Chapter 3
Linear Second-Order Equations

In this chapter, we derive the heat equation in general dimension, the equation of wave propagation in an elastic membrane, and the Poisson's equation. We present the properties of each equation and then classify the linear second-order partial differential equations into elliptic, parabolic, and hyperbolic equations. The main reason for this classification is that each class of equation has distinct solutions that behave differently. For instance, elliptic equations have solutions that are smooth and have no singularities, whereas hyperbolic equations have solutions that have singularities and can exhibit wave-like behavior. Understanding these differences is crucial for solving and interpreting the solutions of partial differential equations in various fields of science and engineering.

3.1 Heat equation

The heat equation is a fundamental equation in physics, engineering, and applied mathematics that is used to model a wide range of phenomena, including heat transfer, thermal diffusion, and mass diffusion. In this section, we will first derive the heat equation for solid conductive continua, and then introduce the advection-diffusion equation for fluids.

3.1.1 Derivation of heat equation

In the first chapter, we demonstrated the derivation of the equation for heat flow in a conductive rod. In this chapter, we present the derivation in three-dimensional space ($\mathbb{R}^3$) to gain a deeper understanding of its physics.

Consider a solid conductive continuum $\Omega \subset \mathbb{R}^n$, which can be a bounded or unbounded set, with or without a boundary. If Ω is a bounded domain, we suppose that the only possibility that Ω exchange heat with the ambient space is through its boundary $\mathrm{bnd}(\Omega)$. Let $u(x,t)$ denote the temperature of an arbitrary point $x \in \Omega$ at time t. For any bounded region $D \subset \Omega$, the thermal energy inside D at time t is defined by the integral:

$$Q_D(t) = \iiint_D q(x,t)\, dV,$$

where q is the density function of the heat energy. Therefore, the increment in energy can be expressed as:

$$Q_D(t+\delta t) - Q_D(t) = \iiint_D \{q(x, t+\delta t) - q(x,t)\}\, dV.$$

In the limiting case as $\delta t \to 0$, we obtain the following relation:

$$\frac{d}{dt} Q_D(t) = \iiint_D \rho c \partial_t u(x,t)\, dV,$$

where ρ is the density and c is the specific heat capacity of the continuum.

On the other hand, we can relate the rate of change of thermal energy inside D to the heat exchange through its boundary $\mathrm{bnd}(D)$. The rate of heat exchange through the boundary of D is proportional to the temperature gradient, according to Fourier's law, which states that the heat flux $J(x,t)$ at a point x in D is given by $J(x,t) = -\alpha(x)\nabla u(x,t)$, where $\alpha(x)$ is the thermal conductivity at x. Note that the negative sign arises because the direction of the temperature gradient is towards increasing temperature, whereas the direction of the heat flux is from hotter to colder regions. The net flux leaving D through $\mathrm{bnd}(D)$ is equal to the surface integral

$$\oiint_{\mathrm{bnd}(D)} J(z,t) \cdot \nu(z)\, dS = -\oiint_{\mathrm{bnd}(D)} \alpha(z)\, \nabla u(z,t) \cdot \nu(z)\, dS,$$

where $\nu(z)$ is the outward-pointing unit normal vector at $z \in \mathrm{bnd}(D)$. By the conservation of energy principle, we have

$$\frac{d}{dt} Q_D(t) = -\oiint_{\mathrm{bnd}(D)} J(z,t) \cdot \nu(z)\, dS,$$

where the negative sign accounts for the fact that the integral measures the flux leaving D at time t. Substituting $\frac{dQ_D}{dt}$ by the triple integral and rearranging, we obtain

$$\iiint_D \rho c \partial_t u(x,t)\, dV = \oiint_{\mathrm{bnd}(D)} \alpha(z)\, \nabla u(z,t) \cdot \nu(z)\, dS.$$

Using Gauss's theorem, we can convert the surface integral in the right-hand side of the previous relation to a volume integral. Applying the theorem yields:

$$\iiint_D \mathrm{div}\,(\alpha(x)\nabla u(x,t))\mathrm{dV} = \oiint_{\mathrm{bnd}(D)} \alpha(x)\nabla u(x,t) \cdot \nu(x)\mathrm{dS},$$

and y rearranging the terms, we arrive at the integral form of the heat equation:

$$\iiint_\Omega [\rho c \partial_t u(x,t) - \mathrm{div}\,(\alpha(x)\, \nabla u(x,t))]\, dV = 0.$$

Since this equation holds for arbitrary Ω, we obtain the following differential form of the heat equation

$$u_t(x,t) = \frac{1}{\rho c} \mathrm{div}\,(\alpha(x)\, \nabla u(x,t)).$$

If α is constant, then we have

$$u_t(x,t) = k\,\Delta u,$$

where $k = \frac{\alpha}{\rho c} > 0$.

Remark 3.1. The heat equation that we derived involves a first-order partial derivative of u with respect to time. Therefore, to fully describe the heat dynamics, we need to specify the initial condition $u(x,0)$ for $x \in \Omega$. This is called the *initial condition* of the heat equation. In fact, the initial condition provides the initial thermal energy distribution for the system, which drives its subsequent behavior. Additionally, in some physical systems, there may be an internal source or sink of thermal energy that is not accounted for in our derivation. In the presence of such a source or sink, the heat equation takes the form:

$$u_t = \frac{1}{\rho c}\operatorname{div}(\alpha \nabla u) + h,$$

where the function h represents the rate at which the internal source or sink produces or absorbs energy.

