Chapter 6
3D Linear Second-Order Equations

In this chapter, we delve into the study of linear problems, encompassing Poisson, heat, and wave equations on Cartesian, cylindrical, and spherical domains. The cylindrical and spherical domains are obtained through linear transformations from a cube in $\mathbb{R}^3$, enabling us to employ the separation of variables technique for their solutions.

6.1 Cartesian coordinate

In this section, we initiate our exploration by solving Laplace's equations defined on cubes $(x_0, x_1) \times (y_0, y_1) \times (z_0, z_1)$ in $\mathbb{R}^3$. This foundational step will enable us to tackle linear heat, wave, and Poisson equations with nonhomogeneous boundary conditions.

6.1.1 Laplace equation

Ler Ω be the cube $(x_0, x_1) \times (y_0, y_1) \times (z_0, z_1)$. We consider the Laplace equation $\Delta u = 0$ subject to the boundary condition:

$$\alpha u + \beta \frac{\partial u}{\partial n} = f,$$

on the boundary $\operatorname{bnd}(\Omega)$. For this problem, we assume that the boundary condition is homogeneous on the sides x and y. By applying the separation of variables method, we assume $u(x, y, z)$ can be represented as $\phi(x, y) Z(z)$, which transforms the Laplace equation into:

$$\frac{\Delta_{(x,y)}\phi}{\phi} = -\frac{-Z}{Z}.$$

To satisfy this equality, we introduce a constant, denoted as $-\lambda$, which leads to the following eigenvalue problem:

$$\begin{cases} \Delta\phi = -\lambda\phi & \text{on } \Omega \\ \alpha\phi + \beta\frac{\partial\phi}{\partial n} = 0 & \text{on } \operatorname{bnd}(\Omega) \end{cases}.$$

The differential equation for $Z(z)$ becomes:

$$Z'' - \lambda Z = 0.$$

Example 6.1. Let's solve the following problem on a unit cube $(0,1)\times(0,1)\times(0,1)$

$$\begin{cases} \Delta u = 0 \\ u(x,y,1) = xy \\ u(1,y,z) = \sin(\pi z) \\ u = 0 \text{ other sides} \end{cases}.$$

According to the boundary condition, we split the problem as

$$\begin{cases} \Delta u = 0 \\ u|_{\Omega_1} = 0 \end{cases} + \begin{cases} \Delta u = 0 \\ u|_{\Omega_3} = 0 \end{cases}.$$

The first problem has the solution of the form

$$u = \sum_{n,m=1}^{\infty} B_{nm} \sinh\left(\sqrt{\lambda_{nm}}\, z\right) \phi_{nm}(x,y),$$

where $\lambda_{nm} = (n^2+m^2)\pi^2$, and

$$B_{nm} = \frac{4}{\sinh(\sqrt{\lambda_{nm}})} \int_0^1 \int_0^1 xy \sin(n\pi x)\sin(m\pi y)\, dy dx = \frac{4\cos(n\pi)\cos(m\pi)}{nm\pi^2 \sinh(\sqrt{\lambda_{nm}})}.$$

For the second problem, we can write the solution as (why?)

$$u = \sum_{n=1}^{\infty} B_n \sinh\left(\sqrt{n^2+1}\,\pi x\right) \sin(n\pi y)\sin(\pi z),$$

where

$$B_n = \frac{2}{\sinh\left(\sqrt{n^2+1}\pi\right)} \int_0^1 \sin(n\pi y)\, dy = \frac{2(1-\cos(n\pi))}{\sinh\left(\sqrt{n^2+1}\pi\right) n\pi},$$

and finally

$$u(x,y,z) = \sum_{n,m=1}^{\infty} \frac{4\cos(n\pi)\cos(m\pi)}{nm\pi^2 \sinh\left(\sqrt{n^2+m^2}\pi\right)} \sinh\left(\sqrt{n^2+m^2}\pi z\right) \sin(n\pi x)\sin(m\pi y) + \\ + \sum_{n=1}^{\infty} \frac{2(1-\cos(n\pi))}{n\pi \sinh\left(\sqrt{n^2+1}\pi\right)} \sinh\left(\sqrt{n^2+1}\pi x\right) \sin(n\pi y)\sin(\pi z).$$

Problem 6.1. Solve the following LAPLACE equation on $\Omega := (0,\pi)\times(0,\pi)\ (0,\pi)$

$$\begin{cases} \Delta u = 0 \\ u(x,\pi,z) = \sin(x)\sin(z) \\ u = 0 \text{ other sides} \end{cases}.$$

Problem 6.2. Solve the following problem on $\Omega := (0,\pi)\times(0,\pi)\ (0,\pi)$

$$\begin{cases} \Delta u + 2\partial_x u - u = 0 \\ u(\pi,y,z) = \sin(y)\sin(z) \\ u = 0 \text{ other sides} \end{cases}$$

6.1.2 Eigenfunction of Laplacian in Cartesian coordinate

We solve the following problem on Ω

$$\begin{cases} \Delta\phi = -\lambda\phi \\ \phi|_{\mathrm{bnd}(\Omega)} = 0 \end{cases}.$$

It is simply seen that $\lambda > 0$. In fact, we have

$$\iiint_\Omega \Delta\phi\phi = -\lambda \iiint_\Omega |\phi|^2,$$

and by the integration by parts formula,

$$\iiint_\Omega \Delta\phi\phi = \oiint_{\mathrm{bnd}(\Omega)} \phi\partial_n\phi - \iiint_\Omega |\nabla\phi|^2,$$

that gives $\lambda > 0$. Now, By the separation $\phi = X(x)\ Y(y)\ Z(z)$, we reach the following eigenvalue problems

$$\begin{cases} X'' = -\mu X \\ X(0) = X(a) = 0 \end{cases}, \begin{cases} Y'' = -\eta Y \\ Y(0) = Y(b) = 0 \end{cases}, \begin{cases} Z'' = -\nu Z \\ Z(0) = Z(c) = 0 \end{cases},$$

and finally

$$\phi_{nmp}(x, y, z) = \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)\sin\left(\frac{p\pi}{c}z\right),$$

with associated eigenvalues

$$\lambda_{nmp} = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2.$$

Example 6.2. Let us solve the following Poisson equation on $\Omega := (0,1)\times(0,1)\times(0,1)$

$$\begin{cases} \Delta u = \sin(\pi y)\sin(2\pi z) \\ u(x, y, 1) = xy \\ u(1, y, z) = \sin(\pi z) \\ u = 0 \text{ other sides} \end{cases}.$$

The solution can be written as $u = v + w$, where v solves the Laplace equation with given boundary conditions that we solved in the previous example. The equation for w reads

$$\begin{cases} \Delta w = \psi_{1,2}(y, z) \\ w|_{\mathrm{bnd}(\Omega)} = 0 \end{cases},$$

for $\psi_{1,2} = \sin(\pi y)\sin(2\pi z)$. The solution can be written as follows (why?)

$$w(x, y, z) = \left[\sum_{n=1}^\infty A_n \sin(n\pi x)\right]\psi_{1,2}(y, z)$$

By substituting the series into the equation, we obtain

$$\sum_{n=1}^\infty -(n^2+5)\pi^2 A_n \sin(n\pi x)\ \psi_{1,2}(y, z) = \psi_{1,2}(y, z),$$

and thus

$$\sum_{n=1}^{\infty} -(n^2+5)\pi^2 A_n \sin(n\pi x) = 1,$$

that results

$$A_n = \frac{-2(1-\cos(n\pi))}{(n^2+5)n\pi^3},$$

and finally,

$$w(x,y,z) = \left[\sum_{n=1}^{\infty} \frac{-2(1-\cos(n\pi))}{(n^2+5)n\pi^3}\sin(n\pi x)\right]\psi_{1,2}(y,z).$$

Remark 6.1. We could also assume the form of w as $w(x, y, z) = W(x)\ \psi_{1,2}(y, z)$ for an unknown function W. By this method, we obtain the following equation for $W(x)$

$$\begin{cases} W'' - 5\pi^2 W = 1 \\ W(0) = W(1) = 0 \end{cases},$$

that is solved for

$$W(x) = \frac{1-\cosh(\sqrt{5}\,\pi\,)}{5\pi^2\sinh(\sqrt{5}\,\pi\,)}\sinh(\sqrt{5}\,\pi x) + \frac{\cosh(\sqrt{5}\pi x) - 1}{5\pi^2},$$

and thus

$$w(x,y,z) = \left[\frac{1-\cosh(\sqrt{5}\,\pi)}{5\pi^2\sinh(\sqrt{5}\,\pi\,)}\sinh(\sqrt{5}\,\pi x) + \frac{1}{5\pi^2}(\cosh(\sqrt{5}\,\pi x) - 1)\right]\psi_{1,2}(y,z).$$

This is the closed form solution of the equation for w. It turns out that the closed form solution is equivalent to the series solution obtained above.

Problem 6.3. Solve the following wave equation on $\Omega := (0,\pi)\times(0,\pi)\times(0,\pi)$

$$\begin{cases} \partial_{tt} u = \Delta u \\ u|_{\Omega} = 0 \\ u(0,x,y,z) = \sin(x)\sin(z) \\ \partial_t u(0,x,y,z) = 0 \end{cases}$$

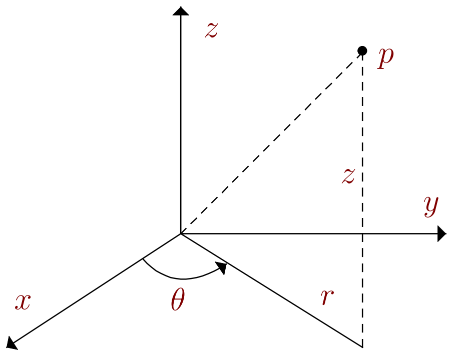
Problem 6.4. Solve the following heat equation with NEUMANN boundary conditions on $\Omega := (0,\pi)\times(0,\pi)\times(0,\pi)$

$$\begin{cases} \partial_t u = \Delta u \\ \partial_n u|_{\Omega} = 0 \\ u(0,x,y,z) = \cos(y)\cos(2z) \end{cases}$$

6.2 Problems in cylindrical domains

6.2.1 Cylindrical coordinate

A point p in spherical coordinate is represented by the triple (r, θ, z), where $r \geq 0$ is the distance of the projection of p in the (x, y)-plane to the origin, $\theta \in [-\pi, \pi]$ is the angle of the projected point on the (x, y)-plane with respect to x-axis, and z is the height of point p.



Let Ω denote the following cylinder

$$\Omega = \{(r,\theta,z), 0 \le r < a, -\pi \le \theta \le \pi, 0 < z < H\}, \tag{6.1}$$

with boundary $\mathrm{bnd}(\Omega) = \{(r,\theta,0)\} \cup \{(r,\theta,H)\} \cup \{(a,\theta,z)\}$. The Laplacian operator in the cylindrical coordinate is

$$\Delta u = \Delta_{(r,\theta)} u + u_{zz} = \partial_{rr} u + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial_{\theta\theta} u + \partial_{zz} u. \tag{6.2}$$

Note that the cube $[0,a) \times [-\pi,\pi] \times (0,H)$ is transformed to Ω by the following transformation

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases},$$

that allows us to use the separation of variables technique for solving linear equations on a cylinder.

