# Chapter 5
# 2D Linear Second-Order Equations

In this chapter, we delve into the realm of linear partial differential equations in 2D domains. Our focus lies on rectangular and disk-shaped domains, as they provide us with the opportunity to employ the spatial variable separation technique. Within this framework, we explore a range of partial differential equations, namely: 1) Laplace equations, 2) Poisson equations, 3) heat equations, and 4) wave equations.

Let $\Omega$ be an open bounded set in $\mathbb{R}^2$ with a smooth or piecewise smooth boundary $\mathrm{bnd}(\Omega)$. We begin by considering the Laplace equation, which takes the general form $\Delta u(x, y) = 0$ for all $(x, y) \in \Omega$. This equation is defined for all smooth functions $u$ that satisfy a prescribed boundary condition, typically of the form $\alpha u + \beta \frac{\partial u}{\partial n} = \gamma$ on the boundary $\mathrm{bnd}(\Omega)$. Here, $\alpha, \beta$, and $\gamma$ are constants.

The Laplace equation is a special case of the Poisson equation, which has the general form $-\Delta u = f$, where $f$ is a function that is independent of $u$ and depends only on the spatial variables $(x, y)$. In this equation, we seek a solution u that satisfies the prescribed boundary conditions.

Moving on, we explore the steady state of a heat equation, which has the form $u\_t = k\Delta u + f$. Here, $u_t$ represents the partial derivative of $u$ with respect to time, $k$ is a constant representing thermal diffusivity, $\Delta u$ is the Laplacian of $u$, and $f$ represents an external source of heat.

Finally, we consider the wave equation or damped wave equation, given by $u_{tt} + 2\xi u_t = c^2 \Delta u + f$. In this equation, $u_{tt}$ represents the second partial derivative of $u$ with respect to time, $\xi$ is a non-negative constant known as the damping factor, $c$ represents the wave speed, $\Delta u$ is the Laplacian of $u$, and $f$ represents an external forcing term.

Throughout this chapter, we aim to study the properties of these equations, investigate their solutions, and understand their physical interpretations in the context of 2D domains.

## 5.1 Overview of the eigenfunction expansion method

The eigenfunction expansion method for problems defined on bounded domains in the plane $\mathbb{R}^2$ is an extension of the approach used for 1D problems on bounded domains in $\mathbb{R}$. To demonstrate this method, let's consider the heat problem:

$$\begin{cases} u_t = k\Delta u + h & \text{on } \Omega, t > 0 \\ \alpha u + \beta \frac{\partial u}{\partial n} = 0 & \text{on } \mathrm{bnd}(\Omega), t > 0 \end{cases},$$

where $\Omega$ is a bounded domain in $\mathbb{R}^2$ with a piecewise smooth boundary $\text{bnd}(\Omega)$, and $h$ is a heat source term on $\Omega$. Here, $\frac{\partial}{\partial n}$ represents the directional derivative along the outward normal vector $n$ on the boundary of $\Omega$. To solve this problem, we first form the associated eigenvalue problem:

$$\begin{cases} \Delta\phi = -\lambda\phi & \text{on } \Omega \\ \alpha\phi + \beta\frac{\partial\phi}{\partial n} = 0 & \text{on } \text{bnd}(\Omega) \end{cases}. \tag{5.1}$$

In this problem, $\lambda$ is referred to as the eigenvalue, and $\phi$ represents the corresponding eigenfunction. The following theorem describes the properties of the eigenvalue problem:

**Theorem 5.1.** *The eigenvalue problem possesses the following properties:*

a) *all eigenvalues are real, and they increases positively unbounded.*

b) *Eigenfunctions associated with different eigenvalues are orthogonal in the following sense:*

$$\langle\phi_{n,m}, \phi_{n',m'}\rangle := \iint_\Omega \phi_{n,m}\,\phi_{n',m'}\,dS = 0,$$

*if* $(n,m) \neq (n',m')$, *where* $dS$ *is the differential area in* $\Omega$.

c) *The set of eigenfunctions* $\{\phi_{n,m}\}$ *is an orthogonal basis for smooth functions defined in* $\Omega$. *In other words, a continuously differentiable function* $f$ *defined on* $\Omega$ *can be represented as:*

$$f = \sum_{n,m} \frac{\langle f, \phi_{n,m}\rangle}{\|\phi_{n,m}\|^2}\phi_{n,m}.$$

**Problem 5.1.** Prove that if $\beta \neq 0$ in the eigenvalue problem (5.1), the eigenvalues are greater than or equal to $\frac{\alpha}{\beta}$. If $\beta = 0$, then all eigenvalues are strictly positive.

**Problem 5.2.** Let $\phi_{n,m}$ and $\phi_{n',m'}$ be eigenfunctions associated with $\lambda_{n,m} \neq \lambda_{n',m'}$. Show the following orthogonality property

$$\langle\phi_{n,m}, \phi_{n',m'}\rangle := \iint_\Omega \phi_{n,m}\,\phi_{n',m'}\,dS = 0.$$

Hint:To prove, write down

$$\begin{cases} \Delta\phi_{n,m} = -\lambda_{n,m}\,\phi_{n,m} \\ \Delta\phi_{n',m'} = -\lambda_{n',m'}\,\phi_{n',m'} \end{cases}.$$

Multiply the first equation and integrate over $\Omega$:

$$\iint_\Omega \Delta\phi_{n,m}\,\phi_{n',m'}dS = -\lambda_{n,m}\iint_\Omega \phi_{n,m}\,\phi_{n',m'}dS$$

Show that $\Delta$ is symmetric over functions satisfying the given boundary condition, i.e.,

$$\iint_\Omega \Delta\phi_{n,m}\,\phi_{n',m'}dS = \iint_\Omega \Delta\phi_{n',m'}\,\phi_{n,m}\,dS.$$

Since the set of eigenfunctions $\{\phi_{n,m}\}$ forms a basis for continuously differentiable functions on $\Omega$, we can express the desired solution to the given heat problem as the series:

$$u = \sum_{n,m} U_{n,m}(t)\,\phi_{n,m}, \tag{5.2}$$

where $U_{n,m}(t)$ are undetermined coefficient functions that need to be determined to ensure that the series is a valid solution to the given problem. Substituting this series into the heat equation yields:

$$\sum_{n,m} U'_{n,m}(t)\, \phi_{n,m}(x,y) = \sum_{n,m} k U_{n,m}(t) \Delta\left[\phi_{n,m}(x,y)\right] + h$$

Using the equality $\Delta[\phi_{n,m}] = -\lambda_{n,m}\phi_{n,m}$, we arrive at the following equation:

$$\sum_{n,m} \left[U'_{n,m}(t) + k\lambda_{n,m} U_{n,m}(t)\right] \phi_{n,m}(x,y) = h.$$

To proceed and determine $U_{n,m}$, we express the source term $h$ in terms of the eigenfunction basis $\{\phi_{n,m}\}$ as well:

$$h = \sum_{n,m} H_{n,m}(t)\, \phi_{n,m},$$

where the coefficients $H_{n,m}(t)$ are determined by the inner production as:

$$H_{n,m} = \frac{\langle h, \phi_{n,m}\rangle}{\|\phi_{n,m}\|^2}.$$

Substituting this into the series equation leads to the following ordinary differential equation for $U_{n,m}(t)$:

$$U'_{n,m} + k\lambda_{n,m} U_{n,m} = H_{n,m}(t).$$

The solution to this equation determines $U_{n,m}(t)$ and consequently, the solution $u(x,y,t)$ in the series form (5.2). It is important to note that this method provides us with a series solution to the problem, and in most cases, this will be an infinite series similar to the 1D problems studied in the previous chapter.

In particular, if the heat equation is homogeneous ($h$ is identically zero), the coefficient functions $U_{n,m}$ will be exponential functions:

$$U_{n,m}(t) = C_{n,m}\, e^{-k\lambda_{n,m}t},$$

which leads to the general series solution for the homogeneous heat equation:

$$u = \sum_{n,m} C_{n,m}\, e^{-k\lambda_{n,m}t}\, \phi_{n,m}(p),$$

where $p \in \Omega$, and $C_{n,m}$ are constants that can be determined through the initial condition $u(p,0) = f$. The series solution must converge to the given initial condition as $t \to 0$, which results in:

$$f = \sum_{n,m} C_{n,m}\, \phi_{n,m}(p).$$

Then, the constants $C_{n,m}$ for the homogeneous equation ($h = 0$) can be determined as:

$$C_{n,m} = \frac{\langle f, \phi_{n,m}\rangle}{\|\phi_{n,m}\|^2}.$$

The outlined method for solving the heat equation can be extended to homogeneous or non-homogeneous general second-order partial differential equations. Consider the following equation:

$$\begin{cases} a(t)u_{tt} + b(t)\,u_t = k\,\Delta u + h & \text{on}\,\Omega \\ \alpha u + \beta \frac{\partial u}{\partial n} = 0 & \text{on}\,\mathrm{bnd}(\Omega) \end{cases}.$$

This equation is general enough to encompass heat, wave, damped wave, and Poisson equations. To find the solution, we represent the desired solution $u$ as a series in terms of the eigenfunctions $\{\phi_{n,m}\}$:

$$u = \sum_{n,m} U_{n,m}(t)\,\phi_{n,m},$$

where $U_{n,m}(t)$ are to be determined to satisfy the given partial differential equation. By substituting the series into the equation, we obtain:

$$\sum_{n,m} [a(t)\,U''_{n,m}(t) + b(t)\,U'_{n,m}(t) + \lambda_{n,m}\,U_{n,m}(t)]\,\phi_{n,m} = h.$$

Expanding the source term $h$ in terms of the basis $\{\phi_{n,m}\}$, we arrive at the following equation for $U_{n,m}$:

$$a(t)\,U''_{n,m}(t) + b(t)\,U'_{n,m}(t) + \lambda_{n,m}\,U_{n,m}(t) = H_{n,m}(t),$$

where $H_{n,m}(t)$ represents the coefficients of the expansion of $h$ in terms of $\{\phi_{n,m}\}$:

$$h = \sum_{n,m} H_{n,m}(t)\,\phi_{n,m}.$$

Solving this resulting second-order ordinary differential equation provides the general series solution for the given partial differential equation.

## 5.2 Rectangular domains and Cartesian coordinate

### 5.2.1 Eigenfunctions of Δ in rectangles

We will now apply the outlined method to solve the eigenvalue problem in the Cartesian coordinate system $(x, y)$, where $\Omega$ is a rectangular domain defined as $\Omega$: $(x_0, x_1) \times (y_0, y_1)$. The eigenvalue problem can be stated as follows:

$$\begin{cases} \Delta\phi(x,y) = -\lambda\,\phi(x,y) & \text{on}\,\Omega \\ \alpha\,\phi + \beta\,\frac{\partial \phi}{\partial n} = 0 & \text{on}\,\mathrm{bnd}(\Omega) \end{cases}.$$

The rectangular geometry of the domain allows us to use the method of separation of variables. We assume that the eigenfunction $\phi(x, y)$ can be expressed as the product of two separate functions, $X(x)$ and $Y(y)$, i.e., $\phi(x,\ y) = X(x)Y(y)$. Substituting this into the eigenvalue problem, we obtain the following equation:

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda.$$

It is evident that this equation holds only if $\frac{X''(x)}{X(x)}$ and $\frac{Y''(y)}{Y(y)}$ are constants. Let us denote $\frac{X''(x)}{X(x)} = -\eta$ and $\frac{Y''(y)}{Y(y)} = -\mu$. We proceed to determine the associated boundary conditions for the functions $X(x)$ and $Y(y)$.

The general boundary condition $\alpha\phi + \beta\frac{\partial\phi}{\partial n} = 0$, where $n$ is the unit normal vector on the boundary of $\Omega$, leads to the following boundary conditions for $X(x)$:

$$\begin{cases} \alpha_1 X(x_0) - \beta_1 X'(x_0) = 0 \\ \alpha_2 X(x_1) + \beta_2 X'(x_1) = 0 \end{cases}.$$

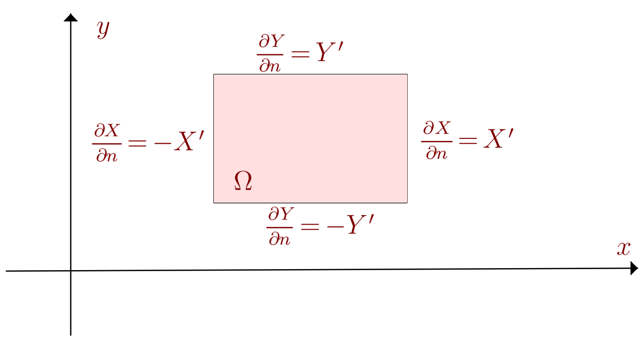
Similarly, for $Y(y)$, the boundary conditions are given by:

$$\begin{cases} \alpha_3 Y(y_0) - \beta_3 Y'(y_0) = 0 \\ \alpha_4 Y(y_1) + \beta_4 Y'(y_1) = 0 \end{cases}.$$

In these equations, the values of $\alpha$ and $\beta$ depend on the specific problem and the boundary conditions imposed on $\Omega$. Note that we also made use of the following relations:

$$\left.\frac{\partial X}{\partial n}\right|_{x=x_0} = -X'(x_0), \left.\frac{\partial X}{\partial n}\right|_{x=x_1} = X'(x_1),$$

and similar ones for $Y$. The figure below depicts these relations:




By applying the separation of variables technique, we can split the original eigenvalue problem for the Laplacian operator $\Delta$ into two separate eigenvalue problems: For the $X(x)$ function:

$$\begin{cases} X'' = -\eta X \\ \alpha_1 X(x_0) - \beta_1 X'(x_0) = 0 \\ \alpha_2 X(x_1) + \beta_2 X'(x_1) = 0 \end{cases},$$

and for the $Y(y)$ function:

$$\begin{cases} Y'' = -\mu Y \\ \alpha_3 Y(y_0) - \beta_3 Y'(y_0) = 0 \\ \alpha_4 Y(y_1) + \beta_4 Y'(y_1) = 0 \end{cases}.$$

Consequently, the original eigenvalue $\lambda$ can be determined by the relation $\lambda = \eta + \mu$, which arises from the separation of variables approach.

**Example 5.1.** Below are the eigenfunctions and eigenvalues of the Laplace operator $\Delta$ for some well-known boundary conditions on the domain $\Omega$: $(0,a)\times(0,b)$:

a) Dirichlet boundary condition:

$$\phi(0,y)=\phi(a,y)=\phi(x,0)=\phi(x,b)=0$$

Eigenfunctions:

$$\phi_{n,m}(x,y)=\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right).$$

Eigenvalues:

$$\lambda_{n,m}=\frac{n^2}{a^2}\pi^2+\frac{m^2}{b^2}\pi^2$$

for $n=1,2,\cdots$, $m=1,2,\cdots$.

b) Neumann boundary condition:

$$\phi_x(0,y)=\phi_x(a,y)=\phi_y(x,0)=\phi_y(x,b)=0.$$

Eigenfunctions:

$$\phi_{n,m}(x,y)=\cos\left(\frac{n\pi}{a}x\right)\cos\left(\frac{m\pi}{b}y\right).$$

Eigenvalues:

$$\lambda_{n,m}=\frac{n^2}{a^2}\pi^2+\frac{m^2}{b^2}\pi^2,$$

for $n=0,1,2,\cdots$, $m=0,1,2,\cdots$.

c) Dirichlet in $x$-direction, and Neumann in $y$-direction

$$\phi(0,y)=\phi(a,y)=\phi_y(x,0)=\phi_y(x,b)=0.$$

Eigenfunctions:

$$\phi_{n,m}(x,y)=\sin\left(\frac{n\pi}{a}x\right)\cos\left(\frac{m\pi}{b}y\right).$$

Eigenvalues:

$$\lambda_{n,m}=\frac{n^2}{a^2}\pi^2+\frac{m^2}{b^2}\pi^2,$$

for $n=1,2,\cdots$, $m=0,1,2,\cdots$.

d) Neumann in $x$-direction and Dirichlet in $x$-direction:

$$\phi_x(0,y)=\phi_x(a,y)=\phi(x,0)=\phi(x,b)=0.$$

Eigenfunctions:

$$\phi_{n,m}(x,y)=\cos\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right).$$

Eigenvalues:

$$\lambda_{n,m}=\frac{n^2}{a^2}\pi^2+\frac{m^2}{b^2}\pi^2,$$

for $n=0,1,2,\cdots$, $m=1,2,\cdots$.

e) Mixed in $x$ and $y$-directions

$$\phi(0,y)=\phi_x(a,y)=\phi(x,0)=\phi_y(x,b)=0.$$

Eigenfunctions:

$$\phi_{n,m}(x,y)=\sin\left(\frac{(2n-1)\pi}{2a}x\right)\sin\left(\frac{(2m-1)\pi}{2b}y\right).$$

Eigenvalues:

$$\lambda_{n,m} = \frac{(2n-1)^2}{4a^2}\pi^2 + \frac{(2m-1)^2}{4b^2}\pi^2,$$

for $n = 1, 2, \cdots$, $m = 1, 2, \cdots$.

f) Mixed in $x$ and $y$-directions:

$$\phi_x(0, y) = \phi(a, y) = \phi_y(x, 0) = \phi(x, b) = 0.$$

Eigenfunctions:

$$\phi_{n,m}(x, y) = \cos\left(\frac{(2n-1)\pi}{2a}x\right)\cos\left(\frac{(2m-1)\pi}{2b}y\right).$$

Eigenvalues:

$$\lambda_{n,m} = \frac{(2n-1)^2}{4a^2}\pi^2 + \frac{(2m-1)^2}{4b^2}\pi^2,$$

for $n = 1, 2, \cdots$, $m = 1, 2, \cdots$.

**Example 5.2.** Consider a smooth function $f(x, y)$ defined on $(0, 1) \times (0, 1)$. Let's fix $y$ for the function $f(x, y)$ and consider $f$ as a pure function of $x$. We can express this function using the basis $\{\sin(n\pi x)\}$ for $n = 1, 2, \cdots$ as:

$$f(x, y) = \sum_{n=1}^{\infty} F_n(y)\sin(n\pi x),$$

where $F_n(y)$ is given by:

$$F_n(y) = 2\int_0^1 f(x, y)\sin(n\pi x)\, dx.$$

Each function $F_n(y)$ can be represented in the basis $\{\sin(m\pi y)\}$ for $m = 1, 2, \cdots$ as:

$$F_n(y) = \sum_{m=1}^{\infty} c_{n,m}\sin(m\pi y),$$

where $c_{n,m}$ is:

$$c_{n,m} = 2\int_0^1 F_n(y)\sin(m\pi y)\, dy.$$

Substituting the series for $F_n(y)$ into the series for $f(x, y)$, we obtain:

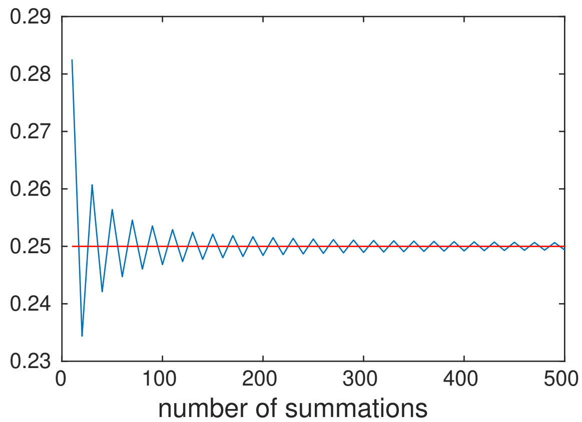
$$f(x, y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{n,m}\sin(m\pi y)\sin(n\pi x).$$

It is important to note that the set $\{\sin(n\pi x)\sin(m\pi y)\}$ for $n = 1, 2, \cdots$ and $m = 1, 2, \cdots$ forms an orthogonal basis for functions defined on the rectangle $\Omega = (0, 1) \times (0, 1)$. Additionally, the functions in this set satisfy the homogeneous Dirichlet boundary condition on $\mathrm{bnd}(\Omega)$.

