x,y,t)=\mathcal{F}^{-1}\{\hat{u}(\omega,y,t)\}=\sum_{n=1}^{\infty}\mathcal{F}^{-1}\{C(\omega)\,e^{-\omega^2 t}\}\,e^{-\frac{n^2\pi^2}{L^2}t}\sin\Big(\frac{n\pi}{L}y\Big).$$

Exercise 7.48. Let Ω be the strip $(-\infty,\infty)\times(0,1)$ in the xy-plane. Consider the following heat problem in Ω

$$\begin{cases} u_t=\Delta u & \text{on }\Omega\\ u(x,0,t)=u(x,1,t)=0 & B.\text{Cs.}\\ u(x,y,0)=\frac{\sin(\pi y)}{1+x^2} & I.C. \end{cases}.$$

a) Find an integral solution for the problem.

b) Show that $u\big(0,\frac{1}{2},\frac{1}{\pi}\big)$ is equal to

$$u\Big(0,\frac{1}{2},\frac{1}{\pi}\Big)=e^{-\pi}\int_0^{\infty}\frac{e^{-\frac{\pi}{4}z^2}}{1+z^2}\,dz.$$

Exercise 7.49. Let Ω be the strip $(0,1)\times(-\infty,\infty)$ in the xy-plane. Determine and integral solution for the following wave equation:

$$\begin{cases} u_{tt}=\Delta u+\delta(t-1)\sin(\pi x) & \text{on }\Omega\\ u(0,y,t)=u(1,y,t)=0 & B.\text{Cs.}\\ u(x,y,0)=0 & I.C.\\ u_t(x,y,0)=0 & I.C. \end{cases}.$$

The Laplace equation defined on a strip in the xy-plane can also be solved using the Fourier transform method. Let Ω be the same strip as above, and consider the following Laplace equation:

$$\begin{cases} \Delta u=0 & \text{on }\Omega\\ u(x,0)=f(x) & \\ u(x,L)=g(x) & \end{cases}.$$

Taking the Fourier transform in x yields:

$$\begin{cases} \hat{u}_{yy}(\omega,y)-\omega^2\,\hat{u}(\omega,y)=0\\ \hat{u}(\omega,0)=\hat{f}(\omega)\\ \hat{u}(\omega,L)=\hat{g}(\omega) \end{cases}.$$

This equation can be solved using the ordinary differential equations method, resulting in:

$$\hat{u}(\omega, y) = \hat{f}(\omega)\cosh(\omega y) + \left[\frac{\hat{g}(\omega)}{\sinh(\omega L)} - \hat{f}(\omega)\coth(\omega L)\right]\sinh(\omega y).$$

The solution $u(x, y)$ is retrieved by performing the inverse Fourier transform $\mathcal{F}^{-1}$.

Exercise 7.50. Solve the heat problem $u_t = \Delta u$ on the strip $0 < y < 1$, $-\infty < x < \infty$ subject to the boundary condition

$$\begin{cases} u(x, 0, t) = 0 \\ u(x, 1, t) = \frac{1}{1+x^2} \end{cases},$$

and the initial condition $u(x, y, 0) = 0$.

Problems

Problem 7.34. Let Ω be the upper half-plane $y > 0$. Solve the following heat problem:

$$\begin{cases} u_t = \Delta u & \text{on } \Omega \\ u(x, 0, t) = e^{-|x|} \\ u(x, y, 0) = e^{-x^2 - y^2} \end{cases}.$$

Problem 7.35. Solve the following heat equation on the upper half plane $y > 0$

$$\begin{cases} u_t = \Delta u \\ u_y(x, 0, t) = 0 \\ u(x, y, 0) = e^{-|x|}\, e^{-y} \end{cases}.$$

Problem 7.36. Let Ω be the half-plane $y > 0$. Consider the Laplace equation

$$\begin{cases} \Delta u = 0 & \text{on } \Omega \\ u(x, 0) = f(x) \end{cases}.$$

The solution in the convolution form is:

$$u(x, y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(z)}{y^2 + (x-z)^2}\, dz.$$

At first glance, it may seem that $\lim_{y \to 0} u(x, y) = 0$. However, this assumption is not true. Sketch the graph of the solution $u(x, y)$ for $y = 0.1, 0.05, 0.01$ if $f(x) = e^{-|x|}$ and explain the converges of the solution to $f(x)$.

Problem 7.37. Write the solution of the following heat problem defined in the quadrant $x > 0$, $y > 0$ in the integral form

$$\begin{cases} u_t = \Delta u \\ u(0, y, t) = f(x) \\ u(x, 0, 0) = 0 \\ u(x, y, 0) = g(x, y) \end{cases}.$$

Problem 7.38. Let Ω be the strip $\{(x, y); 0 < y < 1\}$

a) Show that the solution to the LAPLACE equation

$$\begin{cases} \Delta u = 0 \\ u(x, 0) = 0, u(x, 1) = \delta(x+1) + \delta(x-1) \end{cases}$$

is

$$u(x, y) = \frac{2}{\pi}\int_0^{\infty} \frac{\sinh(\omega y)}{\sinh(\omega)}\cos(\omega)\cos(\omega x)\, d\omega.$$

b) Find a solution if the boundary condition changes to $u(x,1)=\delta(x+1)-\delta(x-1)$.