Problem 6.5. The unit vectors $\hat{r}, \hat{\theta}, \hat{z}$ in the spherical coordinate are respectively

$$\hat{r} = \left(\frac{dx}{dr}, \frac{dy}{dr}, \frac{dz}{dr}\right) = (\cos\theta, \sin\theta, 0)$$

$$\hat{\theta} = \frac{1}{r}\left(\frac{dx}{d\theta}, \frac{dy}{d\theta}, \frac{dz}{d\theta}\right) = (-\sin\theta, \cos\theta, 0)$$

$$\hat{z} = \left(\frac{dx}{dz}, \frac{dy}{dz}, \frac{dz}{dz}\right) = (0,0,1)$$

a) Show that the nabla operator ∇ in the coordinate is

$$\nabla = \hat{r}\,\partial_r + \frac{1}{r}\hat{\theta}\,\partial_\theta + \hat{z}\,\partial_z.$$

b) Find the form of Δ through the relation $\Delta = \nabla \cdot \nabla$.

6.2.2 LAPLACE equation: Type I

Consider the following problem

$$\begin{cases} \Delta u = 0 \\ u(a,\theta,z) = 0 \\ u(r,\theta,0) = f(r,\theta), u(r,\theta,H) = g(r,\theta) \end{cases}. \tag{6.3}$$

Note that u is homogeneous on the side surface of Ω and is equal to functions f, g on the bottom and upper caps respectively. We relax this condition later and study the case $u=h$ on the side surface. By the separation of variable $u(r,\theta,z)=\Phi(r,\theta)\,Z(z)$, we reach the following equation

$$\frac{\Delta_{(r,\theta)}\phi}{\phi}+\frac{Z''}{Z}=0, \tag{6.4}$$

where $\Delta_{(r,\theta)}$ stands for the Laplacian on the disk. The associate eigenvalue of Eq.6.4 is

$$\begin{cases}\Delta_{(r,\theta)}\phi=-\lambda\phi\\ \phi(a,\theta)=0\end{cases}.$$

As we saw in the last chapter, the set of eigenpairs is as follows

$$\Phi=\{J_{0p}(r), J_{np}(r)\cos(n\theta), J_{np}(r)\sin(n\theta)\}_{n,p=1}^{\infty}, \lambda_{np}=\frac{z_{np}^2}{a^2} \tag{6.5}$$

The equation for Z becomes accordingly

$$Z''-\lambda_{np}Z=0,$$

and therefore

$$Z=\left\{\cosh\left(\sqrt{\lambda_{np}}\,z\right),\sinh\left(\sqrt{\lambda_{np}}\,z\right)\right\}.$$

Finally, the series solution u is

$$u(r,\theta,z)=\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\cosh\left(\sqrt{\lambda_{np}}z\right)J_{np}(r)\{A_{np}\cos(n\theta)+B_{np}\sin(n\theta)\}+$$
$$+\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sinh\left(\sqrt{\lambda_{np}}z\right)J_{np}(r)\{C_{np}\cos(n\theta)+D_{np}\sin(n\theta)\}.$$

The boundary conditions at $z=0$ and $z=H$ determine constants A, B, C, D.

Example 6.3. Let $a=1$ and $H=1$. Consider the problem

$$\begin{cases}\Delta u=0,\\ u(1,\theta,z)=0\\ u(r,\theta,0)=0\\ u(r,\theta,1)=(1-r)\cos\theta\end{cases}.$$

According to the boundary condition, it makes sense to write the solution as

$$u(r,\theta,z)=\sum_{p=1}^{\infty}\{A_p\cosh(z_{1p}z)+B_p\sinh(z_{1p}z)\}\,J_{1p}(r)\cos(\theta)$$

The condition at $z=0$ determines $A_p=0$ for all p. Therefore we write

$$u(r,\theta,z)=\left[\sum_{p=1}^{\infty}B_p\sinh(z_{1p}\,z)\,J_{1p}(r)\right]\cos(\theta).$$

Substituting the condition at $z=1$ determines B_p as

$$B_p = \frac{1}{\|J_{1p}\|^2 \sinh(z_{1p})} \int_0^1 J_{1p}(r)\,(1-r)\,r dr,$$

where

$$\|J_{1p}\|^2 = \int_0^1 |J_1(z_{1p}\, r)|^2\, r dr.$$

6.2.3 Laplace equation: Type II

Now, we solve the LAPLACE equation when $u(a,\theta,z)=f(\theta,z)$ but $u(r,\theta,0)=u(r,\theta,H)=0$. According to Eq.6.4, we solve first the following boundary value problem for Z

$$\begin{cases} Z'' = -\lambda Z \\ Z(0) = Z(H) = 0 \end{cases}.$$

The solution is

$$Z_p(z) = \sin\!\left(\frac{p\pi}{H} z\right), \quad \lambda_p = \frac{p^2\pi^2}{H^2}.$$

Therefore the equation for ϕ becomes

$$\Delta_{(r,\theta)}\phi - \frac{p^2\pi^2}{H^2}\,\phi = 0.$$

Using the separation of variables $\phi(r,\theta)=R(r)\,\Theta(\theta)$, we reach the following equation

$$\frac{r^2 R'' + r R'}{R} + \frac{\Theta''}{\Theta} = \frac{p^2\pi^2}{H^2} r^2, \tag{6.6}$$

and thus

$$\Theta_n(\theta) = \{\cos(n\theta), \sin(n\theta)\}.$$

Substituting Θ_n into (6.6) yields

$$r^2 R'' + r R' + \left(-\frac{p^2\pi^2}{H^2} r^2 - n^2\right) R = 0. \tag{6.7}$$

Note that the equation is similar to the BESSEL equation except the negative sign of r^2. If we take $x = \sqrt{-\frac{p^2\pi^2}{H^2}}\, r = i\frac{p\pi}{H} r$, then the equation reduces to the standard BESSEL equation

$$x^2 R''(x) + x R'(x) + (x^2 - n^2) R(x) = 0.$$

Therefore, the solution of Eq.(6.7) is

$$R_{np}(r) = J_n\!\left(\frac{i\pi p}{H} r\right).$$

What is the series of $J_n(ix)$? From the series of $J_n(x)$, we have

$$J_n(ix) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!}\left(\frac{ix}{2}\right)^{2k+n} = (i)^n \sum_{k=0}^{\infty} \frac{1}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n}.$$

The series

$$I_n(x) = \sum_{k=0}^{\infty} \frac{1}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n},$$

is called the *modified* Bessel functions of the first kind. For simplicity, we denote $I_n\left(\frac{\pi p}{H} r\right)$ by $I_{np}(r)$ and write

$$\phi_{np}(r,\theta) = \{I_{np}(r)\cos(n\theta), I_{np}(r)\sin(n\theta)\}.$$

Using the superposition principle, $u(r,\theta,z)$ can be represented as follows

$$u(r,\theta,z) = \sum_{n=0}^{\infty}\sum_{p=1}^{\infty} \sin\left(\frac{\pi p}{H} z\right) I_{np}(r)(A_{np}\cos(n\theta) + B_{np}\sin(n\theta)). \tag{6.8}$$

The coefficients A_{np}, B_{np} are determined by the aid of the boundary data.

Example 6.4. Let Ω denote a solid cylinder with the radius $a=1$ and height $H=1$. Let us solve the following equation defined on $\bar{\Omega}$

$$\begin{cases} \Delta u = 0 \\ u(1,\theta,z) = z\cos(\theta) \\ u(r,\theta,0) = u(r,\theta,1) = 0 \end{cases}.$$

Based on the boundary condition, u has the series representation

$$u(r,\theta,z) = \left[\sum_{p=1}^{\infty} U_p\, I_{1p}(r)\sin(\pi p z)\right]\cos(\theta).$$

For $r=1$, we have

$$z = \sum_{p=1}^{\infty} U_p \sin(\pi p z)\, I_{1p}(1),$$

and thus

$$U_p = \frac{2\,(-1)^p}{p\pi\, I_{1p}(1)}.$$

The solution u is as follows

$$u(r,\theta,z) = \left[\sum_{p=1}^{\infty} \frac{2\,(-1)^p\, I_{1p}(r)}{p\pi\, I_{1p}(1)} \sin(\pi p z)\right]\cos(\theta).$$

6.2.4 Eigenfunctions of the Laplacian

Consider the following eigenvalue problem

$$\begin{cases} \Delta\phi = -\lambda\phi \\ \phi|_{\mathrm{bnd}(\Omega)} = 0 \end{cases}, \tag{6.9}$$

where Ω denotes a solid cylinder with radius a and height H. Since $\Delta\phi = \Delta_{(r,\theta)}\phi + \phi_{zz}$, we use the separation of variables $\phi(r,\theta,z) = \psi(r,\theta)\,Z(z)$, and write

$$\frac{\Delta_{(r,\theta)}\psi}{\psi} + \frac{Z''}{Z} = -\lambda. \tag{6.10}$$

The ordinary differential equation for Z is solved for sine functions

$$Z_m(z) = \sin\left(\frac{m\pi}{H}z\right).$$

Substituting Z_m into (6.10) results to

$$\Delta_{(r,\theta)}\psi = -\left(\lambda - \frac{m^2\pi^2}{H^2}\right)\psi. \tag{6.11}$$

Take μ as $\mu = \lambda - \frac{m^2\pi^2}{H^2}$, and write the problem as follows

$$\left\{\begin{array}{l} \Delta_{(r,\theta)}\psi = -\mu\psi \\ \psi(a,\theta) = 0 \end{array}\right. . \tag{6.12}$$

As we saw before, the eigenfunctions of the above problem are

$$\psi_{np}(r,\theta) = \{J_{np}(r)\cos(n\theta), J_{np}(r)\sin(n\theta)\}.$$

Therefore, the eigenfunctions of Δ on a cylinder are

$$\phi_{mnp}(r,\theta,z) = \left\{\sin\left(\frac{m\pi}{H}z\right)J_{np}(r)\cos(n\theta), \sin\left(\frac{m\pi}{H}z\right)J_{np}(r)\sin(n\theta)\right\}, \tag{6.13}$$

with eigenvalues $\lambda_{mnp} = \frac{m^2\pi^2}{H^2} + \frac{z_{np}^2}{a^2}$.

Remark 6.2. For problems defined on the side surface of a cylinder (when $r = a$), we solve the eigenvalue problem $\Delta\phi = -\lambda\phi$ on the following set

$$D := \{(\theta,z), -\pi \le \theta \le \pi, 0 \le z \le H\}.$$

It is simply seen that in this case, the eigenfunctions are

$$\phi_{mn}(\theta,z) = \left\{\sin\left(\frac{m\pi}{H}z\right)\cos(n\theta), \sin\left(\frac{m\pi}{H}z\right)\sin(n\theta)\right\}, \tag{6.14}$$

with eigenvalues $\lambda_{nm} = \frac{m^2\pi^2}{H^2} + n^2$.

6.2.5 Linear problems on the surface of a cylinder

Let D denote the side surface of a cylinder of radius a and height H (without top and bottom caps).