As a numerical example, let's consider $f(x, y) = xy$ defined on the rectangle $(0, 1) \times (0, 1)$. The series representation of the function in the basis $\{\sin(n\pi x)\sin(m\pi y)\}$ is:

$$xy = \sum_{n,m=1}^{\infty} \frac{4(-1)^{n+m}}{nm\pi^2}\sin(m\pi y)\sin(n\pi x).$$

To observe the convergence of the double series to $f(x, y)$ for $(x, y)$ in the open rectangle, let's consider the point $x = \frac{1}{2}, y = \frac{1}{2}$. The figure below depicts the convergence of the series in terms of the number of iterations:




As we observe, the double series converges to the true value very slowly. The reason for this becomes apparent. If we represent the function as a single series:

$$xy = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n\pi} y \sin(n\pi x),$$

then the convergence at $y = \frac{1}{2}$ requires fewer summations. The single series converges with an error of less than $0.0033$ after $50$ summations. Achieving the same accuracy for each function $F\_n(y) = \frac{-2(-1)^n}{n\pi} y$ requires a similar number of iterations. Consequently, the double summation needs a significantly larger number of iterations to provide the same level of convergence accuracy.

**Exercise 5.1.** Let $\Omega$ be the square $(-\pi, \pi) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ in the $xy$-plane. Find the eigenvalues and eigenfunctions of the eigenvalue problem $\Delta\phi = -\lambda\phi$ subject to the following eigenvalue problem:

$$\begin{cases} \phi(-\pi, y) = \phi(\pi, y) \\ \phi_x(-\pi, y) = \phi_x(\pi, y) \\ \phi\left(x, -\frac{\pi}{2}\right) = \phi\left(x, \frac{\pi}{2}\right) \\ \phi_y\left(x, -\frac{\pi}{2}\right) = \phi_y\left(x, \frac{\pi}{2}\right) \end{cases}.$$

Use these eigenfunctions to represent the function $f(x, y) = y\sin(2x)$ defined on $\Omega$. What is the geometry of the domain with the given boundary conditions of the eigenfunctions $\phi_{n,m}$?

**Exercise 5.2.** Let $\Omega$ be the unit square $(-1, 1) \times (0, 1)$.

a) Find the eigenfunctions and eigenvalues of the eigenvalue problem $\Delta\phi = -\lambda\phi$ on $\Omega$ with the following boundary conditions

$$\begin{cases} u(-1, y) = u(1, y) \\ u_x(-1, y) = u_x(1, y) \\ u(x, 0) = u(x, 1) = 0 \end{cases}.$$

b) Show that these eigenfunctions are orthogonal with respect to the weight function $\sigma = 1$. Use these eigenfunctions to approximate the function $f(x, y) = x\sin(\pi y)$. How many terms do you need to use to obtain an approximation with accuracy of $10^{-3}$?

**Exercise 5.3.** Let $\Omega$ be the square $(0, \pi) \times (0, \pi)$. Find the eigenfunctions and eigenvalues of the eigenvalue problem $\Delta\phi = -\lambda\phi$ satisfying the following boundary conditions

$$\begin{cases} u(0, y) = 0, u(1, y) + u_x(1, y) = 0 \\ u(x, 0) = u_y(x, 1) = 0 \end{cases}.$$

Here you may need to find the roots of the trigonometric function $\sin(z) + z\cos(z)$. Use an online application to determine a few roots of the function.

### 5.2.2 Heat, wave and Poisson equations on rectangles

In this section, we will utilize the eigenfunction expansion method to express the solution of heat, wave, and Poisson equations as series expansions using the eigenfunctions of the operator $\Delta$, subject to specific boundary conditions on rectangular domains. To illustrate this method, we will solve several examples and provide detailed explanations.

#### Poisson's equation

The solution to Poisson's equation $-\Delta u = f$ defined on the rectangle $(x_0, x_1) \times (y_1, y_2)$ in the $xy$-plane can be expressed as a series in terms of the eigenfunction basis $\{\phi_{n,m}(x, y)\}$ as:

$$u(x, y) = \sum_{n,m} U_{n,m}\, \phi_{n,m}(x, y),$$

where $U_{n,m}$ are undetermined constants. Substituting this series into the Poisson equation yields:

$$\sum_{n,m} \lambda_{n,m} U_{n,m}\, \phi_{n,m}(x, y) = f(x, y),$$

and then $U_{n,m}$ can be determined using the inner product as:

$$U_{n,m} = \frac{\langle f, \phi_{n,m}\rangle}{\lambda_{n,m}\, \|\phi_{n,m}\|^2},$$

provided that $\lambda_{n,m} \neq 0$.

**Example 5.3.** As a numerical example, let $\Omega = (0, \pi) \times (0, \pi)$, and consider the Poisson equation:

$$\begin{cases} -\Delta u = f & \text{on } \Omega \\ u = 0 & \text{on bnd}(\Omega) \end{cases},$$

where $f(x, y) = y\sin(x)$. The eigenfunctions of $\Delta$ on $\Omega$ subject to the homogeneous Dirichlet boundary condition are given by:

$$\phi_{n,m}(x, y) \in \{\sin(nx)\sin(my)\}.$$

Since $\langle f, \phi_{n,m}\rangle = 0$ for $n \neq 1$, we can seek a solution in the single series form:

$$u(x, y) = \sum_{m=1}^{\infty} U_m \sin(my)\sin(x).$$

Substituting this series into the Poisson equation yields:

$$\sum_{m=1}^{\infty} \lambda_{1,m} U_m \sin(my) = y,$$

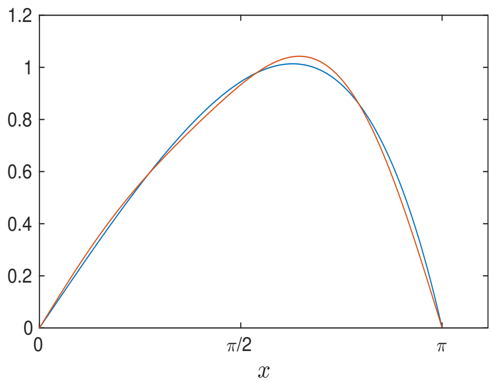
which determines $U_m$ as: $U_m = \frac{2(-1)^m}{\lambda_{1,m} m}$. Therefore, the series solution to the equation is obtained as:

$$u(x, y) = \sin(x) \sum_{m=1}^{\infty} \frac{-2(-1)^m}{m(1 + m^2)} \sin(my).$$

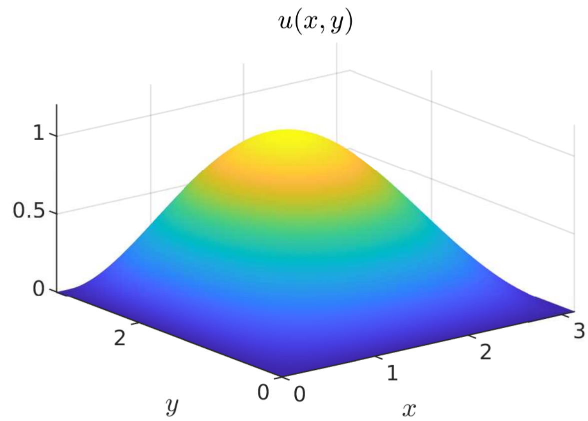
**Exercise 5.4.** The solution obtained above is a series solution to the given Poisson problem. In some cases, it is possible to derive a closed-form solution. For the given problem, consider the solution as $u(x,y) = Y(y)\sin(x)$. Substitute this into the equation and show that $Y(y)$ must be equal to:

$$Y(y) = y - \frac{\sinh(y)}{\sinh(1)}$$

The series representation of $Y(y)$ in terms of the functions $\{\sin(m\pi y)\}$ is as the derived in the solution of the example. The figure below depicts the function $Y(y)$ and its series truncated up to three terms:



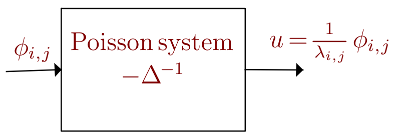

The surface solution is illustrated below:




**Remark 5.1.** Consider the Poisson equation

$$\begin{cases} -\Delta u = \phi_{i,j} & \text{on } \Omega \\ u = 0 & \text{on bnd}(\Omega) \end{cases},$$

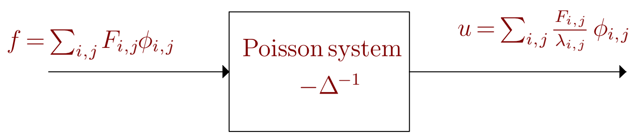
where $\phi_{i,j}$ is the $(i,j)^{\text{th}}$ eigenfunction of $\Delta$ on $\Omega$. This equation can be interpreted in the system interpretation as shown in the following block-diagram:



Since $\phi_{i,j}$ is the eigenfunction of $\Delta$ with the given boundary condition, the response of the system to this input is equal to $u = \frac{1}{\lambda_{i,j}}\phi_{i,j}$. Since the set $\{\phi_{n,m}\}$ is a basis for functions define don $\Omega$, we can use the superposition principle to determine the solution to the equation

$$-\Delta u = f := \sum_{n,m} F_{n,m}\phi_{n,m}.$$

The above equation is equivalent to the following system




**Exercise 5.5.** Let $\Omega$ be the square $(0,\pi)\times(0,\pi)$. Solve the Poisson equation $-\Delta u = y\sin\left(\frac{x}{2}\right)$ on $\Omega$ where $u$ satisfies the following boundary conditions

$$\begin{cases} u(0,y) = u_x(\pi,y) = 0 \\ u(x,0) = u_y(x,\pi) = 0 \end{cases}.$$

**Exercise 5.6.** Let $\Omega$ be the square $(-1,1)\times(-1,1)$. Solve the Poisson equation $-\Delta u = (1+x)\sin(\pi y)$ on $\Omega$ where $u$ satisfies the following boundary conditions

$$\begin{cases} u(0,y) = u(1,y) \\ u_x(0,y) = u_x(1,y) \\ u(x,0) = u(x,1) = 0 \end{cases}.$$

**Exercise 5.7.** Let $\Omega$ be the unit square $(0,1)\times(0,1)$, and consider the Poisson equation $-\Delta u = f$ where $u$ satisfies the following boundary conditions

$$\begin{cases} u_x(0,y) = u_x(1,y) = 0 \\ u_y(x,0) = u_y(x,1) = 0 \end{cases}$$

a) Find the solution to the problem if $f = y\cos(\pi x)$.

b) Show that there is not a solution to the problem if $f = y\sin(\pi x)$. Explain the reason for the non-existence of the solution.

## Heat problems on rectangles

Let's solve the following heat problem on $\Omega$: $(0,1)\times(0,1)$

$$\begin{cases} u_t = \Delta u + h & \text{on } \Omega \\ u = 0 & \text{on bnd}(\Omega) \\ u(x,y,0) = 0 \end{cases}.$$

Since the set $\phi_{n,m} \in \{\sin(n\pi x)\sin(m\pi y)\}$ forms a basis for the functions defined on $\Omega$, we express the solution $u(x,y,t)$ as

$$u = \sum_{n,m=1}^{\infty} U_{n,m}(t)\phi_{n,m},$$

for undetermined coefficients $U_{n,m}(t)$. This series is a valid solution if it satisfies the equation:

$$\sum_{n,m=1}^{\infty} U'_{n,m}(t)\,\phi_{n,m}(x,y) = \sum_{n,m=1}^{\infty} -\lambda_{n,m}U_{n,m}(t)\,\phi_{n,m}(x,y) + h(x,y,t),$$

where $\lambda_{n,m} = (n^2 + m^2)\pi^2$. Moving the second summation to the left side, we can write

$$\sum_{n,m=1}^{\infty} [U'_{n,m}(t) + \lambda_{n,m} U_{n,m}(t)]\, \phi_{n,m}(x, y) = h(x, y, t).$$

To proceed, we use the representation of $h$ as a series in terms of eigenfunctions $\{\phi_{n,m}\}$ as:

$$h(x, y, t) = \sum_{n,m=1}^{\infty} H_{n,m}(t)\, \phi_{n,m}(x, y),$$

where $H_{n,m}(t)$ are determined by the inner product

$$H_{n,m}(t) = 4 \int_0^1 \int_0^1 h(x, y, t)\, \phi_{n,m}(x, y)\, dx\, dy.$$

For instance, let $h = \phi_{1,1}(x, y)$. In this case, we have:

$$H_{n,m}(t) = \begin{cases} 1 & (n, m) = (1, 1) \\ 0 & (n, m) \neq (1, 1) \end{cases}.$$

According to the given initial condition, we can write the initial value problems for $U_{n,m}$ as:

$$\begin{cases} U'_{n,m}(t) + \lambda_{n,m} U_{n,m}(t) = 0 \\ U_{n,m}(0) = 0 \end{cases}, (n, m) \neq (1, 1),$$

and

$$\begin{cases} U'_{n,m}(t) + \lambda_{n,m} U_{n,m}(t) = 1 \\ U_{n,m}(0) = 0 \end{cases}, (n, m) = (1, 1).$$

Solving the above system yields: $U_{n,m}(t) = 0$ for $(n, m) \neq (1, 1)$ and

$$U_{1,1}(t) = \frac{1}{2\pi^2}(1 - e^{-2\pi^2 t}).$$

Finally, we arrive at:

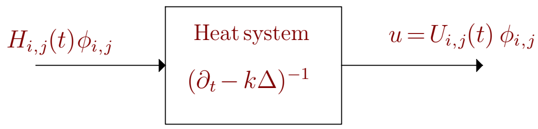
$$u(x, y, t) = \frac{1}{2\pi^2}(1 - e^{-2\pi^2 t}) \sin(\pi x) \sin(\pi y).$$

### System interpretation

Consider the equation

$$\begin{cases} u_t = k\Delta u + H_{i,j}(t)\, \phi_{i,j}(x, y) & \text{on } \Omega \\ u = 0 & \text{on bnd}(\Omega) \\ u(x, y, 0) = 0 \end{cases},$$

where $\phi_{i,j}(x, y)$ is an eigenfunction of $\Delta$ with the given boundary condition. From a system perspective, the heat system is triggered by the source terms $H_{i,j}(t)\, \phi_{i,j}(x, y)$. This can be depicted in the following diagram:




In this diagram, the function $U_{i,j}(t)$ is the solution to the ordinary differential equation

$$U_{i,j}' + k\lambda_{i,j} U_j = H_{i,j}.$$

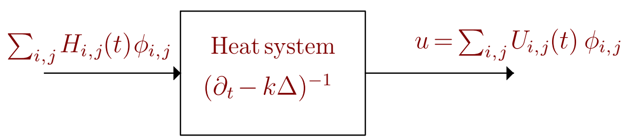
In the above example, the heat system is triggered by the source $\phi_{1,1}(x,y)$, resulting in the response:

$$u_{1,1} = \frac{1 - e^{-\lambda_{1,1} t}}{\lambda_{1,1}} \phi_{1,1}(x,y).$$

Now, if the heat system is triggered by the source term

$$h = \sum_{i,j} H_{i,j}(t)\,\phi_{i,j}(x,y),$$

then by the superposition principle, the response can be written as the summation depicted in the following diagram:




A heat system can be triggered by the initial condition itself. Consider the following problem on $\Omega = (0,1) \times (0,1)$:

$$\begin{cases} u_t = \Delta u & \text{on } \Omega \\ u = 0 & \text{on bnd}(\Omega) \\ u(x,y,0) = \phi_{i,j} \end{cases}, \tag{5.3}$$

where $\phi_{ij} = \sin(i\pi x)\sin(j\pi x)$. It can be shown that the response of the heat system to this initial condition is given by:

$$u_{ij} = e^{-\lambda_{ij} t}\,\phi_{ij}(x,y).$$

Now, let's consider a slightly different problem:

$$\begin{cases} u_t = \Delta u + \delta(t)\,\phi_{ij} & \text{on } \Omega \\ u = 0 & \text{on bnd}(\Omega) \\ u(x,y,0) = 0 \end{cases},$$

where the initial condition appears as an external source multiplied by the Dirac delta function $\delta(t)$. In this case, we seek the solution in the form:

$$u_{ij} = U(t)\,\phi_{ij}(x,y).$$

Substituting this into the equation yields:

$$U' + \lambda_{ij} U = \delta(t).$$

This ordinary differential equation can be solved using the Laplace transform method, resulting in: $U(t) = e^{-\lambda_{ij}t}u(t)$, where $u(t)$ is the Heaviside function defined as:

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}.$$

Thus, the solution $u_{ij}(x, y, t)$ for $t > 0$ is the same as the solution to the problem (5.3), namely $e^{-\lambda_{ij}t}\phi_{ij}(x, y)$.

**Exercise 5.8.** What is the difference between the solution to the problem

$$\begin{cases} u_t = \Delta u + \phi_{ij} & \text{on } \Omega \\ u = 0 & \text{on bnd}(\Omega) \\ u(x, y, 0) = 0 & \end{cases},$$

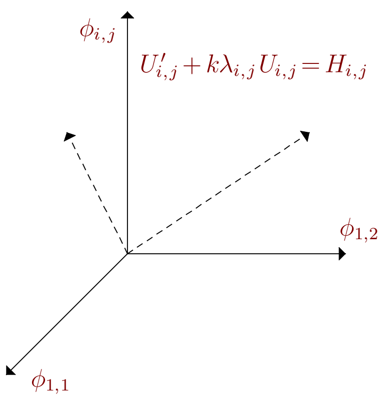
and the solution to the following one

$$\begin{cases} u_t = \Delta u + u(t)\phi_{ij} & \text{on } \Omega \\ u = 0 & \text{on bnd}(\Omega) \\ u(x, y, 0) = 0 & \end{cases},$$

where $u(t)$ is the Heaviside function.

## Geometrical interpretation

From a geometric perspective, we can rephrase the argument in the above remark as follows: each eigenfunction $\phi_{i,j}$ defines a direction in an infinite-dimensional vector space. Along each direction (eigenfunction), the heat partial differential equation reduces to an ordinary differential equation, as depicted in the following diagram:




**Exercise 5.9.** Write down the series solution of the following heat problem defined on $\Omega$: $(0, \pi) \times (0, \pi)$

$$\begin{cases} u_t = \Delta u \\ u(0, y, t) = u(\pi, y, t) = 0 \\ u(x, 0, t) = u_y(x, \pi, t) = 0 \\ u(x, y, 0) = x\sin\left(\frac{y}{2}\right) \end{cases}.$$

**Exercise 5.10.** Write down the series solution of the following heat problem defined on $\Omega$: $(0, \pi) \times (0, \pi)$

$$\begin{cases} u_t = \Delta u + e^{-t} \\ u_x(0, y, t) = u_x(\pi, y, t) = 0 \\ u_y(x, 0, t) = u_y(x, \pi, t) = 0 \\ u(x, y, 0) = 0 \end{cases}.$$

**Exercise 5.11.** Consider the heat problem

$$\begin{cases} u_t = \Delta u + \sin(\pi x)\sin(\pi y) \\ u(0,y,t) = u(1,y,t) = 0 \\ u(x,0,t) = u(x,1,t) = 0 \end{cases}.$$

a) The source terms is independent of time $t$. For this, we can consider the solution as

$$u(x,y,t) = V(x,y) + w(x,y,t).$$

Substituting $u$ into the equation leads to a Poisson equation for $V$.

b) The equation for a $w$ will be a homogeneous equation with the general solution

$$w(x,y,t) = \sum_{n,m} C_{n,m}\, e^{-\lambda_{n,m}t}\, \phi_{n,m}(x,y),$$

where $\phi_{n,m} = \sin(n\pi x)\sin(m\pi y)$. Find the solution of the problem if $u(x,y,0) = 0$. This solution is the same the solution we obtained by employing the eigenfunction expansion method.