3.1.2 Convection-diffusion equation

The convection-diffusion equation is a partial differential equation that is widely used to model heat and mass transfer in fluids, such as in chemical engineering and environmental science. The equation takes into account both convection, which is the transport of heat or mass by the fluid flow, and diffusion, which is the transport of heat or mass due to a concentration gradient.

In the modeling of the heat problem, we assumed that heat is transferred only by conductivity. This assumption makes sense if the medium is solid. If the medium is a fluid or gas, the heat can be transferred in the form of convection in addition to diffusion. In this scenario, the heat flux J consists of two terms, the conductivity term and the convection term:

$$J = J_{\text{cond}} + J_{\text{conv}}.$$

If the fluid moves with velocity V, then $J_{\text{conv}} = uV$, and thus

$$\oiint_{\text{bnd}(D)} J \cdot \nu\, dS = -\oiint_{\text{bnd}(D)} \alpha \nabla u \cdot \nu\, dS + \oiint_{\text{bnd}(D)} uV \cdot \nu\, dS.$$

Therefore, the heat equation reads

$$\rho c u_t + \operatorname{div}(uV) = \operatorname{div}(\alpha \nabla u).$$

The above equation is called the convection-diffusion equation. The first term on the left-hand side of the equation represents the change in temperature with respect to time. The second term represents the convective transport of heat by the fluid flow, while the third term represents the diffusive transport due to a concentration gradient. In the presence of a source or sink, the heat equation takes the form:

$$\rho c u_t + \operatorname{div}(uV) = \operatorname{div}(\alpha \nabla u) + h.$$

The convection-diffusion is a fundamental equation in many areas of science and engineering and has many applications, such as modeling air and water pollution, heat transfer in pipes, and drug delivery in biological systems.

3.1.3 Boundary conditions for bounded domains

Continuing with our discussion of the heat equation on bounded domain, it is worth noting that the solution to a partial differential equation, and especially to the heat equation, depends heavily on the boundary conditions imposed on the bounded domain Ω.

Recall that the Newton cooling law relates the temperature inside and outside a region Ω. Specifically, if T denotes the temperature outside Ω, the law states that the heat flux through $\mathrm{bnd}(\Omega)$, is proportional to the difference $T-u$, where u is the temperature function inside Ω. This relationship is expressed mathematically by the Robin boundary condition

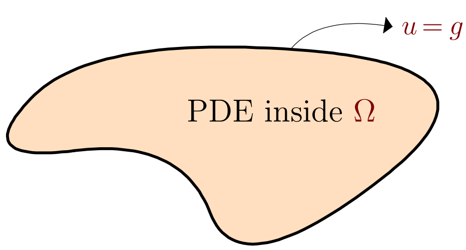
$$\frac{\partial u}{\partial n} = \kappa(T-u),$$

where ν denotes the outward normal to the boundary, and κ is the conductivity factor of the boundary. When $\kappa = 0$, the boundary is perfectly insulated, and the condition reduces to the homogeneous Neumann boundary condition $\frac{\partial u}{\partial n} = 0$. More generally, the Neumann boundary condition takes the form $\frac{\partial u}{\partial n} = g$ on $\mathrm{bnd}(\Omega)$, where g is a given function. In the limiting case where κ is very large, the Robin condition can be rewritten as

$$T - u = \frac{1}{\kappa}\frac{\partial u}{\partial n},$$

and taking the limit $\kappa \to \infty$ yields the condition $u = T$ on $\mathrm{bnd}(\Omega)$, which is called the Dirichlet boundary condition.

Dirichlet boundary condition. The Dirichlet boundary condition is used when the temperature at the boundary of the domain Ω is known, such that $u(z,t) = g(z,t)$ for $z \in \mathrm{bnd}(\Omega)$. This condition is named after the German mathematician Peter Gustav Lejeune Dirichlet. For a heat equation, this means that the temperature at $\mathrm{bnd}(\Omega)$ is kept at the known value $g(z,t)$. If g is identically zero, then the condition is called the homogeneous Dirichlet condition.

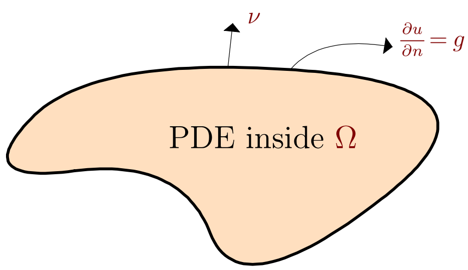


In one-dimensional problems, such as a conductive rod extending from $x = 0$ to $x = L$, the condition becomes $u(0,t) = g_0$ and $u(L,t) = g_L$, where g_0 and g_L can be constants or functions of time.

Neumann boundary condition. This condition is named after the German mathematician Carl Neumann. Mathematically, the Neumann boundary condition is expressed as

$$\frac{\partial u}{\partial n} := \nabla u \cdot \nu = g,$$

where $\frac{\partial u}{\partial n}$ is the directional derivative of u along the unit normal direction ν on $\mathrm{bnd}(\Omega)$. In the context of a heat problem, this condition specifies the heat flux through the boundary $\mathrm{bnd}(\Omega)$ in terms of function g. In particular, if g is identically zero, the boundary is perfectly insulated.