Example 6.5. Let us solve the following POISSON equation

$$\left\{\begin{array}{l} \Delta u = z \\ u(\theta,0) = 0, u(\theta,\pi) = \cos(\theta) \end{array}\right. ,$$

where u is defined in the side surface of a cylinder of radius 1 and height π. By superposition principle, the solution consists two terms, the solution that is contributed by the boundary data, and the solution that is contributed by the source term. For the contribution from boundary, we solve the following equation

$$\begin{cases} \Delta v = 0 \\ v(\theta, 0) = 0, v(\theta, \pi) = \cos(\theta) \end{cases}.$$

Since $\cos(\theta)$ is a part of the eigenfunction $\sin\left(\frac{m\pi}{H} z\right)\cos(\theta)$, we write $v = V(z)\cos(\theta)$, for some unknown function $V(z)$. Substituting this into the equation leads to the following one

$$\begin{cases} V'' - V = 0 \\ V(0) = 0, V(\pi) = 1 \end{cases},$$

that is solved for

$$V(z) = \frac{1}{\sinh(\pi)} \sinh(z),$$

and thus $v = \frac{1}{\sinh(\pi)} \sinh(z)\cos(\theta)$. For the contribution by the source z, we solve the following POISSON equation

$$\begin{cases} \Delta w = z \\ w(\theta, 0) = 0, w(\theta, \pi) = 0 \end{cases}.$$

Note that the source term is independent f θ and thus we can write $w = W(z)$. Substituting w into the equation leads to the following equation

$$\begin{cases} W'' = z \\ W(0) = W(\pi) = 0 \end{cases},$$

that is solved for $W(z) = \frac{1}{6} z(z^2 - \pi^2)$. Finally, the solution is

$$u(\theta, z) = \frac{1}{\sinh(\pi)} \sinh(z)\cos(\theta) + \frac{1}{6} z(z^2 - \pi^2).$$

Remark 6.3. In the above example, we could solve the POISSON equation for w by eigenfunction series as follows

$$w(\theta, z) = \sum_{m=1}^{\infty} W_m \sin(mz), \tag{6.15}$$

that leads to

$$w(z) = \sum_{m=1}^{\infty} \frac{2\cos(m\pi)}{m^3} \sin(mz).$$

It is simply verified that the above series is the series expansion of $\frac{1}{6} z(z^2 - \pi^2)$.

Example 6.6. Let us solve the following wave problem on the side surface of a cylinder of radius $a = 1$ and height $H = 1$

$$\begin{cases} \partial_{tt} u = \Delta u \\ u(t, \theta, 0) = u(t, \theta, 1) = 0 \\ u(0, \theta, z) = 0, \partial_t u(0, \theta, z) = \sin(\theta) \end{cases}.$$

According to the initial data, the solution can be written in the following form

$$u(t,\theta,z)=\sum_{m=1}^{\infty} U_m(t)\sin(m\pi z)\sin(\theta).$$

Substituting this into the wave equation, we derive the following equation for $U_m(t)$

$$\begin{cases} U_m''(t)=-(m^2\pi^2+1)U_m(t) \\ U_m(0)=0, U_m'(0)=\frac{2(1-\cos(m\pi))}{m\pi} \end{cases},$$

which is solved for

$$U_m(t)=\frac{2(1-\cos(m\pi))}{m\pi\sqrt{\lambda_m}}\sin(\sqrt{\lambda_m}\,t),$$

where $\lambda_m=(m^2\pi^2+1)$. Therefore we reach the solution as the following series

$$u(t,\theta,z)=\left[\sum_{m=1}^{\infty}\frac{2(1-\cos(m\pi))}{m\pi\sqrt{\lambda_m}}\sin(\sqrt{\lambda_m}\,t)\sin(m\pi z)\right]\sin(\theta).$$

6.2.6 Linear problems on a solid cylinder

We solve a few examples to show the method how linear problems can be solve by the eigenfunction expansion method.

Example 6.7. Consider the the following problem in the solid cylinder of radius $a=1$ and height $H=1$

$$\begin{cases} \Delta u=r\sin(\pi z) \\ u(1,\theta,z)=0 \\ u(r,\theta,0)=u(r,\theta,1)=0 \end{cases}.$$

Since the source term is independent of θ (the eigenfunction associated to $n=0$), we write the solution as follows

$$u(r,\theta,z)=\sin(\pi z)\sum_{p=1}^{\infty} U_p\,J_{0p}(r).$$

Substituting u into the equation and using $\Delta u=-(z_{0p}^2+\pi^2)u$, yields

$$\sum_{p=1}^{\infty} -(z_{0p}^2+\pi^2)U_p\,J_{0p}(r)=r.$$

The above equation determines U_p as follows

$$U_p=\frac{-1}{(z_{0p}^2+\pi^2)\|J_{0p}\|^2}\int_0^1 J_{0p}(r)\,r^2\,dr,$$

where

$$\|J_{0p}\|^2 = \int_0^1 |J_0(z_{0p}\, r)|^2\, r\, dr.$$

Example 6.8. Consider the the following problem in the solid cylinder D of radius $a=1$ and height $H=1$

$$\begin{cases} \partial_t u = \Delta u \\ u(t,1,\theta,z) = 0 \\ u(t,r,\theta,0) = 0 \\ u(t,r,\theta,1) = \sin(\theta) \\ u(0,r,\theta,z) = 0 \end{cases}.$$

First, we find the steady state solution of the problem through solving the following one

$$\begin{cases} \Delta v = 0 \\ v(1,\theta,z) = 0 \\ v(r,\theta,0) = 0 \\ v(r,\theta,1) = \sin(\theta) \end{cases}.$$

Regarding the boundary data, the solution can be written in the following form

$$v(r,\theta,z) = \sum_{p=1}^{\infty} A_p \sinh(z_{1p}\, z) J_{1p}(r) \sin(\theta),$$

where A_p is

$$A_p = \frac{1}{\|J_{1p}\|^2 \sinh(z_{1p})} \int_0^1 J_{1p}(r)\, r\, dr.$$

Now, we write $u(t,\, r,\, \theta,\, z) = v(r,\, \theta,\, z) + w(t,\, r,\, \theta,\, z)$ where w satisfies the following homogeneous equation

$$\begin{cases} \partial_t w = \Delta w \\ w|_{\mathrm{bnd}(D)} = 0 \end{cases}.$$

It is simply seen that w has the series form

$$w(t,r,\theta,z) = \sum_{n,m,p} e^{-\lambda_{nmp} t} \sin(m\pi z)\, J_{np}(r)\, \{A_{nmp} \cos(n\theta) + B_{nmp} \sin(n\theta)\}.$$

Applying the initial condition of the problem leads to the following equality

$$-\sum_{p=1}^{\infty} A_p \sinh(z_{1p}\, z) J_{1p}(r) \sin(\theta) = \sum_{n,m,p} \sin(m\pi z)\, J_{np}(r) \{A_{nmp} \cos(n\theta) + B_{nmp} \sin(n\theta)\},$$

that implies in turn $A_{nmp} = 0$ and

$$-A_p \sinh(z_{1p}\, z) = \sum_{m=1}^{\infty} B_{1mp} \sin(m\pi z)\,.$$

We obtain

$$B_{1mp} = -2A_p \int_0^1 \sinh(z_{1p}\, z) \sin(m\pi z)\, dz,$$

and finally

$$u = \sum_{p=1}^{\infty} A_p \sinh(z_{1p}\, z)\, J_{1p}(r) \sin(\theta) + \sum_{m,p=1}^{\infty} A_{1mp}\, e^{-\lambda_{1mp} t}\, J_{1p}(r) \sin(m\pi z) \sin(\theta).$$

Problems

Problem 6.6. Show that functions $I_{np}(r)$ are orthogonal with respect to $\sigma(r) = r$, that is,

$$\langle I_{np}, I_{nq} \rangle_r = 0, p \neq q.$$

Problem 6.7. Let Ω denote a solid cylinder with radius $a = 1$ and height $H = 1$. Find a series solution to the following problem on Ω

$$\begin{cases} \Delta u = 0 \\ u(1, \theta, z) = 0 \\ u(r, \theta, 0) = \cos\theta \\ u(r, \theta, 1) = \sin\theta \end{cases}.$$

Problem 6.8. Let Ω denote a solid cylinder with radius $a = 1$ and height $H = 1$. Find a series solution to the following problem on Ω

$$\begin{cases} \Delta u = 0 \\ u(1, \theta, z) = 0 \\ u(r, \theta, 0) = r \\ u(r, \theta, 1) = \sin\theta + \cos\theta \end{cases}.$$

Problem 6.9. Let Ω denote a solid cylinder with radius $a = 1$ and height $H = 1$. Find a series solution to the following equation on Ω

$$\begin{cases} \Delta u = 0 \\ u(1, \theta, z) = 0 \\ u(r, \theta, 0) = \theta \\ u(r, \theta, 1) = r \end{cases}.$$

Problem 6.10. Let Ω denote a solid cylinder with radius $a = 1$ and height $H = 1$. Find a series solution to the following equation on Ω

$$\begin{cases} \Delta u = 0 \\ u(1, \theta, z) = z\cos\theta \\ u(r, \theta, 0) = 0 \\ u(r, \theta, 1) = 0 \end{cases}.$$

Draw solution at $\theta = \pi/4$, $z = 1/2$ with respect to r.

Problem 6.11. Let Ω denote a solid cylinder with radius $a = 1$ and height $H = 1$. Find a series solution to the following equation on Ω

$$\begin{cases} \Delta u = 0 \\ u(1, \theta, z) = \sin(2\pi z)\sin\theta \\ u(r, \theta, 0) = u(r, \theta, 1) = 0 \end{cases}.$$

Draw solution at $\theta = \pi/4$, $z = 1/2$ with respect to r.

Problem 6.12. Let Ω denote a solid cylinder with radius $a=1$ and height $H=1$. Find a series solution to the following equation on Ω

$$\begin{cases} \Delta u = 0 \\ u(1,\theta,z) = \sin(2\pi z)\,\theta \\ u(r,\theta,0) = \cos\theta, u(r,\theta,1) = 0 \end{cases}.$$

Problem 6.13. Let D denote the side surface of a cylinder with radius a and height H. Note that the cylinder does not include top and button caps. Solve the following equation on D for $u = u(\theta, z)$

$$\begin{cases} \Delta u = 0 \\ u(\theta,0) = f(\theta), u(\theta,H) = g(\theta) \end{cases}.$$

Problem 6.14. Let D denote the side surface of a cylinder with radius a and height H. Note that the cylinder does not include top and button caps. Solve the following equation on D for $u = u(\theta, z)$

$$\begin{cases} \Delta u + 2\partial_z u = 0 \\ u(\theta,0) = f(\theta), u(\theta,H) = g(\theta) \end{cases}.$$

Problem 6.15. Let D denote the side surface of a cylinder with radius $a=1$ and height $H=1$. Note that the cylinder does not include top and button caps. Find a closed form solution to the following problem on D

$$\begin{cases} \Delta u = z \\ u(\theta,0) = \cos\theta, u(\theta,1) = 0 \end{cases}$$

Problem 6.16. Let D denote the side surface of a cylinder with radius $a=1$ and height $H=1$. Note that the cylinder does not include top and button caps. Consider the following problem on D

$$\begin{cases} \Delta u = z\sin(2\theta) \\ u(\theta,0) = u(\theta,1) = 0 \end{cases}.$$

a) Find a series solution to the problem.

b) Find a closed form solution to the problem and verify it is equal to the series solution obtained in (a).

Problem 6.17. Let D denote the side surface of a cylinder with radius $a=1$ and height $H=1$. Note that the cylinder does not include top and button caps. Find a series solution to the following problem on D

$$\begin{cases} \Delta u = z\theta \\ u(\theta,0) = \sin\theta, u(\theta,1) = \cos\theta \end{cases}.$$

Problem 6.18. Let D denote the side surface of a cylinder with radius $a=1$ and height $H=1$. Note that the cylinder does not include top and button caps. Consider the following problem on D

$$\begin{cases} \Delta u = \sin(\pi z)\theta \\ u(\theta,0) = \sin(\theta), u(\theta,1) = 0 \end{cases}.$$

a) Find a series solution to the problem.

b) Find a closed form solution to the problem and verify it is equal to the series solution obtained in (a).

