Chapter 4

1D Linear Second-Order Equations

In this chapter, our primary focus is on introducing the series solution approach for 1D heat and wave differential equations. Building upon the foundations laid in the preceding chapters, where we explored the derivation of these equations in higher dimensions and analyzed their overarching behavior, we now delve into the specific context of solving heat and wave equations within bounded media extending along the x-axis, defined as the interval (x_0, x_1).

As we navigate this chapter, we encounter a crucial factor that heavily influences the nature of the solution: the conditions at the boundary points $x=x_0$ and $x=x_1$. This prompts the emergence of boundary value problems as a pivotal concept in this context. To enhance our understanding, we introduce a renowned theorem, the Sturm-Liouville theorem, which assumes a critical role in our exploration of the eigenfunction expansion method. This method becomes instrumental in representing the solutions of 1D partial differential equations.

4.1 From PDE to eigenvalue problem

This section serves as a bridge between partial differential equations (PDEs) and eigenvalue problems.

4.1.1 Outline of the method

We'll focus on the context of an open interval $\Omega=(x_0, x_1)$, where we encounter a PDE that governs the function $u=u(x,t)$:

$$u_t = L[u]$$

where L takes the form of a 1D differential operator:

$$L := a(x)\frac{\partial^2}{\partial x_2} + b(x)\frac{\partial}{\partial x} + c(x). \tag{4.1}$$

In our scenarios, a, b, and c are continuous functions defined within Ω. This equation finds practical applications, like modeling temperature distribution along an extended conductive rod within Ω. However, to establish a complete formulation, we need to determine conditions at both endpoints, $x=x_0$ and $x=x_1$. This is because the differential equation is confined to the open interval Ω.

We proceed by assuming that the function u adheres to a particular condition at these endpoints:

$$\alpha u + \beta u_x = 0.$$

Additionally, the initial temperature distribution, $u(x, 0)$, along the rod must be specified. The interplay between the thermal energy distribution along the rod and heat diffusivity generates the dynamic heat behavior of the system. By combining these elements, we present a comprehensive initial-boundary value problem:

$$\begin{cases} u_t = L[u] & \text{on } \Omega \\ \alpha u + \beta u_x = 0 & \text{on } \mathrm{bnd}(\Omega) \\ u(x, 0) = f(x) & \text{initial condition} \end{cases}.$$

This synthesis of the differential operator L, boundary conditions, and initial condition lays the foundation for solving various heat and wave equations, gradually unraveling the intricate dynamics of these systems.

As L is a linear operator that operates solely on the spatial variable x, we can treat the solution of the equation $u_t = L[u]$ as an ordinary differential equation for u and express it as: $u_t = L[u]$, as an ordinary differential equation for u and write it as: $u(x, t) = \phi(x)\, e^{-\lambda t}$ where $\phi(x)$ is an unknown function and λ is a constant. To verify its validity, substituting this into the equation gives:

$$-\lambda \phi(x)\, e^{-\lambda t} = e^{-\lambda t} L[\phi],$$

which leads to: $L[\phi] = -\lambda\phi$. To satisfy the prescribed boundary conditions for u, we arrive at the subsequent boundary value problem, often termed the eigenvalue problem:

$$\begin{cases} L[\phi] = -\lambda\phi & \text{on } \Omega \\ \alpha\phi + \beta\phi' = 0 & \text{ob } \mathrm{bnd}(\Omega) \end{cases}$$

Remark 4.1. The use of terminology can be justified by drawing parallels with the eigenvalue problem in linear algebra. Remember that a vector v in $\mathbb{R}^n$ is an eigenvector of a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ if $T[v] + \lambda v = 0$. In the context of our problem, the operator L acts linearly on the space of smooth functions defined on Ω. Consequently, a function $\phi: \Omega \to \mathbb{R}$ is termed an eigenfunction if $L[\phi](x) + \lambda\phi(x) = 0$ holds for all x in Ω. Following the analogy with the eigenvalue problem in linear algebra, the value λ is referred to as the associated eigenvalue of the eigenfunction ϕ. It's important to note that if ϕ is an eigenfunction of the problem, then $c\phi(x)$ for any non-zero constant c is also an eigenfunction. Just like eigenvectors in linear algebra, eigenfunctions are typically considered as non-trivial functions.

Solving an eigenvalue problem involves determining the eigenfunctions $\phi(x)$ and eigenvalues λ. When the problem can be solved, the solution of the equation $u_t = L[u]$ while adhering to the boundary condition $\alpha u + \beta u_x = 0$ can be expressed as follows: $u(x, t) = \phi(x)\, e^{-\lambda t}$.

Example 4.1. Consider the interval $(0, 1)$ and let Ω represent this interval. Let's explore the eigenvalue problem $\phi'' = -\lambda\phi$, where $\phi(x)$ is defined on Ω. The boundary conditions are set as $\phi(0) = 0$ and $\phi(1) = 0$. We want to find values of λ and corresponding non-trivial functions $\phi(x)$ that satisfy this equation.

First, let's show that the problem has a solution when λ is greater than zero. To do this, we multiply the equation by $\phi(x)$ and integrate it over the interval $(0,1)$. This gives us an equation involving integrals and the boundary term:

$$-\int_0^1 |\phi'(x)|^2\,dx + \phi'(x)\,\phi(x)\Big|_0^1 = -\lambda\int_0^1 |\phi(x)|^2\,dx,$$

where we used the integration by parts formula for the integral at the left-hand side. The boundary term $\phi'(x)\,\phi(x)|_0^1$ is equal zero due to the given boundary conditions. This gives the following formula for λ:

$$\lambda = \frac{\displaystyle\int_0^1 |\phi'(x)|^2\,dx}{\displaystyle\int_0^1 |\phi(x)|^2\,dx} \geq 0.$$

If we were to assume $\lambda = 0$, we'd find that $\phi(x)$ turns out to be a constant function. But considering the boundary conditions and continuity, we'd be left with the trivial function $\phi(x) = 0$. Therefore, λ must be greater than zero.

With this understanding, we proceed to solve the equation $\phi'' = -\lambda\phi$ as an ordinary differential equation. The general solution for $\phi(x)$ can be expressed as:

$$\phi(x) = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x).$$

Plugging in the boundary condition $\phi(0) = 0$, we find that $A = 0$. Applying the second boundary condition, we determine that the eigenvalues λ are given by $\lambda = n^2\pi^2$ for $n = 1, 2, \cdots$.

For each eigenvalue λ_n, we obtain an associated eigenfunction $\phi_n(x) = B_n\,\sin(n\pi x)$, where B_n is a non-zero constant. This set of eigenvalues and eigenfunctions forms an infinite collection: $\phi_1(x) = \sin(\pi x)$, $\lambda_1 = \pi^2$, $\phi_2(x) = \sin(2\pi x)$, $\lambda_2 = 4\pi^2$, $\phi_3(x) = \sin(3\pi x)$, $\lambda_3 = 9\pi^2$ and so on. These eigenfunctions provide the building blocks for solving more complex differential equations and understanding the behavior of solutions on the interval $(0,1)$.

Exercise 4.1. Let's explore another approach to demonstrate that $\lambda > 0$ in the previously discussed eigenvalue problem. This will involve direct calculations. Begin by considering the algebraic characteristic equation for the differential equation $\phi'' = -\lambda\phi$, which results in $r = \pm\sqrt{-\lambda}$.

Task 1: Suppose $\lambda < 0$. Proceed by establishing the general solution for this ordinary differential equation (ODE). Afterward, apply the given boundary conditions and deduce that the only possible outcome is $\phi = 0$.

Task 2: Assume that $\lambda = 0$. Derive the general solution for this ODE and arrive at the conclusion that $\phi = 0$ is again the only feasible solution.

Task 3: Now focus on the case where $\lambda > 0$. Determine the non-trivial eigenfunctions that correspond to these positive eigenvalues.

Exercise 4.2. Building upon the insights gained from the previous example, let's apply that knowledge to draw conclusions about the heat problem given as:

$$\begin{cases} u_t = u_{xx} & x \in (0,1) \\ u(0,t) = u(1,t) = 0 & \end{cases}.$$

Task: Utilize the outcomes from the previous example and deduce that this heat problem possesses an infinite number of solutions, denoted as $u_n(x,t) = e^{-n^2\pi^2 t}\sin(n\pi x)$ for any positive integer $n = 1, 2, \cdots$. These solutions are often referred to as "separated solutions" due to their form.

Additionally, employing the superposition principle, you can express the general solution as:

$$u(x,t) = \sum_{n=1}^{\infty} c_n\, u_n(x,t),$$

Here, the constants c_n can take any arbitrary values, provided that the infinite series converges to a valid function.

4.1.2 Separation of variables

The derivation of the eigenvalue problem discussed above can also be expressed in terms of the separation of variables technique. By taking $u(x,t)$ as the separated function:

$$u(x,t) = \phi(x)\, U(t),$$

and substituting this into the heat equation, we arrive at:

$$\frac{U'}{U} = \frac{L[\phi]}{\phi},$$

where $L[\phi]$ represents the operator applied to $\phi(x)$. This equation is possible only when the ratios are equal to the same dimensionless constant. Let's denote this constant by $-\lambda$. The minus sign is used for historical reasons and does not carry a physical meaning, but it will become clear why this choice is more appropriate for subsequent calculations.

By introducing $-\lambda$ as the eigenvalue, we obtain the following ordinary differential equation for $U(t)$:

$$U' = -\lambda U,$$

which is solved as $U(t) = \mathrm{C}e^{-\lambda t}$, where C is a constant. The eigenvalue λ and eigenfunction $\phi(x)$ are determined by solving the second ordinary differential equation:

$$L[\phi] = -\lambda \phi,$$

subject to the boundary conditions: $\alpha\phi + \beta\phi' = 0$ at the boundary points.

The derivation holds true for general 1D partial differential equations. Consider the following second-order PDE for $u(x,t)$ on the interval $x \in [x_0, x_1]$:

$$\begin{cases} a_1(t) u_{tt} + a_2(t)\, u_t = L[u] \\ \alpha_1 u(x_0,t) + \beta_1\, u_x(x_0,t) = 0 \\ \alpha_2 u(x_1,t) + \beta_2\, u_x(x_1,t) = 0 \end{cases},$$

where $a_1(t)$ and $a_2(t)$ are continuous functions. The equation is general enough to encompass all types of 1D heat and wave equations. To find a solution to this PDE, we consider the solution in the separated form as $u(x,t) = U(t)\phi(x)$. By substituting this separated function into the PDE, we obtain the equality

$$\frac{a_1(t) U'' + a_2(t) U'}{U} = \frac{L[\phi]}{\phi}.$$

Obviously, the equality holds only if the ratio is a dimensionless constant since x and t are independent variables. Therefore, we can write

$$\frac{a_1(t)U'' + a_2(t)U'}{U} = \frac{L[\phi]}{\phi} = -\lambda,$$

where λ is a constant. Here, the negative sign does not carry any significant physical meaning; it is merely a historical convention. Therefore, the separated form of the solution leads to the following ordinary differential equations:

1. For $U(t)$, we have the equation:

$$a_1(t)\, U'' + a_2(t)\, U' = -\lambda U,$$

2. For $\phi(x)$, we have the eigenvalue problem:

$$\begin{cases} L[\phi] = -\lambda\phi & \text{on } \Omega \\ \alpha_1\phi(x_0) + \beta_1\, \phi'(x_0) = 0 & \\ \alpha_2\phi(x_0) + \beta_2\, \phi'(x_1) = 0 & \end{cases}.$$

4.1.3 Series solution

Superposition solutions hold value when the coefficients c_n are carefully chosen to ensure that the infinite series converges into a smooth function $u(x,t)$. The convergence of infinite function series is a complex topic beyond this book's scope. Nevertheless, we provide illustrative examples to convey the concept of function series convergence.

Example 4.2. Consider the interval $(0,1)$, where we delve into the following infinite series:

$$\sum_{n=1}^{\infty} \frac{1}{n} \sin(n\pi x).$$

Upon observation, it's evident that this series doesn't converge to a continuous function within the interval $(0,1)$. A clear instance arises when $x = \frac{1}{2}$, resulting in the series becoming a numerical series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}.$$

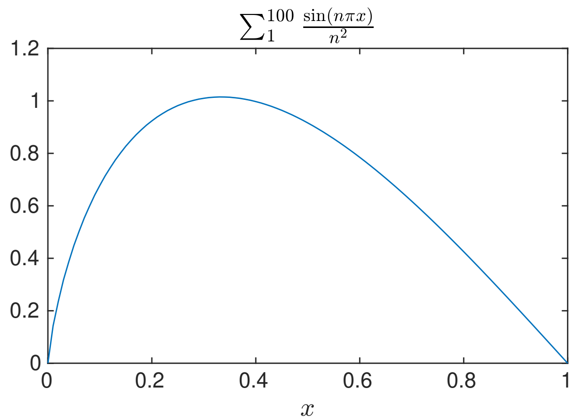
This numerical series exhibits conditional convergence, lacking absolute convergence. Now explore the series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \sin(n\pi x).$$

We ascertain that this series demonstrates absolute convergence. This is evident from the inequality:

$$\left|\sum_{n=1}^{\infty} \frac{1}{n^2} \sin(n\pi x)\right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

The figure below depicts the truncated series of the above infinite series up to 100 terms:



These examples underscore the intricate convergence behaviors of infinite series and their implications on the continuity of functions within a specified interval.

In the following sections, we will delve into other types of convergence. However, the discussed form of convergence remains fundamentally crucial for comprehending initial value problems. Consider the subsequent heat problem as an example:

$$\begin{cases} u_t = u_{xx} & x \in (0,1) \\ u(0,t) = u(1,t) = 0 \end{cases},$$

alongside the initial condition $u(x,0) = f(x)$. Given the general series solution for this heat problem as:

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} \sin(n\pi x).$$

The significance of the convergence of this infinite series as t approaches 0 becomes evident. This convergence property implies:

$$\lim_{t\to 0} u(x,t) = \lim_{t\to 0} \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} \sin(n\pi x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) = f(x).$$

Consequently, the coefficients c_n must be chosen carefully, ensuring that:

$$\lim_{N\to\infty} \sum_{n=1}^{N} c_n \sin(n\pi x) = f(x).$$

As an illustration, consider the case where $f(x) = x(1-x)$. For this scenario, appropriate values for c_n are determined as follows:

$$c_n = \begin{cases} \frac{8}{n^3\pi^3} & n\text{: odd} \\ 0 & n\text{: even} \end{cases}.$$

With this specific selection of c_n, the series:

$$\sum_{n=1}^{\infty} c_n e^{-\lambda_n t} \sin(n\pi x),$$

successfully converges to the initial condition as t approaches 0, aligning with the prescribed values of c_n.

Exercise 4.3. Let Ω be the interval Ω: $(0,1)$:

a) Solve the following eigenvalue problem

$$\begin{cases} \phi'' = -\lambda\phi & \text{on } \Omega \\ \phi'(0) = \phi'(1) = 0 & \text{on } \mathrm{bnd}(\Omega) \end{cases}.$$

b) Use this eigenvalue problem and solve the following heat problem

$$\begin{cases} u_t = k u_{xx} & \text{on } \Omega \\ u_x(0,t) = u_x(1,t) = 0 & \end{cases},$$

for $k > 0$.

Exercise 4.4. Let Ω be the interval Ω: $(0,1)$:

a) Solve the following eigenvalue problem

$$\begin{cases} \phi'' = -\lambda\phi & \text{on } \Omega \\ \phi(0) = \phi'(1) = 0 & \text{on } \mathrm{bnd}(\Omega) \end{cases}.$$

b) Use this eigenvalue problem and solve the following heat problem

$$\begin{cases} u_t = k u_{xx} & \text{on } \Omega \\ u(0,t) = u_x(1,t) = 0 & \end{cases},$$

for $k > 0$.

c) Find the steady state solution

$$\lim_{t\to\infty} u(x,t).$$

Exercise 4.5. Let Ω be the interval Ω: $(0,1)$:

a) Solve the following eigenvalue problem

$$\begin{cases} \phi'' = -\lambda\phi & \text{on } \Omega \\ \phi'(0) = \phi(1) = 0 & \text{on } \mathrm{bnd}(\Omega) \end{cases}.$$

b) Use this eigenvalue problem and solve the following heat problem

$$\begin{cases} u_t = k u_{xx} & \text{on } \Omega \\ u_x(0,t) = u(1,t) = 0 & \end{cases},$$

for $k > 0$.

c) Find the steady state solution

$$\lim_{t\to\infty} u(x,t).$$

Exercise 4.6. Let Ω be the interval Ω: $(0,1)$:

a) Find eigenvalues and eigenfunctions of the following eigenvalue problem

$$\begin{cases} \phi'' = -\lambda\phi \\ \phi\left(-\frac{\pi}{2}\right) = 0 \\ \phi\left(\frac{\pi}{2}\right) = 0 \end{cases}.$$

b) Use these eigenfunctions and solve the following wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx} & x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ u\left(-\frac{\pi}{2}, t\right) = u\left(\frac{\pi}{2}, t\right) = 0 & \\ u(x,0) = \cos(x) + \sin(2x) & \\ u_t(x,0) = 0 & \end{cases}.$$

4.1.4 Problems

Problem 4.1. Find the eigenvalues and eigenfunctions of the following eigenvalue problem

$$\begin{cases} \phi'' = -\lambda\phi \\ \phi\left(\frac{1}{2}\right) = \phi\left(\frac{3}{4}\right) = 0 \end{cases}.$$

Problem 4.2. Find the eigenvalues and eigenfunctions of the following eigenvalue problem

$$\begin{cases} \phi'' = -\lambda\phi \\ \phi(-1) = \phi(1) \\ \phi'(-1) = \phi'(1) \end{cases}.$$

Problem 4.3. Find the eigenvalues and eigenfunctions of the following eigenvalue problem

$$\begin{cases} \phi'' = -\lambda\phi \\ \phi\left(\frac{-1}{2}\right) = 0 \\ \phi'\left(\frac{1}{2}\right) = 0 \end{cases}.$$

Problem 4.4. Find the general series solution of the following wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx} & x \in (0,1) \\ u(0,t) = u(1,t) = 0 \end{cases}.$$

Problem 4.5. Consider the following damped wave equation

$$\begin{cases} u_{tt} + 2\xi u_t = c^2 u_{xx} & x \in (0,1) \\ u(0,t) = u(1,t) = 0 \end{cases}.$$

a) For what value of damping factor ξ, the system is in the underdamped state? The underdamped state is the state where the t-component of the solution is in the form

$$U(t) = e^{\sigma t}[A\cos(\omega t) + \sin(\omega t)],$$

for some real parameters σ and ω.

b) Find the general series solution of the problem when the system is under damped.

Problem 4.6. Consider the following eigenvalue problem on $x \in (1, e)$

$$\begin{cases} L[\phi] = -\lambda\phi \\ \phi(1) = 0 \\ \phi(e) = 0 \end{cases},$$

where L stands for the differential operator

$$L := x^2 \frac{d^2}{dx^2} + x\frac{d}{dx}.$$

a) Use the transformation $x = e^s$ and convert the given eigenvalue problem to the the following one

$$\begin{cases} \varphi''(s) = -\lambda\varphi(s) \\ \varphi(0) = 0 \\ \varphi(1) = 0 \end{cases}.$$

Solve the transformed problem and find eigenvalues and eigenfunctions of the operator L.

b) Use this results and solve the following problem

$$\begin{cases} u_t = x^2 u_{xx} + x u_x \\ u(1,t) = 0 \\ u(e,t) = 0 \end{cases}.$$

Problem 4.7. Consider the following eigenvalue problem on $x \in (0,1)$

$$\begin{cases} \phi'' = -\lambda\phi \\ \phi(0) = 0 \\ \phi(1) + \phi'(1) = 0 \end{cases}.$$

a) Prove that the problem is solvable for a non-trivial function ϕ only if $\lambda > 0$.

b) Verify that there are infinitely many eigenvalues $\lambda_1 < \lambda_2 < \cdots$. Find the smallest eigenvalue λ_1 by a numerical method and the correspond eigenfunction $\phi_1(x)$.