**Exercise 5.12.** Consider the heat problem

$$\begin{cases} u_t = \Delta u - y\sin(x) \\ u(0,y,t) = u(\pi,y,t) = 0 \\ u(x,0,t) = u(x,\pi,t) = 0 \\ u(x,y,0) = 0 \end{cases}.$$

a) The source terms is independent of time $t$. For this, we can consider the solution as

$$u(x,y,t) = V(x,y) + w(x,y,t).$$

Substituting $u$ into the equation leads to a Poisson equation for $V$. Find a closed form solution for $V(x,y)$.

b) The equation for a $w$ will be a homogeneous equation. Choose a single series for $w$ based on the form of the source term. Use the initial condition for $u$ and obtain the valued series solution of the problem.

## Wave problem

Let's solve the following wave equation defined on the unit square $\Omega = (0,1)\times(0,1)$:

$$\begin{cases} u_{tt} = c^2\Delta u + h & \text{on } \Omega \\ u = 0 & \text{on bnd}(\Omega) \\ u(x,y,0) = 0 \\ u_t(x,y,0) = 0 \end{cases}.$$

Representing the desired solution $u$ in terms of the eigenfunctions $\{\phi_{n,m}\}$, we reach the following equation:

$$\sum_{n,m=1}^{\infty} [U''_{n,m}(t) + c^2\lambda_{n,m} U_{n,m}]\, \phi_{n,m}(x,y) = h(x,y,t).$$

Using the representation of $h$ as a series in terms of $\phi_{n,m}$ as

$$h = \sum_{n,m=1}^{\infty} H_{n,m}(t)\, \phi_{n,m}(x,y),$$

we arrive at the following equation for the coefficients $U_{n,m}$:

$$U''_{n,m} + c^2 \lambda_{n,m} U_{n,m} = H_{n,m}(t).$$

For example, let $h = \sin(i\pi x)\sin(j\pi y)$. Since $H_{ij}(t) = 1$, and $H_{n,m}(t) = 0$ for $(n,m) \neq (i,j)$, we obtain the following system for $U_{n,m}$:

$$\begin{cases} U''_{ij} + \lambda_{ij} c^2 U_{ij} = 1 \\ U_{ij}(0) = U'_{ij}(0) = 0 \end{cases}, \begin{cases} U''_{n,m} + \lambda_{n,m} c^2 U_{n,m} = 0 \\ U_{n,m}(0) = U'_{n,m}(0) = 0 \end{cases}, (n,m) \neq (i,j).$$

The solution to the second system is $U_{n,m}(t) = 0$, and for the first one is

$$U_{ij}(t) = \frac{1}{\lambda_{ij} c^2}\left[1 - \cos\left(\sqrt{\lambda_{ij}}\, ct\right)\right],$$

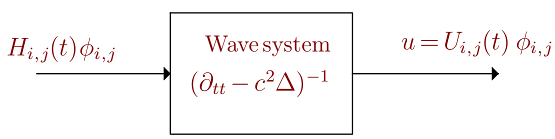
Therefore, the solution to the given wave problem is:

$$u_{ij}(x,y,t) = \frac{1}{\lambda_{ij} c^2}\left[1 - \cos\left(\sqrt{\lambda_{ij}}\, ct\right)\right] \phi_{ij}(x,y).$$

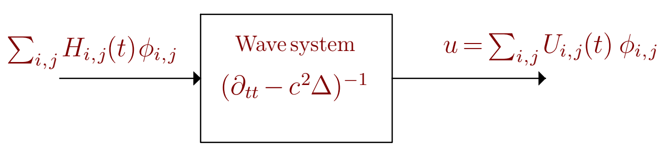
**Remark 5.2.** The logic behind the solution of the above example is clear. From a system perspective, the response of the wave system $(\partial_{tt} - c^2\Delta)^{-1}$ to the input $H_{i,j}(t)\ \phi_{i,j}(x,\ y)$ is given by $U_{i,j}(t)\ \phi_{i,j}(x,\ y)$, where $U_{i,j}(t)$ satisfies the second-order ordinary differential equation:

$$U''_{i,j} + c^2 \lambda_{i,j} U_{i,j} = H_{i,j}(t).$$

Thus, we can represent the solution as shown in the block-diagram below:




For general source terms $h = \sum_{i,j} H_{i,j}\phi_{i,j}$, the response of the system will be smmation as depicted in the diagram below:




An equivalent interpretation, which is geometric, is that the wave partial differential equation reduces to a second-order ordinary differential equation along each direction defined by $\phi_{ij}$ in an infinite-dimensional vector space spanned by the set $\{\phi_{n,m}(x,y)\}$ for $n = 1, 2, \cdots$, $m = 1, 2, \cdots$.

**Exercise 5.13.** Solve the following problem

$$\begin{cases} u_{tt} = c^2\Delta u + \sin(t)\sin(\pi x)\sin(\pi y) & \text{on } \Omega \\ u = 0 & \text{on bnd}(\Omega) \\ u(x,y,0) = 0 & \\ u_t(x,y,0) = 0 & \end{cases},$$

where $\Omega$ is the unit square $\Omega=(0,1)\times(0,1)$.

**Exercise 5.14.** Solve the following wave problem defined on $\Omega=(0,1)\times(0,1)$

$$\begin{cases} u_{tt}=\Delta u-\sin(t)y\sin\left(\frac{\pi}{2}x\right) \\ u(0,y)=u_x(1,y)=0 \\ u(x,0)=u_y(x,1)=0 \\ u(x,y,0)=0 \\ u_t(x,y,0)=0 \end{cases}$$

**Exercise 5.15.** Consider the following wave equation

$$\begin{cases} u_{tt}=\Delta u-y\sin\left(\frac{\pi}{2}x\right) \\ u(0,y)=u_x(1,y)=0 \\ u(x,0)=u_y(x,1)=0 \\ u(x,y,0)=0 \\ u_t(x,y,0)=0 \end{cases}.$$

a) Since the source term is independent of time, we can consider the solution $u$ as follows

$$u(x,y,t)=V(x,y)+w(x,y,t).$$

Substituting $u$ into the equation results to a Poisson equation for $V$. Find the closed form solution for this equation.

b) Write a single summation series solution for $w$ based on the form of the source term. Use the initial condition and determine the solution $u$.

**Exercise 5.16.** Consider the following wave equation $\Omega=(0,1)\times(0,1)$

$$\begin{cases} u_{tt}=\Delta u-\sin(\pi x)\sin(\pi y) \\ u(0,y)=u(1,y)=0 \\ u(x,0)=u(x,1)=0 \\ u(x,y,0)=\sin(\pi x)\sin(\pi y) \\ u_t(x,y,0)=0 \end{cases}.$$

**Exercise 5.17.** Solve the following damped wave equation

$$\begin{cases} u_{tt}+0.2u_t=\Delta u \\ u(0,y)=u(1,y)=0 \\ u(x,0)=u(x,1)=0 \\ u(x,y,0)=\sin(\pi x)\sin(\pi y) \\ u_t(x,y,0)=0 \end{cases}.$$

### 5.2.3 Laplace equation

So far, we have solved partial differential equations with homogeneous boundary conditions. To illustrate the solution to problems with non-homogeneous boundary conditions, let's consider the following heat problem:

$$\begin{cases} u_t=\Delta u & \text{on } \Omega \\ u=f & \text{on } \operatorname{bnd}(\Omega) \\ u(x,y,0)=0 \end{cases}.$$

Since the source term in the heat equation is the time-independent boundary condition $f$, we can express the solution $u$ as:

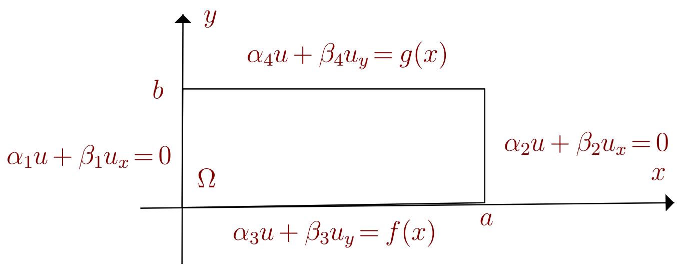
$$u(x, y, t) = V(x, y) + w(x, y, t).$$

By substituting $u$ into the equation, we obtain the Laplace equation:

$$\begin{cases} \Delta V = 0 & \text{on } \Omega \\ V = f & \text{on } \mathrm{bnd}(\Omega) \end{cases}.$$

In this section, our focus is on solving Laplace's equation.

Let $\Omega$ be the rectangle $(0, a) \times (0, b)$. Consider the Laplace equation $\Delta u = 0$ subject to the boundary condition shown below




Using the separation $u(x, y) = X(x)\,Y(y)$, we arrive at the equation:

$$\frac{X''}{X} + \frac{Y''}{Y} = 0.$$

Since $u$ satisfies the homogeneous boundary condition in the $x$-direction as:

$$\begin{cases} \alpha_1 u + \beta_1 u_x = 0 \\ \alpha_2 u + \beta_2 u_x = 0 \end{cases},$$

we arrive at the following eigenvalue problem for $X(x)$:

$$\begin{cases} X'' = -\lambda X \\ \alpha_1 X(0) + \beta_1 X'(0) = 0 \\ \alpha_2 X(a) + \beta_2 X'(a) = 0 \end{cases}.$$

**Problem 5.3.** Prove that eigenvalue $\lambda$ of the problem are non-negative.

Let this eigenvalue problem to be solved for eigenfunctions $\phi_n(x)$ and eigenvalues $\lambda_n$ for $n = 1, 2, \cdots$. The equation for $Y$ satisfies the second-order ODE:

$$Y'' - \lambda_n Y = 0.$$

If $\lambda_0 = 0$ is an eigenvalue, the is solved for $Y_0(y) = A_0 + B_0\,y$. For $n \geq 1$, the equation is solved for the following functions:

$$Y_n(y) = A_n \cosh(\sqrt{\lambda_n}\, y) + B_n \sinh(\sqrt{\lambda_n}\, y).$$

The general solution of the Laplace equation is then obtained as:

$$u(x,y) = (A_0 + B_0 y)\,\phi_1(x) + \sum_{n=1}^{\infty} [A_n \cosh(\sqrt{\lambda_n}\, y) + B_n \sinh(\sqrt{\lambda_n}\, y)]\,\phi_n(x).$$

The parameters $A_n$, $B_n$ are determined by applying the boundary conditions at $y=0$ and $y=b$.

**Example 5.4.** Let's solve the following Laplace's equation on the rectangular domain $\Omega$: $(0,1)\times(0,1)$:

$$\begin{cases} \Delta u = 0 \\ u(0,y) = u(1,y) = 0 \\ u(x,0) = \sin(2\pi x), u(x,1) = -\sin(2\pi x) \end{cases}.$$

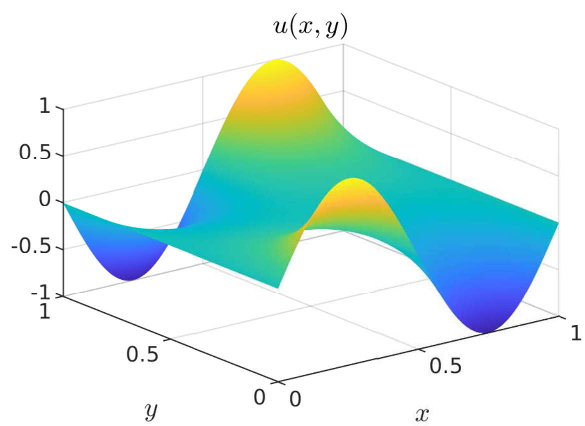
Since the boundary condition is Dirichlet homogeneous in the $x$-direction, the eigenfunctions corresponding to this condition are given by $\phi_n(x) = \sin(n\pi x)$, with associated eigenvalues $\lambda_n = n^2\pi^2$. We can express the general series solution as:

$$u(x,y) = \sum_{n=1}^{\infty} [A_n \cosh(n\pi y) + B_n \sinh(n\pi y)] \sin(n\pi x).$$

To determine the coefficients $A_n$ and $B_n$, we apply the boundary conditions at $y=0$ and $y=1$. From $u(x,0)=\sin(\pi x)$, we find that $A_2=1$, and $A_n=0$ for all $n\neq 2$. From $u(x,1)=-\sin(2\pi x)$, we obtain: $B_2 = -\frac{1+\cosh(2\pi)}{\sinh(2\pi)}$, and $B_n=0$ for $n\neq 1$. Thus, we obtain

$$u(x,y) = \frac{\sinh(2\pi(1-y)) - \sinh(2\pi y)}{\sinh(2\pi)} \sin(2\pi x).$$

The figure below illustrate the surface of $u(x,y)$:




**Example 5.5.** Let's solve the Laplace equation $\Delta u=0$ on the unit square $(0,1)\times(0,1)$ with the following boundary conditions:

$$\begin{cases} u(0,y) = -1, u(1,y) = 1 \\ u(x,0) + u_y(x,0) = 0, u_y(x,1) = 0 \end{cases}.$$

Since the problem is homogeneous in the $y$-direction, we first solve the following eigenvalue problem for $\phi(y)$:

$$\begin{cases} \phi'' = -\lambda\phi \\ \phi(0) + \phi'(0) = 0 \\ \phi'(1) = 0 \end{cases}.$$

It can be shown that the eigenvalues of the problem are strictly positive and satisfy the equation:

$$\sqrt{\lambda}\sin(\sqrt{\lambda}) + \cos(\sqrt{\lambda}) = 0.$$

Accordingly, the eigenfunctions are obtained as:

$$\phi_n(y) = \sin(\sqrt{\lambda_n}\, y) - \sqrt{\lambda_n}\cos(\sqrt{\lambda_n}\, y)$$

The general series solution to the problem is:

$$u(x,y) = \sum_{n=1}^{\infty} \left[A_n \cosh(\sqrt{\lambda_n}\, x) + B_n \sinh(\sqrt{\lambda_n}\, x)\right] \phi_n(y).$$

To determine the coefficients $A_n$ and $B_n$, we apply the boundary conditions. At $x=0$, we obtain:

$$-1 = \sum_{n=1}^{\infty} A_n\, \phi_n(y),$$

which gives

$$A_n = \frac{-1}{\|\phi_n\|^2}\int_0^1 \phi_n(y)\, dy = \frac{-1}{\sqrt{\lambda_n}\, \|\phi_n\|^2}$$

At $x=1$, we reach the equation

$$1 = \sum_{n=1}^{\infty} \left[\frac{-1}{\sqrt{\lambda_n}\, \|\phi_n\|^2}\cosh(\sqrt{\lambda_n}\,) + B_n \sinh(\sqrt{\lambda_n}\,)\right] \phi_n(y),$$

and thus:

$$B_n = \frac{1 + \cosh(\sqrt{\lambda_n})}{\sqrt{\lambda_n}\, \|\phi_n\|^2 \sinh(\sqrt{\lambda_n})}$$

The final series solution is obtained as:

$$u(x,y) = \sum_{n=1}^{\infty} \left[\frac{-1}{\sqrt{\lambda_n}\, \|\phi_n\|^2}\cosh(\sqrt{\lambda_n}\, x) + \frac{1 + \cosh(\sqrt{\lambda_n})}{\sqrt{\lambda_n}\, \|\phi_n\|^2}\, \frac{\sinh(\sqrt{\lambda_n}\, x)}{\sinh(\sqrt{\lambda_n})}\right] \phi_n(y).$$

**Example 5.6.** Consider the following Laplace equation on $\Omega = (0,1) \times (0,1)$

$$\begin{cases} \Delta u = 0 \\ u(0,y) = 0, u(1,y) = y \\ u(x,0) = 0, u(x,1) = x \end{cases}.$$

Since the boundary conditions are non-homogeneous in both directions, we can split the problem into two sub-problems, each homogeneous in one direction:

$$(1)\begin{cases}\Delta u=0\\ u(0,y)=u(1,y)=0\\ u(x,0)=0, u(x,1)=x\end{cases}, (2)\begin{cases}\Delta u=0\\ u(0,y)=0, u(1,y)=y\\ u(x,0)=u(x,1)=0\end{cases}.$$

The solution to the first problem is given by:

$$u_1(x,y)=\sum_{n=1}^{\infty}\frac{-2(-1)^n}{n\pi\sinh(n\pi)}\sinh(n\pi y)\sin(n\pi x).$$

And the solution to the second sub-problem is:

$$u_2(x,y)=\sum_{n=1}^{\infty}\frac{-2(-1)^n}{n\pi\sinh(n\pi)}\sinh(n\pi x)\sin(n\pi y).$$

The superposition solution to the original equation is

$$u(x,y)=\sum_{n=1}^{\infty}\frac{-2(-1)^n}{n\pi\sinh(n\pi)}[\sinh(n\pi y)\sin(n\pi x)+\sinh(n\pi x)\sin(n\pi y)].$$

On the other hand, it can be verified that the function $u=\mathrm{xy}$ satisfies the given Laplace equation. According to the uniqueness theorem, this closed-form solution and the series solution must be equal, resulting in the equation:

$$xy=\sum_{n=1}^{\infty}\frac{-2(-1)^n}{n\pi\sinh(n\pi)}[\sinh(n\pi y)\sin(n\pi x)+\sinh(n\pi x)\sin(n\pi y)].$$

**Exercise 5.18.** Solve the LAPLACE equation $\Delta u=0$ satisfying boundary conditions given below

a)

$$u(x,0)=-1, u(\pi,y)=0, u(x,\pi)=1, u(0,y)=0$$

b)

$$u(x,0)=1, u(1,y)=1, u(x,\pi)=0, u(0,y)=0$$

c)

$$u(x,0)=-1, u(2\pi,y)=1, u(x,\pi)=-1, u(\pi,y)=1$$

d)

$$u_y(x,0)=0, u(2\pi,y)=1, u_y(x,\pi)=0, u(\pi,y)=1$$

**Exercise 5.19.** Consider the Laplace equation $\Delta u=0$ for function $u(x,y)$ defined on the domain $\Omega$: $(0,1)\times(0,1)$ with the following boundary conditions

$$u(0,y)=0, u(1,y)=\sin\left(\frac{\pi y}{2}\right), u(x,0)=0, u(x,1)=\sin\left(\frac{\pi x}{2}\right)$$

a) Find a series solution to the given equation.

b) The solution derived in part a) is a series form of the closed form solution $u(x,y) = \sin\left(\frac{\pi x}{2}\right)\sin\left(\frac{\pi y}{2}\right)$. verify that this function satisfies the Laplace equation and the given boundary conditions.

c) To make sure that these two solutions are the same, choose some arbitrary point or points inside $\Omega$ and verify that the series and the closed form solution are the same.