Problem 6.19. Let Ω denote a solid cylinder with radius $a=1$ and height $H=1$. Solve the following Poisson equation on Ω.

$$\begin{cases} \Delta u = rz \\ u(1,\theta,z) = 0 \\ u(r,\theta,0) = 0, u(r,\theta,1) = r \end{cases}$$

Problem 6.20. Let D denote the side surface of a cylinder with radius $a=1$ and height $H=1$. Solve the following heat problem on D

$$\begin{cases} \partial_t u = \Delta u \\ u(0,\theta,z) = \cos(\theta)\sin(\pi z) \\ u(t,\theta,0) = u(t,\theta,1) = 0 \end{cases}.$$

Problem 6.21. Let D denote the side surface of a cylinder with radius $a=1$ and height $H=1$. Solve the following heat problem on D

$$\begin{cases} \partial_t u = \Delta u - t\sin\theta \\ u(0,\theta,z) = 0 \\ u(t,\theta,0) = \cos\theta, u(t,\theta,1) = \sin\theta \end{cases}.$$

Problem 6.22. Let D denote the side surface of a cylinder with radius $a=1$ and height $H=1$. Solve the following heat problem on D

$$\begin{cases} \partial_{tt} u = c^2\Delta u \\ u(0,\theta,z) = \sin\theta\sin(\pi z), \partial_t u(0,\theta,z) = 0 \\ u(t,\theta,0) = 0, u(t,\theta,1) = 0 \end{cases}.$$

Problem 6.23. Let Ω denote a solid cylinder with radius $a=1$ and height $H=1$. Solve the following problem on Ω.

$$\begin{cases} \partial_t u = \Delta u - e^{-t} \\ u(t,1,\theta,z) = 0 \\ u(t,r,\theta,0) = u(t,r,\theta,1) = 0 \\ u(0,r,\theta,z) = 0 \end{cases}.$$

Problem 6.24. Let Ω denote a solid cylinder with radius $a=1$ and height $H=1$. Solve the following problem on Ω.

$$\begin{cases} \partial_t u = \Delta u \\ u(t,1,\theta,z) = z\sin\theta \\ u(t,r,\theta,0) = u(t,r,\theta,1) = 0 \\ u(0,r,\theta,z) = 0 \end{cases}.$$

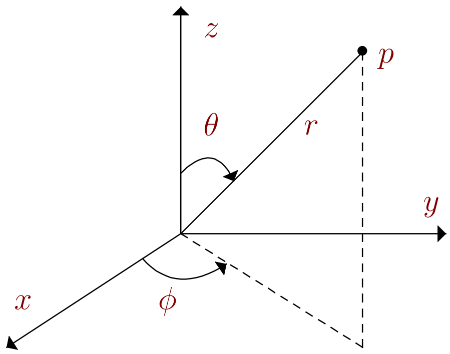
Problem 6.25. Let Ω denote a solid cylinder with radius $a=1$ and height $H=1$. Solve the following problem on Ω.

$$\begin{cases} \partial_{tt} u = c^2\Delta u \\ u(t,1,\theta,z) = 0 \\ u(t,r,\theta,0) = u(t,r,\theta,1) = 0 \\ u(0,r,\theta,z) = 0, \partial_t u(0,r,\theta,z) = (1-r)z \end{cases}.$$

6.3 Problems on spheres

6.3.1 Spherical coordinate

A point p in the spherical coordinate is represented by the triple (r,θ,ϕ), where $r \geq 0$ is the distance of p to the origin, $\theta \in [0,\pi]$ is the angle it makes with z-axis, and $\phi \in [-\pi,\pi]$ is the angle the projection of p on the (x,y)-plane makes with x-axis.



If Ω denote a ball of radius a, then a closed solid ball is represented by the following set of points

$$\Omega = \{(r, \theta, \phi), 0 \le r \le a, 0 \le \theta \le \pi, -\pi \le \phi \le \pi\}.$$

The Laplacian operator Δ in the spherical coordinate has the following form

$$\Delta u = \frac{1}{r^2}\partial_r(r^2 \partial_r u) + \frac{1}{r^2 \sin\theta}\partial_\theta(\sin\theta\,\partial_\theta u) + \frac{1}{r^2 \sin^2\theta}\partial_{\phi\phi} u. \tag{6.16}$$

To distinguish between a solid ball and a sphere, we use word *sphere* for the shell and denote it by S^2, and word *ball* for a solid ball. The inner product between two function $f(r, \theta, \phi)$ and $g(r, \theta, \phi)$ in the spherical coordinate is as follows

$$\langle f, g\rangle = \int_{-\pi}^{\pi}\int_0^{\pi}\int_0^a f(r, \theta, \phi)\, g(r, \theta, \phi)\, r^2 \sin\theta\, dr d\theta d\phi. \tag{6.17}$$

Remember that $dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$ is the volume element in the spherical coordinate.

Remark 6.4. The spherical coordinate is specified by the following transformation

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}.$$

In the first chapter, we asked the reader to derive the form of nabla operator ∇ by the aid of unit vectors

$$\hat{r} = \frac{1}{\sqrt{(\partial_r x)^2 + (\partial_r y)^2 + (\partial_r z)^2}}(\partial_r x, \partial_r y, \partial_r z) = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k},$$

and similarly for $\hat{\theta}, \hat{\phi}$ as

$$\hat{\theta} = \frac{1}{\sqrt{(\partial_\theta x)^2 + (\partial_\theta y)^2 + (\partial_\theta z)^2}}(\partial_\theta x, \partial_\theta y, \partial_\theta z) = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$$

$$\hat{\phi} = \frac{1}{\sqrt{(\partial_\phi x)^2 + (\partial_\phi y)^2 + (\partial_\phi z)^2}}(\partial_\phi x, \partial_\phi y, \partial_\phi z) = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}.$$

It is simply seen that

$$\nabla = \hat{r}\,\partial_r + \frac{1}{r\sin\theta}\,\hat{\phi}\,\partial_\phi + \frac{1}{r}\,\hat{\theta}\,\partial_\theta,$$

and thus

$$\Delta = \nabla\cdot\nabla = \left(\hat{r}\,\partial_r + \frac{1}{r\sin\theta}\,\hat{\phi}\,\partial_\phi + \frac{1}{r}\,\hat{\theta}\,\partial_\theta\right)\cdot\left(\hat{r}\,\partial_r + \frac{1}{r\sin\theta}\,\hat{\phi}\,\partial_\phi + \frac{1}{r}\,\hat{\theta}\,\partial_\theta\right),$$

after after direct calculation gives the form of Δ in the spherical coordinate as given in (6.16).

Problem 6.26. Show with the aid of following relations that the differential volume in spherical coordinate is $dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$

$$dx = \sin\theta\cos\phi\, dr + r\cos\theta\cos\phi\, d\theta - r\sin\theta\sin\phi\, d\phi,$$

$$dy = \sin\theta\sin\phi\, dr + r\cos\theta\sin\phi\, d\theta + r\sin\theta\cos\phi\, d\phi,$$

$$dz = \cos\theta\, dr - r\sin\theta\, d\theta.$$

6.3.2 Laplace equation and Legendre polynomials

Let Ω denote a ball in $\mathbb{R}^3$ of radius a. Consider the following equation on Ω

$$\begin{cases} \Delta u = 0 \\ u(a,\theta,\phi) = f(\theta,\phi) \end{cases}. \tag{6.18}$$

First, assume that f is independent of ϕ, that is, $f = f(\theta)$. This leads to an important class of *special functions* called the Legendre polynomials. Consider the equation

$$\begin{cases} \Delta u = 0 \\ u(a,\theta,\phi) = f(\theta) \end{cases}. \tag{6.19}$$

Since the boundary data is independent of ϕ, (that is, f is axially symmetric about the z-axes), we can assume that the solution is axially symmetric in Ω, that is, $u = u(r,\theta)$. Due to the form of Δ in the spherical coordinate, the separated solution $u(r,\theta) = R(r)\,\Theta(\theta)$ leads to the following equation

$$\frac{r^2 R'' + 2r R'}{R} + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}(\sin\theta\,\Theta') = 0. \tag{6.20}$$

The equation for Θ leads to the following eigenvalue problem:

$$\frac{d}{d\theta}(\sin\theta\,\Theta') = -\lambda_\theta \sin\theta\,\Theta, \tag{6.21}$$

for some eigenvalue λ_θ. It is observed that a singularity in Θ may happen at $\theta = 0, \pi$, where the coefficient of the highest derivative Θ'' become zero. To avoid the possible blow up at these points, we impose the following condition for the solution $\Theta(\theta)$

$$\lim_{\theta\to 0,\pi} |\Theta(\theta)| < \infty. \tag{6.22}$$

If we take $x = \cos(\theta)$, the eigenvalue problem (6.21) reduces to the following familiar form which is called the Legendre's equation

$$\frac{d}{dx}[(1-x^2)\,\Theta'] = -\lambda_\theta\,\Theta(x), \quad -1 \le x \le 1. \tag{6.23}$$

The LEGENDRE equation is discussed in our book on ordinary differential equations and readers are referred to it. The solution of the LEGENDRE equation can be derived by the power series as follows

$$\Theta(x) = \sum_{k=0}^{\infty} c_k x^k,$$

where c_k satisfy the following recursive formula

$$c_{k+2} = \frac{k(k+1) - \lambda_\theta}{(k+2)(k+1)} c_k.$$

Note that if λ_θ is an integer of the form $n(n+1)$ for $n = 0, 1, 2, \cdots$, then one solution to (6.23) is a polynomial of order n. This polynomial is called the LEGENDRE polynomial of order n after the French mathematician A.M. LEGENDRE, and is denoted by $P_n(x)$. A closed form representation of this polynomial is given by the O. RODRIGUES formula

$$P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} (1 - x^2)^n. \tag{6.24}$$

The following table shows $P_n(x)$ for $n = 0, \cdots, 6$.

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2 - 1)$
3	$\frac{1}{2}x(5x^2 - 3)$
4	$\frac{1}{8}(35x^4 - 30x^2 + 3)$
5	$\frac{1}{8}x(63x^4 - 70x^2 + 15)$
6	$\frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$

Table 6.1.

It turns out that if $\lambda_\theta \neq n(n+1)$, the infinite series solution blows up at $x = \pm 1$. Therefore,, to keep the solutions bounded, we assume $\lambda_\theta = n(n+1)$, where n is a non-negative integer. Consequently, $\Theta_n(\theta) = P_n(\cos\theta)$ for $n \geq 0$. Note that $P_n(\cos\theta)$ for $n = 0, 1, \ldots$, are solutions to (6.21) for $\lambda_\theta = n(n+1)$ and therefore they are orthogonal with respect to the weight function $\sigma = \sin\theta$

$$\int_0^\pi P_n(\cos\theta)\, P_m(\cos\theta) \sin(\theta)\, d\theta = 0, n \neq m. \tag{6.25}$$

In addition, the set $\{P_n(\cos\theta)\}_{n=0}^\infty$ is a basis for smooth functions in $\theta \in [0, \pi]$. At the end of this chapter, we will prove the following proposition for $P_n(\cos\theta)$.