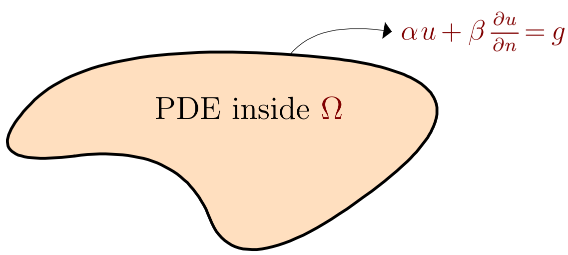
For a one-dimensional problem where the conductive rod extends from $x = 0$ to $x = L$, the condition reads:

$$u_x(0, t) = g_0, u_x(L, t) = g_L.$$

Robin's or mixed boundary condition. As we observed above, the Robin boundary condition is a straightforward derivation of the Newton cooling law in the context of a heat problem. Mathematically, it can be expressed as

$$\alpha u(z, t) + \beta \frac{\partial u}{\partial n}(z, t) = g(z, t).$$

where α and β are some coefficients, and g is a given function. In particular, if g is identically zero, the condition is called the homogeneous Robin boundary condition. The Robin boundary condition is commonly used in problems that involve heat transfer between a solid and a fluid, where the heat transfer coefficient (i.e., α / β) depends on the properties of the fluid and the surface of the solid. It is also used to model heat transfer in biological tissues, where the heat transfer coefficient can depend on the blood flow rate and other physiological factors. The Robin boundary condition is named after the French mathematician Victor Gustave Robin.



For a one-dimensional problem, the condition reads

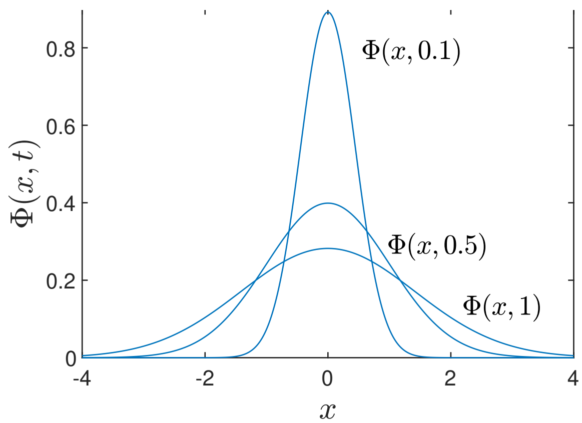
$$\begin{cases} \alpha_1 u(0, t) + \beta_1 u_x(0, t) = g_0 \\ \alpha_2 u(L, t) + \beta_2 u_x(L, t) = g_L \end{cases}.$$

3.1.4 Fundamental solution of heat equation

Consider a conductive rod of diffusivity factor $k>0$, extended over the entire real line $(-\infty, \infty)$. Suppose that an infinitely small portion of the rod at $x=0$ has a temperature of $u=1$, and the temperature is zero everywhere else. As time progresses, the heat diffuses across the rod, but the total thermal energy remains constant as it was at $t=0$. It can be shown that the heat profile evolves according to the function:

$$\Phi(x,t)=\frac{1}{\sqrt{4\pi kt}}\,e^{-\frac{x^2}{4kt}},$$

for $t>0$. In fact, one can verify that $\Phi_{tt}=k\Phi_{xx}$ through direct calculation. The graph of $\Phi(x,t)$ demonstrates the evolution of heat profile for $t>0$. The following figure shows $\Phi(x,t)$ for $t=0.1, 0.5$ and 1 where $k=1$:



That the total thermal energy remains constant over time follows from the fact that the integral of Φ for any fixed value of t is constant:

$$\int_{-\infty}^{\infty}\Phi(x,t)dx=1.$$

In fact, using the change of variable $v=\frac{x}{\sqrt{4kt}}$, we have:

$$\int_{-\infty}^{\infty}\Phi(x,t)dx=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-v^2}dv=1.$$

This confirms the conservation of thermal energy since the initial heat source is a unit pointwise source at $x=0$. The function Φ is called the Gaussian or heat kernel of a heat equation.

Exercise 3.1. By direct calculation, verify that $\Phi(x,t)$ satisfies the heat equation $u_t=ku_{xx}$.

Now, consider the heat problem given by the partial differential equation

$$u_t=ku_{xx}, x\in(-\infty,\infty), t>0,$$

with initial condition $u(x,0)=f(x)$, where $f(x)$ is a continuous and integrable function in $(-\infty,\infty)$. The following theorem gives the solution to this problem:

Theorem 3.1. *The solution to the given problem is given by the convolution*

$$u(x,t)=f*\Phi:=\int_{-\infty}^{\infty}\Phi(x-y,t)\,f(y)\,dy.$$

Proof. We first show that u defined as above satisfies the partial differential equation. We have

$$u_t(x,t)=\int_{-\infty}^{\infty}\Phi_t(x-y,t)\,f(y)\,dy,$$

and u_{xx} is

$$u_{xx}(x,t)=\int_{-\infty}^{\infty}\Phi_{xx}(x-y,t)\,f(y)\,dy.$$

The above representations of the partial derivatives are justified by the assumption of integrability of f, that is,

$$\int_{-\infty}^{\infty}|f(x)|\,dx<\infty,$$

and the Dominated Convergence Theorem which allows us to take the limits from outside the integrals to inside the integrals. Accordingly, we have

$$u_t-k u_{xx}=\int_{-\infty}^{\infty}[\Phi_t(x-y,t)-k\,\Phi_{xx}(x-y,t)]\,f(y)\,dy.$$