Problem 4.8. Consider the following eigenvalue problem on $x \in (0,1)$

$$\begin{cases} \phi'' = -\lambda\phi \\ \phi(0) = 0 \\ \phi(1) - \phi'(1) = 0 \end{cases}.$$

a) Prove that the problem is solvable for a non-trivial function ϕ only if $\lambda \geq 0$.

b) Verify that there are infinitely many eigenvalues $0 = \lambda_0 < \lambda_1 < \cdots$. What is the first eigenfunction $\phi_0(x)$.

Problem 4.9. Consider the following eigenvalue problem on $x \in (0,1)$:

$$\begin{cases} \phi'' + 2\phi' = -\lambda\phi \\ \phi(0) = 0 \\ \phi(1) = 0 \end{cases}.$$

Show that the eigenvalues of the problem is strictly greater than 1. The root of the characteristic equation of the ordinary differential equation $\phi + 2\phi + \lambda\phi = 0$ are

$$r_{1,2} = -1 \pm \sqrt{1-\lambda}.$$

Give argument why $\lambda > 0$ is the condition for the solvability of the eigenvalue problem for a non-trivial eigenfunction $\phi(x)$.

Problem 4.10. We Consider the eigenvalue problem

$$\begin{cases} \phi'' + 2\phi' = -\lambda\phi \\ \phi(0) = 0 \\ \phi(1) = 0 \end{cases}.$$

a) Multiply the equation by ϕ and integrate the equation in $[0,1]$. Use the boundary condition and show

$$\int_0^1 \phi'(x)\, \phi(x)\, dx = 0.$$

b) The part a) gives the result

$$\lambda = \frac{\displaystyle\int_0^1 |\phi'(x)|^2\, dx}{\displaystyle\int_0^1 |\phi(x)|^2\, dx}.$$

Prove the inequality

$$\int_0^1 |\phi(x)|^2\, dx < \int_0^1 |\phi'(x)|^2\, dx,$$

if $\phi(0) = \phi(1) = 0$. Hint: write

$$\phi(x) = \int_0^x \phi'(x)\, dx,$$

and conclude

$$|\phi(x)| \leq \int_0^1 |\phi'(x)|\, dx.$$

Use the Cauchy-Schwarz inequality

$$\int_0^1 |\phi'(x)|\,dx < \left(\int_0^1 |\phi'(x)|^2\,dx\right)^{\frac{1}{2}},$$

and prove the claim.

4.2 Sturm-Liouville problem

As we observed above, the solution of a general linear 1D partial differential equation strongly relies on the solvability of the following eigenvalue problem

$$\begin{cases} L[\phi] = -\lambda\phi \\ \alpha_1\phi(x_0) + \beta_1\,\phi'(x_0) = 0 \\ \alpha_2\phi(x_0) + \beta_2\,\phi'(x_1) = 0 \end{cases},$$

where L is the differential operator (4.1).

While there is no general method to solve this problem, the Sturm-Liouville theorem provides valuable insights into its solutions. This theorem is a powerful tool in mathematical physics and plays a significant role in understanding the behavior of differential equations.

To gain a rigorous understanding of the assertions of the theorem, it is essential to familiarize ourselves with some advanced mathematical concepts. These include function spaces, the notion of inner product, and the concepts of convergence of function series or sequences.

4.2.1 Inner product

Recall the concept of the dot product between two vectors in $\mathbb{R}^n$. Extending this operation to functions, we define the inner product as a natural generalization. Let $f(x)$ and $g(x)$ be two real-valued functions defined on the interval $[x_0, x_1]$. The inner product $\langle f, g\rangle$ is defined as the integral

$$\langle f, g\rangle = \int_{x_0}^{x_1} f(x)\,g(x)\,dx. \tag{4.2}$$

The defined inner product exhibits the following properties:

1. Positivity: $\langle f, f\rangle \geq 0$, and $\langle f, f\rangle = 0$ only if $f \equiv 0$.
2. Commutativity: $\langle f, g\rangle = \langle g, f\rangle$
3. Homogeneity: $\langle \lambda f, g\rangle = \lambda\langle f, g\rangle$ for any $\lambda \in \mathbb{R}$
4. Additivity: $\langle f, g + h\rangle = \langle f, g\rangle + \langle f, h\rangle$, as long as the associated integrals exist.

To grasp the natural extension of the dot product, let us recall its definition between two arbitrary vectors $\vec{u} = (u_1, ..., u_n)$ and $\vec{v} = (v_1, ..., v_n)$, which is given by:

$$\vec{u}\cdot\vec{v} = \sum_{j=1}^{n} u_j\,v_j.$$

It is worth noting that the Riemann sum of the integral (4.2) can be expressed as:

$$\langle f\,,g\rangle = \lim_{n\to\infty}\sum_{j=1}^{n} f(x_j)\,g(x_j)\,\delta x_j.$$

Exercise 4.7. Another approach to defining the inner product, which we will utilize in our subsequent discussions, is known as the weighted inner product. In this case, we introduce a positive function $\sigma(x)$. The weighted product, denoted as $\langle f\,,g\rangle_\sigma$, is defined as:

$$\langle f\,,g\rangle_\sigma := \int_{x_0}^{x_1} f(x)\,g(x)\,\sigma(x)\,dx$$

Verify that this weighted inner product also satisfies the properties of positivity, commutativity, homogeneity, and additivity, just like the previously defined inner products.

Now, we can define the notion of orthogonality between two functions using the inner product.

Definition 4.1. *Two non-trivial real-valued functions f and g defined on the interval $[x_0, x_1]$ are called orthogonal if their inner product satisfies $\langle f\,,g\rangle = 0$.*

Example 4.3. Consider the set of functions $\left\{\sin\left(\frac{n\pi}{L}x\right), x\in[0,L]\right\}$ for $n=1,2,3,...,$ which are defined on the interval $[0,L]$. It is evident that these functions are mutually orthogonal, meaning that their inner product satisfies:

$$\left\langle \sin\left(\frac{n\pi}{L}x\right), \sin\left(\frac{m\pi}{L}x\right)\right\rangle = \int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx = 0,$$

for $n\neq m$. However, when $n=m$, we have:

$$\left\|\sin\left(\frac{n\pi}{L}x\right)\right\|^2 = \int_0^L \sin^2\left(\frac{n\pi}{L}x\right)dx = \frac{L}{2}.$$

Similarly, the functions in the set $\left\{\cos\left(\frac{n\pi}{L}x\right), x\in[0,L]\right\}$ for $n=0,1,2,3,...$ are also mutually orthogonal. Additionally, the inner product satisfies:

$$\left\langle \sin\left(\frac{n\pi}{L}x\right), \cos\left(\frac{m\pi}{L}x\right)\right\rangle = 0,$$

for all integer values of n and m.

In addition to the orthogonality, we use the inner production notion and define the magnitude of a function. Let $f(x)$ be a function defined on the interval $[x_0, x_1]$. The magnitude or norm of f, compatible with the inner product $\langle,\rangle$, is defined as $\|f\| = \sqrt{\langle f\,,f\rangle}$. In general, the norm of a function should satisfy the following properties

1. $\|f\|\geq 0$, and if $\|f\|=0$, then $f\equiv 0$.
2. $\|\lambda f\| = |\lambda|\,\|f\|$ for any $\lambda\in\mathbb{R}$.
3. $\|f+g\|\leq\|f\|+\|g\|$.

Exercise 4.8. Prove that functions in the set

$$\{1, \cos(nx), \sin(nx)\}$$

defined in the interval $[-\pi, \pi]$ are mutually orthogonal:

$$\langle \cos(nx), \cos(mx) \rangle = 0, n \neq m$$

$$\langle \cos(nx), \sin(mx) \rangle = 0, \forall n, m$$

Find the norm of each function.

Exercise 4.9. Show that the functions in the set

$$\left\{ \sin\left(\frac{2n-1}{2} \pi x \right) \right\},$$

are mutually orthogonal. What are the norms of these functions?

4.2.2 Approximating a function by orthogonal functions

The significance of the orthogonality condition in our subsequent discussions lies in the following fact: Assume that $f(x)$ is a function defined on the interval $[x_0, x_1]$. Furthermore, assume that the functions in the set $\{\phi_n(x), x \in [x_0, x_1]\}$ are mutually orthogonal with respect to the inner product $\langle , \rangle$, and that $f(x)$ is approximated in terms of these functions as:

$$f(x) \approx c_1 \, \phi_1(x) + c_2 \, \phi_2(x) + \cdots,$$

for some undetermined coefficients c_n. The orthogonality condition enables us to determine these parameters as:

$$c_n = \frac{\langle f, \phi_n \rangle}{\|\phi_n\|^2}.$$

Note that for any fixed j, we have

$$\langle f, \phi_j \rangle = c_1 \langle \phi_1, \phi_j \rangle + \cdots + c_j \langle \phi_j, \phi_j \rangle + \cdots + c_n \, \langle \phi_n, \phi_j \rangle + \cdots = c_j \langle \phi_j, \phi_j \rangle,$$

and thus

$$c_j = \frac{\langle f, \phi_j \rangle}{\|\phi_j\|^2},$$

for any fixed j, where $\|\phi_j\|^2 = \langle \phi_j, \phi_j \rangle$.

Example 4.4. The functions in the set $\{\sin(n\pi x)\}_{n=1}^{\infty}$ defined on the domain $x \in [0, 1]$ are mutually orthogonal. We can now find the best approximation of the function $f(x) = x$, $x \in [0, 1]$ in terms of the functions in the set $\{\sin(n\pi x)\}_{n=1}^{N}$ for some number $N \geq 1$. The best approximation function $\hat{f}(x)$ is given by

$$\hat{f}(x) = \sum_{n=1}^{N} \frac{\langle f, \sin(n\pi x) \rangle}{\|\sin(n\pi x)\|^2} \sin(n\pi x).$$

We have

$$\langle f, \sin(n\pi x) \rangle = \int_0^1 x \sin(n\pi x) \, dx = \frac{-\cos(n\pi)}{n\pi}$$

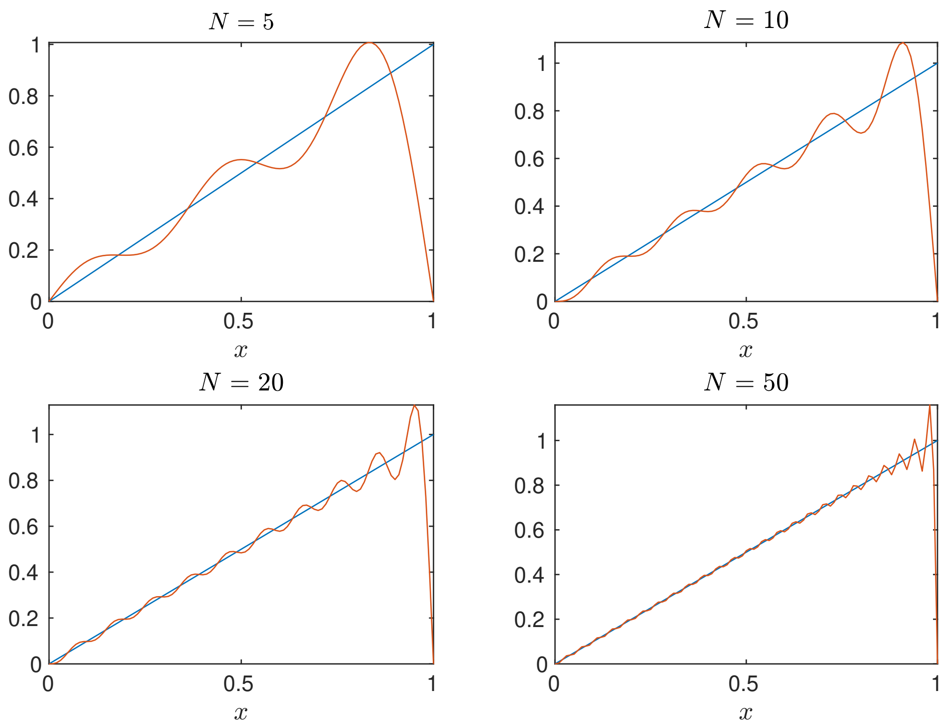
and

$$\|\sin(n\pi x)\|^2 := \langle \sin(n\pi x), \sin(n\pi x) \rangle = \int_0^1 |\sin(n\pi x)|^2 \, dx = \frac{1}{2},$$

and then, the best approximation is obtained as

$$\hat{f}(x) = \sum_{n=1}^{N} \frac{-2\cos(n\pi)}{n\pi} \sin(n\pi x).$$

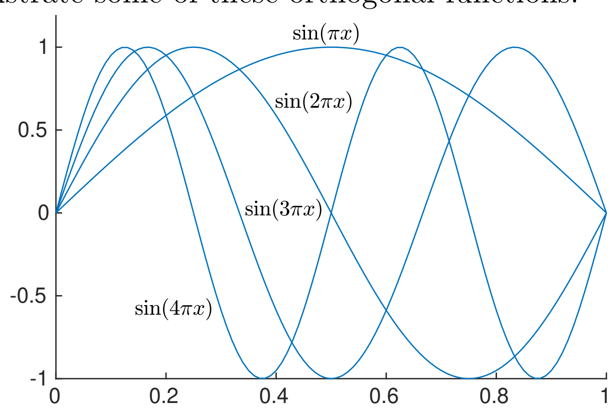
The figures below show the graph of $f(x)$ and $\hat{f}(x)$ for some values of N:



Remark 4.2. (Fourier sine series) In the above example, we used the set of eigenfunctions $\phi_n = \{\sin(n\pi x)\}$ of the eigenvalue problem

$$\begin{cases} \phi'' = -\lambda\phi \\ \phi(0) = \phi(1) = 0 \end{cases}.$$

The series in terms of these eigenfunctions is known as Fourier sine series. The figure below illustrate some of these orthogonal functions:



Example 4.5. The functions in the set $\{\cos(n\pi x)\}_{n=0}^{\infty}$ are mutually orthogonal. The best approximation of the function $f(x)=x$ in the set $\{\cos(n\pi x)\}_{n=0}^{N}$ is given by

$$\hat{f}(x)=\sum_{n=0}^{N} c_n \cos(n\pi x).$$

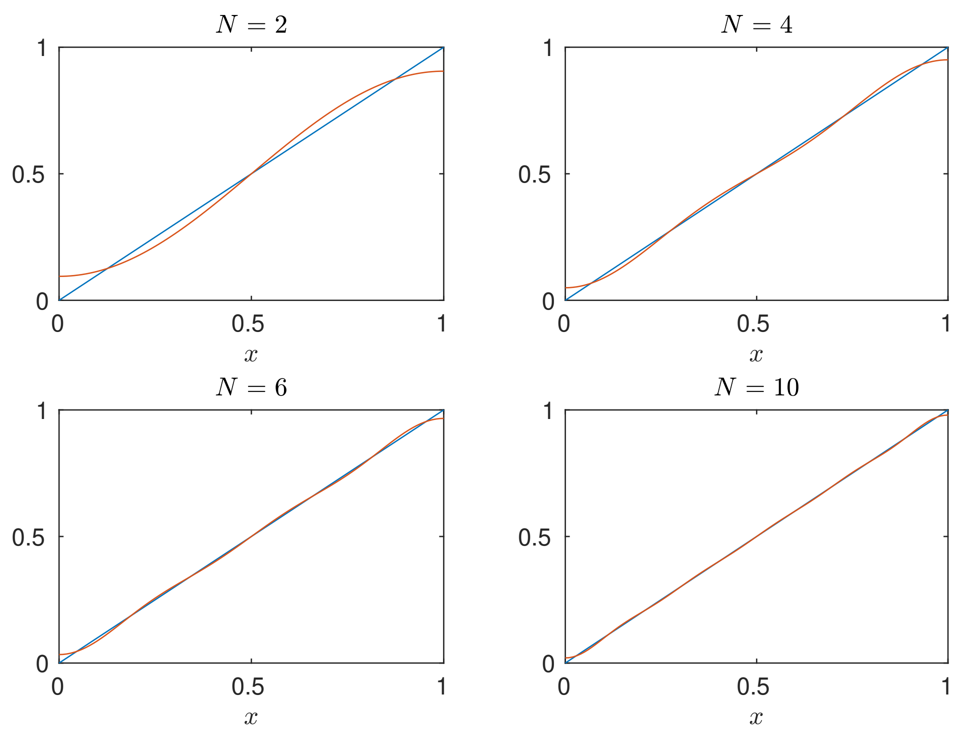
For $n=0$, we have

$$c_0=\int_0^1 x\,dx=\frac{1}{2},$$

and for $n=1,2,\cdots$, the coefficients c_n are

$$c_n=2\int_0^1 x\cos(n\pi x)=\frac{2(1-\cos(n\pi))}{n^2\pi^2}.$$

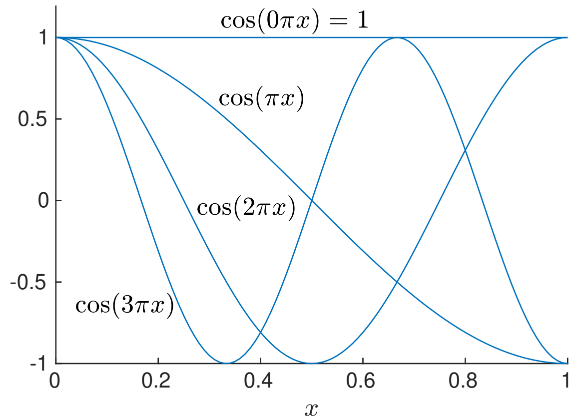
The figures below show the graph of $f(x)=x$ and its best approximation for some values of N:



Remark 4.3. The functions $\phi_n=\cos(n\pi x)$ that we have used to approximate the function $f(x)$ are the eigenfunctions of the eigenvalue problem

$$\begin{cases} \phi''=-\lambda\phi \\ \phi'(0)=\phi'(1)=0 \end{cases}.$$

The series in terms of these eigenfunctions is known as Fourier cosine series. The figure below illustrate some of these orthogonal functions:



Exercise 4.10. Express the function $f(x)=x$ defined on $x\in[0,1]$ in terms of mutually orthogonal functions $\phi_n=\sin\left(\frac{2n-1}{2}\pi x\right)$.

Exercise 4.11. Express the function $f(x)=x$ defined on $x\in[0,1]$ in terms of mutually orthogonal functions $\phi_n=\cos\left(\frac{2n-1}{2}\pi x\right)$.

Exercise 4.12. Express the function $f(x)=1+x$ defined on $x\in[-1,1]$ in terms of mutually orthogonal functions $\phi_n=\{1,\cos(n\pi x),\sin(n\pi x)\}$. These functions are the eigenfunction of the eigenvalue problem

$$\begin{cases}\phi''=-\lambda\phi\\ \phi(-1)=\phi(1)\\ \phi'(-1)=\phi'(1)\end{cases}.$$

The series in terms of these functions are known as Fourier series.

4.2.3 Sturm-Liouville theorem

The theorem for the eigenvalue problem can be stated as follows:

Theorem 4.1. *Consider the eigenvalue problem defined by*

$$\begin{cases}a(x)\phi''+b(x)\phi'+c(x)\phi=-\lambda\phi\\ \alpha_1\,\phi(x_0)+\beta_1\,\phi'(x_0)=0\\ \alpha_2\,\phi(x_1)+\beta_2\,\phi'(x_1)=0\end{cases}, \tag{4.3}$$

where a,b, and c are continuous functions with $a(x)>0$ in the interval $[x_0,x_1]$.