Proposition 6.1. *The following relations hold for* LEGENDRE *polynomials* $P_n(\cos\theta)$

$$\int_0^\pi P_n(\cos\theta) \sin(\theta)\, d\theta = 0,$$

$$\|P_n\|^2 = \int_0^\pi |P_n(\cos\theta)|^2 \sin(\theta)\, d\theta = \frac{2}{2n+1}.$$

Now, we back to the equation for R. From (6.20), we reach

$$r^2R'' + 2rR' - n(n+1)R = 0,$$

which is a Cauchy-Euler equation. The solution set of the above equation is $R_n(r) = \{r^n, r^{-n-1}\}$ where we ignore term r^{-n-1} for problems defined inside a ball. The reason is that r^{-n-1} is unbounded at $r=0$. For the *exterior* Laplace equations (equations defined in the exterior of a ball), we ignore the term r^n and write $R_n(r) = r^{-n-1}$. Therefore, for problems inside a ball, the solution of (6.19) has the following form

$$u(r,\theta) = \sum_{n=0}^{\infty} U_n r^n P_n(\cos\theta),$$

where U_n are determined through the boundary condition and the following inner product

$$U_n = \frac{\langle f, P_n\rangle_{\sin\theta}}{a^n \|P_n\|^2} = \frac{2n+1}{2a^n}\int_0^{\pi} f(\theta)\, P_n(\cos\theta)\sin(\theta)\, d\theta.$$

Example 6.9. Consider the following problem

$$\begin{cases} \Delta u = 0 \\ u(1,\theta,\phi) = 1 \end{cases}.$$

Let us write the series solution as follows

$$u(r,\theta,\phi) = \sum_{n=0}^{\infty} U_n r^n P_n(\cos\theta).$$

The boundary condition at $r=1$ gives

$$1 = \sum_{n=0}^{\infty} U_n P_n(\cos\theta),$$

and since $P_0 = 1$, we derive $U_n = 0$ for $n \geq 1$ and $U_0 = 1$ and thus $u = 1$ is the solution to the problem. This solution confirms the maximum principle for harmonic functions.

Example 6.10. We solve the following Laplace equation

$$\begin{cases} \Delta u = 0 \\ u(1,\theta,\phi) = \cos(\theta) \end{cases}.$$

Since the boundary condition is independent of ϕ then we take $u = u(r,\theta)$ and write

$$u(r,\theta) = \sum_{n=0}^{\infty} U_n r^n P_n(\cos\theta).$$

Applying the boundary condition results to

$$\cos\theta = \sum_{n=0}^{\infty} U_n P_n(\cos\theta) \Rightarrow U_n = \begin{cases} 1 & n=1 \\ 0 & n\neq 1 \end{cases},$$

and finally $u(r,\theta)=r\cos(\theta)$.

6.3.3 Associated Legendre polynomials

Now we relax the restriction that f is axially symmetric. Let us write the separated solution as $u=R(r)\,Y(\theta,\phi)$ and substitute that into (6.18) to obtain

$$\frac{r^2 R''+2r R'}{R}+\frac{1}{Y}\Delta_{(\theta,\phi)}Y=0, \tag{6.26}$$

where $\Delta_{(\theta,\phi)}Y$ stands for

$$\Delta_{(\theta,\phi)}Y=\frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta Y)+\frac{1}{\sin^2\theta}\partial_{\phi\phi}Y. \tag{6.27}$$

The above equation implies that $\frac{1}{Y}\Delta_{(\theta,\phi)}Y$ is a constant. Let us denote this constant again by λ_θ. Therefore

$$\Delta_{(\theta,\phi)}Y=-\lambda_\theta Y. \tag{6.28}$$

To solve the above problem, we write $Y(\theta,\phi)=\Theta(\theta)\Phi(\phi)$ and obtain

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=-\lambda\sin^2\theta. \tag{6.29}$$

This equation implies in turn that $\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}$ is constant. Let us denote this constant by λ_ϕ, that is, $\Phi''=-\lambda_\phi\Phi$. Notice that $\Phi(\phi)$ must satisfy the periodicity condition

$$\Phi(-\pi)=\Phi(\pi), \Phi'(-\pi)=\Phi'(\pi).$$

Regarding this periodicity condition, we derive the following list of solutions for Φ

$$\Phi_m(\phi)\in\{1,\cos(m\phi),\sin(m\phi)\}_{m=1}^{\infty},$$

and $\lambda_\phi=m^2$. Substituting this constants into (6.29) gives

$$\frac{d}{d\theta}(\sin(\theta)\,\Theta')-\frac{m^2}{\sin(\theta)}\Theta=-\lambda_\theta\sin(\theta)\,\Theta.$$

Observe that the derived equation is similar to a Legendre equation except the extra term $\frac{m^2}{\sin(\theta)}\Theta$. Taking $x=\cos(\theta)$, will transforms the equation into the following one which is called the *associated* Legendre equation:

$$\frac{d}{dx}[(1-x^2)\,\Theta']-\frac{m^2}{1-x^2}\Theta=-\lambda_\theta\,\Theta. \tag{6.30}$$

The admissible values for λ_θ are again $\lambda_\theta=n(n+1)$, $n\geq 0$ to guarantee that the solution remains bounded at $x=-1,1$.

Proposition 6.2. *If $\lambda_\theta=n(n+1)$, then the solution of (6.30) are*

$$P_{n,m}(x)=(1-x^2)^{\frac{m}{2}}\,P_n^{(m)}(x), \quad m=0,\dots,n,$$

where $P_n^{(m)}(x) = \frac{d^m}{dx^m} P_n(x)$.

Proof. Note that we have

$$(1-x^2)P_n'' - 2x P_n' = -n(n+1)P_n.$$

Differentiating the above equation m-times leads to the following one

$$(1-x^2)P_n^{(m+2)} - 2x(m+1)P_n^{(m+1)} = [-n(n+1) + m(m+1)]\, P_n^{(m)}. \tag{6.31}$$

Now for the function $P_{n,m}(x) = (1-x^2)^{\frac{m}{2}}\, P_n^{(m)}(x)$, one needs to show

$$\frac{d}{dx}\left[(1-x^2)\,\frac{d}{dx}\,P_{n,m}\right] - \frac{m^2}{1-x^2}P_{n,m} = -n(n+1)\,P_{n,m}. \tag{6.32}$$

A simple calculation shows

$$\frac{d}{dx}\left[(1-x^2)\,\frac{d}{dx}\,P_{n,m}\right] = (1-x^2)^{\frac{m}{2}}\{(1-x^2)P_n^{(m+2)} - 2(m+1)x\,P_n^{(m+1)}\} + \\ + \left(\frac{m^2}{1-x^2} - m(m+1)\right)P_{n,m}.$$

Now, by the relation (6.31), we have

$$(1-x^2)^{m/2}\{(1-x^2)P_n^{(m+2)} - 2(m+1)x\,P_n^{(m+1)}\} = [-n(n+1) + m(m+1)]\, P_{n,m}.$$

This implies in turn

$$\frac{d}{dx}\left[(1-x^2)\,\frac{d}{dx}\,P_{n,m}\right] = \left[-n(n+1) + m(m+1) + \left(\frac{m^2}{1-x^2} - m(m+1)\right)\right]P_{n,m} + \\ = \left[-n(n+1) + \frac{m^2}{1-x^2}\right]P_n,$$

that completes the proof. □

Since $P_n(x)$ is a polynomial of order n, $P_{n,m}(x)$ are zero for $m > n$. It is evident also that $P_{n,m}(1) = 0$ for all $n > 0$ and $P_{n,0}(x) = P_n(x)$.

Proposition 6.3. *The function* $P_{n,m}(\cos\theta)$ *are orthogonal in the following sense*

$$\int_0^\pi P_{n,m}(\cos\theta)\, P_{n',m}(\cos\theta)\sin(\theta)\, d\theta = 0, n \neq m,$$

and each list of the following class

$$\{P_n(\cos\theta)\}_{n=0}^\infty, \{P_{n,1}(\cos\theta)\}_{n=1}^\infty, \{P_{n,2}(\cos\theta)\}_{n=2}^\infty, \cdots,$$

is a basis for smooth functions $f(\theta)$ *defined for* $\theta \in [0,\pi]$.

The figure (6.1), shows the convergence of a series in terms of the functions in $\{P_{n,2}(x)\}_{n=2}^N$ to the functions $f(x) = x^2$, $-1 < x < 1$ for $N = 4, 10$.

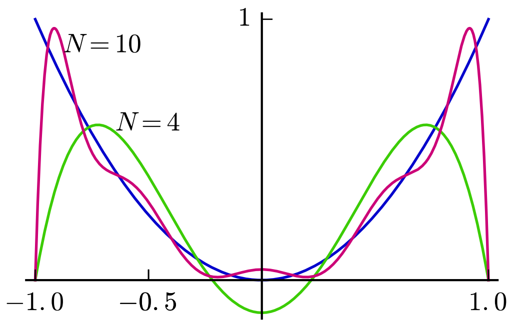


Figure 6.1.

The table (6.2) shows some associated LEGENDRE functions $P_{n,m}(x)$.

m	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
0	1	$P_1(x)$	$P_2(x)$	$P_3(x)$	$P_4(x)$
1	0	$\sqrt{1-x^2}$	$3x\sqrt{1-x^2}$	$\frac{3}{2}\sqrt{1-x^2}(5x^2-1)$	$\frac{5}{2}x\sqrt{1-x^2}(7x^2-3)$
2	0	0	$3(1-x^2)$	$15x(1-x^2)$	$\frac{15}{2}(1-x^2)(7x^2-1)$
3	0	0	0	$15(1-x^2)^{3/2}$	$105x(1-x^2)^{3/2}$
4	0	0	0	0	$105(1-x^2)^2$

Table 6.2.

Now back to (6.26), the equation for R is

$$\frac{r^2 R'' + 2r R'}{R} - n(n+1) = 0,$$

which is a CAUCHY-EULER equation with the solution $\{r^n, r^{-n-1}\}$. Solving an equation inside a ball, we reject the solution r^{-n-1} and keep r^n, that is, $R_n(r) = r^n$. Finally, the solution u in this case is written in the series form as

$$u(r,\theta,\phi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} r^n P_{n,m}(\cos\theta)\,(A_{nm}\cos(m\phi) + B_{nm}\sin(m\phi)).$$

The constants A_{nm}, B_{nm} are determined by the aid of the boundary condition.

Example 6.11. Let us solve the equation in the exterior of the unit ball

$$\begin{cases} \Delta u = 0 \\ u(1,\theta,\phi) = \sin(\phi) \end{cases}.$$

According to the boundary condition, we write the solution in the series form as

$$u = \sum_{n=1}^{\infty} U_n r^{-n-1} P_{n,1}(\cos\theta)\sin\phi.$$

Applying the boundary condition, we obtain

$$\sin\phi = \sum_{n=1}^{\infty} U_n P_{n,1}(\cos\theta)\sin\phi,$$

and thus

$$1=\sum_{n=1}^{\infty} U_n P_{n,1}(\cos\theta).$$

The above equation determines constants U_n as

$$U_n=\frac{1}{\|P_{n,1}\|^2}\int_0^{\pi} P_{n,1}(\cos\theta)\sin(\theta)\,d\theta=\frac{1}{\|P_{n,1}\|^2}\int_{-1}^{1} P_{n,1}(x)\,dx.$$

6.3.4 Eigenvalue problem

In order to solve linear problems in a ball or on a sphere, one needs to solve the following eigenvalue problem

$$\begin{cases} \Delta u=-\lambda u \\ u(a,\theta,\phi)=0 \end{cases}.$$

The separation of variable $u(r,\theta,\phi)=R(r)\,Y(\theta,\phi)$ leads to the eigenvalue problem

$$\frac{1}{R}\frac{d}{dr}(r^2 R')+\frac{1}{Y}\Delta_{(\theta,\phi)}Y=-\lambda r^2. \tag{6.33}$$

Thus $\frac{1}{Y}\Delta_{(\theta,\phi)}Y$ is $\lambda_\theta=n(n+1)$ (otherwise the solution goes unbounded), and hence

$$r^2 R''+2r R'+(\lambda r^2-n(n+1))R=0. \tag{6.34}$$

The equation (6.34) is called the *spherical* Bessel equation (because of the term $n(n+1)$ instead of n^2 and $2r$ instead of r). Since $\lambda>0$, and we take $x=\sqrt{\lambda}\,r$ to transform (6.34) into the following standard form

$$x^2y''+2xy'+(x^2-n(n+1))y=0. \tag{6.35}$$

The bounded solution to the above equation is called j-*spherical* Bessel functions and is denoted by $j_n(x)$.