By the above exercise, we have $\Phi_t-k\,\Phi_{xx}=0$, and this completes the claim. It remains to show that the convolution satisfies the initial condition for $t\to 0$. We have

$$\lim_{t\to 0}u(x,t)=\lim_{t\to 0}\frac{1}{\sqrt{4\pi k t}}\int_{-\infty}^{\infty}e^{-\frac{(x-y)^2}{4kt}}\,f(y)dy=\lim_{t\to 0}\frac{1}{\sqrt{4\pi k t}}\int_{-\infty}^{\infty}e^{-\frac{y^2}{4kt}}\,f(x-y)dy.$$

Using the substitution $v=\frac{y}{\sqrt{4kt}}$, we can write

$$\lim_{t\to 0}u(x,t)=\lim_{t\to 0}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-v^2}\,f(x-\sqrt{4kt}v)dv.$$

If we pass the limit $t\to 0$ inside the integral (which is justified by the Dominant Convergence Theorem), we obtain

$$\lim_{t\to 0}u(x,t)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-v^2}\lim_{t\to 0}f(x-\sqrt{4kt}v)dv.$$

Since f is continuous, we obtain

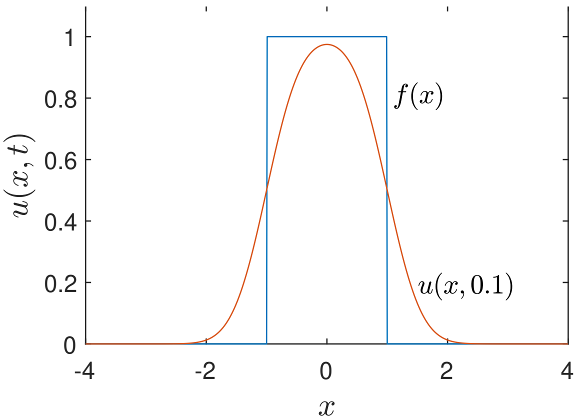
$$\lim_{t\to 0}u(x,t)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-v^2}\,f(x)dv=\frac{f(x)}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-v^2}\,dv=f(x),$$

which completes the proof. □

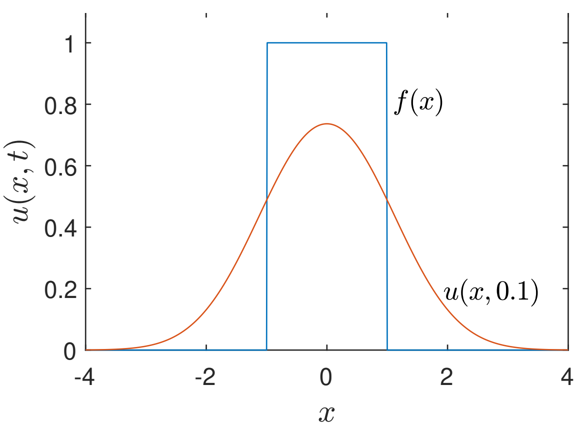
Example 3.1. Consider the following problem:

$$\begin{cases} u_t=u_{xx} \\ u(x,0)=\left\{\begin{smallmatrix} 1 & -1<x<1 \\ 0 & \text{otherwise}\end{smallmatrix}\right. \end{cases}.$$

The initial heat profile, denoted by $f(x)$, is 1 for x in the interval $(-1,1)$, and zero elsewhere. The figure below shows the solution at time $t=0.1$. Note that the solution is smooth, despite the initial condition having jumps at $x=\pm 1$. This is a characteristic property of the heat equation, where it immediately smooths out any discontinuities after $t=0$.



In this equation, we have taken $k=1$. Remember that k is proportional to the diffusivity of the rod. Therefore, for larger values of k, the solution $u(x,t)$ spreads out faster, as shown in the figure below for $k=4$.



Exercise 3.2. Another interesting property of the convolution solution of the heat equation is that the initial temperature of distant points can have an immediate impact on the heat evolution of a fixed point. This is because the solution involves an integral over the entire domain, and the contribution from faraway points is not necessarily negligible. This is in contrast to some other types of differential equations, where the evolution of a point is only influenced by its immediate neighbors. This property of the heat equation makes it particularly useful for modeling systems where heat can quickly spread throughout a medium, such as in heat transfer problems in engineering or physics.

For the simple heat equation $u_t=u_{xx}$, suppose the initial condition is given by $f(x)=\delta(x)+\delta(x-3)$, that is, two point heat sources at $x=0$ and $x=3$. Find the convolution solution and then use the following code in Matlab to draw the solution at $t=0.5$:

```
x=-10:0.01:10;
k=1;t=0.5;
Fi=@(z) exp(-z.^2/(4*k*t))/sqrt(4*pi*k*t);
plot(x,Fi(x)+Fi(x-3))
```