1. *The problem has infinitely many real eigenvalues arranged in increasing order: $\lambda_1<\lambda_2<\lambda_3<\cdots$. Furthermore, the eigenvalues diverge to infinity as n tends to infinity:*

$$\lim_{n\to\infty}\lambda_n=\infty.$$

2. *For each eigenvalue* λ_n*, there exists a unique eigenfunction* $\phi_n(x)$ *(up to a constant multiplication).*

3. *The eigenfunctions* $\phi_n(x)$ *for* $n=1,2,\cdots$ *are mutually orthogonal with respect to the weight function* $\sigma(x)$ *defined as*

$$\sigma(x)=\frac{1}{a(x)}e^{\int\frac{b(x)}{a(x)}}.$$

This orthogonality is represented by the equation $\langle\phi_n,\phi_m\rangle_\sigma=0$ *for* $n\neq m$*.*

4. *The set of functions* $\{\phi_n(x)\}$ *for* $n=1,2,\cdots$ *forms a basis for piecewise continuously differentiable functions defined in the interval* $[x-0,x_1]$*. This implies that any piecewise continuously differentiable function* $f(x)$ *defined in* $[x_1,x_1]$ *can be approximated as*

$$\hat{f}_N(x)\approx\sum_{n=1}^{N}\frac{\langle f,\phi_n\rangle_\sigma}{\|\phi_n\|_\sigma^2}\phi_n(x)$$

in the following sense:

$$\lim_{N\to\infty}|f(x)-\hat{f}_N(x)|=0,$$

as long as x *is a continuity point for* f*.*

This theorem establishes the existence of eigenvalues and eigenfunctions, their orthogonality, and their role in approximating piecewise continuously differentiable functions.

The result of this theorem is crucial in solving linear second-order PDEs. As an example, let's consider the equation:

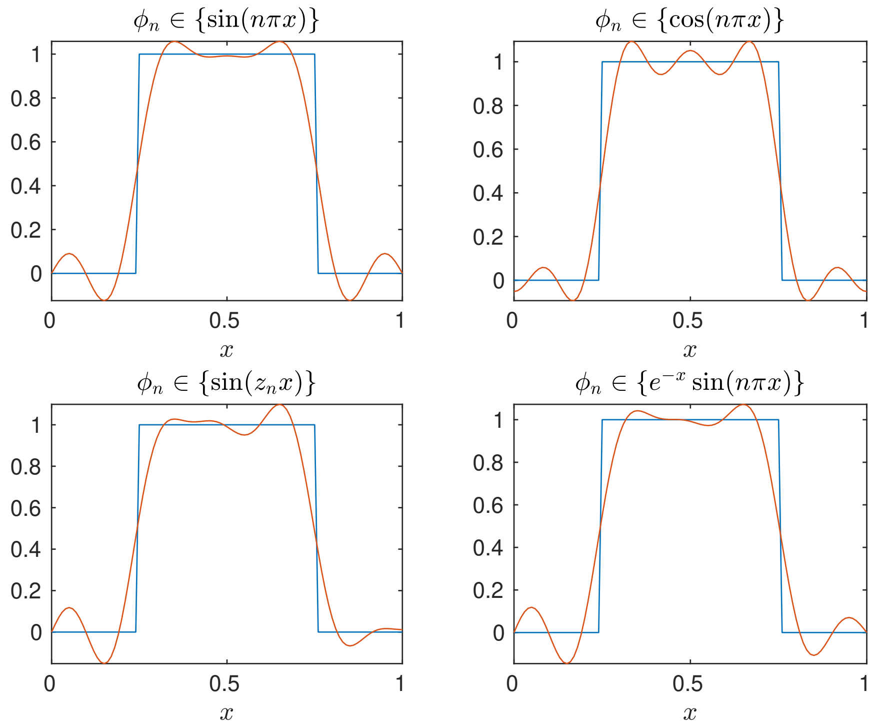
$$\begin{cases} u_t=L[u] \\ \alpha_1\,\phi(x_0)+\beta_1\,\phi'(x_0)=0 \\ \alpha_2\,\phi(x_1)+\beta_2\,\phi'(x_1)=0 \end{cases},$$

where L is the differential operator (4.1). By assuming a separable solution of the form $u(x,t)=\phi(x)e^{-\lambda t}$, we arrive at the eigenvalue problem (4.3). For each pair (λ_n,ϕ_n), we obtain a solution $u_n(x,t)=e^{-\lambda_n t}\phi_n(x)$, which is referred to as the separated solution.

Example 4.6. Consider the function $f(x)$ defined as:

$$f(x)=\begin{cases} 1 & \frac{1}{4}\leq x\leq\frac{3}{4} \\ 0 & \text{otherwise} \end{cases}$$

defined for $x\in(0,1)$. We can expand this function using different basis sets generated from various eigenvalue problems. The figure below illustrates the representation of this function using four sets of eigenfunctions, truncated to $N=10$ terms.



1. The first figure at the left is generated using the set of functions $\{\sin(n\pi x)\}_{n=1}^{10}$ which are eigenfunctions of the following problems:

$$\begin{cases} \phi'' = -\lambda\phi \\ \phi(0) = \phi(1) = 0 \end{cases}.$$

   The weight function for this set is $\sigma(x) = 1$.

2. The second figure at the right is generated using the set of functions $\{\cos(n\pi x)\}_{n=1}^{10}$ which are eigenfunctions of the following problems:

$$\begin{cases} \phi'' = -\lambda\phi \\ \phi'(0) = \phi'(1) = 0 \end{cases}.$$

   The weight function for this set is $\sigma(x) = 1$.

3. The third figure is generated using the set of functions $\{\sin(z_n x)\}_{n=0}^{10}$ where z_n are the roots of the equation

$$\sin(z) + z\cos(z) = 0.$$

   These functions are eigenfunction of the eigenvalue problem

$$\begin{cases} \phi'' = -\lambda\phi \\ \phi(0) = 0 \\ \phi(1) + \phi'(1) = 0 \end{cases}.$$

   The weight function for this set is $\sigma(x) = 1$.

4. The last figure is generated using the set of functions $\{e^{-x}\sin(n\pi x)\}_{n=1}^{10}$ which are eigenfunctions of the following problems:

$$\begin{cases} \phi'' + 2\phi' = -\lambda\phi \\ \phi(0) = \phi(1) = 0 \end{cases}.$$

The weight function for this set is $\sigma(x) = e^{2x}$.

It is worth noting that in all series representations of the function $f(x)$, the series converge to the average of the right and left limits of $f(x)$ at its discontinuity points located at $x = \frac{1}{4}$ and $x = \frac{3}{4}$. This convergence behavior is a characteristic feature of series expansions for functions with discontinuities.

Exercise 4.13. Consider the following eigenvalue problem on $x \in (1, e)$

$$\begin{cases} x^2\phi'' + x\phi' = -\lambda\phi \\ \phi(1) = \phi(e) = 0 \end{cases}.$$

a) Determine the weight function $\sigma(x)$ that makes the eigenfunctions of the problem orthogonal.

b) The eigenfunctions of the problem are $\phi_n(x) = \sin(n\pi \ln x)$. Calculate directly that these eigenfunctions are orthogonal with respect to $\sigma(x)$.

Exercise 4.14. Consider the following eigenvalue problem

$$\begin{cases} \phi'' + 2\phi' = -\lambda\phi \\ \phi(0) = 0 \\ \phi'(1) = 0 \end{cases}.$$

Determines the eigenfunctions of the problem and the weight function $\sigma(x)$ for the orthogonality of the these eigenfunctions. Note that the first eigenvalue is $\lambda = 1$, and for $\lambda > 1$, you derive an equation of the form:

$$z\cos(z) - \sin(z) = 0,$$

which has infinitely many roots.

4.2.4 Problems

Problem 4.11. There are alternative forms of the inner product beyond the one defined in equation (4.2). Let's consider the set of continuously differentiable functions defined on the interval $[x_0, x_1]$, denoted by $C^1(x_0, x_1)$. We can explore a different definition of the inner product, denoted as $\langle f, g\rangle$, which is given by:

$$\langle f, g\rangle = \int_{x_0}^{x_1} f(x)\, g(x)\, dx + \int_{x_0}^{x_1} f'(x)\, g'(x)\, dx.$$

As an exercise, we can show that this alternative definition satisfies the fundamental properties of positivity, commutativity, homogeneity, and additivity, just like the inner product defined in equation (4.2).

Problem 4.12. Let $\mathbb{T}_2$ be the space of all 2×2 matrices.

i. Show that the following operation is an inner product in T_2.

$$\langle A, B\rangle = \operatorname{tr} A^t B,$$

ii. Generalize this fact for $\mathbb{T}_n$, the space of all $n \times n$ matrices.

iii. Is the operation $\langle A, B\rangle = \operatorname{tr} AB$ an inner product in $\mathbb{T}_n$?

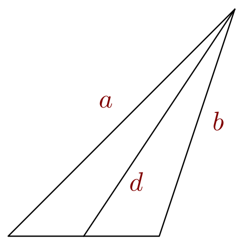
Problem 4.13. Several theorems in plane geometry can be proved by the concept of dot product. Here are two of such problems:

a) Prove the cosine law in triangles, i.e.,

$$c^2 = a^2 + b^2 - 2ab\cos(\theta),$$

where θ is the angle between sides a and b.

b) Consider the triangle shown in the figure where the side d bisects the side c. Use vector operations to prove the Apollonius law $a^2 + b^2 = \frac{1}{2}c^2 + 2d^2$



Problem 4.14. Propose a norm for $\mathbb{T}_2$, the set of 2×2 matrices.

Exercise 4.15. (Cauchy inequality) Cauchy's inequality states the following inequality:

$$\langle f, g\rangle \leq \|f\|\,\|g\|.$$

To prove this inequality, we start by expanding the right-hand side of the following inequality for any complex number λ:

$$0 \leq \langle f + \lambda g, f + \lambda g\rangle.$$

Using this inequality, we can then show the Cauchy inequality.

$$\|f + g\| \leq \|f\| + \|g\|.$$

Use this inequality and prove the triangle inequality

$$\|f + g\| \leq \|f\| + \|g\|$$

Problem 4.15. Prove the following version of the triangle inequality for arbitrary positive number $\varepsilon > 0$.

$$\|f + g\| \leq \frac{1}{\varepsilon}\|f\| + \varepsilon\,\|g\|.$$

Problem 4.16. If $x_1, \ldots, x_n$ are positive numbers, prove the following identities:

a) $n^2 \leq (x_1 + \cdots + x_n)\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right)$

b) $(x_1 + \cdots + x_n)^2 \leq n(x_1^2 + \cdots + x_n^2)$.

Problem 4.17. If the relation $\|\lambda_1 f + \lambda_2 g\| = \|\lambda_2 f + \lambda_1 g\|$ holds for all real numbers λ_1, λ_2, show $\|f\| = \|g\|$.

Exercise 4.16. Let $C([x_0, x_1])$ be the set of all continuous functions defined on $[x_0, x_1]$.

a) Show that the operation $\|f\|_\infty$ defines as

$$\|f\|_\infty = \max_{x\in[x_0,x_1]} |f(x)|,$$

is a norm for $f \in C([x_0, x_1])$. For this, you need to show that this operation satisfies all properties of a norm.

b) For any inner product, show that the following equality holds

$$\|f + g\|^2 - \|f - g\|^2 = 4\langle f, g\rangle.$$

Use this to prove that there is no inner product $\sqrt{\langle f, f\rangle} = \|f\|_\infty$.

Problem 4.18. Assume that $\{\phi_n(x)\}$ for $n = 1, 2, \cdots$ is a set of mutually orthogonal functions. The best approximation of a functions $f(x)$ in the set $\{\phi_n(x)\}_{n=1}^N$ is defined as

$$\hat{f}(x) = \sum_{n=1}^{N} c_n \phi_n(x),$$

such that

$$\|f - \hat{f}\| \leq \|f - g\|,$$

for any other $g(x)$ constructed by a linear combination of functions in the set $\{\phi_n(x)\}_{n=1}^N$. If the norm is compatible with the inner product $\langle,\rangle$, prove that $\hat{f}$ is the best approximation of f if c_n are

$$c_n = \frac{\langle f, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle}.$$

Hint: Define $h = g - \hat{f}$, and use the inequality $\|f - g\| \geq 0$.

Problem 4.19. Let $\hat{f}$ is the best approximation of function f in terms of the mutually orthogonal functions in the set $\{\phi_n\}_{n=1}^N$.

a) show that $f - \hat{f}$ is orthogonal to all functions ϕ_n.

b) For any function g as

$$g = \sum_{n=1}^{N} b_n \phi_n,$$

for arbitrary b_n, show that $\langle f - \hat{f}, g\rangle = 0$.

Problem 4.20. As we observed in previous sections, the set of orthogonal functions $\left\{\sin\left(\frac{n\pi}{L}x\right)\right\}$ for $n = 1, 2, \cdots$, and $x \in [0, L]$ are eigenfunctions of the eigenvalue problems

$$\begin{cases} \phi'' = -\lambda\phi \\ \phi(0) = \phi(L) = 0 \end{cases}.$$

Similarly, the set of orthogonal functions $\left\{\cos\left(\frac{n\pi}{L}x\right)\right\}$ for $n = 0, 1, 2, \cdots$ are generated by the eigenvalue problem

$$\begin{cases} \phi'' = -\lambda\phi \\ \phi'(0) = \phi'(L) = 0 \end{cases}.$$

Consider the following eigenvalue problem:

$$\begin{cases} \phi'' = -\lambda\phi \\ \phi(0) = 0 \\ \phi(1) + \phi'(1) = 0 \end{cases}.$$

a) Use a numerical method and find first six eigenvalues of the problem $\lambda_1, \ldots, \lambda_6$. You can verify that the eigenfunctions $\phi_n(x) = \sin(\sqrt{\lambda_n}\, x)$ are mutually orthogonal for the values of $\lambda_1, \ldots, \lambda_6$.

b) Find the bets approximation of the function $f(x) = \sin(\pi x)$ for $x \in [0, 1]$ in terms of $\{\phi_n(x)\}_{n=1}^6$. You can use the following code in Matlab:

```
g=@(s) sin(s)+s.*cos(s);
z=zeros(1,6);
for n=1:6
z(n)=fzero(g,3*n);
end
C=integral(@(x) sin(pi*x).*sin(z(1:6)'.*x),0,1,'arrayvalue',true)./
integral(@(x) sin(z(1:6)'.*x).^2,0,1,'arrayvalue',true);
x=0:0.01:1;
fhat=sum(C.*sin(z(1:6)'.*x));
plot(x,sin(pi*x),x,fhat)
```

In the above code, the set z are equal to $\sqrt{\lambda}$.

Problem 4.21. In the book of ODEs, we saw that Legendre polynomials $P_n(x)$ are orthogonal functions in $[-1,1]$. Find the best approximation of $f(x)=\sin(\pi x)$, $x\in[-1,1]$ in the set $\{P_n(x)\}_{n=1}^4$. You can use the following code in Matlab to draw $f(x)$ and its best approximation in the same coordinate. The code provides also the error:

$$\|f-\hat{f}\|:=\sqrt{\int_{-1}^{1}|\sin(\pi x)-\hat{f}(x)|^2}.$$

```
N=4;
C=integral(@(x) sin(pi*x).*legendreP(1:N,x),-1,1,'arrayvalued',true).*(2*(1:N)+1)/
2;
x=-1:0.01:1;
fhat=0;
for i=1:N
 fhat=fhat+C(i).*legendreP(i,x);
end
plot(x,sin(pi*x),x,sum);
err=sqrt(trapz((sin(pi*x)-fhat).^2));
title('error=',err)
```

Problem 4.22. Consider the set of functions $\{\varphi_0=1,\varphi_1=x,\varphi_2=x^2\}$ defined in the domain $[-1,1]$.

a) Find the best approximation of function $f(x)=\sin(\pi x)$ in terms of $\{\varphi_n\}_{n=0}^2$. It is worth noting that these functions are orthogonal. To solve this, let's write $\tilde{f}=a_0+a_1x+a_2x^2$, and minimize the norm $\|f-\tilde{f}_2\|$ by assuming that $f-\tilde{f}_2$ is orthogonal to functions in the set $\{\varphi_n\}$, i.e.,

$$\int_{-1}^{1}(\sin(x)-a_0\,x^2-b_0\,x-c_0)x^n=0,$$

for $n=0,1,2$.

b) Determine the square error defined by the following formula:

$$\int_{-1}^{1}|f(x)-\tilde{f}_2(x)|^2\,dx.$$

c) Repeat parts a) and b) for the set $\{\varphi_0=1,\varphi_1=x,\varphi_2=x^2,\varphi_3=x^3\}$. As you can see, we need to recalculate all the coefficients. However, this is not the case for orthogonal functions. This is one of the reasons why orthogonality is highly valued in applied sciences.

4.3 From Strum-Liouville problem to PDEs

4.3.1 Homogeneous 1D problems

To demonstrate the application of the Sturm-Liouville problem in solving partial differential equations, we consider the homogeneous heat equation given by:

$$\begin{cases} u_t=L[u] & x\in(x_0,x_1)\\ \alpha_1\,u(x_0,t)+\beta_1 u_x(x_0,t)=0 & \\ \alpha_2\,u(x_1,t)+\beta_2 u_x(x_1,t)=0 & \\ u(x,0)=f(x) & \end{cases},$$

where L represents a linear second-order partial differential operator in the variable x:

$$L[u]=a(x)\,u_{xx}+b(x)\,u_x+c(x)\,u.$$

The associated eigenvalue problem for the spatial variable x is:

$$\begin{cases} L[\phi(x)] = -\lambda\,\phi(x), \\ \alpha_1\,\phi(x_0) + \beta_1\phi'(x_0) = 0 \\ \alpha_2\,\phi(x_1) + \beta_2\phi'(x_1) = 0 \end{cases}.$$

According to the Sturm-Liouville theorem, there exists an infinite set of eigenvalues $\lambda_1 < \lambda_2 < \cdots$ with corresponding orthogonal eigenfunctions $\phi_1(x)$, $\phi_2(x)$, $\cdots$, which are orthogonal with respect to the weight function $\sigma(x)$. Furthermore, the set $\{\phi_n(x)\}$ forms a basis for piecewise continuously differentiable functions defined on the interval $(x_0,\ x_1)$. Therefore, the desired solution $u(x,t)$ can be expressed as a series in terms of this set:

$$u(x,t) = \sum_{n=1}^{\infty} U_n(t)\,\phi_n(x),$$

where $U_n(t)$ are coefficient functions that need to be determined appropriately for the series to be a valid solution to the problem. Substituting this series into the equation yields:

$$\sum_{n=1}^{\infty} U_n'(t)\,\phi_n(x) = \sum_{n=1}^{\infty} U_n(t)\,L\left[\phi_n(x)\right] = \sum_{n=1}^{\infty} -\lambda_n U_n(t)\,\phi_n(x).$$

This equality leads to the following ordinary differential equation for $U_n(t)$:

$$U_n' + \lambda_n U_n = 0,$$

which can be solved to obtain the function

$$U_n(t) = C_n\,e^{-\lambda_n t}.$$

Finally, we arrive at the solution:

$$u(x,t) = \sum_{n=1}^{\infty} C_n\,e^{-\lambda_n t}\,\phi_n(x).$$

To ensure that this series satisfies the initial condition at $t=0$, we require:

$$\lim_{t\to 0} u(x,t) = u(x,0) = \sum_{n=1}^{\infty} C_n\,\phi_n(x).$$

By the given initial condition, we obtain the following equation:

$$f(x) = \sum_{n=1}^{\infty} C_n\,\phi_n(x),$$

where the orthogonality of eigenfunctions $\phi_{\mathrm{n}}(x)$ determines C_{n} as:

$$C_n = \frac{\langle f, \phi_n \rangle_\sigma}{\|\phi_n\|^2} = \frac{\displaystyle\int_{x_0}^{x_1} f(x)\,\phi_n(x)\,\sigma(x)\,dx}{\displaystyle\int_{x_0}^{x_1} \phi_n^2(x)\,\sigma(x)\,dx}.$$

Finally, the true solution of the given problem is:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{\langle f, \phi_n \rangle_\sigma}{\|\phi_n\|^2}\, e^{-\lambda_n t}\,\phi_n(x).$$

This series solution known as the *eigenfunction series solution* provides converges to a smooth solution for $t > 0$ even if the initial condition is not continuous. This method can be employed to solve a wide range of 1D second-order PDEs in t and x for a two-variable function $u(x,t)$.