Proposition 6.4. *The function* $j_n(x)=\sqrt{\frac{\pi}{2x}}\,J_{n+\frac{1}{2}}(x)$ *is the bounded solution to the equation* (6.35).

The proof is left as an exercise to the reader. Applying the zero boundary condition, determines λ as

$$j_n(\sqrt{\lambda}\,a)=0\Rightarrow\lambda_{np}=\frac{z^2_{\left(n+\frac{1}{2}\right)p}}{a^2}.$$

We denote the zeroth of $j_n(x)$ by ζ_{np}, that is , $j_n(\zeta_{np})=0$. It is simply seen (and it is left a an exercise to the reader again) that the following orthogonality condition holds for spherical Bessel functions

$$\int_0^a j_{np}(r)\,j_{nq}(r)\,r^2\,dr=0,\, p\neq q,$$

where $j_{np}(r)$ stands for the functions $j_n(\zeta_{np}r\,/\,a)$. The table (6.3) shows the zeros of the function $j_n(x)$.

p	ζ_{0p}	ζ_{1p}	ζ_{2p}	ζ_{3p}
1	3.1415	4.4934	6.7635	6.9880
2	6.2832	7.7252	9.0950	10.4171
3	9.4248	10.9041	12.3230	13.6980
4	12.5664	14.0662	15.5146	16.9236
5	15.7080	17.2208	18.6890	20.1218
6	18.8496	20.3713	21.8539	23.3042
7	21.9911	23.5195	25.0129	26.4768
8	25.1327	26.6661	28.1678	29.6426
9	28.2743	29.8116	31.3204	32.8037
10	31.4159	32.9564	34.4705	35.9614

Table 6.3.

The above discussion implies that the eigenfunctions of Δ on a ball are:

$$\varphi_{nmp}(r,\theta,\phi) \in \{j_{np}(r)P_{n,m}(\cos\theta)\cos(m\phi), j_{np}(r)P_{n,m}(\cos\theta)\sin(m\phi)\}. \tag{6.36}$$

In addition, we have $\langle\varphi_{nmp}, \varphi_{n'm'p'}\rangle = 0$ if at least one of indices are not equal. Note also that

$$\Delta\varphi_{nmp} = -\frac{{\zeta_{np}}^2}{a^2}\,\varphi_{nmp},$$

and in particular for $m=0$ we have

$$\Delta\{j_{np}(r)\,P_n(\cos\theta)\} = -\frac{\zeta_{np}^2}{a^2} j_{np}(r)\,P_n(\cos\theta).$$

Remark 6.5. The above discussion gives also the eigenvalues and eigenfunctions of $\Delta_{(\theta,\phi)}$ on S^2. In fact, the solution to the problem

$$\Delta_{(\theta,\phi)}Y = -\lambda Y,$$

are

$$Y_{n,m}(\theta,\phi) \in \{P_{n,m}(\cos\theta)\cos(m\phi), P_{n,m}(\cos\theta)\sin(m\phi)\}. \tag{6.37}$$

Notice that S^2 has no boundary, and thus $\Delta_{(\theta,\phi)}$ is symmetric operator on $C^\infty(S^2)$. Furthermore, if $f(\theta,\phi)$ is a smooth function defined on S^2, one can represent it as a series in terms of functions in $Y_{nm}(\theta,\phi)$. The appropriate inner product between two functions $f(\theta,\phi)$ and $g(\theta,\phi)$ is

$$\langle f, g\rangle = \int_{-\pi}^{\pi}\int_0^{\pi} f(r,\theta)\,g(r,\theta)\sin\theta\, d\theta\, d\phi.$$

6.3.5 Linear problems on the shell

Let us solve the Poisson equation $\Delta u = f(\theta,\phi)$ on the shell of a ball. Since the solution should be 2π-periodic with respect to ϕ, the solution should satisfied the following conditions

$$u(\theta,-\pi) = u(\theta,\pi), \partial_\phi(\theta,-\pi) = \partial_\phi(\theta,\pi).$$

But, according to the divergence theorem, we have

$$\iint_S \Delta u = \oint_{\mathrm{bnd}(S)} \nabla u \cdot \hat{n}\, dA = 0,$$

(since $\mathrm{bnd}(S) = \emptyset$) and thus the equation $\Delta u = f(\theta, \phi)$ has a solution only if

$$\iint_S f(\theta, \phi)\, dA = 0.$$

Example 6.12. Let us solve the equation $\Delta u = \phi$ on the unit sphere. Since the function ϕ is odd in $(-\pi, \pi)$, we write the solution as

$$u(\theta, \phi) = \sum_{n=1}^{\infty} \sum_{m=1}^{n} U_{nm} P_{n,m}(\cos\theta) \sin(m\phi).$$

Substituting this into the equation and using the relation

$$\Delta_{(\theta,\phi)}\{P_{n,m}(\cos\theta)\sin(m\phi)\} = -n(n+1)P_{n,m}(\cos\theta)\sin(m\phi),$$

we obtain

$$\sum_{n=1}^{\infty} \sum_{m=1}^{n} -n(n+1)U_{nm}P_{n,m}(\cos\theta)\sin(m\phi) = \phi.$$

The above equation determines U_{nm} as

$$U_{nm} = \frac{-1}{\pi n(n+1)\|P_{n,m}\|^2} \int_{-\pi}^{\pi}\int_0^{\pi} \phi P_{n,m}(\cos\theta)\sin(m\phi)\sin(\theta)d\theta d\phi =$$
$$\frac{-1}{\pi n(n+1)\|P_{n,m}\|^2}\left(\int_{-1}^{1} P_{n,m}(x)dx\right)\left(\int_{-\pi}^{\pi} \phi\sin(m\phi)\, d\phi\right) =$$
$$= \frac{2(-1)^m}{mn(n+1)\|P_{n,m}\|^2}\left(\int_{-1}^{1} P_{n,m}(x)dx\right).$$

Example 6.13. Let us solve the following heat problem on the unit sphere

$$\begin{cases} \partial_t u = \Delta_{(\theta,\phi)} u - \cos(\theta) \\ u(0, \theta, \phi) = \sin(\theta) \end{cases}.$$

We first solve the Poisson equation $\Delta u = \cos\theta$ on the sphere. The solution is independent of ϕ and thus we write

$$u(\theta, \phi) = \sum_{n=0}^{\infty} U_n\, P_n(\cos\theta).$$

Substitution into the Poisson equation gives

$$\Delta_{(\theta,\phi)} u = \sum_{n=0} -n(n+1)U_n\, P_n(\cos\theta) = \cos(\theta).$$

Hence $n=1$ and $U_n=\frac{-1}{2}$ and consequently $u(\theta,\phi)=-\frac{1}{2}\cos(\theta)$. Now, we write the solution to the original equation as

$$u(t,\theta,\phi)=-\frac{1}{2}\cos(\theta)+w(t,\theta,\phi),$$

where w satisfies the homogeneous heat problem

$$\begin{cases}\partial_t w=\Delta_{(\theta,\phi)}w\\ w(0,\theta,\phi)=\sin(\theta)+\frac{1}{2}\cos(\theta)\end{cases}.$$

We represent the solution as the series

$$w(t,\theta,\phi)=\sum_{n=1}^{\infty} W_n(t)\,P_n(\cos\theta).$$

The equation for $W_n(t)$ is $W_n'=-n(n+1)W_n$ and therefore

$$u(t,\theta,\phi)=\sum_{n=1}^{\infty} W_n e^{-n(n+1)t}\,P_n(\cos\theta).$$

Applying the initial condition gives

$$\sin(\theta)+\frac{1}{2}\cos(\theta)=\sum_{n=1}^{\infty} W_n\,P_n(\cos\theta).$$

This implies $W_1=\frac{1}{2}$, $W_n=0$ for $n=2k+1$, $k=0,1,2,\cdots$. For n even, we have

$$W_n=\frac{1}{\|P_n\|^2}\int_{-1}^{1} P_n(x)\,\sqrt{1-x^2}\,dx.$$

Finally, we write the solution as

$$u(t,r,\theta)=-\frac{1}{2}\cos(\theta)+\frac{1}{2}\cos(\theta)\,e^{-2t}+\sum_{n:\text{even}} W_n e^{-n(n+1)t}\,P_n(\cos\theta).$$

Example 6.14. We solve the following wave problem on a unit sphere

$$\begin{cases}\partial_{tt}u=c^2\Delta_{(\theta,\phi)}u\\ u(0,\theta,\phi)=0\\ \partial_t u(0,\theta,\phi)=\cos(\theta)\sin(2\phi)\end{cases}.$$

According to the initial condition, we write the solution as

$$u(t,\theta,\phi)=\sum_{n=2}^{\infty} U_n(t)P_{n,2}(\cos\theta)\sin(2\phi).$$

Substituting the series into the equation gives

$$\sum_{n=2}^{\infty} U_n''(t)P_{n,2}(\cos\theta)\sin(2\phi)=\sum_{n=2}^{\infty} -c^2\,n(n+1)U_n(t)P_{n,2}(\cos\theta)\sin(2\phi).$$

Hence the following equation holds for $U_n(t)$

$$U_n''(t) = -c^2\, n(n+1) U_n(t).$$

Thus u has the series form

$$u(t,\theta,\phi) = \sum_{n=2}^{\infty} [A_n \cos(c\sqrt{\lambda_n}\, t) + B_n \sin(c\sqrt{\lambda_n}\, t)] P_{n,2}(\cos\theta) \sin(2\phi).$$

To determine A_n, B_n, we use the initial data. It is simply seen that $A_n = 0$ and

$$\cos(\theta) = \sum_{n=2}^{\infty} c\sqrt{\lambda_n}\, B_n\, P_{n,2}(\cos\theta),$$

that gives

$$B_n = \frac{1}{\|P_{n,2}\|^2} \int_{-1}^{1} x\, P_{n,2}(x)\, dx.$$

6.3.6 Linear Problems on a solid ball

Example 6.15. Let us solve the following POISSON equation on a unit ball

$$\begin{cases} \Delta u = \cos(\theta) \\ u(1,\theta,\phi) = 0 \end{cases}.$$

Note that the forcing term is independent of ϕ. Since $P_n(\cos\theta) = \cos(\theta)$, we write the solution as the series

$$u(r,\theta,\phi) = \sum_{p=1}^{\infty} U_p\, j_{1p}(r) \cos(\theta).$$

Regarding the relation

$$\Delta(\, j_{1p}(r) \cos\theta) = -\zeta_{1p}^2\, j_{1p}(r) \cos(\theta),$$

we obtain

$$\cos(\theta) \sum_{p=1}^{\infty} -\zeta_{1p} U_p\, j_{1p}(r) = \cos(\theta).$$

Constants U_p are determined by the formula

$$U_p = \frac{-1}{\zeta_{1p}^2 \|j_{1p}\|^2} \int_0^1 j_{1p}(r)\, r^2\, dr.$$

An alternative method is to write the solution as $u(r,\theta,\phi) = U(r)\cos(\theta)$ and substitute that into the equation to obtain

$$r^2 U'' + 2r U' - 2U = r^2.$$

The above CAUCHY-EULER equation is solved for $U(r) = Ar + \frac{1}{4} r^2$. The boundary condition $u(1,\theta,\phi) = 0$ implies $A = -\frac{1}{4}$ and finally

$$u(r,\theta,\phi) = \frac{1}{4} r(r-1)\cos(\theta).$$

The series solution of the equation is just the series form of the later closed form solution.