Exercise 3.3. From the above exercise, we learned that the heat equation is a diffusion equation, which means that heat is gradually spread throughout the rod. This spreading process is affected by the diffusivity factor k and the initial heat distribution, but it also depends on the position of the fixed point relative to all other points on the rod. In other words, the heat at distant points has an immediate impact on the heat evolution of a fixed point. In addition to its smoothing-out property, another important feature of the heat equation is its time-invariance. This means that the solutions to the system:

$$\begin{cases} u_t = k u_{xx} & t > 0 \\ u(x,0) = f(x) \end{cases},$$

and

$$\begin{cases} u_t = k u_{xx} & t > t_0 \\ u(x,t_0) = \Phi(x,t_0) * f(x) \end{cases},$$

are the same. In other words, we can define a *flow* as

$$\phi_t(f(x)) = \Phi(x,t) * f(x),$$

that satisfies the property

$$\phi_{t+s}(f(x)) = \phi_t(\phi_s(f(x))).$$

This means that the heat evolution of a fixed point depends only on the initial heat distribution over the entire rod and the diffusivity factor k, and not on the specific time at which we consider the system. The flow ϕ_t thus provides a way to study the time-evolution of the heat equation in a systematic way, by tracking how initial heat distributions evolve over time. We will prove this fact in subsequent chapters using the Fourier transform. However, we can verify the fact for the Dirac delta function, $f(x) = \delta(x)$, here. Consider the heat equation $u_t = u_{xx}$ with initial condition $u(x,0) = \delta(x)$. Show that the solution of this problem at time $t + s$ is equal to the solution of the system

$$\begin{cases} u_t = u_{xx} \\ u(x,s) = \Phi(x,s) \end{cases},$$

at time $t > 0$. In other words, the following relation holds:

$$\Phi(x,t) * \Phi(x,s) = \Phi(x,t+s),$$

for $t, s > 0$.

Exercise 3.4. The above exercise helps us to gain a better understanding of the solution to a heat equation as a convolution integral. Specifically, a heat problem can be viewed as a system with the response $\phi_t(f(x))$ for the initial condition f. When $f(x) = \delta(x)$, the impulse response of the system is $\Phi(x,t)$. Recall that any continuous function $f(x)$ can be expressed as the convolution

$$f(x) = \int_{-\infty}^{\infty} f(y)\, \delta(x-y)\, dy.$$

We can rewrite this integral as a Riemann sum

$$f(x) \approx \sum_n f(y_i)\, \delta(x-y_i)\, \delta y_i.$$

The response of the heat system to this function is

$$\phi_t(f(x)) \approx \sum_n f(y_i)\, \phi_t[\delta(x-y_i)]\, \delta y_i = \sum_n f(y_i)\, \Phi(x-y_i,t)\, \delta y_i,$$

Returning to the integral formulation, we obtain

$$\phi_t(f(x)) = \int_{-\infty}^{\infty} f(y)\, \Phi(x-y,t)\, dy.$$

In the above proof, we have used the following properties

a) The solution of the system

$$\begin{cases} u_t = u_{xx} \\ u(x,0) = \alpha\delta(x) \end{cases},$$

for a constant α is $u = \alpha\Phi(x,t)$.

b) The solution of the system

$$\begin{cases} u_t = u_{xx} \\ u(x,0) = \delta(x-y) \end{cases},$$

is $u = \Phi(x-y,t)$.

Justify both properties through direct calculation.

Exercise 3.5. We can compare the heat kernel $\Phi(x,t)$ to the Gaussian or normal probability density function of a random variable X with expected value $\mu = 0$ and variance σ^2, given by

$$N(x,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}.$$

Upon comparison, we see that the standard deviation of the Gaussian distribution is related to time as $\sigma = \sqrt{2t}$.

a) Use a Z-table and find $a > 0$ in terms of σ such that

$$P(-a < X < a) = 0.95.$$

b) Since 95 percent energy of Φ is concentrated in $(-a,a)$, we can use this value to approximate the convolution integral $\Phi(x,t) * f(x)$ as

$$\Phi(x,t) * f(x) \approx \int_{-a}^{a} \Phi(x-y,t)\, f(y)\, dy.$$

c) Suppose $|f(x)| \leq M$, show the inequality

$$\left|\Phi(x,t) * f(x) - \int_{-a}^{a} \Phi(x-y,t)\, f(y)\, dy\right| \leq 0.05\, M.$$

Exercise 3.6. Verify that the function Φ_n given by the function

$$\Phi_n(x_1, \ldots, x_n, t) = (4\pi k t)^{-n/2} e^{-(x_1^2 + \cdots + x_n^2)/4kt},$$

solves the heat equation

$$u_t = k\Delta u,$$

on $\mathbb{R}^n$, and furthermore the convolution

$$u(x,t) = \Phi_n(x,t) * f(x),$$

solve the heat problem

$$\begin{cases} u_t = k\Delta u \\ u(x,0) = f(x) \end{cases},$$

where f is a continuous and integrable function in $\mathbb{R}^n$. For the proof, we need the result

$$\int_{\mathbb{R}^n} \Phi_n(x_1, \ldots x_n, t) dV = 1,$$

which we accept without prof.

Exercise 3.7. Assume that f is a piecewise continuous function and integrable in $(-\infty,\infty)$. Suppose that $\lim_{x\to 0^+} f(x) = a$ and $\lim_{x\to 0^-} f(x) = -a$. Show the following relation

$$\lim_{t\to 0} \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} f(x)\, e^{-\frac{x^2}{4kt}}\, dx = 0.$$

Exercise 3.8. Consider the equation $u_t = u_{xx}$ for $x \in (-\infty,\infty)$, and $t > 0$. If $u(x,t)$ and $u_x(x,t)$ are bounded and furthermore $\lim\limits_{x\to\pm\infty} u(t,x) = 0$, then show the following integral is decreasing

$$P(t) = \int_{-\infty}^{\infty} |u(x,t)|^2 dx.$$

Conclude that the following problem

$$\begin{cases} u_t = u_{xx} \\ u(0,x) = f(x) \end{cases},$$

has a unique solution under the above assumptions.