Example. We will solve the standard heat problem with the given conditions:

$$\begin{cases} u_t = k u_{xx} & x \in (0,1) \\ u(0,t) = 0 \\ u(1,t) = 0 \\ u(x,0) = f(x) \end{cases}.$$

The associated eigenvalue problem is

$$\begin{cases} \phi'' = -\lambda\phi \\ \phi(0) = \phi(1) = 0 \end{cases},$$

Solving the eigenvalue problem, we obtain the eigenpairs $\lambda_n = n^2\pi^2$ and $\phi_n = \sin(n\pi x)$. Using these eigenfunctions, the superposition solution can be expressed as:

$$u(x,t) = \sum_{n=1}^{\infty} c_n\, e^{-kn^2\pi^2 t} \sin(n\pi x).$$

The coefficients c_n can be determined by solving the equation:

$$f(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x).$$

The orthogonality of the eigenfunctions ϕ_n with respect to the weight function $\sigma = 1$ leads to the coefficients:

$$c_n = 2\int_0^1 f(x) \sin(n\pi x)\, dx.$$

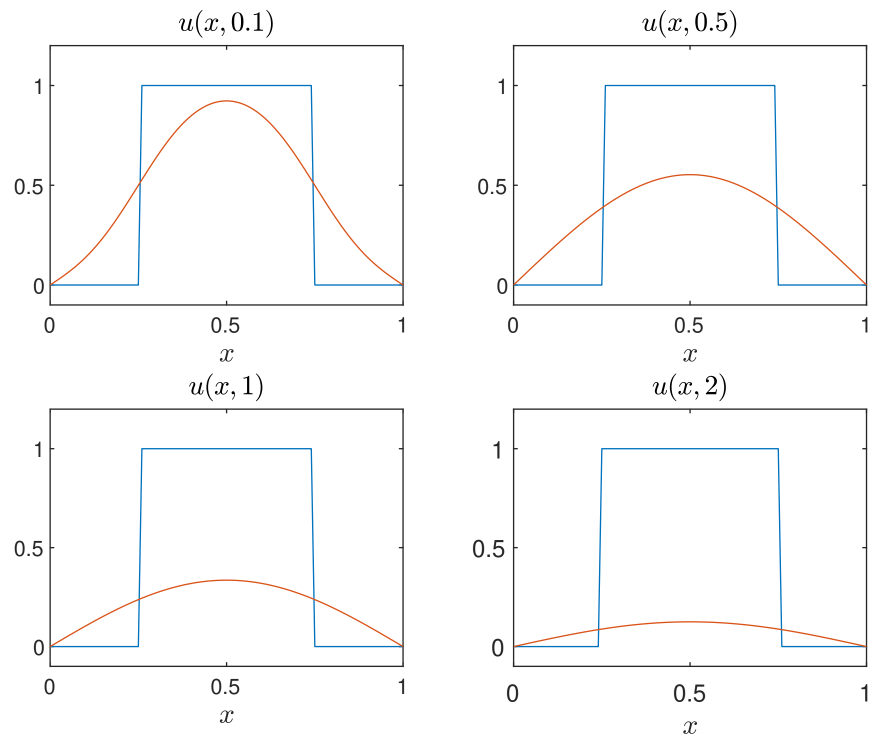
Let's consider the specific initial condition given as:

$$f(x) = \begin{cases} 1 & \frac{1}{4} \le x \le \frac{3}{4} \\ 0 & \text{otherwise} \end{cases}.$$

The coefficients c_n can be calculated as:

$$c_n = \frac{2}{n\pi}\left[\cos\left(\frac{n\pi}{4}\right) - \cos\left(\frac{3n\pi}{4}\right)\right].$$

The figures below illustrate the initial condition and the solution at different instances of time for $k = 0.1$.



Example 4.7. Let us solve the following parabolic equation

$$\begin{cases} \frac{1}{k}u_t = u_{xx} + 2u_x \\ u(0,t) = 0 \\ u(1,t) = 0 \\ u(x,0) = f(x) \end{cases},$$

for $k > 0$ a constant. The associated eigenvalue problem for the given PDE is

$$\begin{cases} \phi'' + 2\phi' = -\lambda\phi \\ \phi(0) = 0 \\ \phi(1) = 0 \end{cases}.$$

The characteristic equation of the ODE is

$$r^2 + 2r + \lambda = 0,$$

which has roots $r_{1,2} = -1 \pm \sqrt{1-\lambda}$. It is observed that the eigenvalue problem is solvable only for $\lambda > 1$. In this case, the solution $\phi(x)$ is given by:

$$\phi(x) = c_1\, e^{-x} \cos(\sqrt{\lambda - 1}\, x) + c_2\, e^{-x} \sin(\sqrt{\lambda - 1}\, x).$$

Applying the given boundary conditions, we find $c_1 = 0$, and $\lambda_n = 1 + n^2\pi^2$ with associated eigenfunctions $\phi_n(x) = e^{-x} \sin(n\pi x)$, which are orthogonal with respect to the weight function $\sigma(x) = e^{2x}$. The superposition solution can be expressed as:

$$u(x,t) = \sum_{n=1}^{\infty} c_n\, e^{-(1+n^2\pi^2)kt}\, e^{-x} \sin(n\pi x).$$

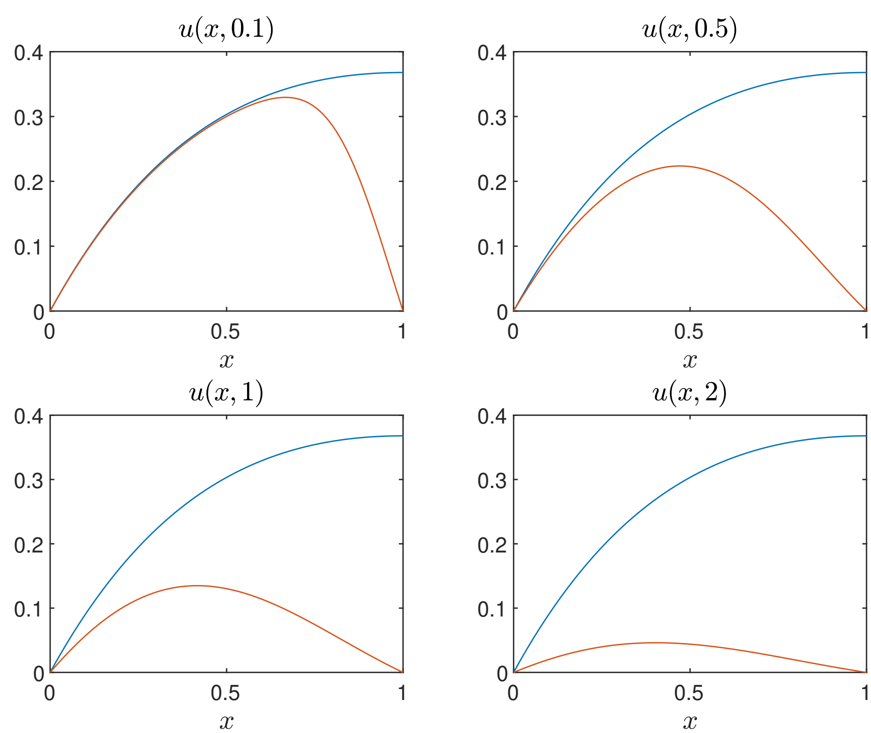
To determine the coefficients c_n, we need to satisfy the initial condition $f(x)$. Thus, the series solution should satisfy the equation:

$$f(x) = \sum_{n=1}^{\infty} c_n\, e^{-x} \sin(n\pi x),$$

and the coefficients can be computed as:

$$c_n = 2\int_0^1 f(x)\, e^{-x} \sin(n\pi x)\, e^{2x}\, dx,$$

where the factor e^{2x} represents the weight function $\sigma(x)$. The figures below depict the graph of the initial heat profile $f(x) = x e^{-x}$ and the solution $u(x,t)$ at different time values for $k = 0.1$.



Example 4.8. Let's solve the damped wave equation

$$\begin{cases} u_{tt} + 2\xi u_t = u_{xx} \\ u(0,t) = 0 \\ u_x(\pi,t) = 0 \end{cases},$$

where $\xi > 0$ is the damping factor. This equation represents an elastic string fixed at $x = 0$ and free at $x = 1$.

To solve the problem, we first form the associated eigenvalue problem:

$$\begin{cases} \phi'' = -\lambda\phi \\ \phi(0) = 0 \\ \phi'(\pi) = 0 \end{cases}.$$

The solutions of the eigenvalue problem are given by the pairs $\lambda_n = \frac{(2n-)^2}{4}$, and $\phi_n = \sin\left(\frac{2n-1}{2}x\right)$ for $n = 1, 2, \cdots$. Since the set $\{\phi_n(x)\}$ forms a basis for functions defined on $x \in [0, \pi]$, we can write the solution of the given PDE as:

$$u(x,t) = \sum_{n=1}^{\infty} U_n(t)\, \phi_n(x),$$

where $U_n(t)$ are undetermined coefficients. This series is known as the eigenfunction expansion of the unknown solution $u(x, t)$. To determine $U_n(t)$, we substitute the eigenfunction series into the PDE:

$$\sum_{n=1}^{\infty} [U_n''(t) + 2\xi U_n'(t)]\, \phi_n(x) = \sum_{n=1}^{\infty} U_n(t)\, \phi_n''(x),$$

and using the relation $\phi_n'' = -\lambda_n \phi_n$, we obtain the following ODE for $U_n(t)$:

$$U_n''(t) + 2\xi U_n'(t) + \lambda_n U_n(t) = 0.$$

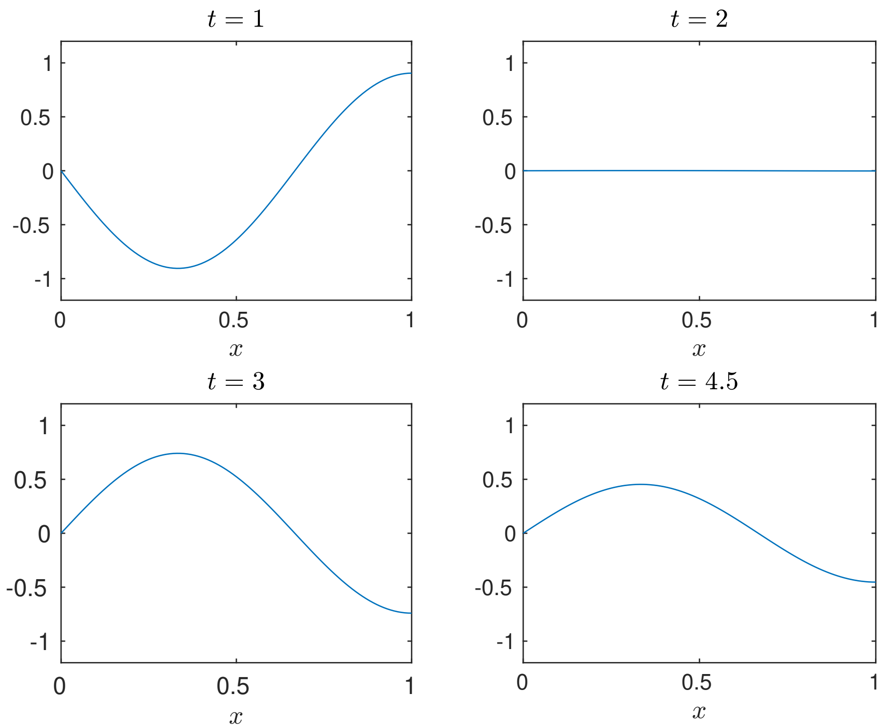
Assuming $\xi < \sqrt{\lambda_1}$, the solution of the above ODE is given by:

$$U_n(t) = e^{-\xi t}[A_n \cos(\omega_n t) + B_n \sin(\omega_n t)],$$

where $\omega_n = \sqrt{\lambda_n - \xi^2}$. The superposition solution of the given PDE is:

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\xi t}[A_n \cos(\omega_n t) + B_n \sin(\omega_n t)]\, \phi_n(x).$$

The coefficients A_n and B_n are determined using the initial conditions $u(x,0)$ and $u_t(x,0)$ for the equation. The figures below show the wave for $u(x,0) = 0$, $u_t(x,0) = \sin\left(\frac{3\pi}{2}x\right)$ and $\xi = 0.1$



Exercise 4.17. Find the series solution of the following parabolic type PDE on the domain $[1,e]$

$$\begin{cases} u_t = x^2 u_{xx} + x u_x \\ u(1,t) = u(e,0) = 0 \\ u(x,0) = \ln(x) \end{cases}.$$

Exercise 4.18. We aim to solve the following wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = 0 \\ u(1,t) + u_x(1,t) = 0 \\ u(x,0) = \sin(\pi x) \\ u_t(x,0) = 0 \end{cases}.$$

a) Determine first 6 eigenvalues and eigenfunctions of the following eigenvalue problem

$$\begin{cases} \phi'' = -\lambda\phi \\ \phi(0) = 0 \\ \phi(1) + \phi'(1) = 0 \end{cases}.$$

b) Use the derived eigenfunctions to find a truncated series solution of the wave equation.

Exercise 4.19. Find the eigenvalues and eigenfunctions of the eigenvalue problem:

$$\begin{cases} \phi'' + 2\phi' = -\lambda\phi \quad x \in (0,1) \\ \phi(0) = 0 \\ \phi'(1) = 0 \end{cases}.$$

Use the result and write the solution of the following heat equation

$$\begin{cases} u_t = u_{xx} + 2u_x \quad x \in (0,1) \\ u(0,t) = 0 \\ u_x(1,t) = 0 \\ u(x,0) = x e^{-x} \end{cases}.$$

4.3.2 Non-homogeneous equations

In this section, we apply the eigenfunction expansion method to solve non-homogeneous linear second-order partial differential equations. To illustrate the method, let us consider the following heat problem:

$$\begin{cases} u_t = L[u] + h(x,t) \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = 0 \end{cases}.$$

The associated eigenvalue problem for the given equation is

$$\begin{cases} L[\phi] = -\lambda\phi \\ \phi(0) = \phi(1) = 0 \end{cases}.$$

Since the set of eigenfunctions $\{\phi_n(x)\}$ forms a basis for functions defined on the interval $[0,1]$, we express the desired solution as a series:

$$u(x,t) = \sum_{n=1}^{\infty} U_n(t)\, \phi_n(x).$$

Substituting this series into the heat equation results in:

$$\sum_{n=1}^{\infty} [U_n'(t) + \lambda_n U_n]\, \phi_n(x) = h(x,t). \tag{4.4}$$

To proceed, we can represent the source term h as the series in terms of $\{\phi_n(x)\}$ as:

$$h(x,t) = \sum_{n=1}^{\infty} H_n(t)\, \phi_n(x),$$

where $H_n(t) = \frac{\langle h, \phi_n \rangle}{\|\phi_n\|^2}$. Substituting the series representation of h into the series equation (4.4), we obtain the following ordinary differential equation for $U_n(t)$:

$$U_n'(t) + \lambda_n U_n = H_n(t).$$

For example, let's consider $h(x,t) = e^{-t}\phi_1(x)$. Expanding h in terms of $\{\phi_n\}$ yields: $H_n(t) = 0$ for $n \neq 1$ and $H_1(t) = e^{-t}$. Therefore, we obtain the following initial value problems:

$$\begin{cases} U_1' + \lambda_1 U_1 = e^{-t} \\ U_1(0) = 0 \end{cases}, \begin{cases} U_n' + \lambda_n U_n = 0 \\ U_n(0) = 0 \end{cases}, n = 2, 3, \cdots.$$

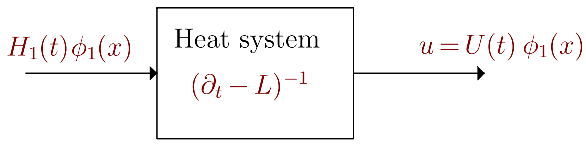
The initial conditions for $U_{\mathrm{n}}(t)$ are chosen to satisfy the zero initial condition $u(x,0)=0$. Solving this system of equations, we find $U_n(t)=0$ for $n=2,3,\cdots$, and $U_1(t)=\frac{1}{\lambda_1-1}(e^{-t}-e^{-\lambda_1 t})$, and finally the solution is given by:

$$u(x,t)=\frac{1}{\lambda_1-1}(e^{-t}-e^{-\lambda_1 t})\,\phi_1(x).$$

Remark 4.4. (System interpretation) Let's examine the above example from a system perspective. We can view the heat equation as a dynamic system that responds to different input sources. In this case, we have a specific input source represented by $h=H_1(t)\sin(\pi x)$. The heat system can be redefined as:

$$u=(\partial_t-L)^{-1}[H_1(t)\,\phi_1(x)].$$

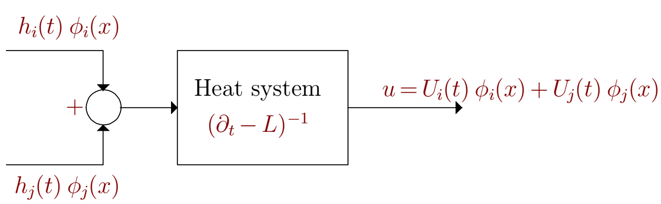
Since ϕ_1 is an eigenfunction of the operator L, the response of the heat system to this input is of the general form $u=U(t)\phi_1(x)$. This can be represented using a block diagram as:



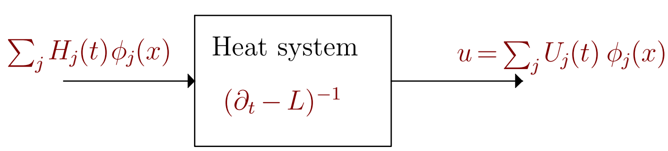
Now, let's consider a more general source term $h(x,t)=h_i(t)\,\phi_i(x)+h_j(t)\,\phi_j(x)$. By applying the superposition principle, we can express the desired solution as:

$$u=U_i(t)\,\phi_i(x)+U_j(t)\,\phi_j(x).$$

This is depicted schematically in the figure below:



This argument extends for arbitrary summations $h=\sum_j H_j(t)\,\phi_j(x)$ as shown below:

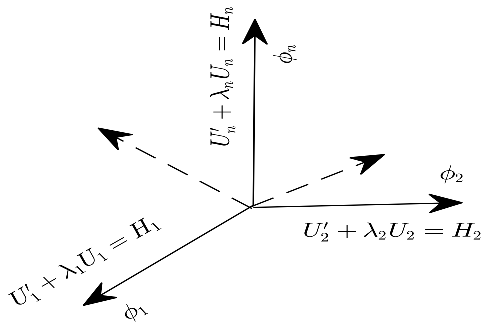


Here each coefficient function $U_j(t)$ is determined by solving the associated ordinary differential equation:

$$U_j'+\lambda_j U_j=H_j(t).$$

Remark 4.5. (Geometric interpretation) Another perspective to consider is the geometric interpretation of the given partial differential equation. Each eigenfunction $\phi_{\mathrm{n}}(x)$ can be seen as a direction in an infinite-dimensional vector space spanned by the functions in the set $\{\phi_{\mathrm{n}}(x)\}$. Along each direction, the given partial differential equation reduces to an ordinary differential equation of the form:

$$U_n'+\lambda_n U_n=H_n(t).$$



In this geometric interpretation, we can view the partial differential equation as an infinite system of ordinary differential equations, where each equation is defined along the direction of ϕ_j. Drawing a parallel to our approach for solving first-order partial differential equations, we can observe similarities in the method. In both cases, we identify characteristic curves along which the given partial differential equation transforms into an ordinary differential equation. This observation highlights the underlying similarity between the two methods and reinforces the understanding of the eigenfunction expansion method.