**Exercise 5.20.** In the domain $\Omega = (0,1) \times (0,1)$:

a) Solve the LAPLACE equation

$$\begin{cases} \Delta u = 0 \\ u(x,0) = 0, u(0,y) = 0 \\ u_x(1,y) = y, u(x,1) = x \end{cases}.$$

b) Solve the POISSON equation $\Delta u = y$ with the boundary conditions given in part a).

### 5.2.4 Problems

**Problem 5.4.** Let $\Omega$ be the unit square $(0,1) \times (0,1)$. Consider the following Poisson equation on $\Omega$

$$\begin{cases} \Delta u = xy & \text{on } \Omega \\ u = 0 & \text{on bnd}(\Omega) \end{cases}.$$

a) Find the double series solution of the problem in terms of the eigenfunctions of $\Delta$ satisfying the given boundary condition.

b) As we observed above, the convergence of double series is usually slower than single series. Let's consider the solution to the problem as the following single series

$$u(x,y) = \sum_{n=1}^{\infty} U_n(y)\sin(n\pi x).$$

Substitute this series into the Poisson equation as obtain an ordinary differential equation of order $2$ for $U_n(y)$.

c) Repeat the argument for the single series

$$u = \sum_{n=1}^{\infty} U_n(x)\sin(n\pi y).$$

**Problem 5.5.** Let $\Omega$ be the rectangle $(0,1) \times (0,1)$. Consider the Poisson equation

$$\Delta u = x\sin(\pi y)$$

on $\Omega$ where $u$ satisfies the homogeneous Dirichlet boundary condition: $u = 0$ on $\text{bnd}(\Omega)$.

a) To solve the equation and determine $u = u(x,y)$, we express the solution $u$ as the series in terms of the eigenfunctions in the set $\{\sin(n\pi x)\sin(m\pi y)\}$. Note that these functions satisfy the given boundary condition:

$$u(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\sin(n\pi x)\sin(m\pi y).$$

Substitute the series into the differential equation and determine the coefficients $C_{n,m}$. Hint: Note that the solution could be expressed as

$$u = \sin(\pi y)\sum_{n=1}^{\infty} C_n\sin(n\pi x),$$

according to the source terms of the equation: $x\sin(\pi y)$.

b) Try a closed form solution of the form $u(x,y)=U(x)\sin(\pi y)$. Substitute this into the equation and show that $U(x)$ is of the form

$$U(x)=\frac{1}{\pi^2}\left(\frac{\sinh(\pi x)}{\sinh(\pi)}-x\right).$$

c) According to the uniqueness theorem of the problem, the series solution derived in part a) must be equal to the closed form solution derived in part b). Chose an arbitrary point on $\Omega$ and compare two solutions at the chosen point.

**Problem 5.6.** Consider the following Poisson equation on $\Omega=(0,a)\times(0,b)$

$$\begin{cases}\Delta u=f(x,y) & \text{on }\Omega\\ \frac{\partial u}{\partial n}=0 & \text{on bnd}(\Omega)\end{cases}.$$

Show that the problem is solvable only if

$$\iint_\Omega f(x,y)\,dx\,dy=0.$$

**Problem 5.7.** Consider the following Poisson equation

$$\begin{cases}\Delta u=y\sin\left(\frac{3}{2}x\right)\\ u(0,y)=u_x(1,y)=0\\ u(x,0)=u(x,1)=0\end{cases}.$$

Find a series solution to the problem and then find a closed form solution to it.

**Problem 5.8.** Find a series or closed form solution to the following Poisson equation

$$\Delta u=\sin(x)$$

on the domain $\Omega$: $(0,\pi)\times(0,\pi)$ satisfying the following boundary conditions

$$u(0,y)=0, u(\pi,y)=0, u_y(x,0)=0, u_y(x,\pi)=0.$$

**Problem 5.9.** Let $\Omega$ be the rectangle $\Omega$: $(0,1)\times(0,1)$. Solve the following heat equation on $\Omega$:

$$\begin{cases}u_t=k\Delta u & \text{on }\Omega\\ \frac{\partial u}{\partial n}=0 & \text{on bnd}(\Omega)\\ u(x,y,0)=\cos(2\pi x)\cos(\pi y)\end{cases}.$$

**Problem 5.10.** Let $\Omega$ be the rectangle $(0,\pi)\times(0,\pi)$. Consider the following damped wave equation on $\Omega$:

$$\begin{cases}u_{tt}+0.2u_t=c^2\Delta u\\ u_x(0,y,t)=u_x(\pi,y,t)=0\\ u(x,0,t)=u(x,\pi,t)=0\end{cases}.$$

a) Write down the general series solution of the equation in terms of the appropriate eigenfunctions of the operator $\Delta$.

b) Determine the solution of the problem if the initial conditions are given by: $u(x,y,0)=0$, and $u_t(x,y,0)=\cos(x)\sin(2y)$.

**Problem 5.11.** Let $\Omega$ be the rectangle $(0,\pi)\times(0,\pi)$. Consider the heat equation

$$\begin{cases} u_t = \Delta u + \sin\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right) \\ u(0,y,t) = u_x(\pi,y,t) = 0 \\ u(x,0,t) = u_y(x,\pi,t) = 0 \\ u(x,y,0) = 0 \end{cases}.$$

The initial condition of the system is zero, and the dynamic is driven by the source term $h = \sin\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right)$. This source term is independent of time $t$. To solve the equation and determine the solution $u(x, y, t)$, we first write $u$ as

$$u(x,y,t) = V(x,y) + W(x,y,t).$$

a) Substitute $u$ into equation. This leads to a Poisson equation for $V(x,y)$. Determine $V$.

b) Write down the general series solution for $W(x,y,t)$.

c) Apply the initial condition for $u$, and determine the solution to the given problem.

d) An alternative and more straightforward way to solve the given equation is as follows: the source term is an eigenfunction of the operator $\Delta$ with the given boundary condition. The response of the heat system to this input is

$$u(x,y,t) = U(t)\sin\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right).$$

Determine $U(t)$ and verify that the result confirms the result of part (c).

**Problem 5.12.** Now, let us transfer the source term $h$ to the initial condition and consider the following equation on $\Omega := (0,\pi)\times(0,\pi)$

$$\begin{cases} u_t = \Delta u \\ u(0,y,t) = u_x(\pi,y,t) = 0 \\ u(x,0,t) = u_y(x,\pi,t) = 0 \\ u(x,y,0) = \sin\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right) \end{cases}$$

a) Determine the solution to the given equation.

b) Show that the solution is equal to the solution to the following system

$$\begin{cases} u_t = \Delta u + \delta(t)\sin\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right) \\ u(0,y,t) = u_x(\pi,y,t) = 0 \\ u(x,0,t) = u_y(x,\pi,t) = 0 \\ u(x,y,0) = 0 \end{cases},$$

where $\delta(t)$ is the Dirac delta function.

**Problem 5.13.** Solve the following wave problem defined on $\Omega = (0,\pi)\times(0,\pi)$

$$\begin{cases} u_t = \Delta u \\ u_x(0,y,t) = u_x(\pi,y,t) = 0 \\ u_y(x,0,t) = u_y(x,\pi,t) = 0 \\ u(x,y,0) = 3\cos(2x) + y \end{cases}.$$

**Problem 5.14.** Use the superposition principle and solve the following heat problem on $\Omega := (0,\pi)\times(0,\pi)$

$$\begin{cases} u_t = \Delta u + \sin\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right) \\ u(0,y,t) = u_x(\pi,y,t) = 0 \\ u(x,0,t) = u_y(x,\pi,t) = 0 \\ u(x,y,0) = \sin\left(\frac{x}{2}\right)\sin\left(\frac{3y}{2}\right) - \sin\left(\frac{3x}{2}\right)\sin\left(\frac{y}{2}\right) \end{cases}$$

**Problem 5.15.** Consider the following heat equation on $\Omega := (0,1)\times(0,1)$

$$\begin{cases} u_t = \Delta u \\ u(0,y,t) = u(1,y,t) = 0 \\ u(x,0,t) = u(x,1,t) = 0 \\ u(x,y,0) = x+y \end{cases}.$$

a) Write the solution of the problem as the series

$$u(x,y,t) = \sum_{n,m=1}^{\infty} U_{n,m}(t)\,\phi_{n,m}(x,y),$$

where $\phi_{n,m}$ are the eigenfunctions of $\Delta$ with given boundary conditions. Derive an ordinary differential equation for undetermined coefficients $U_{n,m}(t)$

b) Solve the obtained differential equation and write the general series solution of the equation.

c) Apply the given initial condition and derive the particular series solution of the problem.

d) Verify that the solution is equivalent to the superposition of the solution of the following equations

$$(1)\begin{cases} u_t = \Delta u \\ u(0,y,t) = u(1,y,t) = 0 \\ u(x,0,t) = u(x,1,t) = 0 \\ u(x,y,0) = x \end{cases}, (2)\begin{cases} u_t = \Delta u \\ u(0,y,t) = u(1,y,t) = 0 \\ u(x,0,t) = u(x,1,t) = 0 \\ u(x,y,0) = y \end{cases}.$$

**Problem 5.16.** Let $\Omega$ be the rectangle $(0,1)\times(0,1)$. Consider the following wave equation

$$\begin{cases} u_{tt} = \Delta u + x\sin(\pi y) & \text{on } \Omega \\ u = 0 & \text{on } \Omega \\ u(x,y,0) = 0 \\ u_t(x,y,0) = 0 \end{cases}$$

a) Since the external source term is independent of time $t$, we can express the solution $u$ as

$$u(x,y,t) = V(x,y) + w(x,y,t),$$

where $V$ satisfies the Poisson equation

$$\begin{cases} -\Delta V = x\sin(\pi y) & \text{on } \Omega \\ V = 0 & \text{on } \operatorname{bnd}(\Omega) \end{cases}.$$

Find a closed form solution for $V(x,y)$.

b) The function $w$ satisfies the following equation

$$\begin{cases} w_{tt} = \Delta w & \text{on } \Omega \\ w = 0 & \text{on } \operatorname{bnd}(\Omega) \\ w(x,y,0) = -V(x,y) \\ w_t(x,y,0) = 0 \end{cases}.$$

Find the series solution for $w(x,y,t)$.

**Problem 5.17.** Let $\Omega$ be the set $\Omega = (0,\pi)\times(0,\pi)$.

a) Solve the Poisson equation

$$\begin{cases} \Delta u = xy & \text{on } \Omega \\ u = 0 & \text{on } \operatorname{bnd}(\Omega) \end{cases}.$$

b) Use the result obtained in the above part and solve the following heat problem

$$\begin{cases} u_t = \Delta u - xy & \text{on } \Omega \\ u = 0 & \text{on bnd}(\Omega) \\ u(x, y, 0) = x \end{cases} .$$

**Problem 5.18.** Solve the following problem on $\Omega = (0, \pi) \times (0, \pi)$

$$\begin{cases} u_t = \Delta u + txy & \text{on } \Omega \\ u = 0 & \text{on bnd}(\Omega) \\ u(x, y, 0) = x. \end{cases}$$

**Problem 5.19.** Let $\Omega$ be the rectangle $(0, 1) \times (0, 1)$. Find a series solution for the following wave equation

$$\begin{cases} u_{tt} = c^2 \Delta u + e^{-t} x \sin(\pi y) & \text{on } \Omega \\ u = 0 & \text{on } \Omega \\ u(x, y, 0) = 0 \\ u_t(x, y, 0) = 0 \end{cases}$$

**Problem 5.20.** We aim to solve the following heat problem on $\Omega := (0, 1) \times (0, 1)$

$$\begin{cases} u_t = L[u] \\ u|_{\text{bnd}(\Omega)} = 0 \\ u(x, y, 0) = e^{-x} \sin(\pi x) \sin(\pi y) \end{cases} ,$$

where $L$ is the following operator

$$L[u] = u_{xx} + 2u_x + u_{yy}.$$

a) Determine the eigenvalues and eigenfunctions of the operator $L$ satisfying the given boundary conditions. In what sense are these eigenfunctions orthogonal?

b) Write the series solution of the given problem in terms of this eigenfunctions.

**Problem 5.21.** We aim to solve the following wave equation on $\Omega = (0, 1) \times (0, 1)$

$$\begin{cases} u_{tt} = \Delta u \\ u(x, 0, t) = u(x, 1, t) = 0 \\ u_x(0, y, t) = u_x(1, y, t) = 0 \end{cases} .$$

a) Find the eigenvalues and eigenfunctions of the associated eigenvalue problem, that is

$$\begin{cases} \Delta\phi = -\lambda\phi \\ \phi(x, 0) = \phi(x, 1) = 0 \\ \phi_x(0, y) = \phi_x(1, y) = 0 \end{cases} .$$

b) Use the eigenfunctions expansion method and write the general series solution of the wave equation as follows:

$$u(t, x, y) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} U_{nm}(t)\, \phi_{nm}(x, y).$$

Find differential equations that $U_{n,m}$ satisfy, and then solve them.

c) Assume that the initial conditions are given by $u(x,y,0)=0$, $u_t(x,y,0)=y\cos(\pi x)$. Show that the solution looks like the following one

$$u(t,x,y)=\cos(\pi x)\sum_{m=1}^{\infty}\beta_m\sin(\sqrt{\lambda_{1m}}\,t)\sin(m\pi y),$$

for some constants $\beta_m$. Determine these parameters.

If you are interested in making a video to visualize your solution, the following code can helps

```
%set time of running
T=5;
[x,y,t]=meshgrid(0:0.01:1,0:0.01:1,0:0.05:T);
u=zeros(size(x));
for m=1:10
    u=u-2*(-1)^m*cos(pi*x).*sin(sqrt(1+m^2*pi^2)*t).*sin(m*pi*y)/
(m*pi*sqrt(1+m^2*pi^2));
end
% The file filename.avi is saved on your system. You can change the name if
you wish
obj=VideoWriter('filename.avi');
open(obj);
for i=1:20*T+1
  surf(x(:,:,1),y(:,:,1),u(:,:,i),'edgecolor','none');
  axis([0 1 0 1 -0.25 0.25]);
   hold off;
      f= getframe(gcf);
      writeVideo(obj,f);
      pause(0.1)
end
obj.close();
```

**Problem 5.22.** Consider the following heat problem on $\Omega=(0,1)\times(0,1)$

$$\begin{cases} u_t=0.1(\Delta u+2u_x) & \text{on }\Omega\\ u=0 & \text{on bnd}(\Omega)\end{cases}.$$

a) Find the eigenfunctions and eigenvalues of the following eigenvalue problem

$$\begin{cases} \Delta\phi+2\phi_x=-\lambda\phi & \text{on }\Omega\\ \phi=0 & \text{on bnd}\,\Omega\end{cases}.$$

b) Write the solution of the given heat equation as follows:

$$u(t,x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}U_{nm}(t)\,\phi_{nm}(x,y).$$

Find a differential equation for $U_{nm}(t)$. Solve it and write down the general solution of the equation.

c) Assume $u(x,y,0)=x\,e^{-x}\sin(2\pi y)$. Find the solution that satisfies the given initial condition.

d) Run the following code and observe the change of the temperature with respect to time:

```
%set time of running
T=1;
[x,y,t]=meshgrid(0:0.01:1,0:0.01:1,0:0.05:T);
u=zeros(size(x));
```

```
    for n=1:10
        l=1+(4+n^2)*pi^2;
        u=u-2*(-1)^n*sin(2*pi*y).*exp(-0.1*l*t).*sin(n*pi*x)/(n*pi);
    end
    % The file filename.avi is saved on your system. You can change the name if
you wish
    obj=VideoWriter('filename.avi');
    open(obj);
    for i=1:20*T+1
      surf(x(:,:,1),y(:,:,1),u(:,:,i),'edgecolor','none'); view(2);colormap(jet);
axis equal;
      caxis([-0.2,0.2]); axis([0 1 0 1]);
       hold off;
          f= getframe(gcf);
          writeVideo(obj,f);
          pause(0.2)
    end
    obj.close();
```

**Problem 5.23.** In the domain $\Omega = (1, e) \times (0, 1)$:

a) Solve the eigenvalue problem

$$\begin{cases} x^2\,\phi_{xx} + x\phi_x + \phi_{yy} = -\lambda\phi & \text{on } \Omega \\ \phi = 0 & \text{on bnd}(\Omega) \end{cases}.$$

b) Solve the wave equation

$$\begin{cases} u_{tt} = x^2\,u_{xx} + x\,u_x + u_{yy} + ty & \text{on } \Omega \\ u = 0 & \text{on bnd}(\Omega) \\ u(x,y,0) = 0 & \\ u_t(x,y,t) = 0 & \end{cases}.$$

**Problem 5.24.** Let $\Omega$ be the set $\Omega = (-\pi, \pi) \times (-\pi, \pi)$. Consider the following eigenvalue problem

$$\begin{cases} \Delta\phi = -\lambda\phi \\ \phi(-\pi, y) = \phi(\pi, y),\, \phi_x(-\pi, y) = \phi_x(\pi, y) \\ \phi(x, -\pi) = \phi(x, \pi),\, \phi_y(x, -\pi) = \phi_y(x, \pi) \end{cases}.$$

a) Show the eigenvalues are non-negative: $\lambda \geq 0$.

b) Show that eigenfunctions of the problem are

$$\phi_{nm}(x, y) = \{\cos(nx + my), \sin(nx + my)\},$$

for $n, m = 0, 1, \cdots$.

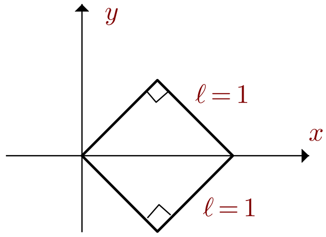
c) Show that

$$\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \phi_{nm}(x,y)\,\phi_{n',m'}(x,y)\,dx\,dy = \begin{cases} 1 & (n,m) = (n',m') \\ 0 & (n,m) \neq (n',m') \end{cases}$$

d) solve the following heat problem:

$$\begin{cases} u_t = \Delta u \\ \phi(-\pi, y, t) = \phi(\pi, y, t),\, \phi_x(-\pi, y, t) = \phi_x(\pi, y, t) \\ \phi(x, -\pi, t) = \phi(x, \pi, t),\, \phi_y(x, -\pi, t) = \phi_y(x, \pi, t) \\ u(x, y, 0) = xy \end{cases}.$$

**Problem 5.25.** Let $D$ be the domain shown below




Consider the following equation on $D$

$$u_t = 2\Delta u - 2u_{xy},$$

with the homogeneous Dirichlet boundary condition. Use a transformation to transform the domain $D$ to an unit square. Derive the appropriate differential equation in the new coordinate. Solve the new equation and then write down the solution of the given equation on $D$.

**Problem 5.26.** Let $\Omega$ be the unit square $(0,1)\times(0,1)$. Consider the following Laplace equation

$$\begin{cases} \Delta u = 0 \\ u(0,y) = -1, u(1,y) = 1 \\ \partial_y u(x,0) = \partial_y u(x,1) = 0 \end{cases}.$$

a) Find a series solution to the equation.

b) Verify that the the function $u = 2x - 1$ solves the equation and thus it is equal to the series solution obtained in part a).

c) Now consider the following Poisson equation

$$\begin{cases} \Delta u = y \\ u(0,y) = -1, u(1,y) = 1 \\ \partial_y u(x,0) = \partial_y u(x,1) = 0 \end{cases}.$$

Consider the solution $u$ as $u = V + w$, where $V$ satisfies the Laplace equation given above. Find a series solution for $w$, and determine $u(x,y)$.