Exercise 3.9. Consider the equation $u_t = \text{div}\,(\alpha\nabla u)$ on $\mathbb{R}^+ \times \mathbb{R}^n$. Suppose $u(x, t)$ and $\nabla u(x, t)$ are bounded, and furthermore $\lim\limits_{|x|\to\pm\infty} u(x,t) = 0$. Show that the following integral is decreasing

$$P(t) = \int_{-\infty}^{\infty} |u(x,t)|^2 dV.$$

Conclude that the following problem

$$\begin{cases} u_x = \text{div}\,(\alpha\nabla u) \\ u(x,0) = f(x) \end{cases},$$

has a unique solution under the above assumptions.

3.1.5 Solution for problems in bounded domains

The convolution solution is only applicable for heat problems defined on domains without boundary. To see this, consider the simple heat problem on the interval $[0, 1]$:

$$\begin{cases} u_t = k u_{xx} & x \in (0,1), t > 0 \\ u(0,t) = u(1,t) = 0 & \\ u(x,0) = f(x) & \end{cases}.$$

Note that the convolution integral does not satisfy the prescribed boundary conditions at $x = 0, 1$. Therefore, the heat kernel $\Phi(x, t)$ cannot be used to obtain the solution of heat problems defined on domains with boundaries. In subsequent chapters, we will study partial differential equations defined on bounded domains, and we will need to develop new tools that allow us to express the solution of PDEs as infinite series or improper integrals.

However, we can still obtain some important properties of the heat equation on bounded domains by solving boundary value problems. The following exercises illustrate some of these properties.

Exercise 3.10. Consider the equation $u_t = k\, u_{xx}$ in $(0, L)$ where $k > 0$ is a constant. Show that the integral

$$P(t) = \int_0^L |u(x,t)|^2\, dx,$$

is decreasing for the following boundary conditions

i. $u(0,t) = u(L,t) = 0$

ii. $u(0,t) - u_x(0,t) = 0$, $u(L,t) + u_x(L,t) = 0$.

Exercise 3.11. Let Ω be a bounded open set with smooth boundary $\text{bnd}(\Omega)$. Consider the following Dirichlet problem

$$\begin{cases} u_t = \text{div}\,(\alpha\nabla u) \\ u|_{\text{bnd}(\Omega)} = 0 \end{cases}.$$

For the power function

$$P(t) = \int_\Omega |u(x,t)|^2\, dV,$$

show $P(t)$ is decreasing.

Exercise 3.12. Consider the following heat problem

$$\begin{cases} u_t = k u_{xx} - r u \\ u_x(0,t) = u_x(L,t) = 0 \\ u(x,0) = f\,(x) \end{cases}.$$

where r is a constant. Let $E(t)$ be the following energy function

$$E(t)=\int_0^L u(x,t)\,dx.$$

Prove the following relation

$$E(t)=\left(\int_0^L f(x)\,dx\right)e^{-rt}.$$

Exercise 3.13. Repeat the above problem for the following equation

$$\begin{cases} u_t = \operatorname{div}(\alpha\nabla u) - ru \\ \frac{\partial u}{\partial n}\Big|_{\mathrm{bnd}(\Omega)} = 0 \\ u(x,0) = f(x) \end{cases}.$$

Exercise 3.14. Let Ω be an open bounded set with smooth boundary $\mathrm{bnd}(\Omega)$. Consider the following heat problem on Ω

$$\begin{cases} u_t = k\Delta u \\ \frac{\partial u}{\partial n}\Big|_{\mathrm{bnd}(\Omega)} = 0 \\ u(x,0) = f(x) \end{cases},$$

a) Show the following relation

$$\int_\Omega u(x,t)\,dV = \int_\Omega f(x)\,dV.$$

b) Now for the problem

$$\begin{cases} u_t = k\Delta u - ru \\ \frac{\partial u}{\partial n}\Big|_{\mathrm{bnd}(\Omega)} = 0 \\ u(x,0) = f(x) \end{cases},$$

where r is a constant, show the following relation

$$\int_\Omega u(x,t)\,dV = \left(\int_\Omega f(x)\,dV\right)e^{-rt}.$$

3.2 Wave equation

3.2.1 Derivation of the equation

In the first chapter, we obtained the equation of motion for a vibrating string by modeling it as an infinite number of small mass-spring systems. We now extend this approach to derive the wave equation for an elastic membrane in two-dimensional Euclidean space. The method we use can be generalized to any dimension $\mathbb{R}^n$, where n is greater than or equal to one.

Let Ω be an elastic membrane in the two-dimensional space, and let D be an infinitesimal element of area $\delta A = \delta x\,\delta y$. For the displacement function $u(x,\ y,\ t)$, the tension on the boundary of D is given by $T = \tau\nabla u$, where τ is called the stress and depends on the physical characteristics of the elastic material. The vector force exerted on D can be defined by the integral

$$F = \oint_D \tau\nabla u\cdot\nu\,dl,$$

where ν is the unit exterior normal vector to the boundary of D. Applying Newton's second law to this small portion of the membrane, we obtain

$$\rho\delta A u_{tt} = F,$$

where ρ is the density of the membrane. By applying the divergence theorem, we can write

$$u_{tt} = \frac{1}{\rho}\lim_{\delta A\to 0}\frac{1}{\delta A}\iint_D \operatorname{div}(\tau\nabla u)\, dS = \frac{1}{\rho}\operatorname{div}(\tau\nabla u).$$

If the medium is homogeneous with constant τ and ρ, we can write the derived equation as $u_{tt} = c^2\,\Delta u$, where $c = \sqrt{\frac{\tau}{\rho}}$ is the wave speed of the elastic medium. This equation is known as the wave equation for elastic membranes. This method can be extended to higher-dimensional spaces, where the wave equation takes a similar form.