Exercise 4.20. Find the series solution of the following heat problem

$$\begin{cases} u_t = u_{xx} + \sin(2\pi x) \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = 0 \end{cases}.$$

Exercise 4.21. Find the solution of the following wave equation

$$\begin{cases} u_{tt} = u_{xx} + \sin(t)\sin\left(\frac{\pi}{2}x\right) \\ u(0,t) = u_x(1,t) = 0 \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}.$$

Exercise 4.22. Consider the following heat problem

$$\begin{cases} u_t = u_{xx} + e^{-t} f(x) \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = 0 \end{cases},$$

where $f(x)$ is the following function

$$f(x) = \sum_{n=1}^{5} \frac{-2(-1)^n}{n\pi} \sin(n\pi x).$$

a) Write the solution to the problem.

b) Suppose $f(x) = x$. Write down the solution to the equation. Hint: write down $f(x)$ as

$$f(x) = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n\pi} \sin(n\pi x),$$

employ the same method you used for the part a).

Exercise 4.23. Consider the following heat equation

$$\begin{cases} u_t = u_{xx} \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = f(x) \end{cases},$$

where $f(x) = \phi_1(x)$, and $\phi_n(x) = \sin(n\pi x)$ for $n = 1, 2, \cdots$.

a) Since the heat system is triggered only by the function $\phi_1(x)$, write the solution $u(x,t)$ as

$$u(x,t) = U(t)\,\phi_1(x).$$

Determine $U(t)$.

b) Now, let $f(x)$ be the following function:

$$f(x) = \sum_{n=1}^{5} \frac{-2(-1)^n}{n\pi}\,\phi_n(x).$$

Find the solution of the equation.

c) Now, let $f(x) = x + \sin(\pi x)$. Determine the solution to the equation.

4.3.3 Non-homogeneous boundary conditions

So far, our discussion has focused on problems with homogeneous boundary conditions. Now, let's consider a case where the boundary conditions are non-homogeneous. The following example will illustrate this scenario.

Example 4.9. Let's continue with the given equation:

$$\begin{cases} u_t = u_{xx} + 2u_x - e^{-2x} \\ u(0,t) = 1 \\ u(1,t) = 1 + \frac{1}{2}e^{-2} \end{cases}.$$

We consider the solution as $u(x,t) = V(x) + w(x,t)$, where $V(x)$ reflects the contribution of the time-independent sources and $w(x,t)$ satisfies the homogeneous equation. Substituting this solution into the heat equation gives:

$$w_t = V'' + 2V' + w_{xx} + 2w_x + e^{-2x}.$$

To keep w as a homogeneous equation, we take $V(x)$ to satisfy the equation:

$$V'' + 2V' = e^{-2x}.$$

We transfer the non-zero boundary conditions to $V(x)$, resulting in the following boundary value problem for V:

$$\begin{cases} V'' + 2V' = e^{-2x} \\ V(0) = 1 \\ V(1) = 1 + \frac{1}{2}e^{-2} \end{cases}.$$

Solving this equation gives us the specific solution for $V(x)$:

$$V(x) = 1 - \frac{1}{2} x e^{-2x}.$$

For $w(x,t)$, we have the following homogeneous equation and boundary conditions:

$$\begin{cases} w_t = w_{xx} + 2w_x \\ w(0,t) = 0 \\ w(1,t) = 0 \end{cases}.$$

The general series solution for w is given by:

$$w(x,t) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n t} \phi_n(x),$$

where $\phi_n = e^{-x}\sin(n\pi x)$, and $\lambda_n = 1 + n^2\pi^2$. Finally, the general solution for the given problem is obtained as:

$$u(x,t) = 1 - \frac{1}{2} x e^{-2x} + \sum_{n=1}^{\infty} C_n e^{-\lambda_n t} \phi_n(x).$$

The parameters C_n can be determined by the initial condition of the problem. For example, if $u(x,0) = 0$, then we have:

$$\frac{1}{2} x e^{-2x} - 1 = \sum_{n=1}^{\infty} C_n \phi_n(x).$$

Using the orthogonality of eigenfunctions with respect to the weight function $\sigma = e^{2x}$, the coefficients C_n are determined as:

$$C_n = \frac{\langle -V, \phi_n \rangle_{e^{2x}}}{\|\phi_n\|^2}.$$

Finally, we obtain the solution as:

$$u(x,t) = 1 - \frac{1}{2} x e^{-2x} - \sum_{n=1}^{\infty} \frac{\langle V, \phi_n \rangle_{e^{2x}}}{\|\phi_n\|^2} e^{-\lambda_n t} \phi_n(x).$$

In general, when the equation or its boundary conditions are functions independent of time, it can be beneficial to split the solution into two terms: 1) a pure function of x denoted as $V(x)$, and 2) a function of both x and t denoted as $w(x,t)$. The desired solution can then be written as the sum of these two terms: $u(x,t) = V(x) + w(x,t)$.

The function $V(x)$ incorporates all the non-homogeneous terms that are independent of time, while $w(x,t)$ satisfies the zero boundary conditions. This decomposition allows us to separate the time-independent part from the time-dependent part of the solution, making the problem more manageable.

Example 4.10. Let's solve the following equation

$$\begin{cases} u_t = u_{xx} - 6x + e^{-t}\sin(\pi x) \\ u(t,0) = 0, u(t,1) = 1 \\ u(0,x) = x^3 \end{cases}.$$

The boundary condition is non-homogeneous. We take $u(x,t)$ as the sum of two terms:

$$u(t,x) = V(x) + w(t,x).$$

By substituting u into the differential equation, we obtain:

$$w_t = V'' + w_{xx} - 6x + e^{-t}\sin(\pi x),$$

and thus the equation for $V(x)$ is:

$$\begin{cases} V'' = 6x \\ V(0) = 0, V(1) = 1 \end{cases}.$$

The differential equation for $V(x)$ is solved and we find $V(x) = x^3$. The equation for $w(t,x)$ becomes:

$$\begin{cases} w_t = w_{xx} + e^{-t}\sin(\pi x) \\ w(t,0) = w(t,1) = 0 \\ w(x,0) = 0 \end{cases}.$$

We observe that $w(x,\ t)$ is triggered only by the term $h(x,\ t) = e^{-t}\ \phi_1(x)$, where $\phi_n(x) = \sin(n\pi x)$. Therefore, we can write the solution as:

$$w(t,x) = W(t)\sin(\pi x),$$

where $W(t)$ is an undetermined function. Substituting this series into the equation for $w(t,x)$, we obtain the following first-order ODE for $W(t)$:

$$W' + \pi^2 W_n = e^{-t}.$$

The solution to this ODE is:

$$W(t) = C e^{-\pi^2 t} + \frac{1}{\pi^2 - 1} e^{-t}.$$

The initial condition $w(x,0) = 0$ implies $W(0) = 0$, which determines $C = -\frac{1}{\pi^2 - 1}$. Therefore, the solution $u(x,t)$ is given by:

$$u(t,x) = x^3 + \frac{1}{\pi^2 - 1}(-e^{-\pi^2 t} + e^{-t})\sin(\pi x).$$

It is observed that the derived solution satisfies the partial differential equation, as well as the given boundary and initial conditions. Additionally, we note that:

$$\lim_{t\to\infty} u(t,x) = x^3 = V(x),$$

and thus the function $V(x)$ is also referred to as the steady-state solution in this case. The function $w(x,t)$ vanishes with time and is called the transient solution.

Exercise 4.24. Solve the following heat problem

$$\begin{cases} u_t = u_{xx} - x \\ u(0,t) = -1, u_x(\pi,t) = 0 \\ u(x,0) = \frac{1}{6}x^3 \end{cases}.$$

Exercise 4.25. Solve the following wave problem

$$\begin{cases} u_{tt} = u_{xx} \\ u(0,t) = 1 \\ u_x(\pi,t) = 0 \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}.$$

4.3.4 Problem

Problem 4.23. Let Ω be the domain $\Omega := (0,\pi)$. We aim to solve the following problem:

$$\begin{cases} u_{tt} = u_{xx} + 2u_x + \sin(t)\, e^{-x} \sin\left(\frac{x}{2}\right) & \text{on } \Omega \\ u(0,t) = 0 \\ u_x(\pi,t) + u(\pi,t) = 0 \end{cases}.$$

a) Find eigenvalues λ and their associated eigenfunctions ϕ of the following eigenvalue problem:

$$\begin{cases} \phi'' + 2\phi' = -\lambda\phi \\ \phi(0) = 0, \phi'(\pi) + \phi(\pi) = 0 \end{cases}.$$

You can assume that $\lambda > 1$. Determine the weight function for the orthogonality of the eigenfunctions too.

b) Use the results from part a) to find the solution of the given wave problem when the initial condition is given by: $u(x,0) = 0$, $u_t(x,0) = 0$.

Problem 4.24. Find the eigenvalues and eigenfunctions of the following eigenvalue problem

$$\begin{cases} \phi'' = -\lambda\phi & x \in (-1,1) \\ \phi(-1) = 0 \\ \phi(1) = 0 \end{cases}.$$

Use the result and find the series solution of the following heat problem

$$\begin{cases} u_t = u_{xx} \\ u(-1,t) = 0 \\ u(1,t) = 0 \\ u(x,0) = x^2 \end{cases}.$$

Problem 4.25. Consider the following heat equation

$$\begin{cases} u_t = \partial_x[(1+x)^2 u_x] \\ u(0,t) = 0 \\ u_x(1,t) = 0 \end{cases}.$$

Find the form of the eigenfunctions and use a numerical method to determine first five eigenvalues and associated eigenfunctions. Use these values and write downs of the series solution of the heat equation. Find the first five coefficients of the series solution if $u(x,0)$ is given as

$$u(x,0) = \frac{100}{1+x}.$$

Problem 4.26. Consider the following heat equation

$$\begin{cases} u_t = x^2 u_{xx} + 3x u_x + u \\ u(1,t) = 0, u(e,t) + u_x(e,t) = 0 \\ u(x,0) = x \end{cases}.$$

a) Use a numerical method to find the first 3 eigenvalues of the associated eigenvalue problem

b) Use these eigenfunctions and write downs an approximate series solution to the heat equation.

Problem 4.27. Consider the wave equation:

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = 0 \end{cases}.$$

a) Write the series solution of the equation.

b) Use the trigonometric identity for terms $\cos(cn\pi t)\,\sin(n\pi x)$ and conclude the solution can be written as

$$u(x,t) = \frac{f_{\text{odd}}(x-ct) + f_{\text{odd}}(x+ct)}{2}.$$

Problem 4.28. (Fourier series) Consider the following eigenvalue problem

$$\begin{cases} \phi''(\theta) = -\lambda\phi(\theta) & \theta \in (-\pi,\pi) \\ \phi(-\pi) = \phi(\pi) & \\ \phi'(-\pi) = \phi'(\pi) & \end{cases}.$$

The physics of the problem is a circle, and $\phi(\theta)$ is the function defined on this circle. The boundary condition of the problem is not Robin condition and then we are not allowed to use the result of the Sturm-Liouville theorem.

a) Show that the eigenvalues of the problem are $\lambda_n = n^2$, $n = 0, 1, 2, \cdots$ and eigenfunctions are $\{\cos(n\theta)\}_{n=0}^{\infty}$ and $\{\sin(n\theta)\}_{n=1}^{\infty}$. These functions are know and the Fourier eigenfunctions.

b) It is known that the eigenfunctions form an orthogonal basis for piecewise continuously differentiable functions defined on $\theta \in [-\pi,\pi]$. Use the orthogonality condition as find an expansion for the function

$$f(\theta) = \begin{cases} 1 & -\pi < \theta < 0 \\ 0 & 0 < \theta < 0 \end{cases}.$$

You can use the following code in Matlab and see the graph of $f(\theta)$ and its approximation for a truncated series.

```
%length
L=pi;
%functon
f=@(x) 1*(x>-L & x<0)+0*(x>0 & x<L);
%number of terms
n=10;
%the coefficients of Fourier series
a0=integral(@(x) f(x),-L,L)/(2*L);
a=integral(@(x) f(x).*cos((1:n)'.*x*pi/L),-L,L,'arrayvalued',true)/L;
b=integral(@(x) f(x).*sin((1:n)'.*x*pi/L),-L,L,'arrayvalued',true)/L;
%making fhat
x=-L:0.01:L;
fhat=a0+sum(a.*cos((1:n)'*x*pi/L))+sum(b.*sin((1:n)'*x*pi/L));
%plofing functions
plot(x,f(x),x,fhat);
```

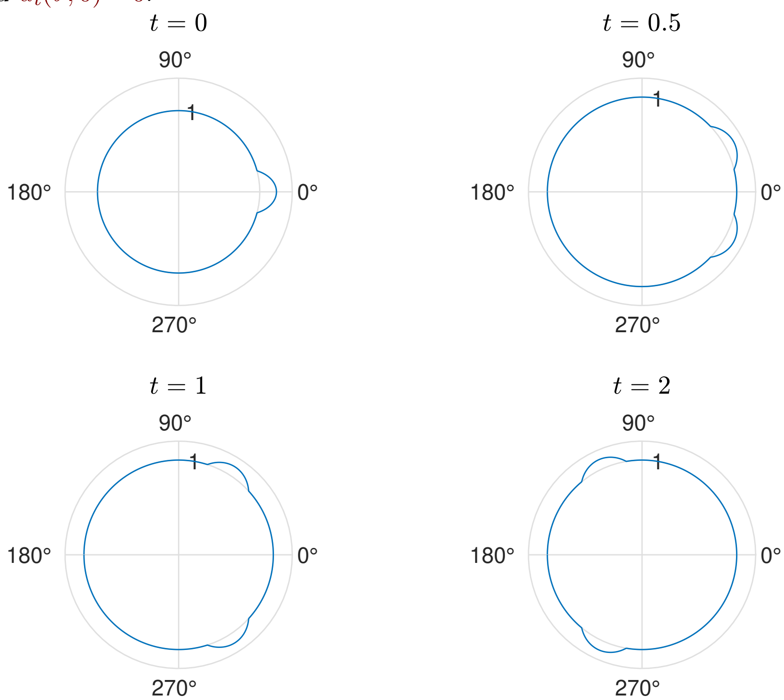
c) Use the above result and solve the following wave equation for $u(\theta,t)$ on the unit circle

$$\begin{cases} u_{tt} = u_{\theta\theta} & -\pi < \theta < \pi \\ u(-\pi,t) = u(\pi,t) & \\ u_\theta(-\pi,t) = u_\theta(\pi,t) & \end{cases}$$

The figures below depicts the graph of the solution at some instances of time for the initial conditions

$$u(\theta,0)=\begin{cases}\left(\frac{\pi}{12}-\theta\right)\left(\theta+\frac{\pi}{12}\right) & -\frac{\pi}{12}<\theta<\frac{\pi}{12}\\ 0 & \text{otherwise}\end{cases},$$

and $u_t(\theta,0)=0$.



Problem 4.29. Solve the following wave problem

$$\begin{cases}u_{tt}=u_{xx}+2u_x+u+tx\\ u(0,t)=u(\pi,t)=0\\ u(x,0)=x,\, u_t(x,0)=0\end{cases}.$$

Problem 4.30. We know that functions $\phi_n(x)=x\sin(n\pi x)$ for $n=1,2,\cdots$ solve the following eigenvalue problem:

$$\begin{cases}x^2\phi''-2x\phi'+2\phi=-\lambda x^2\phi\\ \phi(1)=\phi(2)=0\end{cases}.$$

a) Determine the associated eigenvalue λ_n.

b) Write down the series solution of the following problem

$$\begin{cases}u_t=u_{xx}-\frac{2}{x}u_x+\frac{2}{x^2}u\\ u(1,t)=u(2,t)=0\\ u(x,0)=\sin(\pi x)\end{cases}.$$

Problem 4.31. Consider the following problem

$$\begin{cases}u_t=u_{xx}\\ u(0,t)=0,\, u(\pi,t)=t\end{cases}.$$

The boundary condition at $x=\pi$ is a function of time t.

a) Take u as: $u(x,t)=V(x,t)+w(x,t)$. Substituting this into the equation results in:

$$w_t+V_t=V_{xx}+w_{xx}.$$

Assume that V satisfies the following boundary value problem

$$\begin{cases} V_{xx} = 0 \\ V(0,t) = 0 \\ V(\pi,t) = t \end{cases}.$$

Determine the function $V(x,t)$.

b) Substituting this function into the equation for w yields:

$$\begin{cases} w_t + \frac{x}{\pi} = w_{xx} \\ w(0,t) = 0 \\ w(\pi,t) = 0 \end{cases}.$$

Solve this equation for w and find the solution to the problem if $u(x,0) = 0$.

Problem 4.32. Consider the following wave equation

$$\begin{cases} u_{tt} = u_{xx} + \frac{1}{4}u \\ u(0,t) = 0, u(\pi,t) = \sin(t) \end{cases}.$$

a) Take u as: $u(x,t) = V(x,t) + w(x,t)$. Substituting this into the equation results in:

$$w_{tt} + V_{tt} = V_{xx} + \frac{1}{4}V + w_{xx} + \frac{1}{4}w.$$

Assume that V satisfies the following boundary value problem

$$\begin{cases} V_{xx} + \frac{1}{4}V = 0 \\ V(0,t) = 0 \\ V(\pi,t) = \sin(t) \end{cases}.$$

Determine the function $V(x,t)$.

b) Substituting this function into the equation for w yields:

$$\begin{cases} w_t - \sin(t)\sin\left(\frac{x}{2}\right) = w_{xx} + \frac{1}{4}w \\ w(0,t) = 0 \\ w(\pi,t) = 0 \end{cases}.$$

Solve this equation for w and find the solution to the problem if $u(x,0) = 0$.

Problem 4.33. Solve the following heat problem

$$\begin{cases} u_t = x^2 u_{xx} + 4u \\ u(1,t) = -1, u(e,t) = 1 \\ u(x,0) = x \end{cases}$$

Problem 4.34. Solve the following heat problem:

$$\begin{cases} u_t = u_{xx} + 2u_x \\ u(0,t) = 0, u(1,t) = 1 \\ u(x,0) = x \end{cases}.$$

Problem 4.35. Solve the following equation

$$\begin{cases} u_t = \partial_x[(1+x)^2 u_x] - 2 \\ u(0,t) = 0, u_x(1,t) = 1 \\ u(x,0) = 102\ln(1+x) \end{cases}$$

Problem 4.36. Solve the following equation

$$\begin{cases} u_t = \partial_x[(1+x)^2 u_x] + \frac{t}{\sqrt{1+x}} \\ u(0,t) = 0, u_x(1,t) = 1 \\ u(x,0) = \frac{4x}{1+x} \end{cases}$$

Problem 4.37. Solve the following equation

$$\begin{cases} u_t = u_{xx} + 2u_x + e^{-t} e^{-x} \sin(\pi x) \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = 0 \end{cases}.$$

Problem 4.38. Consider the following problem

$$\begin{cases} u_t = u_{xx} \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = f(x) \end{cases}.$$

Show that the solution to the problem is equal to the solution to the following problem

$$\begin{cases} u_t = u_{xx} + \delta(t)\, f(x) \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = 0 \end{cases},$$

where $\delta(t)$ is the Dirac delta function.