Example 6.16. Let us solve the following heat problem

$$\begin{cases} \partial_t u = \Delta u - e^{-t} \\ u(t, 1, \theta, \phi) = \cos(\theta) \\ u(0, r, \theta, \phi) = 0 \end{cases}.$$

Since the boundary condition is nonzero, we write the solution as

$$u(t, r, \theta, \phi) = v(r, \theta, \phi) + w(t, r, \theta, \phi).$$

The equation for v is the following LAPLACE equation

$$\begin{cases} \Delta v = 0 \\ v(1, \theta, \phi) = \cos(\theta) \end{cases},$$

with the solution $v = r\cos(\theta)$. The equation for w is

$$\begin{cases} \partial_t w = \Delta w - e^{-t} \\ w(t, 1, \theta, \phi) = 0 \\ w(0, r, \theta, \phi) = -r\cos(\theta) \end{cases}.$$

Note that the initial condition for w is independent of ϕ and also the forcing term is independent of ϕ and θ. According to this information, we write the solution as the series in terms of $\{j_{1p}(r)P_1(\cos\theta)\}$ as

$$w(t, r, \theta, \phi) = \sum_{p=1}^{\infty} W_p(t)\, j_{1p}(r)\cos(\theta).$$

Remember that $P_1(\cos\theta) = \cos(\theta)$. Substituting w into the equation gives

$$\sum_{p=1}^{\infty} W_p'(t)\, j_{1p}(r)\cos(\theta) = \sum_{p=1}^{\infty} -\zeta_{1p}^2 W_p(t)\, j_{1p}(r)\cos(\theta) - e^{-t}.$$

In order to merge summations and find the desired equation for W_p', we have to expand the source term e^{-t} in terms of eigenfunctions $\{j_{1p}(r)\, P_1(\cos\theta)\}$. But this is impossible because the source term is independent of θ. For this, we split the equation of w into two sub-problems

$$(a)\begin{cases} \partial_t w = \Delta w - e^{-t} \\ w(t, 1, \theta, \phi) = 0 \\ w(0, r, \theta, \phi) = 0 \end{cases} + (b)\begin{cases} \partial_t w = \Delta w \\ w(t, 1, \theta, \phi) = 0 \\ w(0, r, \theta, \phi) = -r\cos(\theta) \end{cases}.$$

The solution to the sub-problem (b) is derived as

$$w_b(t, r, \theta, \phi) = \sum_{p=1}^{\infty} W_p\, e^{-\zeta_{1p}^2 t}\, j_{1p}(r)\cos(\theta),$$

where W_p are determined as

$$W_p = \frac{-1}{\|j_{1p}\|^2}\int_0^1 j_{1p}(r)\, r^3\, \mathrm{d}r.$$

To solve the sub-problem (a), we expand the source terms as

$$e^{-t} = e^{-t}\sum_{p=1}^{\infty} R_p\, j_{0p}(r),$$

where

$$R_p = \frac{1}{\|j_{0p}\|^2}\int_0^1 j_{0p}(r)\, r^2\, \mathrm{d}r.$$

We write the solution to (a) as

$$w_a(t,r,\theta,\phi) = \sum_{p=1}^{\infty} \frac{R_p}{\zeta_{0p}^2 - 1}\left(e^{-t} - e^{-\zeta_{0p}^2 t}\right) j_{0p}(r).$$

Finally, the solution is

$$u(t,r,\theta,\phi) = r\cos(\theta) + \sum_{p=1}^{\infty} \frac{R_p}{\zeta_{0p}^2 - 1}\left(e^{-t} - e^{-\zeta_{0p}^2 t}\right) j_{0p}(r) + \sum_{p=1}^{\infty} W_p\, e^{-\zeta_{1p}^2 t} j_{1p}(r)\cos(\theta).$$

Example 6.17. Let us solve the following wave equation in a unit solid ball

$$\begin{cases} \partial_{tt} u = \Delta u - 10r\cos(\theta) \\ u(1,\theta,\phi) = 0 \\ u(0,r,\theta,\phi) = r(r^2-1)\cos(\theta), \partial_t u(0,r,\theta,\phi) = \sin(\theta)\sin(\phi) \end{cases}.$$

Write the solution as

$$u(t,r,\theta,\phi) = v(r,\theta,\phi) + w(t,r,\theta,\phi),$$

where v satisfies the POISSON equation

$$\begin{cases} \Delta v = 10r\cos(\theta) \\ v(1,\theta,\phi) = 0 \end{cases}.$$

If we write $v = V(r)\cos\theta$, then $V(r)$ satisfies the ordinary equation

$$\begin{cases} r^2 V'' + 2rV' - 2V = 10r^3 \\ V(1) = 0, V(0)\text{: bounded} \end{cases}.$$

This is a CAUCHY-EULER equation and is solved for $V(r) = \frac{1}{c^2} r(r^2 - 1)$. Therefore, we obtain

$$u(t,r,\theta,\phi) = r(r^2-1)\cos(\theta) + w(t,r,\theta,\phi),$$

where w satisfies the equation

$$\begin{cases} \partial_{tt} w = \Delta w \\ w(1,\theta,\phi) = 0 \\ w(0,r,\theta,\phi) = 0, \partial_t w(0,r,\theta,\phi) = \sin(\theta)\sin(\phi) \end{cases}.$$

Note that $P_1^1(\cos\theta) = \sin\theta$ and thus the solution of the above equation is written in the series form as

$$w(t, r, \theta, \phi) = \sum_{p=1}^{\infty} W_p(t)\, j_{1p}(r)\, P_{1,1}(\cos\theta)\sin(\phi),$$

where $W_p(t)$ satisfies the equation $W_p'' = -\zeta_{1p}^2 W_p$. Since $W_p(0) = 0$, the solution $W(t)$ is

$$W_p(t) = B_p \sin(\zeta_{1p}\, t).$$

Therefore, w is

$$w(t, r, \theta, \phi) = \sum_{p=1}^{\infty} B_p \sin(\zeta_{1p}\, t)\, j_{1p}(r)\, P_{1,1}(\cos\theta)\sin(\phi),$$

where constants B_p satisfy the relation

$$1 = \sum_{p=1}^{\infty} \zeta_{1p}\, B_p\, j_{1p}(r),$$

and

$$B_p = \frac{1}{\zeta_{1p} \|j_{1p}\|^2} \int_0^1 j_{1p}(r)\, r^2\, \mathrm{d}r.$$

Finally, the series solution to the problem is the series

$$u(t, r, \theta, \phi) = r(r^2 - 1)\cos(\theta) + \sum_{p=1}^{\infty} B_p \sin(\zeta_{1p}\, t)\, j_{1p}(r)\sin(\theta)\sin(\phi).$$

Problems

Problem 6.27. Show that $P_n^m(x)$ are orthogonal in the following sense

a)

$$\int_{-1}^{1} P_{n,m}(x)\, P_{n',m}(x)\, dx = 0, \qquad n \neq n'.$$

b)

$$\int_{-1}^{1} \frac{1}{1 - x^2} P_{n,m}(x)\, P_{n,m'}(x)\, dx = 0, \qquad m \neq m'.$$

Problem 6.28. Find series of the following functions in terms of LEGENDRE polynomials $\{P_n(x)\}$ for $x \in [-1, 1]$ and draw the first 5 terms of each series.

a) $f(x) = \sqrt{1 - x^2}$

b) $f(x) = \sin(\pi x)$

Problem 6.29. Write the series representation of the following functions in terms of functions in the list $\{P_n^1(x)\}$ for $-1 \leq x \leq 1$. Draw each series for first 5 terms.

a) $f(x) = 1 - x^2$

b) $f(x) = x^3$

Problem 6.30. Repeat the above problem in terms of $\{P_{n,2}(x)\}$.

Problem 6.31. Solve the LAPLACE equation $\Delta u = 0$ inside the unit ball with the following boundary conditions

a) $u(1, \theta, \phi) = 3\cos^2\theta - 1$

b) $u(1, \theta, \phi) = \theta + 1$

c) $u(1, \theta, \phi) = \sin(3\phi) + \cos(2\theta)$

Problem 6.32. Solve the LAPLACE equation $\Delta u = 0$ in the exterior of the unit ball with the following boundary conditions

a) $u(1, \theta, \phi) = P_n(\cos\theta)$

b) $u(1, \theta, \phi) = \phi$

Problem 6.33. Solve the LAPLACE equation $\Delta u = 0$ defined on the space $1 < r < 2$ in $\mathbb{R}^3$ (the space between two sphere $r = 1$, $r = 2$) with the following boundary condition:

a) $u(1, \theta, \phi) = -1, \quad u(2, \theta, \phi) = 1.$

b) $u(1, \theta, \phi) = \sin(\phi), \quad u(2, \theta, \phi) = \sin(\phi).$

Problem 6.34. Solve the following LAPLACE equation inside the unit ball

$$\begin{cases} \Delta u = 0 \\ u(1, \theta, \phi) = \sin(\phi) \end{cases}.$$

Use a calculator to find the coefficients of first 4 terms.

Problem 6.35. Verify that the spherical BESSEL functions $j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x)$ satisfies the spherical BESSEL equation

$$x^2 y'' + 2xy' + (x^2 - n(n+1))y = 0,$$

where $J_n(x)$ is the solution to the BESSEL equation

$$x^2\, y'' + xy' + (x^2 - n^2)y = 0.$$

Problem 6.36. Write the series representation of the following functions in terms of $\{j_{0p}(r)\}$ in $0 \le r \le 1$ and plot each

a) $f(r) = 1$

b) $f(r) = r$

c) $f(r) = r^2$.

Problem 6.37. Repeat the above problem for functions in the list $\{j_{1p}(r)\}$.

Problem 6.38. Solve the POISSON equation $\Delta_{(\theta,\phi)} u = \theta$ on the surface of the unit ball. Use a calculator to determine the coefficients of the first 4 terms.

Problem 6.39. Solve the POISSON equation $\Delta_{(\theta,\phi)} u = \sin^2\phi$ on the surface of the unit ball.

Problem 6.40. Explain why the eigenfunction expansion method does not work for the POISSON equation $\Delta_{(\theta,\phi)} u = 1$ defined on the surface of the unit ball. Is there a unique solution to the problem?

Problem 6.41. Solve the following heat problem on the surface of the unit ball

$$\begin{cases} \partial_t u = \Delta_{(\theta,\phi)} u - P_k(\cos\theta) \\ u(0, \theta, \phi) = 0 \end{cases},$$

where $k \ge 1$ is a fixed integer. Draw the temperature of the circle $(1, \pi/4, \phi)$ in time for $n = 2$.

Problem 6.42. Solve the following heat problem on the surface of the unit ball

$$\begin{cases} \partial_t u = \Delta_{(\theta,\phi)} u \\ u(0, \theta, \phi) = P_n(\cos\theta) \end{cases},$$

where $n \ge 0$.