The derived formulation is incomplete without specifying the initial conditions for the equation. Note that the equation contains a second-order partial derivative of u with respect to time. Therefore, the equation should be accompanied by two initial conditions: one to specify the initial displacement $u(x, y, 0)$ and the second to specify the initial velocity $u_t(x, y, 0)$. These initial conditions correspond, respectively, to the initial stretch potential energy of the membrane and the initial kinetic energy. These energies cause a dynamic for the membrane that forces it to oscillate like a wave function. Evidently, if $u(x, y, 0) = 0$ and $u_t(x, y, 0) = 0$, then $u = 0$ for all t. Therefore, we can write the wave equation in general dimension as follows

$$\begin{cases} u_{tt} = c^2\,\Delta u \\ u(x, y, 0) = f(x, y) \\ u_t(x, y, 0) = g(x, y) \end{cases},$$

for $x\in\Omega\subset\mathbb{R}^n$, where f and g are initial displacement and velocity respectively.

If Ω is a bounded domain, such as a string, then the natural boundary condition for the equation is the homogeneous Dirichlet condition. Physically, this condition means that the endpoints of the string are fastened and fixed. However, other possible boundary conditions include the Neumann and Robin conditions, which we have seen previously for the heat equation. The wave equation with homogeneous Robin's boundary condition on a bounded domain Ω is

$$\begin{cases} u_{tt} = c^2\,\Delta u & \text{on } \Omega \\ \alpha u + \beta\frac{\partial u}{\partial n} = 0 & \text{on } \operatorname{bnd}(\Omega) \\ u(x, y, 0) = f(x, y) \\ u_t(x, y, 0) = g(x, y) \end{cases}.$$

3.2.2 Damped wave problem

In the formulation of the wave equation, we neglected the drag or resistance force that the membrane may experience. To include such a possible forces, we assume that the drag forces is proportional to the velocity of the displacement, that is,

$$f_{\text{drag}}\propto u_t,$$

and write the damped wave equation as

$$u_{tt} + \xi u_t = c^2 \Delta u,$$

where $\xi > 0$ is a constant. If in addition, the membrane is under an external force h, the equation reads

$$u_{tt} + \xi u_t = c^2 \Delta u + h.$$

3.2.3 Solution of 1D wave equation

The one-dimensional wave equation $u_{tt} = c^2 u_{xx}$ has a simple solution. By denoting the time derivative with the operator ∂_t and the space derivative with ∂_x, we can represent the equation as $(\partial_t - c\partial_x)(\partial_t + c\partial_x)[u] = 0$ or equivalently $(\partial_t + c\partial_x)(\partial_t - c\partial_x)[u] = 0$. This representation allows us to decompose the wave equation into two uncoupled first-order PDEs:

$$\begin{cases} u_t + cu_x = 0 \\ u_t - cu_x = 0 \end{cases},$$

which can be solved for arbitrary functions $F(x - ct)$ and $G(x + ct)$. Thus, the general solution to the equation is $u(x,t) = F(x - ct) + G(x + ct)$.

Now, consider the wave equation with initial conditions $u(x,0) = f(x)$, and $u_t(x,0) = g(x)$. It is possible to determine F, G such that the general solution satisfies these initial conditions. To find F and G, we can use the initial conditions to set up a system of equations. From $u(x, 0) = f(x)$, we have $F(x) + G(x) = f(x)$. Taking the derivative with respect to t and evaluating at $t = 0$, we get $u_t(x,0) = F'(x) - G'(x) = \frac{1}{c} g(x)$, and finally we obtain the following system

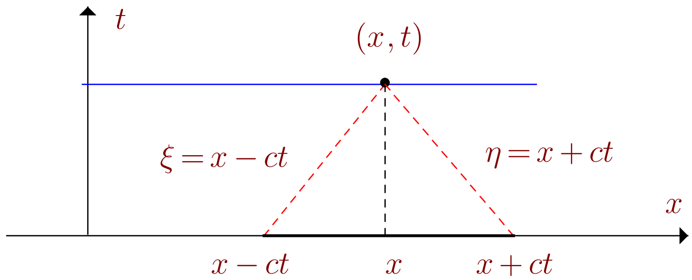
$$\begin{cases} F(x) + G(x) = f(x) \\ F'(x) - G'(x) = \frac{1}{c} g(x) \end{cases}.$$

A simple manipulation gives the following solution

$$u(x,t) = \frac{1}{2}[f(x - ct) + f(x + ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds.$$

The above formula is commonly known as the D'Alembert formula for solving 1D wave equations, named after the French mathematician Jean-Baptiste Rond d'Alembert. According to the formula, the value of u at an arbitrary point (x,t) is determined by the values of f and g in the segment $[x - ct, x + ct]$. This segment is referred to as the wave equation's influence domain, as illustrated in the figure below. It is worth noting that if $g \equiv 0$, then the value of $u(x,t)$ represents the average of the right wave $f(x - ct)$ and the left wave $f(x + ct)$, both propagating with the same speed c.