Problem 4.39. Consider the following problem

$$\begin{cases} u_t = u_{xx} + h(t)\, f(x) \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = 0 \end{cases},$$

where $h = 0$ f or $t \leq 0$. Show that the series solution of the problem is

$$u(x,t) = \sum_{n=1}^{\infty} F_n \left(h(t) * e^{-\lambda_n t}\right) \phi_n(x),$$

where $\phi_n = \sin(n\pi x)$ are the eigenfunctions of the associated eigenvalue problem, λ_n are the associated eigenvalues, F_n are the coefficients of the expansion of $f(x)$ in terms of the functions in $\{\phi_n\}$, and $h(t) * e^{-\lambda_n t}$ is the convolution of $e^{-\lambda_n t}$ and $h(t)$ defined as

$$h(t) * e^{-\lambda_n t} = \int_0^t h(\tau)\, e^{-\lambda_n (t-\tau)}\, d\tau.$$

Problem 4.40. Consider the following problem

$$\begin{cases} u_{tt} = u_{xx} \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = 0, u_t(x,0) = g(x) \end{cases}.$$

Show that the solution to the problem is equal to the solution to the following problem

$$\begin{cases} u_t = u_{xx} + \delta(t)\, g(x) \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = 0, u_t(x,0) = 0 \end{cases},$$

where $\delta(t)$ is the Dirac delta function.

Problem 4.41. Solve the following damped wave equation

$$\begin{cases} u_{tt} + 2u_t = u_{xx} + \sin(x) \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = 0, u_t(x,0) = 1 \end{cases}$$

Problem 4.42. Consider the following equation

$$\begin{cases} u_t = u_{xx} + 2u_x \\ u(0,t) = 0 \\ u_x(1,t) = 0 \end{cases}.$$

a) The associated eigenvalue problem is

$$\begin{cases} \phi'' + 2\phi' = -\lambda\phi \\ \phi(0) = \phi(1) = 0 \end{cases}.$$

Use a numerical method and find first five eigenvalues and the associated eigenfunctions.

b) Use these set and approximate the solution of the heat problem if $u(x,0) = 1$.

Problem 4.43. Consider the following problem

$$\begin{cases} u_t = u_{xx} + u \\ u(0,t) = 0 \\ u(\pi,t) = 1 \end{cases}.$$

Let us take $u(x,t) = V(x) + w(x,t)$, and deriving the following equation for V:

$$\begin{cases} V'' + V = 0 \\ V(0) = 0, V(\pi) = 1 \end{cases}.$$

Show that this equation is not solvable. What is your suggestion to solve the given heat problem?

4.4 Theoretical aspects of the method

The method we introduced above is known as the eigenfunction expansion method, which is based on expanding the desired solution in terms of the eigenfunctions of an eigenvalue problem.

4.4.1 Function spaces and convergence

In linear algebra, we learn that a set of vectors can form a vector space when equipped with the operations of vector addition $(+)$ and scalar multiplication. If the set is closed under these operations, it creates a vector space that can serve as the domain or range of linear mappings.

Similarly, in the context of differential operators, the domain and range are vector function spaces. Consider two functions, f and g, defined on the same domain (x_0, x_1). The vector addition of f and g is defined as follows:

$$(f+g)(x) = f(x) + g(x),$$

for all $x \in (x_0, x_1)$. Similarly, scalar multiplication λf is defined as:

$$(\lambda f)(x) = \lambda f(x).$$

These operations maintain closure within the function space.

One important class of function spaces is the C^r spaces. Recall that a function $f(x)$ is continuously differentiable in the interval (x_0, x_1) if its derivative $f'(x)$ is a continuous function in the same interval. We denote this as $f \in C^1(x_0, x_1)$. A function f is said to be continuously differentiable of order r in the interval (x_0, x_1), denoted as $C^r(x_0, x_1)$, if its k^{th} derivative $f^{(k)}(x)$ is in the vector space $C(x_0, x_1)$ of continuous functions, for all k up to r.

Exercise 4.26. Consider the function

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

Show that the function is not differentiable at $x = 0$. The functions

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is differentiable at $x = 0$ but not in $C^1(-1, 1)$.

Indeed, function spaces, such as the space spanned by $\{\sin(n\pi x)\}$ for $n = 1, 2, \ldots$, are not of finite dimension. They contain linearly independent functions (which are orthogonal in this case), making it tricky to define convergence in such spaces. The meaning of a series like:

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x),$$

One approach is to define the norm of a function $f \in C^r(x_0, x_1)$ as:

$$\|f\| = \sqrt{\sum_{k=0}^{r} \int_a^b |u^{(k)}(x)|^2 \, dx},$$

where the norm measures the square root of the sum of squared derivatives. Convergence of the series in this norm would imply:

$$\lim_{N \to \infty} \left\| f(x) - \sum_{n=1}^{N} c_n \phi_n(x) \right\| = 0.$$

Another notion of norm for a function $f \in C^r(x_0, x_1)$ denoted by $\|f\|_\infty$, can be defined as:

$$\|f\|_\infty = \sum_{k=0}^{r} \sup_{x \in (x_0, x_1)} |f^{(k)}(x)|,$$

which measures the supremum (or maximum) value of the derivatives. Consequently, the notion of convergence changes based on this new definition.

In this book, the focus is mainly on pointwise convergence, which means that for any $\xi \in (x_0, x_1)$, and any $\varepsilon > 0$, there exists an $N = N(\xi,\varepsilon)$ such that:

$$\left| f(\xi) - \sum_{n=1}^{N} c_n \phi_n(\xi) \right| < \varepsilon.$$

This means that the series converges to the function $f(x)$ at each point ξ within the interval (x_0, x_1) with an arbitrary small error ε.

Exercise 4.27. Show that the sequence of functions $f_n(x) = x^n$ for $x \in [0,1]$ and $n = 1, 2, \cdots$ converges pointwise to the function

$$f(x) = \begin{cases} 1 & x = 1 \\ 0 & 0 \le x < 1 \end{cases}.$$

However, if we define the norm

$$\|u\|_\infty = \sup_{x \in [0,1]} |u(x)|,$$

the sequence $f_n(x)$ does not converges to $f(x)$ in this norm. Show that if we change the norm to

$$\|u\| = \sqrt{\int_0^1 |u(x)|^2 \, dx},$$

the sequence $f_n(x)$ converges to $f(x)$ in this norm.

Exercise 4.28. Consider the sequence of functions

$$f_n(x) = \begin{cases} \sqrt{n} & 0 < x < \frac{1}{n} \\ 0 & \frac{1}{n} \le x < 1 \end{cases}.$$

Show that $f_n(x)$ converges pointwise to the zero functions $f(x) = 0$ for $x \in (0,1)$. However, $f_n(x)$ does not converges to the zero function in norm compatible with the inner product.

Exercise 4.29. Assume $\phi_n(x)$ are mutually orthogonal functions and the sequence of functions

$$f_n(x) = \sum_{k=1}^{n} \frac{\langle f, \phi_n \rangle}{\|\phi_n\|^2} \phi_n,$$

converges in norm to functions $f(x)$, where the norm is compatible with the inner product $\langle, \rangle$. Show the equality

$$\|f\|^2 = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{\langle f, \phi_n \rangle^2}{\|\phi_n\|^2}.$$

Exercise 4.30. Consider the sequence of functions $f_n(x) = \frac{1}{\sqrt{n}} \sin(nx)$ on $0 \le x \le \pi$ for $n = 1, 2, \cdots$.

i. Show that $f_n(x) \to 0$ in the norm compatible with the inner product:

$$\langle f, g \rangle = \int_0^\pi f(x) g(x) dx.$$

ii. If the inner product changes to the following one, show that the sequence $f_n(x)$ diverges:

$$\langle f, g \rangle = \int_0^\pi f(x) g(x) dx + \int_0^\pi f'(x) g'(x) dx.$$

Exercise 4.31. Consider the sequence of functions

$$f_n(x) = \sum_{k=0}^{n} \frac{1}{k!} x^k,$$

for $n=1,2,\cdots$ in the interval $[0,1]$. Show that the sequence converges pointwise and in norm to the function $f(x)=e^x$.

In this book, we also consider the function spaces of piecewise continuous functions $\mathrm{PC}(x_0,x_1)$ and piecewise continuously differentiable functions $\mathrm{PC}^1(x_0,x_1)$.

Definition 4.2. *A function $f(x)$, defined on the interval $(x_0,\,x_1)$, is called a piecewise continuous function if:*

a) *f is continuous everywhere in the interval, except possibly at some finite points.*

b) *If z is a discontinuous point for f, then the left and right limits of f at z exist.*

c) *The right limit of f exists at x_0, and the left limit exists at x_1.*

Definition 4.3. *A function $f(x)$ belongs to the space $\mathrm{PC}^1(x_0,\,x_1)$ if $f'(x)$ belongs to the space $\mathrm{PC}(x_0,\,x_1)$, except possibly at some finite points. In other words, $f(x)$ is piecewise continuously differentiable, and its derivative $f'(x)$ is a piecewise continuous function.*

These function spaces allow us to work with functions that may have discontinuities at certain points within the interval (x_0,x_1). Piecewise continuous functions are characterized by their continuity properties and the existence of left and right limits at potential discontinuities. Piecewise continuously differentiable functions, on the other hand, additionally possess piecewise continuous derivatives, except at a finite number of points.

Example 4.11. Let's consider the function:

$$f(x)=\begin{cases}1 & x\in Q\\ 0 & x\in Q^c\end{cases},$$

where Q is the set of rational numbers. This function is not piecewise continuous in any interval $(a,\,b)$ because it has infinitely many discontinuities or jumps within the interval. It is known as the Dirichlet function and serves as a classic example of a function that is discontinuous almost everywhere.

Example 4.12. Now, let's examine the function:

$$f(x)=\begin{cases}\frac{1}{x} & x\neq 0\\ 0 & x=0\end{cases},$$

This function is not in the space $\mathrm{PC}(-1,\,1)$ because it exhibits an infinite jump at $x=0$. Similarly, consider the function:

$$f(x)=\begin{cases}\sin\left(\frac{\pi}{x}\right) & x\neq 0\\ 0 & x=0\end{cases}.$$

This function is also not in the space $\mathrm{PC}(-1,1)$ since it oscillates infinitely near $x=0$.

Example 4.13. Let's examine the function $f(x)=x^{\frac{2}{3}}$. This function belongs to the space $\mathrm{PC}(-1,1)$ since it is continuous in the interval $(-1,1)$. However, it does not belong to the space $\mathrm{PC}^1(-1,1)$ because its derivative, $f'(x)$, is not piecewise continuous within the interval.

Similarly, the function

$$f(x) = \begin{cases} x^2 \sin\left(\frac{\pi}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is continuous and differentiable in the interval $(-1, 1)$. However, its derivative, $f'(x)$, is not piecewise continuous at $x = 0$ as it exhibits infinite oscillation near this point.

Exercise 4.32. Which one of the following functions belong to CP^1 or CP?

i. $f(x) = x^{2/3}$, $x \in [0, 1]$

ii. $f(x) = x|x|$, $x \in -1, 1$

iii. $f(x) = \sqrt{|x|}$, $x \in [-1, 1]$

iv. $f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$.

4.4.2 Proof of the orthogonality of eigenfunctions

The complete proof of the Sturm-Liouville theorem requires advanced tools that are beyond the scope of this book. However, we can establish two key aspects of the theorem through an elementary approach. The crucial step in achieving the proof is to reframe the problem in a symmetric form. To accomplish this, we need to introduce the concept of symmetry for a differential operator.

The notion of inner product extends to another important concept known as symmetric operators. This concept can be related to the concept of symmetric matrices in linear algebra. In linear algebra, a square matrix $A = [a_{ij}]$ is considered symmetric if $a_{ij} = a_{ji}$. It is known that symmetric matrices have real eigenvalues and orthogonal eigenvectors.

We can extend this concept to linear differential operators using the inner product.

Definition 4.4. *A linear differential operator L is said to be symmetric with respect to the inner product $\langle , \rangle$ if the following equality holds for all functions f and g in its domain:*

$$\langle L[f], g \rangle = \langle f, L[g] \rangle.$$

This definition establishes a symmetry relationship between the operator L and the inner product, similar to the symmetry observed in symmetric matrices.

Example 4.14. Consider the differential operator $L := \frac{d^2}{dx^2}$ defined on the set of functions $\phi(x)$ in $C^2(0, 1)$ which satisfies the boundary conditions $\phi(0) = \phi(1) = 0$. Let f and g be any two functions in this set. We have

$$\langle L[f], g \rangle = \int_0^1 f''(x)\, g(x)\, dx,$$

and through integration by parts:

$$\int_0^1 f''(x)\, g(x)\, dx = f'(x)\, g(x)|_0^1 - \int_0^1 f'(x)\, g'(x)\, dx = -\int_0^1 f'(x)\, g'(x)\, dx.$$

The last equality is obtained by using the boundary conditions for $g(x)$. Another integration by parts and use the boundary condition for $f(x)$ yields

$$-\int_0^1 f'(x)\, g'(x)\, dx = \int_0^1 f'(x)\, g''(x)\, dx = \langle f, L[g]\rangle$$

and thus L is symmetric on the defined set of functions.

Consider the differential equation:

$$a(x)\,\phi'' + b(x)\,\phi' + c(x)\phi = -\lambda\phi,$$

where $x \in (x_0, x_1)$, $a(x) > 0$ in $[x_0, x_1]$, and $a(x), b(x)$ and $c(x)$ are continuous functions. By dividing the equation by the term

$$\sigma = \frac{1}{a(x)}\, e^{\int \frac{b(x)}{a(x)}},$$

the equation can be rewritten as

$$\frac{d}{dx}\left[e^{\int \frac{b(x)}{a(x)}}\,\phi'\right] + \frac{c(x)}{a(x)}\, e^{\int \frac{b(x)}{a(x)}}\,\phi = -\lambda\,\frac{1}{a(x)}\, e^{\int \frac{b(x)}{a(x)}}\,\phi.$$

Taking $p(x) = e^{\int \frac{b(x)}{a(x)}}$, $q(x) = \frac{c(x)}{a(x)}\, e^{\int \frac{b(x)}{a(x)}}$, the equation can be expressed as:

$$\frac{d}{dx}[p(x)\,\phi'] + q(x)\,\phi = -\lambda\sigma(x)\,\phi.$$

Theorem 4.2. *The operator*

$$L[\phi] := \frac{d}{dx}(p(x)\,\phi') + q(x)\,\phi,$$

acting on functions in the vector set $C^2(x_0, x_1)$ *that satisfy the boundary condition:*

$$\begin{cases} \alpha_1\,\phi(x_0) + \beta_1\phi'(x_0) = 0 \\ \alpha_2\,\phi(x_1) + \beta_2\phi'(x_1) = 0 \end{cases},$$

is symmetric with respect to the inner product $\langle , \rangle$.

Proof. For any functions ϕ, ψ is the defined set of functions, we have

$$\langle L[\phi], \psi\rangle = \int_{x_0}^{x_1} \frac{d}{dx}(p(x)\,\phi')\,\psi + \int_{x_0}^{x_1} q(x)\,\phi\psi.$$

Applying integration by parts to the first integral yields:

$$\int_{x_0}^{x_1} \frac{d}{dx}(p(x)\,\phi')\,\psi = p(x)\,\phi'\,\psi|_{x_0}^{x_1} - p(x)\,\phi\psi'|_{x_0}^{x_1} + \int_{x_0}^{x_1} \frac{d}{dx}(p(x)\,\psi')\,\phi.$$

We need to show that the boundary terms are zero. The boundary term at $x = x_1$ is:

$$p(x_1)[\phi'(x_1)\,\psi(x_1) - \phi(x_1)\,\psi'(x_1)] = p(x_1)\,W(\phi, \psi)(x_1),$$

where $W(\phi,\ \psi)$ represents the Wronskian of functions ϕ and ψ. Due to the boundary conditions:

$$\begin{cases} \alpha_2\,\phi(x_1)+\beta_2\phi'(x_1)=0 \\ \alpha_2\,\psi(x_1)+\beta_2\psi'(x_1)=0 \end{cases},$$

we have $W(\phi,\psi)(x_1)\neq 0$, otherwise $\alpha_2=\beta_2=0$, which is impossible. The same argument applies to the boundary term at $x=x_0$. Therefore, we can conclude that:

$$\int_{x_0}^{x_1}\frac{d}{dx}(p(x)\,\phi')\,\psi=\int_{x_0}^{x_1}\frac{d}{dx}(p(x)\,\psi')\,\phi,$$

which leads to:

$$\langle L[\phi],\psi\rangle=\langle\phi,L[\psi]\rangle,$$

completing the proof. □

Theorem 4.3. *The eigenvalues of L in the defined vector space of functions are real.*

Proof. Assume $L[\phi]=\lambda\phi$. If λ is a complex eigenvalue, then by taking complex conjugate of the equation, we obtain $L[\bar{\phi}]=\bar{\lambda}\ \bar{\phi}$, where $\bar{\lambda},\bar{\phi}$ represents the complex conjugate of λ,ϕ respectively. Now, by symmetricity, we have

$$-\lambda\int_{x_0}^{x_1}|\phi(x)|^2\,dx=\langle L[\phi],\bar{\phi}\rangle=\langle\phi,L[\bar{\phi}]\rangle=-\bar{\lambda}\int_{x_0}^{x_1}|\phi(x)|^2\,dx,$$

and thus $\lambda=\bar{\lambda}$, which implies λ is a real value. □

Theorem 4.4. *Suppose the eigenfunctions ϕ_n, ϕ_m are associated with distinct eigenvalues λ_n and λ_m, respectively, in the eigenvalue problem:*

$$L[\phi]=-\lambda\sigma\,\phi,$$

Then ϕ_n and ϕ_m are orthogonal with respect to the weight function σ.

Proof. By assumption, we have:

$$\begin{cases} L[\phi_n]=-\lambda_n\,\sigma(x)\,\phi_n \\ L[\phi_m]=-\lambda_m\,\sigma(x)\,\phi_m \end{cases},$$

where $\lambda_n\neq\lambda_m$. Using the property of symmetricity, we can write:

$$-\lambda_n\,\langle\sigma\phi_n,\phi_m\rangle=\langle L[\phi_n],\phi_m\rangle=\langle\phi_n,L[\phi_m]\rangle=-\lambda_m\langle\phi_n,\sigma\phi_m\rangle_\sigma.$$

From this, we have $(\lambda_n-\lambda_m)\langle\phi_n,\phi_m\rangle_\sigma=0$ which implies that $\langle\phi_n,\phi_m\rangle_\sigma=0$, satisfying the orthogonality condition. □

Exercise 4.33. Write down the following eigenvalue problems in the symmetric Sturm-Liouville form. Use the energy method in each case and show that their eigenvalues are strictly positives

a)

$$\begin{cases} \phi''+2x\phi'=-\lambda\phi \\ \phi(0)=0,\,\phi'(1)=0 \end{cases}$$

b)

$$\begin{cases} (1+x)\phi''+x\phi'=-\lambda\phi \\ \phi(0)=\phi'(0),\,\phi'(0)=\phi'(1) \end{cases}$$

c)

$$\begin{cases} \phi'' + 2\phi' = -\lambda e^{-2x}\phi \\ \phi(0) = 0, \phi'(1) = 0 \end{cases}$$

Exercise 4.34. Consider the operator

$$L[\phi] = \frac{d}{dx}(p(x)\phi') + q(x)\phi,$$

with the domain of functions ϕ in the vector space

$$\mathbb{U} = \{\phi \in C^2(x_0, x_1); \phi(x_0) = \phi(x_1), \phi'(x_0) = \phi'(x_1)\}.$$

Prove that if $p(x_0) = p(x_1)$, then L is symmetric with respect to the inner product $\langle , \rangle$ and conclude the that eigenvalues of the following problem are real, and their eigenfunctions are orthogonal with respect to the weight functions $\sigma > 0$

$$\begin{cases} \frac{d}{dx}(p(x)\phi') + q(x)\phi = -\lambda\sigma\phi \\ \phi(x_0) = \phi(x_1) \\ \phi'(x_0) = \phi'(x_1) \end{cases}.$$

Prove that if $p(x_0) = p(x_1)$, then the operator which is defined in the equation is symmetric and therefore its eigenvalues are real and eigenfunctions are orthogonal with respect to weight function σ.