Problem 6.43. Solve the following wave problem on the surface of the unit sphere

$$\begin{cases} \partial_{tt}u = \Delta_{(\theta,\phi)}u \\ u(0,\theta,\phi) = 0, \partial_t u(0,\theta,\phi) = f(\theta,\phi) \end{cases}.$$

Plot the solution at $\theta = \pi/4$, if $f(\theta,\phi) = 0.1\cos\theta$ in terms of time t.

Problem 6.44. Even though the DIRAC delta $\delta(x)$ is not a function in usual sense, let us write a symbolic series of $\delta(x)$ in terms of $P_n(x)$ as

$$\delta(x) = \sum_{n=0}^{\infty} D_n P_n(x).$$

The coefficients D_n are determined by the aid of inner product as

$$D_n = \frac{P_n(0)}{\|P_n\|^2} = \frac{2n+1}{2} P_n(0).$$

a) Plot the series of $\delta(x)$ for 30 terms in the interval $(-1,1)$.

b) Solve the following equation on the surface of the unit ball

$$\begin{cases} \partial_{tt}u = \Delta_{(\theta,\phi)}u - \delta(t-1)\,\delta(\theta-\pi/2) \\ u(0,\theta,\phi) = \partial_t u(0,\theta,\phi) = 0 \end{cases}.$$

Problem 6.45. Consider the following POISSON equation inside the unit ball

$$\begin{cases} \Delta u = 1 \\ u(1,\theta,\phi) = 0 \end{cases}.$$

a) Find a series solution to the problem.

b) Find the solution in closed form.

Problem 6.46. Consider the following POISSON equation inside the unit ball

$$\begin{cases} \Delta u = P_n(\cos\theta) \\ u(1,\theta,\phi) = 0 \end{cases}.$$

a) Find a series solution to the problem.

b) Find the solution in closed form.

Problem 6.47. Solve the following POISSON equation inside the unit ball

$$\begin{cases} \Delta u = \phi \\ u(1,\theta,\phi) = 1 + \cos(\theta) \end{cases}.$$

Problem 6.48. Solve the following heat equation inside the unit ball

$$\begin{cases} \partial_t u = \Delta u - \cos(\theta) \\ u(t,1,\theta,\phi) = \frac{5}{4}\cos(\theta) \\ u(0,r,\theta,\phi) = \frac{r^2}{4}\cos(\theta) \end{cases}.$$

and find first 3 terms of the transient solution.

Problem 6.49. Solve the following heat problem inside the unit ball

$$\begin{cases} \partial_t u = k\,\Delta u - P_k^1(\cos\theta)\sin(\phi) \\ u(t,1,\theta,\phi) = 0 \\ u(0,r,\theta,\phi) = 0 \end{cases},$$

and draw few terms of the series solution at the point $r = 0.5, \theta = \pi/2, \phi = \pi/2$ with respect to time t for $k = 0.001$.

Problem 6.50. (10 points) Let Ω denote the space between the spheres $r = 1$ and $r = 2$.

a) Solve the LAPLACE equation $\Delta u = 0$ in Ω with the boundary conditions

$$u(1, \theta, \phi) = 1 + \cos(\theta), u(2, \theta, \phi) = 2 + 2\cos(\theta)$$

b) Use $P_2^2(\cos\theta)$ and find a closed form solution to the following POISSON equation

$$\begin{cases} \Delta u = 18r \sin^2\theta \sin(2\phi) \\ u(1, \theta, \phi) = 1 + \cos(\theta) \\ u(2, \theta, \phi) = 2 + 2\cos(\theta) \end{cases}.$$

Problem 6.51. Solve the following heat problem inside the unit ball

$$\begin{cases} \partial_t u = 0.001\,\Delta u - \delta(t-1) P_k^1(\cos\theta)\sin(\phi) \\ u(t, 1, \theta, \phi) = 0 \\ u(0, r, \theta, \phi) = 0 \end{cases},$$

and draw few terms of the series solution at the point $r = 0.5, \theta = \pi/2, \phi = \pi/2$ with respect to time t.

Problem 6.52. Solve the following damped wave equation inside the unit ball

$$\begin{cases} \partial_{tt} u + 2\xi \partial_t u = c^2\,\Delta u \\ u(t, 1, \theta, \phi) = 0 \\ u(0, r, \theta, \phi) = 0, \partial_t u(0, r, \theta, \phi) = \phi \end{cases}.$$

Assume that $\xi > 0$ is small enough and the system is in the under-damped mode.

Problem 6.53. Solve the following wave problem inside the unit ball

$$\begin{cases} \partial_{tt} u = c^2\,\Delta u - e^{-t} \\ u(t, 1, \theta, \phi) = 0 \\ u(0, r, \theta, \phi) = r \end{cases}$$

Problem 6.54. Let D be the following section of the unit ball:

$$D = \left\{(r, \theta, \phi), 0 < r < 1, 0 \le \theta \le \pi, 0 \le \phi \le \frac{\pi}{2}\right\}.$$

Solve the following heat equation on D:

$$\begin{cases} \partial_t u = \Delta u \\ u(t, r, \theta, 0) = u(t, r, \theta, \pi/2) = 0 \\ u(t, 1, \theta, \phi) = \cos(\theta)\sin(2\phi) \\ u(0, r, \theta, \phi) = 0 \end{cases}.$$

6.4 LEGENDRE polynomials

The force field generated by a mass m in $\mathbb{R}^3$ is

$$\vec{f} = -\frac{mG}{|r|^2}\, e_r,$$

where e_r is the unit vector in the direction of r, the line connecting a point (x, y, z) to the location of m. P. S. LAPLACE observed that the above force field is the gradient of a scalar function which is equal to

$$V(r) = \frac{mG}{|r|},$$

and moreover, V satisfies the equation $\Delta V = 0$ if $r \neq 0$. The British Mathematician G. GREEN was the first person who called V a *potential function* for $\vec{f}$. For this reason, the LAPLACE equation is sometimes called a *potential equation*. The French mathematician, J. LEGENDRE gave a series representation of the potential generated by a mass which is located at the point $(0, 0, R)$; see Fig.6.2.

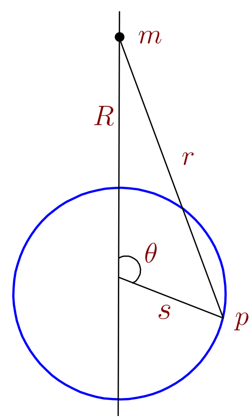


Figure 6.2.

The potential (assuming $m = 1$ and omitting the constant G) is equal to

$$V = \frac{1}{|r|} = \frac{1}{\sqrt{R^2 + s^2 - 2sR\cos(\theta)}}.$$

Substituting $\rho = \frac{s}{R}$ and $x = \cos(\theta)$, we reach

$$V = \frac{1}{R} \frac{1}{\sqrt{1 + \rho^2 - 2\rho x}}.$$

Let us write the series of the right hand side of $V(\rho)$ in terms of ρ as follows (omitting $\frac{1}{R}$)

$$\frac{1}{\sqrt{1 + \rho^2 - 2\rho x}} = \sum_{n=0}^{\infty} P_n(x)\, \rho^n, \tag{6.38}$$

for some functions $P_n(x)$. From (6.38) it is immediately obtained that P_n is of the following form

$$P_n(x) = \frac{1}{n!} \frac{\partial^n}{\partial \rho^n} (1 + \rho^2 - 2\rho x)^{-1/2}|_{\rho=0}.$$

On the other hand, it turns out that $P_n(x)$ satisfies a differential equation called the LEGENDRE equation. In fact, since V is a potential, we have $\Delta V = 0$ or equivalently $\Delta(P_n(\cos\theta)\rho^n) = 0$. We have

$$\Delta(P_n(\cos\theta)\,\rho^n) = \frac{n(n+1)}{R^n} s^{n-2}\,P_n(\cos\theta) + \frac{s^{n-2}}{R^n\sin(\theta)}\frac{d}{d\theta}(\sin\theta\,P_n(\cos\theta)) = 0.$$

Simplifying the above relation gives the following equation:

$$n(n+1)\,P_n + \frac{1}{\sin(\theta)}\frac{d}{d\theta}(\sin\theta\,P_n) = 0.$$

For $x = \cos\theta$, the above equation reads

$$\frac{d}{dx}\left((1-x^2)\frac{dP_n}{dx}\right) + n(n+1)\,P_n = 0. \tag{6.39}$$

From (6.38) we have

$$\frac{1}{1+\rho^2-2\rho x} = \sum_{n=0}^{\infty} P_n^2(x)\,\rho^{2n} + \sum_{n\neq m} P_n(x)\,P_m(x)\,\rho^{n+m},$$

and then

$$\int_{-1}^{1}\frac{dx}{1+\rho^2-2\rho x} = \sum_{n=1}^{\infty}\rho^{2n}\int_{-1}^{1}P_n^2(x)\,\mathrm{d}x.$$

But

$$\int_{-1}^{1}\frac{dx}{1+\rho^2-2\rho x} = \frac{1}{\rho}\log\left|\frac{\rho+1}{\rho-1}\right| = \sum_{n=0}^{\infty}\frac{2}{2n+1}\rho^{2n}.$$

and we obtain

$$\int_{-1}^{1}P_n^2(x)\,\mathrm{d}x = \frac{2}{2n+1}.$$

Again, from (6.38) we obtain for $x=1$

$$\frac{1}{\sqrt{1+\rho^2-2\rho}} = \frac{1}{1-\rho} = \sum_{n=0}^{\infty}\rho^n,$$

and thus $P_n(1) = 1$ for all $n \geq 0$. For $x = -1$ we have

$$\frac{1}{\sqrt{1+\rho^2+2\rho}} = \frac{1}{1+\rho} = \sum_{n=0}^{\infty}(-1)^n\rho^n$$

and therefore $P_n(-1) = (-1)^n$. Also changing ρ to $-\rho$ gives

$$\sum_{n=0}^{\infty}P_n(x)\,(-\rho)^n = \frac{1}{\sqrt{1+\rho^2+2\rho x}} = \frac{1}{\sqrt{1+\rho^2-2\rho\,(-x)}} = \sum_{n=0}^{\infty}P_n(-x)\,\rho^n$$

Therefore $P_n(-x) = (-1)^n\,P_n(x)$. The list $\{P_n(x)\}_{n=0}^{\infty}$ is an orthogonal basis for smooth functions in $[-1,1]$. Therefore if $f(x)$, $x\in[-1,1]$ is a smooth function, it can be represented by the series

$$f(x) \sim \sum_{n=0}^{\infty} f_n\,P_n(x),$$

where the coefficients f_n are calculated by the aid of inner product as

$$f_n = \frac{2n+1}{2} \int_{-1}^{1} f(x)\, P_n(x)\, \mathrm{d}x. \tag{6.40}$$

It is simply seen that the above series can be written in an integral form. In fact, we have

$$f(x) \sim \sum_{n=0}^{\infty} \left(\frac{2n+1}{2} \int_{-1}^{1} f(y)\, P_n(y)\, dy \right) P_n(x) = \int_{-1}^{1} f(y) \left(\sum_{n=0}^{\infty} \frac{2n+1}{2} P_n(x) P_n(y) \right) \mathrm{d}y$$

and if we denote the *kernel* $K(x, y)$ as

$$K(x, y) = \sum_{n=0}^{\infty} \frac{2n+1}{2} P_n(x) P_n(y), \tag{6.41}$$

then the function $f(x)$ can be represented as the integral

$$f(x) \sim \int_{-1}^{1} K(x, y) f(y)\, \mathrm{d}y. \tag{6.42}$$