**Problem 5.27.** Solve the following Poisson equations

a)

$$\begin{cases} \Delta u = \sin(\pi y) \\ u(0,y) = 0, u(1,y) = 0 \\ u(x,0) = -1, u(x,1) = 1 \end{cases}$$

b)

$$\begin{cases} \Delta u = \sin(2x) + \sin(3y) \\ u(0,y) = -1, u(\pi,y) = 1 \\ u(x,0) = -1, u(x,\pi) = 1 \end{cases}$$

c)

$$\begin{cases} \Delta u = xy \\ u_x(0,y) = 0, u_x(\pi,y) = 0 \\ u(x,0) = 0, u(x,\pi) = 1 \end{cases}$$

**Problem 5.28.** We aim to solve the following equation

$$u_t = \Delta u + 2u_x + u + y, \qquad (x,y) \in (0,1)\times(0,1),$$

with the boundary conditions

$$u(0,y,t)=u(1,y,t)=u(x,1,t)=0;\, u(x,0,t)=1,$$

and the initial condition $u(x,y,0)=0$.

a) Show that the steady state solution $V(x,y)$ is

$$V(x,y)=\sum_{n=1}^{\infty}\left(B_n\sinh(n\pi y)+\frac{2((-1)^n-1)}{n^3\pi^3}y\right)e^{-x}\sin(n\pi x).$$

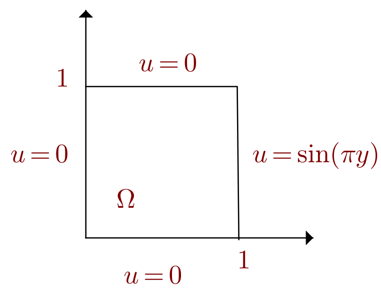
Determine the value of $B_n$.

b) Show that the transient solution has the form

$$w(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{nm}e^{-(n^2+m^2)\pi^2 t}e^{-x}\sin(n\pi x)\sin(m\pi y)$$

Determine the value of $A_{n,m}$.

**Problem 5.29.** Let $\Omega$ be the rectangle $(0,1)\times(0,1)$




a) Find a closed form solution to the LAPLACE equation $\Delta u=0$ with the boundary conditions given in the figure.

b) Solve the following wave problem

$$\begin{cases} u_t=\Delta u-\sin(t) & \text{on } \Omega \\ u|_{\text{bnd}(\Omega)}: \text{given in the figure} & \\ u(x,y,0)=\frac{\sinh(\pi x)}{\sinh\pi}\sin(\pi y) & \\ u_t(x,y,0)=0 & \end{cases}.$$

**Problem 5.30.** In the domain $\Omega=(0,1)\times(0,1)$:

a) Solve the eigenvalue problem

$$\begin{cases} \Delta\phi+2\phi_x-2\phi_y=-\lambda\phi & \text{on } \Omega \\ \phi=0 & \text{on bnd}(\Omega) \end{cases},$$

b) Find $\sigma(x,y)$ such that the obtained eigenfunctions are orthogonal with respect to $\sigma$.

c) Solve the heat equation

$$\begin{cases} u_t=\Delta u+2u_x-2u_y & \text{on } \Omega \\ u=3 & \text{on bnd}(\Omega) \\ u(x,y,0)=xye^{y-x}+3 & \end{cases}.$$

d) Find the steady state solution.

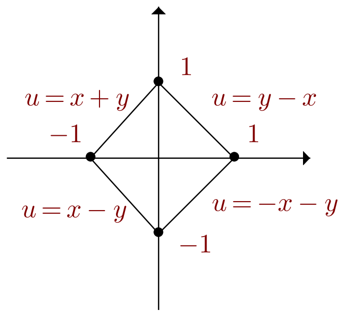
**Problem 5.31.** Solve the following heat problem

$$\begin{cases} u_t = \Delta u \\ u_x(0,y,t) = 0, u(1,y,t) = 1 \\ u(x,0,t) = x, u_y(x,1,t) = 0 \\ u(x,y,0) = xy \end{cases}.$$

**Problem 5.32.** Let $\Omega$ be the square $(0,1) \times (0,1)$. Solve the following wave equation on $\Omega$

$$\begin{cases} u_{tt} = 4\Delta u + 4xy & \text{on } \Omega \\ u = 1 & \text{on bnd}(\Omega) \\ u(x,y,0) = 1; u_t(x,y,0) = 0 & \end{cases}$$

**Problem 5.33.** Consider the equation $\partial_{xx}u + \partial_{yy}u - \partial_{xy}u = 0$ in the domain shown in the following figure




Use an appropriate transformation to rewrite the equation in the normal form. Note that this transformation will rotate the domain too. Find a solution to the new problem, and then obtain the solution to the original problem.

## 5.3 Disk domain and polar coordinate

In this section, we focus on studying partial differential equations defined on a disk or a semi-disk. Unlike the rectangular domain, the geometry of a disk does not lend itself easily to the separation of variable technique when using Cartesian coordinates. However, we can overcome this challenge by introducing polar coordinates, which involves a transformation from the rectangular domain $[-\pi, \pi] \times [0, a)$ to the $(\theta, r)$-plane. By employing the separation of variable technique in polar coordinates, we can effectively solve the eigenvalue problems associated with these domains.

### 5.3.1 Eigenfunctions of $\Delta$ in a disk

Let $\Omega$ be a disk of radius $a$. We consider the following eigenvalue problem for $\phi(r, \theta)$ on $\Omega$:

$$\begin{cases} \Delta\phi = -\lambda\phi \\ \phi(a,\theta) = 0 \end{cases}.$$

Recall that in polar coordinates, the Laplacian operator $\Delta$ takes the form:

$$\Delta_{(r,\theta)}\phi = \phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta}.$$

By using the separation of variables $\phi(r,\theta)=R(r)\Theta(\theta)$, the eigenvalue problem for $\phi$ reduces to:

$$\frac{r^2R''+rR'}{R}+\frac{\Theta''}{\Theta}=-\lambda r^2.$$

This equation holds true only if the ratio $\frac{\Theta''}{\Theta}$ is a constant, denoted as $-\alpha$. Therefore, we have the eigenvalue problem for $\Theta(\theta)$:

$$\Theta''=-\alpha\Theta.$$

To determine suitable boundary conditions for $\Theta(\theta)$, we take into account the geometry of the disk. We impose the following boundary conditions:

$$\begin{cases}\Theta(-\pi)=\Theta(\pi)\\ \Theta'(-\pi)=\Theta'(\pi)\end{cases}.$$

These conditions ensure that the eigenfunctions of the eigenvalue problem $\Theta"=-\alpha\Theta$ are smoothly $2\pi$-periodic, as required by the geometry of the disk.

The eigenpairs of the eigenvalue problem $\Theta''=-\alpha\Theta$ for the disk are given by:

$$\Theta_n(\theta)\in\{1,\cos(n\theta),\sin(n\theta)\},$$

for $n=1,2,\cdots$. This set of eigenfunctions forms the Fourier orthogonal basis for functions defined on the interval $[-\pi,\pi]$. Let $f(\theta)$ be a continuously differentiable function defined on $[-\pi,\pi]$. This function can be expressed in terms of the eigenfunctions as:

$$f(\theta)=a_0+\sum_{n=1}^{\infty}a_n\cos(n\theta)+\sum_{n=1}^{\infty}b_n\sin(n\theta),$$

where the coefficients are given by:

$$a_0=\frac{1}{2\pi}\int_0^1 f(\theta)\,d\theta,\, a_n=\frac{1}{\pi}\int_0^1 f(\theta)\cos(n\theta)\,d\theta,\, b_n=\frac{1}{\pi}\int_0^1 f(\theta)\sin(n\theta)\,d\theta.$$

These coefficients represent the contributions of the different eigenfunctions to the function $f(\theta)$.

**Problem 5.34.** Solve the eigenvalue problem for $\Theta(\theta)$ with the periodic boundary condition and show that the eigenfunctions are $1,\cos(n\theta),\sin(n\theta)$ with associated eigenvalues $n\in\mathbb{N}$.

For $\alpha=n^2$ ($n=0,\ 1,\ 2,\ \cdots$), the equation for $R(r)$ reduces to the following ordinary eigenvalue problem:

$$r^2R''+rR'-n^2R=-\lambda r^2R.$$

The boundary condition that $R(r)$ satisfies is $R(a)=0$ in accordance with the condition $\phi(a,\theta)=0$.

Since $\lambda>0$, we can use the substitution $x=\sqrt{\lambda}r$ to rewrite the above equation in the following standard form known as a Bessel equation:

$$x^2\frac{d^2R}{dx^2}+x\frac{dR}{dx}+(x^2-n^2)R=0. \tag{5.4}$$

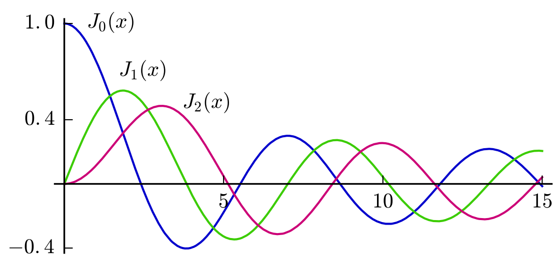
In our book on ordinary differential equations, we have studied this equation in detail using the power series method. If we represent the solution $R(x)$ in the form:

$$R(x)=x^n\sum_{k=0}^{\infty} c_k\, x^k,$$

substituting the series into the differential equation and performing some algebraic manipulations, we arrive at the following series:

$$R(x)=\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!}\left(\frac{x}{2}\right)^{2k+n}.$$

This series is known as the first-type Bessel function and is denoted by $J_n(x)$. The figure below depicts $J_n(x)$ for $n=0,1,2$




For $x=\sqrt{\lambda}\, r$, the solution to the equation for $R(r)$ is:

$$R_n(r)=J_n(\sqrt{\lambda}\, r).$$

The eigenvalues $\lambda$ are determined by the boundary condition $R(a)=0$. This condition leads to the equation $J_n(\sqrt{\lambda}\, a)=0$. This equation can not be solved in closed form, we have to use numerical methods to draw the zeros of the equation $J_n(x)=0$. As we observe from the above graph, there are infinitely many zeros for each fixed $n$. Let us denote these zeros as $z_{n,p}$, i.e., $J_n(z_{n,p})=0$ for $p=1,2,\cdots$ for each fixed $n$. The table below includes some of zeros of $J_n(x)$:

| $p$ | $z_{0p}$ | $z_{1p}$ | $z_{2p}$ | $z_{3p}$ |
|---|---|---|---|---|
| 1 | 2.4048 | 3.8317 | 5.1356 | 6.3802 |
| 2 | 5.5201 | 7.0156 | 8.4172 | 9.7610 |
| 3 | 8.6537 | 10.1735 | 11.6198 | 13.0152 |
| 4 | 11.7915 | 13.3237 | 14.7960 | 16.2235 |
| 5 | 14.9309 | 16.4706 | 17.9598 | 19.4094 |
| 6 | 18.0711 | 19.6159 | 21.1170 | 22.5827 |
| 7 | 21.2116 | 22.7601 | 24.2701 | 25.7482 |
| 8 | 24.3525 | 25.9037 | 27.4206 | 28.9083 |
| 9 | 27.4935 | 29.0468 | 30.5692 | 32.0648 |
| 10 | 30.6346 | 32.1897 | 33.7165 | 35.2187 |

The eigenvalues $\lambda$ are determined in terms of these zeros as:

$$\lambda_{n,p}=\frac{z_{n,p}^2}{a^2}.$$

Therefore, the eigenfunctions for the eigenvalue problem for $R(r)$ are obtained as: $J_n\left(\frac{z_{n,p}}{a}r\right)$. For sake of simplicity, we denote these functions by $J_{n,p}(r)$.

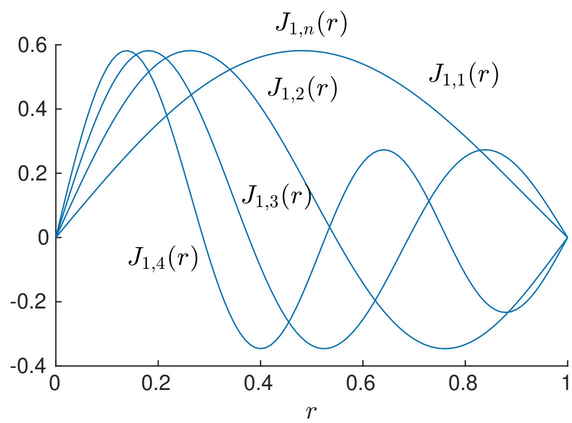
**Remark 5.3.** For each fixed $n$, we obtain a set of infinite eigenfunctions $\{J_{n,p}(r)\}$ for $p=1, 2, \cdots$. For example, for $n=0$ we obtain

$$\{J_{0,1}(r), J_{0,2}(r), J_{0,3}(r), \cdots\},$$

and for $n=1$:

$$\{J_{1,1}(r), J_{1,2}(r), J_{1,3}(r), \cdots\}.$$

For each fixed $n$, the set $\{J_{n,p}(r)\}$ forms a basis for piecewise continuously differentiable functions $f(r)$ defined on $[0, a]$. The figure below illustrate the set $\{J_{1,n}\}$ for $n=1, \cdots, 4$.




**Problem 5.35.** Prove that for each fixed $n$, the eigenfunction $J_{n,p}(r)$ are orthogonal with respect to the weight function $\sigma=r$, i.e.,

$$\langle J_{n,p}, J_{n,p'}\rangle_r := \int_0^a J_{n,p}(r)\, J_{n,p'}(r)\, r\, dr = 0,$$

for $p \neq p'$.

To understand how each set in the family of sets $\{J_{n,p}(r)\}$ is a basis, let's express the function $f(r)=r^2$ in terms of two bases: $\{J_{0,1}, J_{0,2}, \ldots\}$ and $\{J_{1,1}, J_{1,2}), \ldots\}$. We will truncate both approximations by summing up to ten terms. For the first basis $\{J_{0,1}, J_{0,2}, \ldots\}$, the approximation of $f(r)$ is given by:

$$f(r) \approx c_1\, J_{0,1}(r) + c_2\, J_{0,2}(r) + \cdots + c_{10}\, J_{0,10}(r),$$

where $c_j$ are coefficients which are determined by the relation

$$c_j = \frac{\langle f, J_{0,p}\rangle_r}{\|J_{0,p}\|^2} = \frac{\displaystyle\int_0^1 f(r)\, J_{0,p}(r)\, r\, dr}{\displaystyle\int_0^1 J_{0,p}^2(r)\, r\, dr}.$$

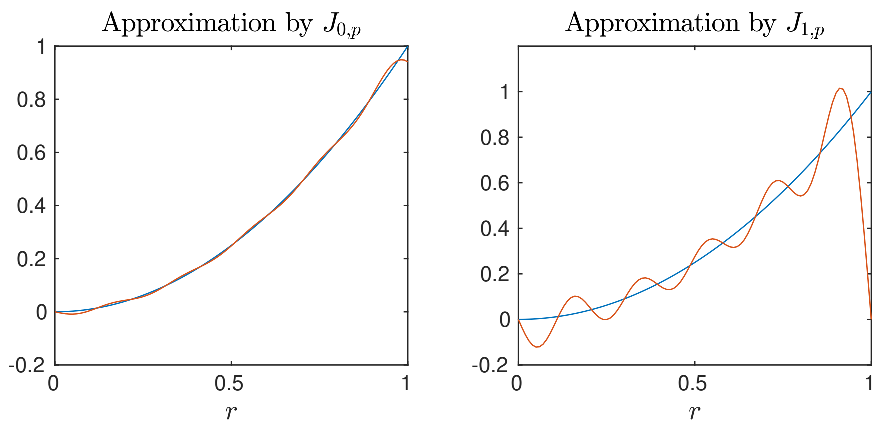
Similarly, for the second basis $\{J_{1,1}, J_{1,2}), \ldots\}$, the approximation of $f(r)$ is given by:

$$f(r) \approx d_1\, J_{1,1}(r) + d_2\, J_{1,2}(r) + \cdots + d_{10}\, J_{1,10}(r),$$

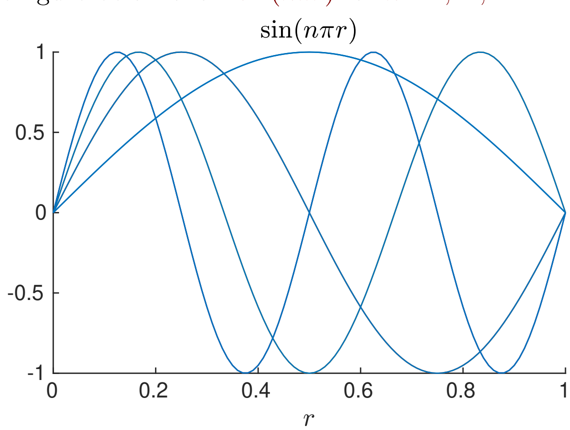
where $d_j$ are coefficients which are determined by the relation

$$d_j = \frac{\langle f, J_{1,p}\rangle_r}{\|J_{1,p}\|^2} = \frac{\int_0^1 f(r)\, J_{1,p}(r)\, r\, dr}{\int_0^1 J_{1,p}^2(r)\, r\, dr}.$$

The resulting approximations will give us an idea of how well the sets $\{J_{0,1}, J_{0,2}, \ldots\}$ and $\{J_{1,1}, J_{1,2}, \ldots\}$ approximate the function $f(r) = r^2$ when truncated to ten terms. The figure below illustrate the result:




**Exercise 5.21.** As we know, the set $\{\sin(n\pi r)\}$ is an orthogonal basis for functions defined on $[0, 1]$. The figure below show $\sin(n\pi r)$ for $n = 1, \cdots, 4$.




Approximate the function $f(r) = r^2$ in terms of $\{\sin(n\pi r)\}$ for $n = 1, \cdots, 10$ and compare the error with the approximation in terms of $\{J_{1,p}(r)\}$ for $p = 1, \cdots, n$. The error is defined as

$$\text{err} := \left(\int_0^1 |f(r) - S_N(r)|^2\, dr\right)^{\frac{1}{2}},$$

where $S_N(r)$ is the truncated series approximation up to $N$ terms.

**Exercise 5.22.** Approximate function $f(r) = \sin(\pi r)$ for $r \in [0, 1]$ in terms of basis $\{J_{0,p}(r)\}$ for $p = 1, \cdots, 10$. Draw $f(r)$ and its approximation in the same coordinate. If you wish, you can use the following code in Matlab:

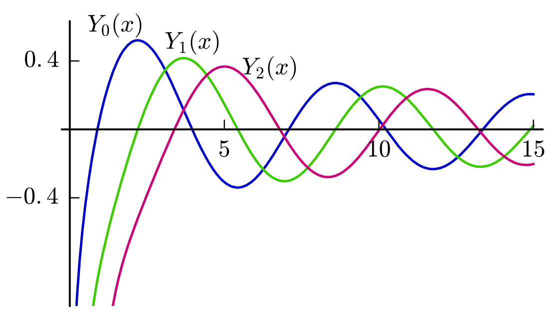
```
Z0=[2.4048 5.5201 8.6537 11.7915 14.9309 18.0711 21.2116 24.3525 27.4935 30.6346];
```

```
f=@(r) sin(pi*r);
C=integral(@(r) r*f(r)*besselj(1,Z0(:)*r),0,1,'arrayvalued',true)./...
integral(@(r) r*besselj(1,Z0(:)*r).^2,0,1,'arrayvalued',true);
r=0:0.01:1;
S=besselj(1,Z0(:)*r);
fhat=C'*S;
plot(r,f(r),r,fhat)
```

**Remark 5.4.** The obtained solutions $J_n(x)$ for the equation (5.4) correspond to the first-type of Bessel functions. In the book on ordinary differential equations, we also encountered the second solution to the Bessel equation, denoted by $Y_n(x)$. The series expansion of $Y_n(x)$ is given by:

$$Y_n(x) = c\,J_n(x)\log(x) + x^{-n}\sum_{k=0}^{\infty} C_k\,x^k,$$

where $c$ and $C_k$ are constants. It can be observed that these solutions are singular at $x=0$, which corresponds to the center of the disk ($r=0$). The figure below illustrates a few examples of $Y_n(x)$:




For partial differential equations defined on the interior of a disk, we typically ignore these singular solutions ($Y_n$) as they are non-physical. However, for equations defined on a annulus shape or the exterior of a disk, we need to consider both $J_n$ and $Y_n$ functions for the solution.