Exercise 4.35. It can be shown that there exists only one eigenfunction (up to a multiplication by constants) associated to each eigenvalue for the Strum-Liouville problem. The scheme of proof is as follows: Let ϕ_1 and ϕ_2 be the eigenfunctions of the Sturm-Liouville eigenvalue problem associated to the same eigenvalue λ. Prove that the Wronskian $W(\phi_1, \phi_2) = 0$ and conclude that ϕ_1 and ϕ_2 are linearly dependent.

Exercise 4.36. The Sturm-Liouville theorem states that the eigenvalues of the problem forms an increasing set of value $\lambda_n \to \infty$. This exercise set a lower bound for the sequence. Show that λ satisfies the inequality

$$\lambda \geq \frac{-\int_{x_0}^{x_1} q(x)|\phi|^2}{\int_{x_0}^{x_1} \sigma\,|\phi|^2},$$

and conclude that if $q < 0$ then $\lambda > 0$.

Exercise 4.37. Show that linear operator $L[\phi] = \phi'' + \phi'$ is not symmetric in the following vector space

$$\mathbb{U} = \{\phi \in C^2(0,1), \phi(0) = \phi(1) = 0\}.$$

Exercise 4.38. Show that linear operator $L[\phi] = e^x\phi'' + e^x\phi'$ is symmetric in the following space

$$\mathbb{U} = \{\phi \in C^\infty(0,1), \phi(0) = \phi(1) = 0\}.$$

Exercise 4.39. Show that linear operator $L[\phi] = \phi'' + e^x\phi$ is symmetric in the following space

$$\mathbb{U} = \{\phi \in C^\infty(0,1), \phi(0) = \phi(1), \phi'(0) = \phi'(1)\}.$$

Exercise 4.40. Show that linear operator $L[\phi] = \phi'$ is anti-symmetric in the following space

$$\mathbb{U} = \{\phi \in C^\infty(0,1), \phi(0) = \phi'(0), \phi(1) = \phi'(1)\}$$

An operator L is anti-symmetric if $\langle L[\phi], \psi\rangle = -\langle \phi, L[\psi]\rangle$.

Exercise 4.41. Suppose $G(x)$ is a continuous function in $[0, 1]$, and $G(-x) = G(x)$. Show that the operator

$$L[\phi](x) = \int_0^1 G(x-t)\phi(t)\,dt,$$

is symmetric.

Exercise 4.42. Consider the following eigenvalue problem:

$$\begin{cases} x^2\phi'' + x\phi' + 2\phi = -\lambda\phi \\ \phi'(1) = \phi'(e) = 0 \end{cases}.$$

a) Find the eigenvalues and eigenfunctions of the problem.
b) Find an expansion of the function $f(x) = 1$, $x \in [1, e]$ in terms of the first four eigenfunctions of the problem.

Exercise 4.43. Consider the linear operator $L[\phi] = \frac{d}{dx}(e^{-x}\,\phi')$ with the domain of functions ϕ in the vector space

$$\mathbb{U} = \{\phi \in C^2(0, \ln 2), \phi(0) = \phi(\ln 2) = 0\}.$$

a) Show that L is symmetric.
b) Find λ_n and $\sigma(x)$ such that the eigenfunctions of the eigenvalue problem

$$L[\phi_n] = -\lambda_n \sigma(x)\,\phi_n,$$

are $\phi_n(x) = \sin(n\pi e^x)$.
c) The set $\{\phi_n\}_{n=1}^{\infty}$ is a basis for $\mathrm{PC}^1(0, \ln 2)$. Find the best approximation of the function $f(x) = 1$ in the space $\mathrm{span}\{\phi_n\}_{n=1}^{5}$. That is, find $f_1, \ldots, f_5$ such that

$$\|1 - f_1\phi_1 - \cdots - f_5\phi_5\| \leq \|f - g\|,$$

for any $g \in \mathrm{span}\{\phi_n\}_{n=1}^{5}$. Draw this approximation and compare it with the function f.
d) Consider the following heat problem

$$\begin{cases} u_t = L[u] \\ u(0,0) = u(\ln 2, t) = 0 \\ u(x, 0) = 1 \end{cases}.$$

Since $\{\phi_n\}_{n=1}^{\infty}$ is a basis, we can approximate the solution u as

$$u(x,0) \approx \sum_{n=1}^{5} U_n(t)\,\phi_n(x).$$

Find the ordinary differential equation of $U_n(t)$ along $\phi_n(x)$. This differential equation describes the change of the temperature along each $\phi_n(x)$ for $n = 1, \cdots 5$.

4.4.3 Operation on infinite series

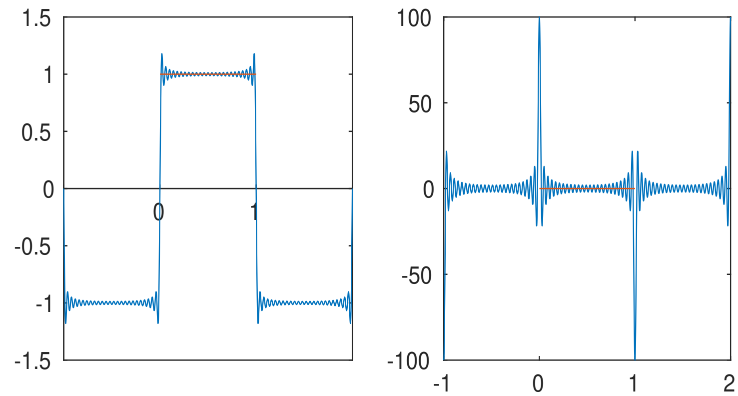
The type of convergence of infinite series imposes certain limitations when working with them. For instance, consider the function $f(x) = 1$, $x \in (0, 1)$ which belongs to the set $\mathrm{PC}^1(0, 1)$ and can be represented in terms of the basis $\{\sin(n\pi x)\}$ for $n = 1, 2, \cdots$ as

$$1 = \sum_{n=1}^{\infty} \frac{2 - 2\cos(n\pi)}{n\pi} \sin(n\pi x).$$

Taking the derivative of this equality yields:

$$0 = \sum_{n=1}^{\infty} [2 - 2\cos(n\pi)] \cos(n\pi x),$$

However, this result is clearly incorrect since the function $g(x) = 0$ is represented as a non-trivial series in terms of the orthogonal set $\{\cos(n\pi x)\}$. The figure below shows the function $f(x)$ in $x \in (0, 1)$ and its series representation in a wider domain.



Where does this error come from? One possible explanation for this discrepancy lies in the behavior of the series representation at the endpoints of the interval $(0, 1)$. Although the function $f(x)$ is continuous on $(0,1)$, its derivative is not continuous at $x=0$ and $x=1$. This leads to a discontinuity in the derivative series representation, which can result in inconsistencies in the equality above.

Furthermore, when extending the series representation of the function $f(x)$ to a wider interval beyond $(0, 1)$, it exhibits an odd extension of $f(x)$. This means that a jump is formed at the endpoints $x=0$ and $x=1$, due to the discontinuity of the original function. Consequently, the derivative of these jumps generates a Dirac delta function at these points.

It is important to note that the Dirac delta function is a distribution that is concentrated at a specific point and has the property of being zero everywhere except at that point, where it behaves like an impulse. The presence of the Dirac delta function-like behavior in the derivative series further contributes to the discrepancies observed in the equality, as it introduces additional irregularities near the endpoints.

Now, let's consider the function $f(x)=x(1-x)$. The series representation of $f(x)$ in terms of the basis $\{\sin(n\pi x)\}$ is given by:

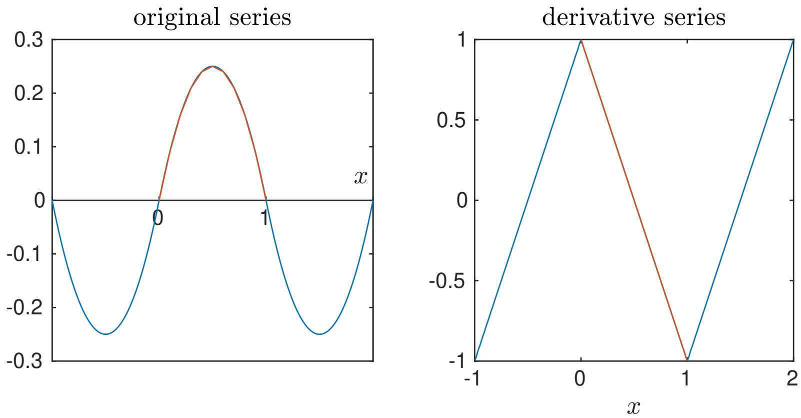
$$x(1-x)=\sum_{n=1}^{\infty}\frac{4-4\cos(n\pi)}{n^3\pi^3}\sin(n\pi x).$$

Taking the derivative of this equation yields the equality:

$$1-2x=\sum_{n=1}^{\infty}\frac{4-4\cos(n\pi)}{n^2\pi^2}\cos(n\pi x).$$

It can be verified that the latter series is a true series representation of the function $f'(x)=1-2x$ in terms of the orthogonal basis $\{\cos(n\pi x)\}$. The figure below shows the original series of the function $f(x)$ and its derivative, both truncated to $N=50$ terms.

It is important to note that since the odd extension of the function $f(x)$ is continuous at the endpoints, the derivative of the series of $f(x)$ converges pointwise to the derivative $f'(x)$ in the interval $(0,1)$. This behavior ensures that the series accurately represents the derivative of the function within the given interval.



Theorem 4.5. *Let f be a function belonging to $C^2(x_0, x_1)$, and let $\{\phi_n(x)\}$ for $n=1,2,...$ be an orthogonal basis generated by a Sturm-Liouville eigenvalue problem. Suppose the series representation of $f(x)$ in the interval (x_0, x_1) is given by:*

$$f(x) = \sum_{n=1}^{\infty} c_n \, \phi_n(x).$$

If $f(x_0) = f(x_1)$, then the derivative of $f(x)$ can be represented as:

$$f'(x) = \sum_{n=1}^{\infty} c_n \, \phi_n'(x).$$

There are no restrictions on other operations such as integration, addition, subtraction, or multiplication when working with series representations. These operations can be applied to series in a straightforward manner.

Exercise 4.44. Let $f(x)$ and $g(x)$ be functions with series representations as follows:

$$f(x) = \sum_{n=1}^{\infty} a_n \, \phi_n(x), \, g(x) = \sum_{n=1}^{\infty} b_n \, \phi_n(x),$$

where $\{\phi_n(x)\}$ is an orthogonal basis generated by a Sturm-Liouville eigenvalue problem.

a) Integration: The integral of the sum of two series can be obtained by integrating each term of the series individually:

$$\int_{x_0}^{x} [f(t) + g(t)] \, dt = \sum_{n=1}^{\infty} [a_n + b_n] \int_{x_0}^{x} \phi_n(t) \, dt,$$

for any $x \in (x_0, x_1)$.

b) Addition and subtraction: The sum or difference of two series can be obtained by adding or subtracting the corresponding terms of the series:

$$f(x) \pm g(x) = \sum_{n=1}^{\infty} [a_n \pm b_n] \, \phi_n(x)$$

c) Multiplication: The product of two series can be obtained by multiplying each term of one series with each term of the other series

$$f(x) \, g(x) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n \, b_m \, \phi_n(x) \, \phi_m(x)$$

In each case, the resulting series is still a representation of the corresponding operation applied to the functions $f(x)$ and $g(x)$.

Appendix A
Fourier series

A.1 Trigonometric and complex Fourier series

Consider the eigenvalue problem defined on a unit circle $-\pi \leq \theta \leq \pi$:

$$\begin{cases} \phi''(\theta) = -\lambda\phi(\theta) \\ \phi(-\pi) = \phi(\pi) \\ \phi'(-\pi) = \phi'(\pi) \end{cases}. \tag{A.1}$$

The periodic boundary condition given in the problem is natural, as $\phi(\theta)$ should be periodic with a period of 2π. Notice that $\phi(\theta)$, the eigenfunction of this problem, is invariant up to multiplication by a constant; that is, if $\phi(\theta)$ solves the equation, so does $c\phi(\theta)$ for any $c \in \mathbb{R}$. Although this problem is not in the standard form of the Sturm-Liouville problem, its eigenfunctions and eigenvalues share resemblances with the properties of the standard Sturm-Liouville problem.

Utilizing the energy method, one can show that the eigenvalues are non-negative:

$$\int_{-\pi}^{\pi} \phi''(\theta)\, \phi(\theta)\, d\theta = -\lambda \int_{-\pi}^{\pi} |\phi(\theta)|^2\, d\theta.$$

Integration by parts yields:

$$\int_{-\pi}^{\pi} |\phi'(\theta)|^2\, d\theta = \lambda \int_{-\pi}^{\pi} |\phi(\theta)|^2\, d\theta,$$

Thus, we have $\lambda \geq 0$. Furthermore, for $\lambda = 0$, the eigenfunction is a constant function, and we can assume, without loss of generality, that $\phi_0(\theta) = 1$. For $\lambda > 0$, the eigenfunctions are given by:

$$\phi_n(\theta) \in \{\cos(n\theta), \sin(n\theta)\}, n \in \mathbb{N}.$$

Furthermore, the set of eigenfunctions $\{1, \cos(nx), \sin(nx)\}$, $n \in \mathbb{N}$ forms a basis for piecewise continuously differentiable functions define don the unit circle. That is, for any $f(\theta), -\pi \leq \theta \leq \pi$, there are constants a_n and b_n such that:

$$f(\theta) = \sum_{n=0}^{\infty} a_n \cos(n\theta) + \sum_{n=1}^{\infty} b_n \sin(n\theta).$$

Since the set $\{1, \cos(n\theta), \sin(n\theta)\}$ is an orthogonal basis with respect to the inner product $\langle , \rangle$ defined as:

$$\int_{-\pi}^{\pi} \phi_n(\theta)\, \phi_m(\theta)\, d\theta = 0,$$

the parameters a_n and b_n are determined by the inner product of f and $\cos(n\theta)$ and $\sin(n\theta)$ respectively.

The eigenvalue problem (A.1) can be modified to represent functions on any symmetric domain $(-L, L)$ as follows:

$$\begin{cases} \phi'' = -\lambda\phi \\ \phi(-L) = \phi(L) \\ \phi'(-L) = \phi'(L) \end{cases}.$$

The set of solutions to this modified eigenvalue problem, with the normalizing factor $c=1$, is given by:

$$\phi_n(x) \in \left\{1, \cos\left(\frac{n\pi}{L}x\right), \sin\left(\frac{n\pi}{L}x\right)\right\}, n \in \mathbb{N}.$$

The associated eigenvalues for this problem are $\lambda_n = \frac{n^2\pi^2}{L^2}$.

Remark A.1. The set $\left\{1, \cos\left(\frac{n\pi}{L}x\right), \sin\left(\frac{n\pi}{L}x\right)\right\}$ is commonly known as the Fourier trigonometric basis. By using the Euler formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, we can express this basis in the complex form as $\left\{e^{i\frac{n\pi}{L}x}\right\}, n \in \mathbb{Z}$. Therefore, a piecewise continuously differentiable function defined on $[-L, L]$ can be represented as:

$$f(x) = \sum_{n=-\infty}^{\infty} F_n e^{in\omega x},$$

where $\omega = \frac{\pi}{L}$, and F_n are constants that can be determined using the complex form of the inner product $\langle\langle, \rangle\rangle$ defined as:

$$\langle\langle f, g\rangle\rangle := \int_{-L}^{L} f(x)\, \bar{g}(x)\, dx,$$

where $\bar{g}$ represents the complex conjugate of g. Utilizing this definition for the orthogonal eigenfunctions $\phi_n(x) = e^{in\omega x}$, the parameters F_n are determined as:

$$F_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-in\omega x}\, dx.$$

While the trigonometric basis $\left\{1, \cos\left(\frac{n\pi}{L}x\right), \sin\left(\frac{n\pi}{L}x\right)\right\}$ and the complex basis $\left\{e^{i\frac{n\pi}{L}x}\right\}$ both represent the same set of functions and are mathematically equivalent, the complex representation is often preferred in theoretical works due to its simplicity and elegance. Thus, for theoretical analysis and calculations, the complex basis offers significant advantages over the trigonometric basis.

Example A.1. Let's find the complex Fourier series of the function $f(x) = x$ in the interval $(-1, 1)$. Using the complex Fourier series representation, we have:

$$x = \sum_{n=-\infty, n\neq 0}^{\infty} \frac{i\cos(n\pi)}{n\pi}\, e^{in\pi x}. \tag{A.2}$$

Now, let's determine the complex Fourier series of the function $f(x)=x^2$ in the interval $(-1,1)$. Using the complex Fourier series representation, we have:

$$x^2=\frac{1}{3}+\sum_{n=-\infty,n\neq 0}^{\infty}\frac{2\cos(n\pi)}{n^2\pi^2}e^{in\pi x}. \tag{A.3}$$

Problem A.1. If $f(x)$ is an odd function, show that F_n are pure imaginary and $F_{-n}=-F_n$ (and thus $F_0=0$). If $f(x)$ is even function, show that F_n are real and $F_{-n}=F_n$.

Exercise A.1. Show that the set of functions $\{\sin(\frac{n\pi}{L}x), \cos(\frac{2n-1}{2L}\pi x)\}$ is a basis for the piecewise continuously differentiable functions defined on $(-L,L)$. Hint: Find an Sturm-Liouville problem that generate the set.

Exercise A.2. Use either the Fourier sine or cosine series of the function $f(x)=x$ on $(0,1)$ to prove the following identity:

$$\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots$$

Exercise A.3. Use either the Fourier sine or cosine series of the function $f(x)=x^2$ on $x\in(0,1)$ to prove the following identity:

$$\frac{\pi^2}{12}=1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\frac{1}{36}+\cdots$$

Exercise A.4. Use either the Fourier sine or cosine series of the function $f(x)=x$ on $x\in(0,2)$ to prove the following identity:

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots.$$

Exercise A.5. Use either the Fourier sine or cosine series of the function $f(x)=x^3$ on $x\in(0,2)$ to prove the following identity:

$$\frac{\pi^3}{32}=1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\cdots.$$

Exercise A.6. Let $f(x)=e^{-x}$ on $(-1,1)$.

a) Write down the FOURIER series of f and draw $S_5(x), S_{10}(x)$.
b) Write down the Fourier sine and cosine series of the function $f(x)=e^{-x}$ defined on $x\in(0,1)$. Draw the partial sums S_5 and S_{10} for each series.