Problem 7.39. Solve the following heat problem in the strip $\Omega=\{(x,y); 0<y<1\}$

$$\begin{cases} u_t=\Delta u \\ u(x,0,0)=u(x,1,t)=1 \\ u(x,y,0)=\left\{\begin{smallmatrix} 1 & -1<x<1 \\ 0 & \text{otherwise} \end{smallmatrix}\right. \end{cases}.$$

Problem 7.40. Find the convolution solution to the following Poisson equation on Ω: $\{(x,y); y\geq 0\}$

$$\begin{cases} \Delta u=f(x) \\ u(x,0)=0 \end{cases},$$

where $f(x)$ is the function

$$f(x)=\begin{cases} -1 & -1<x<0 \\ 1 & 0<x<1 \end{cases}.$$

Problem 7.41. Let $f(x), x\in(-\infty,\infty)$ be a continuous function with the Fourier transform $\hat{f}(\omega)$. Consider the following heat equation

$$\begin{cases} u_t=u_{xx}+e^t f(x) \\ u(0,x)=\frac{1}{2} f(x)*e^{-|x|} \end{cases}.$$

a) Show that the solution of the above equation can be in the form

$$u(t,x)=f(x)*g(t,x).$$

What is $g(t,x)$?

b) Assume that $f(x)=\left\{\begin{smallmatrix} 1 & -1<x<1 \\ 0 & \text{otherwise} \end{smallmatrix}\right.$. What is $u(t,x)$ for $x>1$?

Problem 7.42. Consider the following equation on $x\in(-\infty,\infty)$

$$\begin{cases} u_{tt}=c^2 u_{xx}+\delta(t) f(x) \\ u(0,x)=0, u_t(0,x)=0 \end{cases},$$

where δ is the Dirac delta function and $f(x)$ is a continuous function.

a) Show that the solution can be written in the following convolution form (as long as the convolution exists)

$$u(t,x)=\alpha\,(f*g)(x),$$

for some constant α and function g. Determine $\alpha, g(x)$.

b) Show that the convolution can be calculated as follows

$$(f*g)(x)=\int_{a(t,x)}^{b(t,x)} f(y)\,dy.$$

What are functions $a(t,x), b(t,x)$?

7.7 Appendix

7.7.1 A proof of the theorem (7.3)

If f is integrable, then $\hat{f}$ is continuous: Fix ω, then for any $\varepsilon \neq 0$, we have

$$\lim_{\varepsilon \to 0} \hat{f}(\omega+\varepsilon) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}} f(x) e^{-i\omega x} e^{-i\varepsilon x}\, dx.$$

Since f is integrable, the dominant convergence theorem allows us to pass the limit inside the integral and write

$$\lim_{\varepsilon \to 0} \hat{f}(\omega+\varepsilon) = \int_{\mathbb{R}} f(x)\, e^{-i\omega x} \lim_{\varepsilon \to 0} e^{-i\omega\varepsilon}\, dx = \hat{f}(\omega).$$

Now we prove that $\hat{f}(\omega)$ is differentiable with respect to ω. Fix ω and write:

$$\frac{\hat{f}(\omega+\varepsilon) - \hat{f}(\omega)}{\varepsilon} = \int_{\mathbb{R}} f(x)\, e^{-i\omega x} \frac{e^{-i\varepsilon x} - 1}{\varepsilon} dx.$$

Using the mean value theorem, we can write

$$\frac{e^{-i\varepsilon x} - 1}{\varepsilon} = x\, e^{-ix\varepsilon_0},$$

for some ε_0 in the interval $(0, \varepsilon)$ if $\varepsilon > 0$ (alternatively, $(\varepsilon, 0)$ is $\varepsilon < 0$). Thus,

$$\frac{\hat{f}(\omega+\varepsilon) - \hat{f}(\omega)}{\varepsilon} = -i \int_{\mathbb{R}} x f(x)\, e^{-i\omega x}\, e^{-ix\varepsilon_0} dx$$

The condition of the exponentially decay of f at infinity implies that there is $R > 0$ such that

$$|f(x)| \leq \alpha e^{-\beta|x|}, |x| > R,$$

for some constants $\alpha, \beta > 0$. Note that this implies the function $g(x) = x f(x)$ to be integrable and piecewise continuously differentiable. We have

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}} x f(x)\, e^{-i\omega x}\, e^{-ix\varepsilon_0} dx = \int_{\mathbb{R}} x f(x)\, e^{-i\omega x} \lim_{\varepsilon \to 0} e^{-ix\varepsilon_0}\, dx = \int_{\mathbb{R}} x f(x)\, e^{-i\omega x} dx.$$

Therefore,

$$\lim_{\varepsilon \to 0} \frac{\hat{f}(\omega+\varepsilon) - \hat{f}(\omega)}{\varepsilon} = -i \int_{\mathbb{R}} x f(x)\, e^{-i\omega x}\, dx = -i\mathcal{F}\{x f(x)\}.$$

Since $x f(x)$ is integrable, $\mathcal{F}\{x f(x)\}$ exists.