In our discussion of equations defined on the interior of a disk, we will focus on the first-type Bessel functions ($J_n$) and disregard the singular solutions ($Y_n$) for their non-physical nature.

Having $\Theta_n(\theta)$ and $J_{n,p}(r)$, the eigenfunctions of the eigenvalue problem $\Delta_{(r,\theta)}\phi = -\lambda\phi$ are obtained as

$$\phi_{n,p}(r,\theta) \in \{J_{n,p}(r)\cos(n\theta), J_{n,p}(r)\sin(n\theta)\},$$

with the associated eigenvalues $\lambda_{n,p} = \frac{z_{n,p}^2}{a^2}$, where $z_{n,p}$ are zeros of the Bessel functions $J_n$, i.e., $J_n(z_{n,p}) = 0$ for $n = 0, 1, 2, \cdots$, and $p = 1, 2, \cdots$.

**Theorem 5.2.** *The set of functions* $\{J_{n,p}(r)\cos(n\theta), J_{n,p}(r)\sin(n\theta)\}$ *for* $n = 0, 1, \cdots$, $p = 1, 2, \cdots$, *forms an orthogonal basis for continuously differentiable functions defined in the region* $[-\pi, \pi] \times [0, a]$. *These functions are orthogonal with respect to the weight function* $\sigma = r$.

This means that any continuously differentiable function $f(r,\theta)$ defined in the region $[-\pi,\pi]\times[0,a]$ can be expressed as a linear combination of these eigenfunctions as follows:

$$f(r,\theta)=\sum_{p=1}^{\infty} A_{0,p}\,J_{0,p}(r)+\sum_{n,p=1}^{\infty} A_{n,p}\,J_{n,p}(r)\cos(n\theta)+\sum_{n,p=1}^{\infty} B_{n,p}\,J_{n,p}(r)\sin(n\theta).$$

The orthogonality property ensures that the expansion coefficients can be determined using the inner product weighted by the weight function $\sigma=r$ as follows:

$$A_{0,p}=\frac{\langle f,J_{0,p}\rangle_r}{\|J_{0,p}\|^2}=\frac{\int_{-\pi}^{\pi}\int_0^a f(r,\theta)\,J_{0,p}(r)\,r\,dr\,d\theta}{2\pi\int_0^a J_{0,p}^2(r)\,r\,dr},$$

$$A_{n,p}=\frac{\langle f,J_{n,p}\cos(n\theta)\rangle_r}{\|J_{n,p}\cos(n\theta)\|^2}=\frac{\int_{-\pi}^{\pi}\int_0^a f(r,\theta)\,J_{n,p}(r)\cos(n\theta)\,r\,dr\,d\theta}{\pi\int_0^a J_{n,p}^2(r)\,r\,dr},$$

$$B_{n,p}=\frac{\langle f,J_{n,p}\sin(n\theta)\rangle_r}{\|J_{n,p}\sin(n\theta)\|^2}=\frac{\int_{-\pi}^{\pi}\int_0^a f(r,\theta)\,J_{n,p}(r)\sin(n\theta)\,r\,dr\,d\theta}{\pi\int_0^a J_{n,p}^2(r)\,r\,dr}.$$

These formulas allow us to determine the expansion coefficients $A_{0,p}$, $A_{n,p}$, and $B_{n,p}$ using the given function $f(r,\theta)$ and the properties of the Bessel functions $J_{n,p}(r)$.

**Example 5.7.** Let us represent the function $f(r,\theta)=r\theta$ defined on $[-\pi,\pi]\times[0,1]$ in terms of $\{\phi_{n,p}(r,\theta)\}$. Since $\theta$ is an odd function in $[-\pi,\pi]$, the series representation of $f$ contains only $\sin(n\theta)$:

$$r\theta=\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} B_{n,p}\,J_{0,p}(r)\sin(n\theta),$$

where

$$B_{n,p}=\frac{\langle f,J_{n,p}\sin(n\theta)\rangle_r}{\|J_{n,p}\sin(n\theta)\|^2}=\frac{\int_{-\pi}^{\pi}\int_0^1 r\theta\,J_{n,p}(r)\sin(n\theta)\,r\,dr\,d\theta}{\pi\int_0^1 J_{n,p}^2(r)\,r\,dr}.$$

The double integral at the top can be calculated as

$$\int_{-\pi}^{\pi}\int_0^a r\theta\,J_{n,p}(r)\sin(n\theta)\,r\,dr\,d\theta=\left(\int_0^1 J_{n,p}(r)\,r^2\,dr\right)\left(\int_{-\pi}^{\pi}\theta\sin(n\theta)\,d\theta\right)=$$

$$=\frac{-2\pi(-1)^n}{n}\int_0^1 J_{n,p}(r)\,r^2\,dr.$$

Therefore, $B_{n,p}$ are simplified as

$$B_{n,p} = \frac{-2(-1)^n}{n} \frac{\displaystyle\int_0^1 J_{n,p}(r)\, r^2\, dr}{\displaystyle\int_0^1 J_{n,p}^2(r)\, r\, dr}.$$

**Exercise 5.23.** Consider the function $f(r,\theta) = \sin(r\theta)$ inside the unit disk.

a) Represent this function in terms of basis $\{J_{n,p}(r)\cos(n\theta), J_{n,p}(r)\sin(n\theta)\}$ for $n = 0, 1, 2, \cdots$. Note the the representation holds only for points inside the disk and is not valid at the boundary $r = 1$.

b) As a function of $\theta$, write the Fourier series of $f$ in terms of the basis $\{\cos(n\theta), \sin(n\theta)\}$, for $n = 0$, $1, 2, \cdots$ as

$$f(r,\theta) = \sum_{n=0}^{\infty} a_n(r)\cos(n\theta) + \sum_{n=0}^{\infty} b_n(r)\cos(n\theta).$$

c) As a function of $r$, write the series representation of $f$ in terms of basis $\{J_{1,p}(r)\}$ as

$$f(r,\theta) = \sum_{n=1}^{\infty} C_p(\theta)\, J_{1,p}(r).$$

### 5.3.2 Heat, wave and Poisson equations on a disk

We utilize the eigenfunction expansion method to solve partial differential equations on a disk. In order to demonstrate the method, we will solve a few examples.

#### Poisson equation on a disk

In this section, we will address the Poisson equation on a disk $\Omega$ of radius $a > 0$:

$$\begin{cases} -\Delta u = f(r,\theta) \\ u(a,\theta) = 0 \end{cases}.$$

To solve this equation, we express the desired solution as a series in terms of the basis $\{J_{n,p}(r)\cos(n\theta), J_{n,p}(r)\sin(n\theta)\}$:

$$u(r,\theta) = \sum_{n=0,p=1}^{\infty} A_{n,p}\, J_{n,p}(r)\cos(n\theta) + \sum_{n,p=1}^{\infty} B_{n,p}\, J_{n,p}(r)\sin(n\theta),$$

where $A_{n,p}$ and $B_{n,p}$ are undetermined coefficients. To determine these coefficients, we substitute this series into the Poisson equation, yielding:

$$\sum_{n=0,p=1}^{\infty} \lambda_{n,p} A_{n,p}\, J_{n,p}(r)\cos(n\theta) + \sum_{n,p=1}^{\infty} \lambda_{n,p} B_{n,p}\, J_{n,p}(r)\sin(n\theta) = f(r,\theta)$$

Next, we find the coefficients $A_{n,p}$ and $B_{n,p}$ by taking the inner product of both sides of the Poisson equation with $J_{n,p}(r)\cos(n\theta)$ and $J_{n,p}(r)\sin(n\theta)$, respectively.

**Example 5.8.** Let's consider the Poisson equation on a disk of radius $a=1$:

$$\begin{cases} -\Delta u = r^2 \sin(3\theta) \\ u(1,\theta) = 0 \end{cases}$$

We seek a series solution of the form:

$$u(r,\theta) = \sum_{n=0,p=1}^{\infty} A_{n,p} J_{n,p}(r)\cos(n\theta) + \sum_{n,p=1}^{\infty} B_{n,p} J_{n,p}(r)\sin(n\theta).$$

Substituting this series into the Poisson equation, we obtain:

$$\sum_{n=0,p=1}^{\infty} \lambda_{n,p} A_{n,p} J_{n,p}(r)\cos(n\theta) + \sum_{n,p=1}^{\infty} \lambda_{n,p} B_{n,p} J_{n,p}(r)\sin(n\theta) = r^2 \sin(3\theta).$$

The equality holds for $A_{n,p}=0$, for all $n$, and for $B_{n,p}=0$ for $n\neq 3$. For $n=3$, we have

$$\sum_{p=1}^{\infty} \lambda_{3,p} B_{3,p} J_{3,p}(r)\sin(3\theta) = r^2\sin(3\theta),$$

This yields the expression for $B_{3,p}$:

$$B_{3,p} = -\frac{\langle r^2, J_{3,p}\rangle_r}{\lambda_{0,3}\|J_{3,p}\|^2}.$$

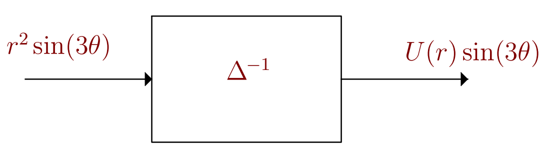
Finally, the solution for $u(r,\theta)$ is given as:

$$u(r,\theta) = \sin(3\theta)\sum_{p=1}^{\infty} \frac{\langle r^2, J_{3,p}\rangle_r}{\lambda_{0,3}\|J_{3,p}\|^2} J_{3,p}(r).$$

In the example above, we obtained the solution in a series form. However, it is also possible to find a closed-form solution for the equation as follows. If we interpret the problem in the system form as:

$$u = -\Delta^{-1}[f(r,\theta)],$$

we can represent this mathematical equation in a block-diagram form as shown below:




.

Since the system $\Delta^{-1}$ is triggered by the function $\sin(3\theta)$, and since $\sin(3\theta)$ is part of the eigenfunctions of $\Delta$, we can assume that the response of this system is of the form:

$$u(r,\theta) = R(r)\sin(3\theta),$$

where $R(r)$ is an undetermined function. We determine this function such that $u(r,\ \theta)$ satisfies the Poisson equation and the boundary condition $R(1)=0$. Substituting $u(r,\theta)$ into the Poisson equation yields:

$$R''(r)\sin(3\theta) + \frac{1}{r}R'(r)\sin(3\theta) - \frac{9R(r)}{r^2}\sin(3\theta) = -r^2\sin(3\theta),$$

leading to the following Cauchy-Euler equation:

$$\begin{cases} r^2 R''(r) + r R'(r) - 9R(r) = -r^4 \\ U(1) = 0 \end{cases}.$$

The general solution to this Cauchy-Euler equation is:

$$U(r) = Ar^3 + Br^{-3} - \frac{1}{7}r^4.$$

Assuming that the solution remains bounded at $r = 0$, $B$ must be equal to zero, and thus:

$$U(r) = Ar^3 - \frac{1}{7}r^4.$$

By performing the boundary condition at $r = 1$, we obtain the solution as:

$$U(r) = \frac{1}{7}r^3(1 - r),$$

and consequently:

$$u(r, \theta) = \frac{1}{7}r^3(1 - r)\sin(3\theta).$$

This solution is known as the closed-form solution to the given problem. By the uniqueness theorem, these two solutions must be the same, i.e.,

$$\frac{1}{7}r^3(1 - r)\sin(3\theta) = \sin(3\theta)\sum_{p=1}^{\infty} \frac{\langle r^2, J_{3,p}\rangle_r}{\lambda_{0,3}\|J_{3,p}\|^2} J_{3,p}(r).$$

**Exercise 5.24.** Find a series solution to the Poisson equation $\Delta u = r$ on the unit disk satisfying the homogeneous Dirichlet boundary conditions. Find a closed form solution for the equation. Draw the functions and a truncated series in the same coordinate. You can use the following code to generate the figure

```
Z0=[2.4048 5.5201 8.6537 11.7915 14.9309 18.0711 21.2116 24.3525 27.4935 30.6346];
f=@(r) r;
C=-integral(@(r) r*f(r)*besselj(0,Z0(:)*r),0,1,'arrayvalued',true)./...
integral(@(r) r*besselj(0,Z0(:)*r).^2,0,1,'arrayvalued',true)./(Z0(:).^2);
r=0:0.01:1;
S=besselj(0,Z0(:)*r);
fhat=C'*S;
plot(r,(r.^3-1)/9,r,fhat)
```

**Exercise 5.25.** Find a series and closed form solution for the Poisson equation $\Delta u = r\sin(3\theta)$ satisfying the boundary condition $u(1, \theta) = 0$.

## Heat problems on a disk

Let's solve the following heat problem

$$\begin{cases} u_t = \Delta u + h(r, \theta, t) \\ u(a, \theta, t) = 0 \\ u(r, \theta, 0) = 0 \end{cases}.$$

Since the set of eigenfunctions $\{\phi_{n,p}(r,\theta)\}$ is a basis for functions define don the unit disk, we can express the desired solution $u(r,\theta,t)$ as:

$$u(r,\theta,t)=\sum_{n=0,p=1}^{\infty} U_{n,p}(t)\, J_{n,p}(r)\cos(n\theta)+\sum_{n=1,p=1}^{\infty} V_{n,p}(t)\, J_{n,p}(r)\sin(n\theta),$$

for undetermined coefficients functions $U_{n,p}(t)$ and $V_{n,p}(t)$. Substituting these series into the heat equation, we reach the following equation:

$$\sum_{n=0,p=1}^{\infty} [U_{n,p}'(t)+\lambda_{n,p}U_{n,p}]\, J_{n,p}(r)\cos(n\theta)+\sum_{n=1,p=1}^{\infty} [V_{n,p}'(t)+\lambda_{n,p}V_{n,p}]\, J_{n,p}(r)\sin(n\theta)=h.$$

To proceed, we write the function $h$ in terms of the basis $\{\phi_{n,p}\}$ as

$$h(r,\theta,t)=\sum_{n=0,p=1}^{\infty} \alpha_{n,p}(t)\, J_{n,p}(r)\cos(n\theta)+\sum_{n=1,p=1}^{\infty} \beta_{n,p}(t)\, J_{n,p}(r)\sin(n\theta),$$

where $\alpha_{n,p}$ and $\beta_{n,p}$ are respectively:

$$\alpha_{n,p}=\frac{\langle h, J_{n,p}(r)\cos(n\theta)\rangle}{\|J_{n,p}(r)\cos(n\theta)\|^2}, \beta_{n,p}=\frac{\langle h, J_{n,p}(r)\sin(n\theta)\rangle}{\|J_{n,p}(r)\sin(n\theta)\|^2}.$$

From these representation, we arrive at the following equations for $U_{n,p}$ and $V_{n,p}$:

$$\begin{cases} U_{n,p}'+\lambda_{n,p}U_{n,p}=\alpha_{n,p} \\ U_{n,p}(0)=0 \end{cases}, \begin{cases} V_{n,p}'+\lambda_{n,p}V_{n,p}=\beta_{n,p} \\ V_{n,p}(0)=0 \end{cases}.$$

Note that the initial condition is in accordance with the initial condition $u(r,\theta,0)=0$.

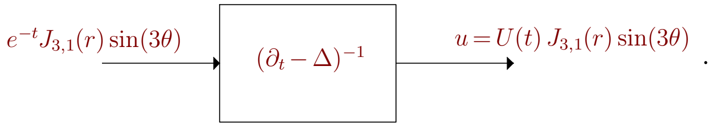
**Example 5.9.** Let's consider the following heat problem on a unit disk

$$\begin{cases} u_t=\Delta u+e^{-t}J_{3,1}(r)\sin(3\theta) \\ u(1,\theta,t)=0 \\ u(r,\theta,0)=0 \end{cases}.$$

From a system point of view, the heat system is triggered by the source term: $h=e^{-t}J_{3,1}(r)\sin(3\theta)$. Since the term $J_{3,1}(r)\sin(3\theta)$ is an eigenfunction of the Laplacian operator, we seek a response of the system in the form:

$$u(r,\theta,t)=U(t)J_{3,1}(r)\sin(3\theta),$$

where $U(t)$ is an undetermined function. The block-diagram below illustrates this approach:




Substituting $u=U(t)\, J_{3,1}(r)\sin(3\theta)$ into the heat equation yields:

$$U'+\lambda_{3,1}U=e^{-t},$$

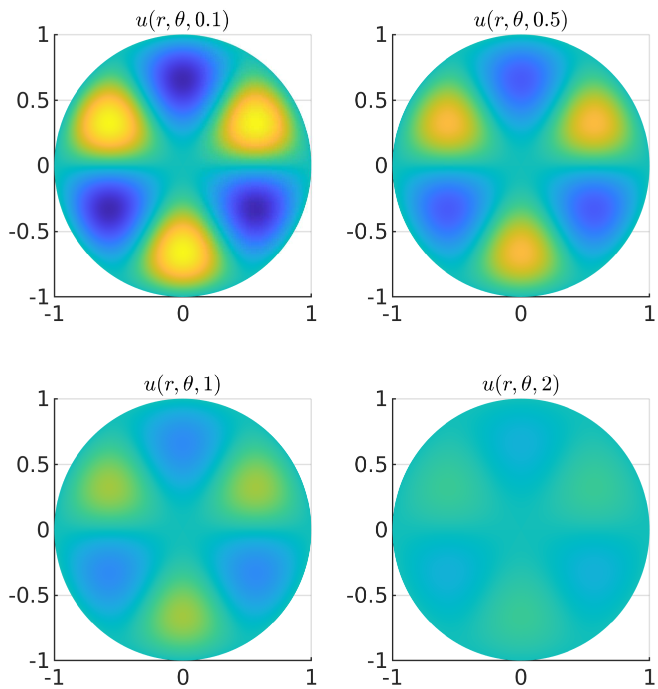
which is solved for the general solution

$$U(t) = Ce^{-\lambda_{3,1}t} + \frac{e^{-t}}{\lambda_{3,1} - 1}.$$

The initial condition $u(r,\,\theta,\,0) = 0$ implies $U(0) = 0$, that in turn determines $C$ as: $C = -\frac{1}{\lambda_{3,1}-1}$. Finally, the given equation is solved for:

$$u(r,\theta,t) = \frac{e^{-t} - e^{-\lambda_{3,1}t}}{\lambda_{3,1} - 1} J_{3,1}(r)\sin(3\theta).$$

The figure below illustrates the change of disk in some instances of time:



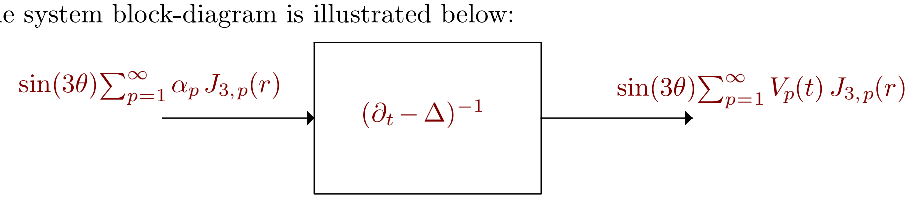

**Exercise 5.26.** Consider the following heat problem:

$$\begin{cases} u_t = \Delta u + e^{-t} r^2 \sin(3\theta) \\ u(1,\theta,t) = 0 \\ u(r,\theta,0) = 0 \end{cases}.$$

The only source terms that triggers the heat system is $h = e^{-t} r^2 \sin(3\theta)$. This source can be represented as the series

$$h = \sum_{p=1}^{\infty} \alpha_p\, e^{-t}\, J_{3,p}(r)\sin(3\theta).$$

The system block-diagram is illustrated below:



Take the solution $u$ as follows:

$$u(r,\theta,t)=\sum_{p=1}^{\infty} V_p(t)\, J_{3,p}(r)\sin(3\theta).$$

Determine the coefficients functions $V_p(t)$ and write down the series solution of the problem.