You can use the following code in Matlab to generate the plot of partial sum S_n of a function $f(x)$ defined in $x\in(-L,L)$

```
%The length L
L=1;
%Determine the functon
f=@(x) exp(-x);
% the number of terms
n=10;
%the coefficients of Fourier sine series
a0=integral(@(x) f(x),-L,L)/(2*L);
a=integral(@(x) f(x).*cos((1:n)'*x*pi/L),-L,L,'arrayvalued',true)/L;
b=integral(@(x) f(x).*sin((1:n)'*x*pi/L),-L,L,'arrayvalued',true)/L;
x=-L:0.01:L;
fhat=a0+a'*cos((1:n)'*x*pi/L)+b'*sin((1:n)'*x*pi/L);
plot(x,fhat,x,f(x);
```

For the Fourier sine series, you can use the following code:

```
%The length L
L=1;
%Determine the functon
```

```
f=@(x) exp(-x);
% the number of terms
n=10;
%the coefficients of Fourier sine series
b=2*integral(@(x) f(x).*sin((1:n)'*x*pi/L),0,L,'arrayvalued',true)/L;
x=0:0.01:L;
fhat=b'*sin((1:n)'*x*pi/L);
plot(x,fhat,x,f(x);
```

For the Fourier cosine series, you can use the following code:

```
%The length L
L=1;
%Determine the functon
f=@(x) exp(-x);
% the number of terms
n=10;
%the coefficients of Fourier sine series
a0=integral(@(x) f(x),0,L)/L;
a=2*integral(@(x) f(x).*cos((1:n)'*x*pi/L),0,L,'arrayvalued',true)/L;
x=0:0.01:L;
fhat=a0+a'*cos((1:n)'*x*pi/L);
plot(x,fhat,x,f(x);
```

Exercise A.7. Consider the function $f(x) = \cos(x)$.

a) Write down the Fourier series of $f(x)$ on $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

b) Write down the Fourier sine and cosine series on $x \in \left(0, \frac{\pi}{2}\right)$

Exercise A.8. In this exercise we use the Fourier series to find the series solution of and boundary value ODE equation. Consider the following problem:

$$\begin{cases} y'' + y = 1 \\ y(0) = y(1) = 0 \end{cases}.$$

a) Assume that the solution to the equation is written as

$$y(x) = \sum_{n=1}^{\infty} Y_n \sin(n\pi x).$$

Substitute $y(x)$ into the equation and find coefficients Y_n.

b) Show that the obtained series is absolutely convergent.

Theorem A.1. (Parseval's Theorem) *Let $f(x)$ be a piecewise continuously differentiable function in $(-L, L)$. The Fourier series of $f(x)$ converges to $f(x)$ in the norm sense, which can be expressed as:*

$$\lim_{N\to\infty} \|f - S_N\|^2 = \lim_{N\to\infty} \int_T |f(x) - S_N(x)|^2 \, dx = 0,$$

where $S_N(x)$ is the partial sum of its Fourier series given by:

$$S_N(x) = \sum_{n=-N}^{N} F_n \, e^{in\omega x},$$

for $\omega = \frac{\pi}{L}$.

Problem A.2. Use the above theorem and prove the following equality

$$\sum_{n=-\infty}^{\infty} |F_n|^2 = \frac{1}{2L} \int_{-L}^{L} |f(x)|^2 \, dx,$$

for piecewise continuously differentiable functions defined on $(-L, L)$.

Hint: Expand the square norm

$$\left\| f - \sum_{n=-N}^{N} F_n e^{in\omega x} \right\|^2 = \left\langle\!\!\left\langle f - \sum_{n=-N}^{N} F_n e^{in\omega x}, f - \sum_{n=-N}^{N} F_n e^{in\omega x} \right\rangle\!\!\right\rangle,$$

and conclude the following inequality known as the Bessel inequality:

$$\sum_{k=-N}^{N} |F_n|^2 \leq \frac{1}{2L} \int_{-L}^{L} |f(x)|^2 \, dx.$$

Then use the Parseval theorem and conclude the equality.

A.2 Operations on Fourier series

Proposition A.1. *If f and g are piecewise continuously differentiable functions, then*

$$f(x) + g(x) = \sum_{n=-\infty}^{\infty} (F_n + G_n) \, e^{in\omega x}. \tag{A.4}$$

Proposition A.2. *If f and g are piecewise continuously differentiable functions, then*

$$f(x) g(x) = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} F_k G_{n-k} \right) e^{in\omega x}. \tag{A.5}$$

Proof. Multiply $f(x)$ by $g(x)$ and write:

$$f(x) g(x) = \sum_{k=-\infty}^{\infty} F_k \, g(x) \, e^{ik\omega x}. \tag{A.6}$$

Insert the series of $g(x)$ into the above series and obtain

$$\sum_{k=-\infty}^{\infty} F_k \left(\sum_{l=-\infty}^{\infty} G_l \, e^{il\omega x} \right) e^{ik\omega x} = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} F_k \, G_l \, e^{i(k+l)\omega x}. \tag{A.7}$$

If we take $k + l = n$, then

$$f(x) g(x) = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} F_k \, G_{n-k} \right) e^{in\omega x}, \tag{A.8}$$

and this complete the proof. □

Proposition A.3. *Let f be a twice continuously differentiable function in $(-L, L)$, and $f(-L) = f(L)$, then*

$$f'(x) = \sum_{n=-\infty}^{\infty} (in\omega) \, F_n \, e^{in\omega x}. \tag{A.9}$$

Proof. We can write

$$f'(x) = \sum_{n=-\infty}^{\infty} G_n \, e^{in\omega x}, \tag{A.10}$$

where by the continuity of f, we can write

$$G_n=\frac{1}{2L}\int_{-L}^{L}f'(x)\,e^{-in\omega x}dx=\frac{1}{2L}f(x)e^{-in\omega x}\Big|_a^b+\frac{in\omega}{2L}\langle\langle f,e^{in\omega x}\rangle\rangle\,. \tag{A.11}$$

The condition $f(-L)=f(L)$ implies $f(x)e^{-in\omega x}|_a^b=0$, and then

$$G_n=\frac{in\omega}{2L}\langle\langle f,e^{in\omega x}\rangle\rangle=in\omega F_n,$$

and this completes the proof. □

The continuity of f, and condition $f(-L)=f(L)$ are crucial for the proof that can not be relaxed. For example, function $f(x)=x$ defined on $[-1,1]$ has the series in basis $\{1,\cos(n\pi x),\sin(n\pi x)\}$ as

$$x=\sum_{n=1}^{\infty}\frac{-2(-1)^n}{n\pi}\sin(n\pi x),$$

while $f'(x)=1$, and it has the trivial series expansion.

Proposition A.4. *Let f be a piecewise continuously differentiable function in $(-L,L)$, then*

$$\int_{-L}^{x}f(t)\,dt=\sum_{n=-\infty}^{\infty}F_n\int_{-L}^{x}e^{in\omega t}dt. \tag{A.12}$$

Proof. Let $F(x)$ be an anti-derivative of f, that is,

$$F(x)=\int_{-L}^{x}f(t)\,dt.$$

First we prove the proposition if $F(L)=0$, that is,

$$F(L)=\int_{-L}^{L}f(t)\,dt=0.$$

Obviously F is a continuously differentiable function, and thus

$$F(x)=\sum_{n=-\infty}^{\infty}\hat{F}_n\,e^{in\omega x}, \tag{A.13}$$

where $\hat{F}_n=\frac{1}{2L}\langle\langle F,e^{in\omega x}\rangle\rangle$. On the other hand, the condition $F(L)=0$ implies $F(-L)=F(L)$, and thus,

$$f(x)=F'(x)=\sum_{n=-\infty}^{\infty}(in\omega)\,\hat{F}_n\,e^{in\omega x}. \tag{A.14}$$

This implies $F_n=(in\omega)\,\hat{F}_n$, and for $n\neq 0$, we can write $\hat{F}_n=\frac{1}{in\omega}F_n$. Therefore,

$$F(x)=\hat{F}_0+\sum_{n=-\infty,n\neq 0}^{\infty}\frac{F_n}{in\omega}\,e^{in\omega x}. \tag{A.15}$$

Since F is continuous and $F(-L)=0$, then

$$\hat{F}_0 = -\sum_{n=-\infty, n\neq 0}^{\infty} \frac{F_n}{in\omega} e^{-in\pi}. \tag{A.16}$$

On the other hand, we have

$$\begin{aligned}\sum_{n=-\infty}^{\infty} F_n \int_{-L}^{x} e^{in\omega t} dt = \sum_{n=-\infty, n\neq 0}^{\infty} \frac{F_n}{in\omega}(e^{in\omega x} - e^{-in\pi}) =\\ -\sum_{n=-\infty, n\neq 0}^{\infty} \frac{F_n}{in\omega} e^{-in\pi} + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{F_n}{in\omega} e^{in\omega x} =\\ = \hat{F}_0 + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{F_n}{in\omega} e^{in\omega x} = F(x).\end{aligned}$$

Hence, we have

$$F(x) = \sum_{n=-\infty}^{\infty} F_n \int_{-L}^{x} e^{in\omega t}\, dt.$$

Now, we relax the condition $F(L)=0$. Define function g as $g(x) = f(x) - F_0$. Since

$$G_0 = \frac{1}{2L}\int_{-L}^{L} g(x)\, dx = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx - F_0 = 0, \tag{A.17}$$

we write $G(x) := \int_{-L}^{x} g(t)\, dt$ as the following series

$$G(x) = \hat{G}_0 + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{G_n}{in\omega} e^{in\omega x},$$

where G_n are:

$$G_n = \frac{1}{2L}\langle\langle g, e^{in\omega x}\rangle\rangle = \frac{1}{2L}\langle\langle f, e^{in\omega x}\rangle\rangle - \frac{1}{2L}\langle\langle F_0, e^{in\omega x}\rangle\rangle = F_n - \frac{1}{2L}\langle\langle F_0, e^{in\omega x}\rangle\rangle.$$

For $n\neq 0$, we have $\langle\langle F_0, e^{in\omega x}\rangle\rangle = 0$, and hence $G_n = F_n$ for $n\neq 0$. On the other hand, we have:

$$G_0 = -\sum_{n=-\infty, n\neq 0}^{\infty} \frac{F_n}{in\omega} e^{-in\pi} = \hat{F}_0,$$

and thus

$$G(x) = \hat{F}_0 + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{F_n}{in\omega} e^{in\omega x}.$$

Note that

$$G(x) = \int_{-L}^{x} g(t)\, dt = \int_{-L}^{x} f(t)\, dt - F_0\,(x+L),$$

which implies

$$\int_{-L}^{x} f(t)\, dt = F_0(x+L) + \hat{F}_0 + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{F_n}{in\omega} e^{in\omega x} = \sum_{n=-\infty}^{\infty} F_n \int_{-L}^{x} e^{in\omega t}\, dt,$$

and this completes the proof. □

A.3 A proof of the Fourier and Parseval theorem

We present a proof of the Fourier theorem for continuously differentiable functions defined on the domain $(-\pi, \pi)$.

Theorem A.2. (Fourier) *Assume that $f \in C^1(-\pi, \pi)$ and continuous on $[-\pi, \pi]$. Then the partial series*

$$S_n(x) = \sum_{k=-n}^{n} F_k\, e^{ikx},$$

converges pointwise to $f(x)$, where $F_k = \frac{1}{2\pi}\langle\langle f, e^{ikx}\rangle\rangle$.

Substituting F_k into the partial sum, gives

$$S_n(x) = \frac{1}{2\pi}\sum_{k=-n}^{n}\left(\int_{-\pi}^{\pi} f(\xi)\, e^{-ik\xi}\, d\xi\right)e^{ikx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)\sum_{k=-n}^{n} e^{ik(x-\xi)}. \tag{A.18}$$

Exercise A.9. Show the following identity

$$\sum_{k=-n}^{n} e^{ikx} = \frac{\sin\left(\left(n+\frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)}, \tag{A.19}$$

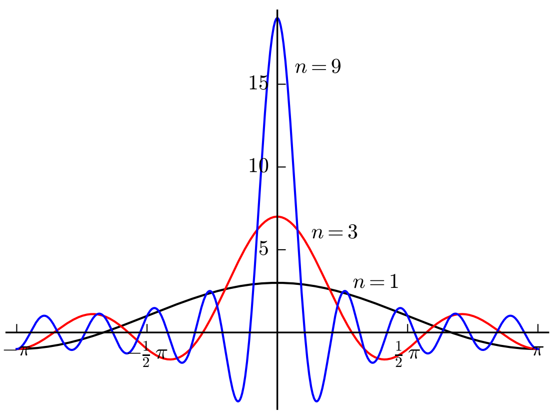
and conclude

$$\int_{-\pi}^{\pi} \frac{\sin\left(\left(n+\frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)}\, dx = 2\pi. \tag{A.20}$$

The 2π-periodic function

$$D_n(x) = \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)}, \tag{A.21}$$

is called the Dirichlet kernel. The figure below depicts this kernel for values $n = 1, 3, 9$.



Therefore, we can rewrite $S_n(x)$ as the following integral

$$S_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)\, D_n(x-\xi). \tag{A.22}$$

The kernel $D_n(x)$ is a Dirac delta sequence for $n \to \infty$.

Proposition A.5. *Let $f \in C^1(-\pi, \pi)$, then for all $x \in (-\pi, \pi)$, we have:*

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)\, D_n(x-\xi) d\xi = f(x). \tag{A.23}$$

In order to prove the proposition, we need the following lemma.

Lemma A.1. *Assume that $f(x)$ is a continuous function in $[-\pi, \pi]$, then*

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx = 0. \tag{A.24}$$

Now using the above lemma, we have

$$S_n(x) = f(x) + \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(\xi) - f(x)) D_n(x-\xi)\, d\xi = f(x) + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(\xi)-f(x)}{\xi - x} \frac{\xi - x}{\sin\left(\frac{x-\xi}{2}\right)} \sin((n+1/2)(x-\xi)) d\xi.$$

Since f is a C^1 function, the function $\frac{f(\xi)-f(x)}{\xi-x}$ is continuous. Similarly, the function $\frac{\xi-x}{\sin\left(\frac{x-\xi}{2}\right)}$ is continuous as well, and thus

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \frac{f(\xi)-f(x)}{\xi - x} \frac{\xi - x}{\sin\left(\frac{x-\xi}{2}\right)} \sin((n+1/2)(x-\xi)) d\xi = 0.$$

Finally, we obtain

$$\lim_{n\to\infty} S_n(x) = f(x).$$

Theorem A.3. (Parseval) *If $f \in \mathrm{PC}^1(-\pi, \pi)$, then partial sum $S_n(x)$ converges to $f(x)$ in norm, that is,*

$$\lim_{n\to\infty} \|f - S_n\| = 0.$$

The proof is based on a fact from analysis that if $f \in \mathrm{PC}^1(T)$, then for every $\varepsilon > 0$, there is a function $g \in C^1(T)$ such that

$$\|f - g\| < \varepsilon.$$

By this fact, we can write:

$$\|f - S_n\| \le \|f - g\| + \|g - S_n\| < \varepsilon + \|g - S_n\|.$$

On the other hand, we have:

$$\|g - S_n\|^2 = 2\pi \sum_{k=-n}^{n} F_k \overline{(F_k - G_k)} \le 2\pi \left(\sum_{k=-n}^{n} |F_k|^2\right)^{1/2} \left(\sum_{k=-n}^{n} |F_k - G_k|^2\right)^{1/2}.$$

The last inequality is the Cauchy-Schwarz inequality. On the other hand, using the Bessel inequality, we have

$$2\pi\sum_{k=-n}^{n}|F_k-G_k|^2\leq\int_{-\pi}^{\pi}|f(x)-g(x)|^2<\varepsilon^2.$$

Similarly, we have

$$\sum_{k=-n}^{n}|F_k|^2\leq\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\leq M^2,$$

for some $M>0$. This implies that $\|g-S_n\|^2\leq\frac{\varepsilon}{M}$ and finally

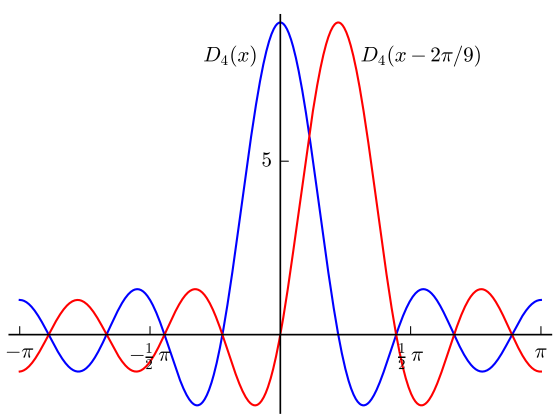
$$\|f-S_n\|<\varepsilon+\frac{\varepsilon}{M}.$$

Since ε is arbitrary, we let $\varepsilon\to 0$ and thus

$$\lim_{n\to\infty}\|f-S_n\|=0.$$

A.3.1 GIBBS phenomena

In this chapter, we have observed the overshoot jump of partial series near discontinuity points, which is commonly known as the Gibbs phenomenon, named after the American physicist J. W. Gibbs. Now, we aim to find an estimate for the maximum value of this jump. For simplicity, we assume that the function f is defined on the interval $[-\pi,\pi]$ and has a discontinuity at $x=0$. From the formula of the Dirichlet kernel $D_n(x)$, we know that $D_n(x)$ vanishes at $z=\pm\frac{2\pi}{2n+1}$. The figure below illustrates $D_4(x)$ and its shift $D_4\big(x-\frac{2\pi}{9}\big)$:



We observe that the maximum of $D_4\big(x-\frac{2\pi}{9}\big)$ occur at the first zero point of $D_4(x)$ which is $z=\frac{2\pi}{9}$. We expect that the jump occurs at the zero points of D_n. For this, we need to calculate $S_n\big(\frac{2\pi}{2n+1}\big)$:

$$S_n\left(\frac{2\pi}{2n+1}\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\xi)D_n\left(\frac{2\pi}{2n+1}-\xi\right)d\xi=\frac{1}{2\pi}\int_{-\pi}^{0}f(\xi)\,D_n\left(\frac{2\pi}{2n+1}-\xi\right)d\xi+$$
$$+\frac{1}{2\pi}\int_{0}^{\pi}f(\xi)D_n\left(\frac{2\pi}{2n+1}-\xi\right)d\xi.$$

According to the shape of the Dirichlet kernel for sufficiently large n, we have:

$$\int_{-\pi}^{0} f(\xi)\, D_n\left(\frac{2\pi}{2n+1}-\xi\right) d\xi \cong f(0^-) \int_{-\pi}^{0} D_n\left(\frac{2\pi}{2n+1}-\xi\right) d\xi. \tag{A.25}$$

Similarly, we have

$$\int_{0}^{\pi} f(\xi)\, D_n\left(\frac{2\pi}{2n+1}-\xi\right) d\xi \cong f(0^+) \int_{0}^{\pi} D_n\left(\frac{2\pi}{2n+1}-\xi\right) d\xi. \tag{A.26}$$

Direct calculation yields the following values for large n:

$$\int_{-\pi}^{0} D_n\left(\frac{2\pi}{2n+1}-\xi\right) d\xi \cong -0.0897, \tag{A.27}$$

and

$$\int_{0}^{\pi} D_n\left(\frac{2\pi}{2n+1}-\xi\right) d\xi = 1.0897. \tag{A.28}$$

Consequently,, we obtain the following estimate for the jump at the right side of the jump point $x=0$:

$$S_n\left(\frac{2\pi}{2n+1}\right) \approx 1.09 f^+(0) - 0.09\, f^-(0). \tag{A.29}$$

A similar estimate holds for the jump at the left side of $x=0$, that is,

$$S_n\left(-\frac{2\pi}{2n+1}\right) \approx 1.09 f^-(0) - 0.09 f^+(0). \tag{A.30}$$