Exercise 7.51. If $f(x)$ decays exponentially at infinity, the function $g_n(x) = x^n f(x)$ is integrable.

7.7.2 A proof of the FOURIER theorem

We need the following FUBINI theorem

Theorem 7.6. *Assume that $f(x,y)$ is an integrable function, i.e.,*

$$I = \iint_{\mathbb{R}^2} |f(x,y)|\, dA < \infty.$$

If integrals

$$I_1 = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)\, dx \right) dy, \quad I_2 = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)\, dy \right) dx,$$

exist, then $I_1 = I_2$.

We also need the following lemma.

Lemma 7.1. (Lebesgue-Riemann) *Assume that $f(x)$ is a piecewise continuous and integrable function. We have the following relation:*

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x) \sin(nx)\, dx = 0.$$

In fact, we have

$$\hat{f}(n) = \int_{-\infty}^{\infty} f(x)\, e^{-inx} dx = \int_{-\infty}^{\infty} \frac{1}{n} f\left(\frac{x}{n}\right) e^{-ix}\, dx.$$

The dominant convergence theorem allows us to write

$$\lim_{n\to\infty} \hat{f}(n) = \lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{1}{n} f\left(\frac{x}{n}\right) e^{-ix}\, dx = \int_{-\infty}^{\infty} \lim_{n\to\infty} \frac{1}{n} f\left(\frac{x}{n}\right) e^{-ix}\, dx = 0.$$

Now we can prove the Fourier theorem.

Theorem 7.7. *Assume that f is an integrable ans piecewise continuously differentiable function. Then we have*

$$\frac{1}{2\pi} \lim_{n\to\infty} \int_{-n}^{n} \hat{f}(\omega)\, e^{i\omega x}\, d\omega = \frac{1}{2}[f(x^+) + f(x^-)].$$

For simplicity, we assume that f is C^1.

Proof. Since $\hat{f}(\omega)$ is continuous, then for any $n > 0$, the integral

$$I_n(x) = \int_{-n}^{n} \hat{f}(\omega)\, e^{i\omega x}\, d\omega,$$

exists. On the other hand, we have

$$I_n(x) = \int_{-n}^{n} \left(\int_{\mathbb{R}} f(\xi)\, e^{-i\xi\omega}\, d\xi \right) e^{i\omega x}\, d\omega = \int_{-n}^{n} \int_{\mathbb{R}} f(\xi)\, e^{i\omega(x-\xi)}\, d\xi d\omega.$$

By FUBINI theorem, we can switch the integrals as

$$I_n(x) = \int_{\mathbb{R}} f(\xi) \left(\int_{-n}^{n} e^{i\omega(x-\xi)}\, d\omega \right) d\xi.$$

Note that:

$$\int_{-n}^{n} e^{i\omega(x-\xi)}\,d\omega = 2\frac{\sin[n(x-\xi)]}{x-\xi},$$

and then

$$I_n(x) = 2\int_{\mathbb{R}} f(\xi)\frac{\sin[n(x-\xi)]}{x-\xi}\,d\xi.$$

By taking $z = x - \xi$, we obtain

$$I_n(x) = 2\int_{\mathbb{R}} f(x-z)\frac{\sin(nz)}{z}\,\mathrm{dz}.$$

We have

$$\int_{\mathbb{R}} f(x-z)\frac{\sin(nz)}{z}\,\mathrm{dz} = \int_{-\infty}^{-1}\frac{f(x-z)}{z}\sin(nz)\,\mathrm{dz} + \int_{-1}^{1} f(x-z)\frac{\sin(nz)}{z}\,dz + \\ + \int_{1}^{\infty}\frac{f(x-z)}{z}\sin(nz)\,dz.$$

By the Lebesgue-Riemann lemma, we have

$$\lim_{n\to\infty}\int_{-\infty}^{-1}\frac{f(x-z)}{z}\sin(nz)\,dz = \lim_{n\to\infty}\int_{1}^{\infty}\frac{f(x-z)}{z}\sin(nz)\,dz = 0.$$

We have also

$$\int_{-1}^{1} f(x-z)\frac{\sin(nz)}{z}\,dz = \int_{-1}^{1}[f(x-z)-f(x)]\frac{\sin(nz)}{z}\,dz + f(x)\int_{-1}^{1}\frac{\sin(nz)}{z}\,dz$$

$$\lim_{n\to\infty} I_n(x) = 2\lim_{n\to\infty}\int_{\mathbb{R}} f\left(x-\frac{z}{n}\right)\frac{\sin z}{z}\,dz = 2\int_{\mathbb{R}}\lim_{n\to\infty} f\left(x-\frac{z}{n}\right)\frac{\sin z}{z}\,dz = \\ = 2f(x)\int_{\mathbb{R}}\frac{\sin z}{z}\,dz = 2\pi f(x),$$

and this completes the proof. □