**Exercise 5.27.** Consider the following heat problem on the unit disk

$$\begin{cases} u_t=\Delta u \\ u(1,\theta,t)=0 \\ u(r,\theta,0)=r\cos(\theta) \end{cases}.$$

The only source that triggers the system is the initial condition $r\cos(\theta)$ which can be represented as

$$r\cos(\theta)=\cos(\theta)\sum_{p=1}^{\infty} \alpha_p\, J_{1,p}(r).$$

Consider the solution of the equation as:

$$u(r,\theta,t)=\sum_{p=1}^{\infty} U_p(t)\, J_{1,2}(r)\cos(\theta).$$

Determine $U_p(t)$ as write down the series solution of the problem.

**Exercise 5.28.** Consider the following heat equation

$$\begin{cases} u_t=\Delta u-r\sin(\theta) \\ u(1,\theta,t)=0 \\ u(r,\theta,0)=0 \end{cases}.$$

a) Take $u$ as the summation

$$u(r,\theta,t)=V(r,\theta)+w(r,\theta,t).$$

Find a closed form solution for the Poisson equation for $V$.

b) What differential equation does $w(r,\theta,t)$ satisfy? Find the series solution of this equation and then write the solution for the original problem.

## Wave problem

Let's solve the following wave equation on the unit disk:

$$\begin{cases} u_{tt}=c^2\Delta u+h(r,\theta,t) \\ u(1,\theta,t)=0 \\ u(r,\theta,0)=0 \\ u_t(r,\theta,0)=0 \end{cases}.$$

Employing the eigenfunction expansion method, the solution $u$ can be expressed as:

$$u(r,\theta,t)=\sum_{n=0,p=1}^{\infty} U_{n,p}(t)\, J_{n,p}(r)\cos(n\theta)+\sum_{n=1,p=1}^{\infty} V_{n,p}(t)\, J_{n,p}(r)\sin(n\theta).$$

The ordinary differential equations for $U_{n,p}$ and $V_{n,p}$ are:

$$\begin{cases} U_{n,p}''+c^2\lambda_{n,p}U_{n,p}=\alpha_{n,p} \\ U_{n,p}(0)=0 \\ U_{n,p}'(0)=0 \end{cases},\ \begin{cases} V_{n,p}''+c^2\lambda_{n,p}V_{n,p}=\beta_{n,p} \\ V_{n,p}(0)=0 \\ V_{n,p}'(0)=0 \end{cases},$$

where $\alpha_{n,p}(t)$ and $\beta_{n,p}(t)$ are respectively the coefficients of the expansion of $h$ in terms of $\{J_{n,p}(r)\cos(n\theta)\}$ and $\{J_{n,p}(r)\sin(n\theta)\}$.

**Example 5.10.** Let's consider the following wave problem

$$\begin{cases} u_{tt} = c^2\Delta u + \sin(t)\sin(\theta) \\ u(1,\theta,t) = 0 \\ u(r,\theta,0) = 0 \\ u_t(r,\theta,0) = 0 \end{cases}.$$

Since the expansion of $h$ is:

$$h = \sin(\theta)\sum_{p=1}^{\infty} \frac{\langle 1, J_{1,p}\rangle_r}{\|J_{1,p}\|^2}\sin(t)\, J_{1,p}(r),$$

we seek for $u$ as:

$$u(r,\theta,t) = \sum_{p=1}^{\infty} U_p(t)\sin(\theta)\, J_{1,p}(r),$$

where $U_p$ satisfies the equation:

$$\begin{cases} U_p'' + c^2\lambda_{1,p}U_p = \frac{\langle 1, J_{1,p}\rangle_r}{\|J_{1,p}\|^2}\sin(t) \\ U_p(0) = 0 \\ U_p'(0) = 0 \end{cases}.$$

For $c \neq \frac{1}{\sqrt{\lambda_{1,p}}}$, the above ODE is solved for

$$U_p(t) = \frac{\langle 1, J_{1,p}\rangle_r}{\|J_{1,p}\|^2(c^2\lambda_{1,p} - 1)}\left(\sin(t) - \frac{1}{c\sqrt{\lambda_{1,p}}}\sin\left(c\sqrt{\lambda_{1,p}}\, t\right)\right).$$

Finally, the solution can be expressed as:

$$u(r,\theta,t) = \sin(\theta)\sum_{p=1}^{\infty} \frac{\langle 1, J_{1,p}\rangle_r}{\|J_{1,p}\|^2(c^2\lambda_{1,p} - 1)}\left(\sin(t) - \frac{1}{c\sqrt{\lambda_{1,p}}}\sin\left(c\sqrt{\lambda_{1,p}}\, t\right)\right) J_{1,p}(r).$$

**Exercise 5.29.** Find a series solution to the following wave equation

$$\begin{cases} u_{tt} = \Delta u + e^{-t}r \\ u(1,\theta,t) = 0 \\ u(r,\theta,0) = 0 \\ u_t(r,\theta,0) = 0 \end{cases}.$$

**Exercise 5.30.** Find a series solution to the following wave problem

$$\begin{cases} u_{tt} = \Delta u \\ u(1,\theta,t) = 0 \\ u(r,\theta,0) = J_{1,2}(r)\sin(\theta) \\ u_t(r,\theta,0) = J_{1,3}(r)\sin(\theta) \end{cases}.$$

**Exercise 5.31.** Consider the following wave equation

$$\begin{cases} u_{tt} = \Delta u - r\sin(\theta) \\ u(1,\theta,t) = 0 \\ u(r,\theta,0) = 0 \\ u_t(r,\theta,0) = 0 \end{cases}.$$

Since the source term is independent of time $t$, we can take the solution $u$ as

$$u(r,\theta,t) = V(r,\theta) + w(r,\theta,t),$$

where $V$ satisfies a Poisson equation.

a) Solve the Poisson equation and determine its solution $V(r,\theta)$.

b) Based on the solution $V$, determine the correct series solution to the wave equation for $w(r,\theta,t)$.

**Exercise 5.32.** Find the series solution for the following wave problem

$$\begin{cases} u_{tt} = c^2\Delta u \\ u(1,\theta,t) = 0 \\ u(r,\theta,0) = r\sin(\theta) \\ u_t(r,\theta,0) = 0 \end{cases}.$$

### 5.3.3 Laplace equation

We consider the following problem on the domain $\Omega$: $[-\pi,\pi]\times[0,a]$:

$$\begin{cases} \Delta u = 0 \\ u(a,\theta) = f(\theta) \end{cases}.$$

To solve this problem, we employ the method of separation of variables and assume a solution of the form $u(r,\theta) = R(r)\Theta(\theta)$. This leads to the equation:

$$\frac{r^2R'' + rR'}{R} + \frac{\Theta''}{\Theta} = 0.$$

Noting that the ratio $\frac{\Theta''}{\Theta}$ should be a dimensionless constant, we express it as $\frac{\Theta''}{\Theta} = -\lambda$. As mentioned before, the function $\Theta(\theta)$ must satisfy periodic boundary conditions:

$$\begin{cases} \Theta(-\pi) = \Theta(\pi) \\ \Theta'(-\pi) = \Theta'(\pi) \end{cases},$$

This gives rise to the following eigenvalue problem:

$$\begin{cases} \Theta'' = -\lambda\Theta \\ \Theta(-\pi) = \Theta(\pi) \\ \Theta'(-\pi) = \Theta'(\pi) \end{cases}.$$

Solving this eigenvalue problem yields the eigenfunctions $\{\cos(n\theta), \sin(n\theta)\}$ for $n = 0, 1, 2, \cdots$. Considering the ratio $\frac{\Theta''}{\Theta} = -n^2$, the equation for $R(r)$ simplifies to the Cauchy-Euler equation:

$$r^2R'' + rR' - n^2R = 0.$$

For $n=0$, the equation is solved as

$$R_0(r) = A_0 + B_0 \ln(r),$$

and for $n \geq 1$, the solution becomes

$$R_n(r) = A_n r^n + B_n r^{-n}.$$

Additionally, the boundedness condition $R(0)$ being bounded implies $B_n = 0$ for $n \geq 0$. Consequently, the separated solution can be written as:

$$u_n(r, \theta) = R_n(r)\,\Theta_n(\theta) = \{r^n \cos(n\theta), r^n \sin(n\theta)\}.$$

The general series solution of the Laplace equation is a linear combination of the separated solutions:

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} r^n [A_n \cos(n\theta) + B_n \sin(n\theta)]$$

The coefficients are determined by the boundary conditions: $u(a, \theta) = f(\theta)$:

$$f(\theta) = A_0 + \sum_{n=1}^{\infty} a^n [A_n \cos(n\theta) + B_n \sin(n\theta)].$$

Using the inner product and orthogonality properties of the functions $\{\cos(n\theta), \sin(n\theta)\}$, we can determine the coefficients as follows:

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta),\ A_n = \frac{1}{\pi a^n} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta),\ B_n = \frac{1}{\pi a^n} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta).$$

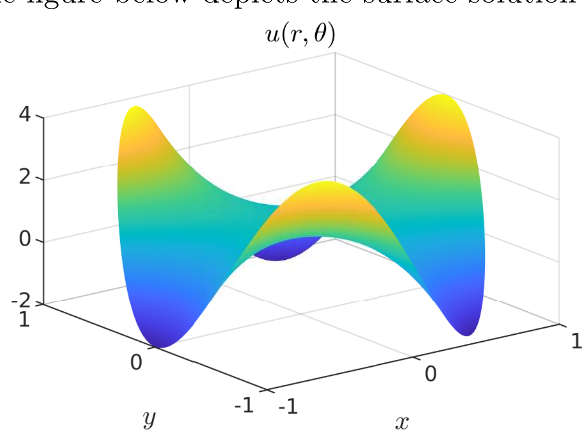
**Example 5.11.** Let's solve the following equation on the disk $r \leq 1$:

$$\begin{cases} \Delta u = 0 \\ u(1, \theta) = 1 + 3\cos(2\theta) \end{cases}.$$

The contribution of the boundary terms $1$ to the solution is the constant function $u = 1$. The contribution of the second boundary term to the solution is: $u = 3r^3 \cos(3\theta)$. Consequently, the solution to the given Laplace equation is given as:

$$u(r, \theta) = 1 + 3r^3 \cos(3\theta).$$

The figure below depicts the surface solution of this Laplace equation:



**Example 5.12.** Let's solve the following equation on the disk $r \leq 1$:

$$\begin{cases} \Delta u = 0 \\ u(1,\theta) = \theta \end{cases}.$$

Since $\theta$ can be expressed by the series:

$$\theta = \sum_{n=1}^{\infty} \frac{-2}{n}(-1)^n \sin(n\theta),$$

the contribution of this term to the solution is

$$u(r,\theta) = \sum_{n=1}^{\infty} \frac{-2}{n}(-1)^n r^n \sin(n\theta).$$

**Exercise 5.33.** Solve the Laplace equation $\Delta u = 0$ inside the unit disk with the boundary conditions given below

a) $u(1,\theta) = \theta^2$

b) $u(1,\theta) = \cos\left(\frac{\theta}{2}\right)$

c) $u(1,\theta) = \sin^2\theta$

**Exercise 5.34.** Solve the following Laplace equation on the unit disk

$$\begin{cases} \Delta u = 0 \\ u(1,\theta) + u_r(1,\theta) = \sin(\theta) \end{cases}.$$

Show that the Laplace equation does not have a solution if the boundary condition is changed to the following one:

$$u(1,\theta) - u_r(1,\theta) = \sin(\theta).$$

**Exercise 5.35.** We aim to prove that the following equation defined inside the unit disk has a unique solution

$$\begin{cases} \Delta u = 0 \\ u(1,\theta) + u_r(1,\theta) = f(\theta) \end{cases}.$$

a) Assume that $u_1(r,\theta)$, $u_2(r,\theta)$ are two solutions to the problem. The function $u = u_1 - u_2$ solves the following equation:

$$\begin{cases} \Delta u = 0 \\ u(1,\theta) + u_r(1,\theta) = 0 \end{cases}.$$

Since $u(r,\theta)$, the solution to the equation, is smooth, we can express it as the following Fourier series:

$$u(r,\theta) = U_0(r) + \sum_{n=1}^{\infty} A_n(r)\cos(n\theta) + B_n(r)\sin(n\theta).$$

Substitute this into the equation $\Delta u = 0$ and conclude $A_n(r) = A_n r^n$, $B_n(r) = B_n r^n$ for $n = 0, 1, \cdots$.

b) Finally use the boundary condition and show $u(r,\theta) = 0$ and conclude $u_1(r,\theta) = u_2(r,\theta)$.

**Exercise 5.36.** Consider the following Laplace equation inside the unit ball

$$\begin{cases} \Delta u = 0 \\ u_r(1,\theta) = f(\theta) \end{cases},$$

where $f(\theta)$ is a smooth function in $\theta$.

a) Is there a unique solution to the problem?

b) Find a solution if $f(\theta) = \cos\theta$.

c) Is there any solution if $f(\theta) = 1$ ?

d) Show that the necessary condition for the problem to have a solution is

$$\int_{-\pi}^{\pi} f(\theta) d\theta = 0.$$

**Exercise 5.37.** In the case of solving a Laplace equation in the exterior of a disk, that is, in the region $\Omega^c := \{(r,\theta), r > a\}$, we keep only the terms $R_n(r) = r^{-n}$ and reject $R_n(r) = r^n$ and $\ln(r)$. The reason is that only the bounded solutions of Laplace equation are meaningful physically. Solve the following Laplace equation in the exterior of a disk of radius $a = 2$

$$\begin{cases} \Delta u = 0 \\ u(2,\theta) = 1 + 2\sin(2\theta) \end{cases}$$

**Exercise 5.38.** Let $\Omega$ be the region outside of the unit disk and interior of the disk of radius $2$. SOlve the following Laplace equation on $\Omega$

$$\begin{cases} \Delta u = 0 \\ u(1,\theta) = \sin(\theta) \\ u(2,\theta) = \sin(2\theta) \end{cases}$$

### 5.3.4 Eigenvalue problem on an annulus

Let $\Omega$ be the region enclosed by two disks $r < b$ and $r < a$, i.e.,

$$\Omega = \{(r,\theta), a < r < b\}.$$

Consider the following eigenvalue problem

$$\begin{cases} \Delta\phi = -\lambda\phi \\ \phi(a,\theta) = \phi(b,\theta) = 0 \end{cases}.$$

For the separated solution $\phi_n(r,\,\theta) = R_n(r)\,\Theta_n(\theta)$, the solution for $R_n(r)$ reduces to the following one:

$$R_n(r) = A_n\,J_n(\sqrt{\lambda}\,r) + B_n\,Y_n(\sqrt{\lambda}\,r).$$

Note that in this case we should keep the second type of the Bessel functions $Y_n(\sqrt{\lambda}\,r)$ as the origin is excluded and thus $Y_n$ is bounded in the annulus domain $\Omega$. The boundary conditions for $R_n(r)$ results to the following equations:

$$\begin{cases} A_n\,J_n(\sqrt{\lambda}\,a) + B_n\,Y_n(\sqrt{\lambda}\,a) = 0 \\ A_n\,J_n(\sqrt{\lambda}\,b) + B_n\,Y_n(\sqrt{\lambda}\,b) = 0 \end{cases}$$

A straightforward algebraic manipulation leads to the following equation:

$$J_n(\sqrt{\lambda}a)\,Y_n(\sqrt{\lambda}b) - J_n(\sqrt{\lambda}b)\,Y_n(\sqrt{\lambda}a) = 0.$$

Taking $x=\sqrt{\lambda}a$, the above equation reduces to the following one:

$$J_n(x)\,Y_n\!\left(\frac{b}{a}x\right)-J_n\!\left(\frac{b}{a}x\right)Y_n(x)=0.$$

If the zeroth of the above equation are denoted by $x=s_{np}$, the eigenvalues are obtained as $\lambda_{np}=\frac{s_{np}^2}{a^2}$. Substituting $\lambda_{np}$ into the above system of equations gives

$$\begin{cases} A_{np}\,J_n(s_{np})+B_{np}\,Y_n(s_{np})=0\\ A_{np}\,J_n\!\left(\frac{s_{np}\,b}{a}\right)+B_{np}\,Y_n\!\left(\frac{s_{np}\,b}{a}\right)=0\end{cases}.$$

Now, constants $A_{np}$, and $B_{np}$ are simply determined by the relation

$$\frac{A_{np}}{B_{np}}=-\frac{Y_n(s_{np})}{J_n(s_{np})}.$$

Hence

$$R_{np}(r)=Y_n(s_{np})J_n\!\left(\sqrt{\lambda_{np}}\,r\right)-J_n(s_{np})\,Y_n\!\left(\sqrt{\lambda_{np}}\,r\right).$$

Note that $R_{np}(r)$ satisfy the equation

$$r^2\frac{d^2R_{np}}{dr^2}+r\frac{dR_{np}}{dr}-n^2R_{np}=-\lambda_{np}\,R_{np},$$

and therefore

$$\langle R_{np},R_{nq}\rangle_r=0,p\neq q.$$

The eigenfunctions of $\Delta$ in $\Omega$ are

$$\phi_{np}(r,\theta)=\{R_{np}(r)\cos(n\theta),R_{np}(r)\sin(n\theta)\},$$

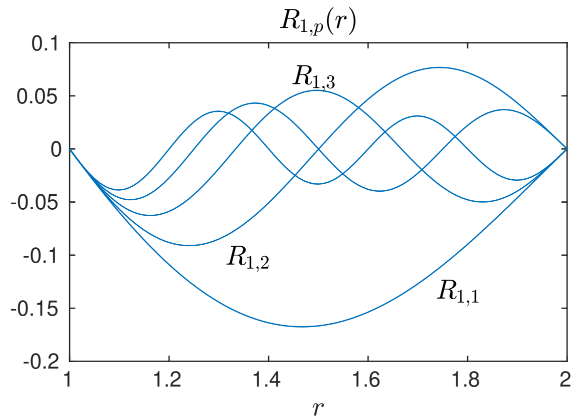
for $n=0,1,2,\cdots$ with associated eigenvalue $\lambda_{n,p}=\frac{s_{n,p}^2}{a^2}$. These eigenparis satisfy the following relation:

$$\Delta\phi_{np}=-\frac{s_{np}^2}{a^2}\,\phi_{np}.$$

The table below gives some values of $s_{np}$ for $b=2a$.

| $p$ | $s_{0p}$ | $s_{1p}$ | $s_{2p}$ |
|---|---|---|---|
| 1 | 3.1230 | 3.1966 | 3.4069 |
| 2 | 6.2734 | 6.3123 | 6.4278 |
| 3 | 9.4182 | 9.4445 | 9.5228 |
| 4 | 12.5614 | 12.5812 | 12.6404 |
| 5 | 15.7040 | 15.7198 | 15.7673 |

The figure below depicts a few of eigenfunctions $R_{1,p}(r)$ in the region $\Omega$: $\{(r,\theta),1<r<r\}$. As we observed they satisfy the homogeneous boundary conditions at $r=1$ and $r=2$




**Example 5.13.** Let's solve the following Poisson equation on $\Omega := \{(r,\theta); 1<r<2\}$

$$\begin{cases} \Delta u = r\sin(\theta) \\ u(1,\theta) = u(2,\theta) = 0 \end{cases}.$$

The series solution of the problem is expressed as

$$u(r,\theta) = \sin(\theta)\sum_{p=1}^{\infty} B_p R_{1,p}(r).$$

Substituting this into the equation gives $B_p$ as

$$B_p = \frac{-1}{s_{1,p}^2}\frac{\langle r, R_{1,p}\rangle_r}{\|R_{1,p}\|^2}.$$

Here, we calculate a few of coefficients $B_p$:

$$B_1 = 1.1314 \quad B_2 = -0.143, B_3 = 0.13416, B_4 = -0.0362, B_5 = 0.0486$$

We also can find a closed form solution as follows. We take the solution of the form $u(r,\theta) = R(r)\sin(\theta)$. Substituting this solution into the Poisson equation gives the following equation for $R(r)$:

$$r^2 R'' + r R' - R = r^3.$$

This equation is solved for

$$R(r) = Ar + Br^{-1} + \frac{1}{8}r^3.$$

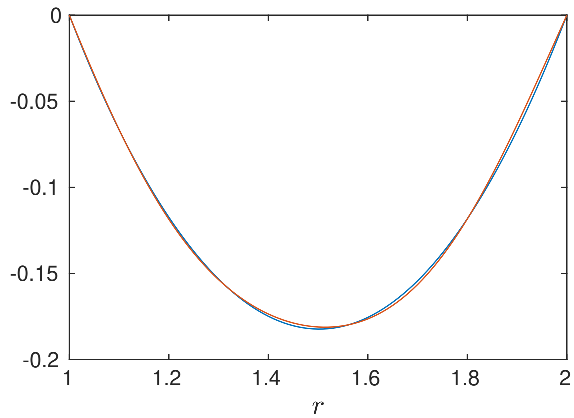
Applying the boundary conditions gives: $A = \frac{-5}{8}$ and $B = \frac{1}{2}$, and consequently:

$$u(r,\theta) = \left(-\frac{5}{8}r + \frac{1}{2}r^{-1} + \frac{1}{8}r^3\right)\sin(\theta).$$

by the uniqueness, we conclude the equality

$$-\frac{5}{8}r + \frac{1}{2}r^{-1} + \frac{1}{8}r^3 = \sum_{p=1}^{\infty} B_p R_{1,p}(r).$$

The figure below depicts This function and its approximation up to only three terms in the same coordinate:




**Exercise 5.39.** Solve the following heat equation on the domain $\Omega := \{(r,\theta); 1 < r < 2\}$

$$\begin{cases} u_t = \Delta u + e^{-t}\sin(\theta) \\ u(1,\theta,t) = 0 \\ u(2,\theta,t) = 0 \\ u(r,\theta,0) = 0 \end{cases}.$$

### 5.3.5 Problems

**Problem 5.36.** Let $\Omega$ denote a disk with radius $a = 1$. Consider the following problem on $\Omega$

$$\begin{cases} \Delta u = r\cos\theta \\ u(1,\theta) = 0 \end{cases}.$$

a) Find a double series solution to the problem in terms of the eigenfunctions $\{J_{n,p}(r)\cos(n\theta), J_{n,p}(r)\sin(n\theta)\}$.

b) Find a closed form solution to the problem and justify it is equal to the series solution.

**Problem 5.37.** Consider the Poisson equation $\Delta u = \theta$ defined on the unit disk satisfying the homogeneous Dirichlet boundary condition.

a) Find a series solution to the problem in terms of the basis $\{J_{n,p}(r)\cos(n\theta), J_{n,p}(r)\sin(n\theta)\}$. This is a double series solution to the problem.

b) Consider the solution $u$ as follows

$$u(r,\theta) = \sum_{n=1}^{\infty} U_n(r)\sin(n\theta).$$

Note that $\theta$ is an odd function on $[-\pi,\pi]$, and its Fourier series contains only sine terms. Obtain a differential equation for $U_n(r)$ and solve it to determine the single series solution.

**Problem 5.38.** Let $\Omega$ denote a disk with radius $a = 1$. Consider the following problem on $\Omega$

$$\begin{cases} \Delta u = \sin(\theta) \\ u_r(1,\theta) = 0 \end{cases}.$$

Here the boundary condition is the homogeneous Neumann boundary condition.

a) Find a closed form solution to the problem by take $u$ as: $u = U(r)\sin(\theta)$.

b) To find a series solution to the problem, we need to solve the eigenvalue problem

$$\begin{cases} \Delta\phi = -\lambda\phi \\ \phi_r(1,\theta) = 0 \end{cases}.$$

Show that the eigenfunctions are

$$\phi_{n,p}(r,\theta) \in \{J_n(\zeta_{n,p} r)\cos(n\theta), J_n(\zeta_{n,p} r)\sin(n\theta)\},$$

for $n = 0, 1, 2, \cdots$ where $\zeta_{n,p}$ are the roots of $J_n'(x)$. Here are a few of them:

$$\zeta_{1,1} = 1.8412, \zeta_{1,2} = 5.3314, \zeta_{1,3} = 8.5363, \zeta_{1,4} = 11.7060, \zeta_{1,5} = 14.8636$$

**Problem 5.39.** Let $\Omega$ be the unit disk. Consider the Poisson problem $\Delta u = f(r,\theta)$ on $\Omega$ satisfying the homogeneous Neumann boundary condition $u_r(1,\theta) = 0$.

a) Show that the condition for solvability of the problem is:

$$\iint_\Omega f(r,\theta)\, r\, dr\, d\theta = 0.$$

Hint: Use the Gauss theorem for the integral

$$\iint_\Omega \Delta u\, dS.$$

b) Show that the eigenvalue problem

$$\begin{cases} \Delta\phi = -\lambda\phi \\ \phi(1,\theta) = 0 \end{cases},$$

accepts the eigenvalue $\lambda = 0$. What is the relationship of this result to the solvability condition of the Poisson equation?

**Problem 5.40.** Find a single series solution to the following problem on the unit disk

$$\begin{cases} \Delta u = \theta \\ u(1,\theta) = \cos\theta \end{cases}.$$

**Problem 5.41.** Let $\Omega^c$ denote the exterior domain of the unit disk in $\mathbb{R}^2$. Find a closed form solution to the following problem on $\Omega^c$

$$\begin{cases} \Delta u = \frac{1}{r^2}\sin\theta \\ u(1,\theta) = 0 \end{cases}.$$

**Problem 5.42.** Consider the following differential equation

$$\begin{cases} \Delta u + \frac{1}{r}u_r = r\sin(\theta) \\ u(1,\theta) = 0 \end{cases}.$$

We do not have any information about the eigenfunctions of the operator $\left(\Delta + \frac{1}{r}\partial_r\right)$. However, we can find a closed form solution to the problem. Take $u$ as $u(r,\theta) = R(r)\sin(\theta)$. Obtain a Cauchy-Euler equation for $R(r)$, solve and and determine the solution $u$.

**Problem 5.43.** Consider the Poisson equation $\Delta u = \sin(\theta)$ on the region $\Omega$: $\{(r,\theta), 1 < r < 2\}$. Assume that $u$ satisfies the boundary conditions $u_r(1,\theta) = 0$ and $u(2,\theta) = 0$.

a) Find eigenvalues and eigenfunctions of the eigenvalue problem

$$\begin{cases} \Delta\phi = -\lambda\phi \quad \text{on } \Omega \\ \phi_r(1,\theta) = 0 \\ \phi(2,\theta) = 0 \end{cases}.$$

Use a numerical method to find first five eigenvalues. Use the results and write a series solution for the given problem.

b) Consider the closed form solution $u$ as $u = R(r)\sin(\theta)$. Determine $R(r)$.

c) With the result in part a), solve the following heat equation on $\Omega$

$$\begin{cases} u_t = \Delta u - e^t \sin(\theta) & \text{on } \Omega \\ u_r(1,\theta,t) = 0 \\ u(2,\theta,t) = 0 \\ u(r,\theta,0) = 0 \end{cases}.$$

**Problem 5.44.** Solve the Poisson equation $\Delta u = \cos(\theta)$ on the domain $\Omega = \{(r,\theta); 0 \le r < 1, 0 \le \theta \le \pi\}$ satisfying the boundary conditions

$$u(r,0) = u(r,\pi) = 0.$$

**Problem 5.45.** Let $\Omega$ be the unit disk. Consider the following heat equation on $\Omega$

$$\begin{cases} u_t = \Delta u - r \\ u(t,1,\theta) = 0 \\ u(0,r,\theta) = \cos\theta \end{cases}.$$

Take the solution $u$ as

$$u(r,\theta,t) = V(r,\theta) + w(r,\theta,t),$$

where $V$ satisfies the following Poisson equation

$$\begin{cases} \Delta u = r \\ u(1,\theta) = 0 \end{cases}.$$

a) Find a closed form solution to the equation for $V$.

b) Use the result in part a) and write downs the value series solution for $w$. Write down the solution for $u$.

**Problem 5.46.** Let $\Omega$ be the unit disk. Consider the following heat problem on $\Omega$:

$$\begin{cases} u_t = \Delta u - r^2 \\ u(1,\theta,t) = \cos(\theta) \\ u(r,\theta,0) = \frac{1}{16}(r^4 - 1) \end{cases}.$$

Consider the solution $u$ as

$$u(r,\theta,t) = V(r,\theta) + w(r,\theta,t),$$

where $V$ satisfies the following Poisson equation

$$\begin{cases} \Delta u = r^2 \\ u(1,\theta) = \cos(\theta) \end{cases}$$

a) Find a closed form solution to the equation for $V$.

b) Use the result in part a) and write downs the value series solution for $w$. Write down the solution for $u$.

**Problem 5.47.** Let $\Omega$ be the unit disk. Consider the following wave equation

$$\begin{cases} u_{tt} = \Delta u - \cos\theta \\ u(1,\theta,t) = 0 \\ u(r,\theta,0) = \frac{1}{3} r^2 \cos\theta \\ u_t(r,\theta,0) = 0 \end{cases}.$$

Take the solution $u$ as

$$u(r,\theta,t) = V(r,\theta) + w(r,\theta,t),$$

where $V$ satisfies a Poisson equation.

a) Solve the Poisson equation for $V$

b) Use the result of part a) and write the series solution for $w$.

**Problem 5.48.** Let $C$ denote the unit circle of radius $1$. As it is known, the Laplacian $\Delta$ on $C$ has the form $\Delta u = u_{\theta\theta}$. Solve the following problem on $C$

$$\begin{cases} u_t = u_{\theta\theta} - e^{-t}\sin\theta \\ u(\theta, 0) = 0 \end{cases}.$$

**Problem 5.49.** Let $C$ denote the unit circle of radius $1$. Solve the following wave equation on $C$

$$\begin{cases} u_{tt} = u_{\theta\theta} \\ u(\theta, 0) = f(\theta) \\ u_t(\theta, 0) = 0 \end{cases},$$

where $f(\theta)$ is the following function

$$f(\theta) = \begin{cases} \theta^2 - \frac{\pi^2}{64} & -\frac{\pi}{8} < \theta < \frac{\pi}{8} \\ 0 & \text{otherwise} \end{cases}.$$

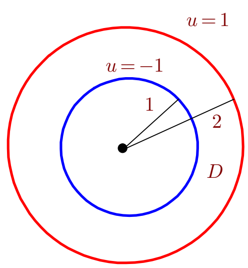
**Problem 5.50.** Let $\Omega$ be the unit disk. Write a series solution to the following problem

$$\begin{cases} u_t = \Delta u - e^{-t}\cos\theta \\ u(1, \theta, t) = \cos(2\theta) \\ u(r, \theta, 0) = \cos(3\theta) \end{cases}.$$

**Problem 5.51.** Let $\Omega$ be the unit disk. Write the series solution to the following problem on $\Omega$ in the series form

$$\begin{cases} u_{tt} = c^2\Delta u + \sin(t)\theta \\ u(1, \theta, t) = 0 \\ u(r, \theta, 0) = 0 \\ u_t(r, \theta, 0) = 0 \end{cases}.$$

**Problem 5.52.** Let $\Omega$ be the domain outside of the disk $r = 1$ and inside the disk $r = 2$. We aim to solve the heat problem $u_t = \Delta u$ on $\Omega$ subject to the boundary condition shown in the figure:




a) Solve the Laplace equation $\Delta V = 0$ subject to the boundary condition given in the figure. Hint: the solution is a pure function of $r$ and independent of $\theta$.

b) Solve the given heat problem if $u(r, \theta, 0) = 0$ by taking $u$ as $u = V(r) + w(r, \theta, t)$.

**Problem 5.53.** Solve the following wave equation on $\Omega := \{(r, \theta), 1 < r < 2\}$

$$\begin{cases} u_{tt} = \Delta u \\ u_r(1, \theta, t) = 0 \\ u(2, \theta, t) = 0 \\ u(r, \theta, 0) = \cos\left(\frac{\pi}{2}(r-1)\right)\sin(\theta) \\ u_t(r, \theta, 0) = 0 \end{cases}.$$

**Problem 5.54.** Let $\Omega$ be the domain $\left\{(r,\theta); 0\leq r<1, 0<\theta<\frac{\pi}{2}\right\}$. We aim to solve the following wave equation for the function $u(r,\theta,t)$:

$$u_{tt}=c^2\,\Delta u+r\sin(3\theta) \quad \text{on}\,\Omega$$

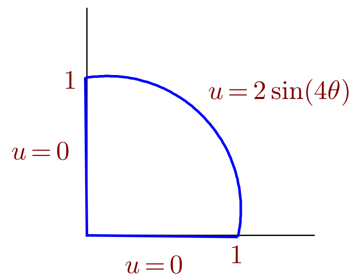
Subject to the following boundary conditions:

$$\begin{cases} u(r,0,t)=0 \\ u_\theta\left(r,\frac{\pi}{2},t\right)=0 \\ u(1,\theta,t)=0 \end{cases}.$$

a) Find the eigenvalues and eigenfunctions of the associated eigenvalue problem.

b) Find a series solution to the problem if the initial conditions are given by: $u(r,\,\theta,\,0)=0$ and $u_t(r,\theta,0)=0$.

**Problem 5.55.** Let $\Omega$ denote the first quarter of a unit disk shown in the following figure




a) Solve the Laplace equation $\Delta u=0$ in $\Omega$ with the boundary conditions given in the figure.

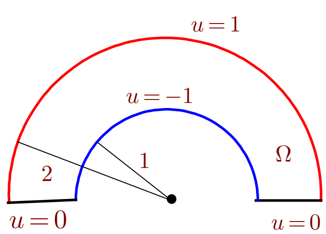
b) Find a closed form solution to the Poisson equation

$$\begin{cases} \Delta u=\sin(2\theta) & \text{on}\,\Omega \\ u=0 & \text{on}\,\text{bnd}(\Omega) \end{cases}.$$

c) Find a series solution to the following heat problem

$$\begin{cases} u_t=\Delta u-\sin(2\theta) \\ u(r,0)=u\left(r,\frac{\pi}{2}\right)=0 \\ u(1,\theta)=2\sin(4\theta) \\ u(r,\theta,t)=2r^4\sin(4\theta)+\frac{7}{6}r^2\ln r\sin(2\theta) \end{cases}$$

**Problem 5.56.** Consider the following figure




a) Solve the Laplace equation $\Delta V=0$ on the domain shown in the figure and associated boundary condition.

b) Solve the Poisson equation $\Delta u=\theta$ on$\Omega$. Hint: Use the single series for $u$ as:

$$u(r,\theta)=\sum_{n=1}^{\infty} U_n(r)\sin(n\theta).$$

**Problem 5.57.** We aim to solve the heat equation $u_t = \Delta u$ inside the domain $\Omega := \{(r,\theta), 0 \le r < 1, 0 \le \theta \le \pi\}$ subject to the boundary conditions

$$\begin{cases} u(1,\theta,t) = 1. \\ u(r,0,t) = u(r,\pi,\pi) = 0 \end{cases}.$$

a) Determine the eigenfunctions and eigenvalues of the Laplacian operator $\Delta$ on $\Omega$ satisfying the homogeneous boundary conditions:

$$\begin{cases} \Delta\phi = -\lambda\phi & \text{on } \Omega \\ \phi = 0 & \text{on bnd}(\Omega) \end{cases}.$$

b) Solve the Laplace equation $\Delta V = 0$ on $\Omega$ adheres to the prescribed boundary conditions.

c) Solve the given heat problem utilizing the separation $u = V(r,\theta) + w(r,\theta,t)$.

**Problem 5.58.** Let $\Omega$ be the semi-disk: $\Omega = \left\{(r,\theta); 0 \le r < 1, \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}\right\}$.

a) Determine the eigenfunctions and eigenvalues of the Laplacian operator $\Delta$ in $\Omega$ subject to toe homogeneous Dirichlet boundary condition.

b) Consider the following heat problem on $\Omega$:

$$\begin{cases} u_t = \Delta u - r\sin(2\theta) & \text{on } \Omega \\ u = 0 & \text{on bnd}(\Omega) \\ u(r,\theta,0) = 0 \end{cases}.$$

Since the source term is independent of time, we can consider the solution $u$ as:

$$u(r,\theta,t) = V(r,\theta) + w(r,\theta,t),$$

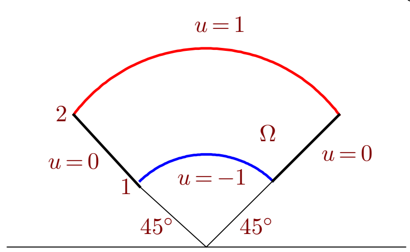
where the function $V$ satisfies the Poisson equitation

$$\begin{cases} \Delta V = r\sin(2\theta) & \text{on } \Omega \\ V = 0 & \text{on bnd}(\Omega) \end{cases}.$$

Take $V$ of the form $V = R(r)\sin(2\theta)$ for an undetermined function $R(r)$. Substitute $V$ into the Poisson equation and derive a Cauchy-Euler equation. Solve this equation and determine $V$. This is a closed form solution for the Poisson equation.

c) Write down the equation for $w$, and determine the correct series solution to it based on the form of the source term $h = r\sin(2\theta)$. Determine the solution $u(r,\theta,t)$.

**Problem 5.59.** Consider the following domain:




a) Determine the eigenfunctions and eigenvalues of the Laplacian operator $\Delta$ on the domain $\Omega$ subject to the homogeneous Dirichlet boundary condition.

b) Solve the Laplace equation $\Delta V = 0$ on the above domain with the given boundary conditions shown in the figure.

c) Solve the heat equation

$$\begin{cases} u_t = \Delta u \\ u(r,\theta,t) = 0 \end{cases},$$

where $u$ satisfies the given boundary conditions in the figure.