In the wave equation, the value of u at an arbitrary point (x,t) is determined not only by the initial conditions but also by the values of u and its derivatives in the segment $[x - ct, x + ct]$. This means that the influence of far distant points on a fixed point is not immediate, but rather it takes time for the values to propagate and reach the fixed point, and this time is determined by the velocity c. So the behavior of the wave equation is very different from that of the heat equation, where the value of u at a point is determined by the convolution over entire axis x.



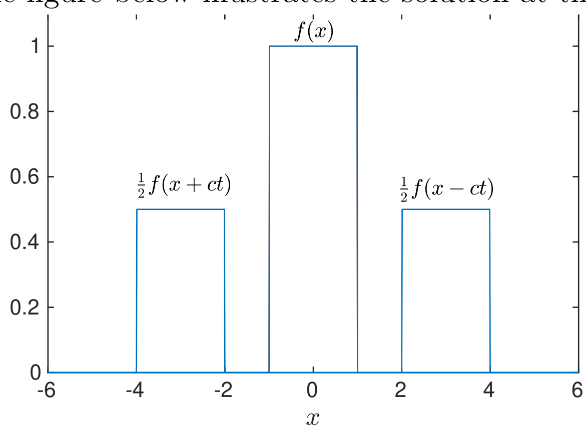
Example 3.2. Consider the following problem

$$\begin{cases} u_{tt} = 9u_{xx} \\ u(x,0) = f(x) \\ u_t(x,0) = 0 \end{cases},$$

where $f(x)$ is the function

$$f(x) = \begin{cases} 1 & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

The figure below illustrates the solution at time $t=0$ and $t=1$ for $c=3$.



As it is observed, the initial condition is split into two branches, one moving to the right and one moving to the left with the same speed $c=3$. This is because the solution to the wave equation is a superposition of two waves propagating in opposite directions. Each wave carries half of the energy of the initial function, which is why we see two identical waves moving away from the origin in the figure. While the heat equation can smooth out discontinuities in the initial condition, wave equations are unable to do so, which can lead to issues with properly defining the partial differential equation at these points.

Exercise 3.15. Consider the equation

$$u_{tt} - c^2 u_{xx} = 0.$$

Take $\xi = x - ct$ and $\eta = x + ct$. Show that the equation under the transformation reduces to

$$\partial_{\xi\eta} u = 0.$$

Solve the obtained equation and conclude that the solution of the equation is

$$u(x,t) = F(x-ct) + G(x+ct),$$

for arbitrary functions F, G.

Exercise 3.16. Consider the non-homogeneous equation

$$\begin{cases} u_{tt} = c^2 u_{xx} + h(t,x) \\ u(x,0) = u_t(x,0) = 0 \end{cases}.$$

a) Use the characteristic change of variable $\xi = x - ct$, $\eta = x + ct$ to change the equation $u_{tt} = c^2 u_{xx}$ to the form $u_{\xi\eta} = 0$.

b) Use this method to show that the solution to the above non-homogeneous problem is

$$u(x,t) = \int\int_{\Delta} h(t', x')\, dt'\, dx',$$

where Δ is the inside of the triangle constructed on the characteristic $\xi = x - ct$, and $\eta = x + ct$

3.2.4 Solution on finite string

We will now derive the solution to a 1D wave problem on a string of length L whose endpoints at $x = 0$ and $x = L$ are fixed. The problem is given by the wave equation:

$$u_{tt} = c^2 u_{xx}$$

with the initial and boundary conditions:

$$\begin{cases} u(t,0) = u(t,L) = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}.$$

To solve this equation, we can use the following formula:

$$u(t,x) = \frac{1}{2}[f_{\text{odd}}(x - ct) + f_{\text{odd}}(x + ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g_{\text{odd}}(s)\, ds. \tag{3.1}$$

Here, f_{odd} and g_{odd} are the odd extensions of f and g, respectively, and are 2L-periodic functions. Recall that the odd extension of a function f defined on $[0, L]$ is

$$f_{\text{odd}}(x) = \begin{cases} f(x) & 0 \le x \le L \\ -f(-x) & -L \le x \le 0 \end{cases}.$$

The solution given by the above formula is obtained by applying the D'Alembert formula to the odd extension of the initial condition $f(x)$ and the odd extension of the initial velocity $g(x)$. The first term in solution represents the contribution of the initial displacement to the solution, and the second term represents the contribution of the initial velocity to the solution.

The boundary conditions $u(t,0) = u(t,L) = 0$ are automatically satisfied by this solution since $\text{fodd}(0) = f_{\text{odd}}(L) = 0$.

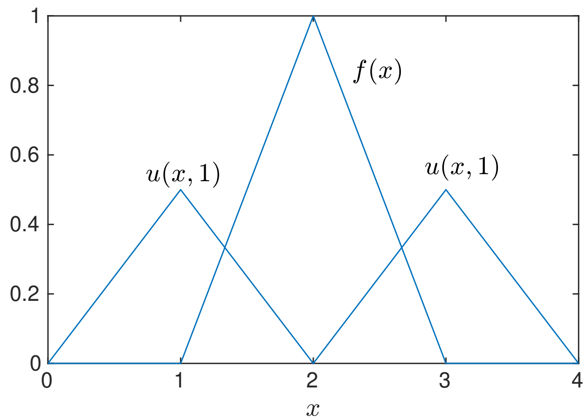
Example 3.3. Consider the following problem

$$\begin{cases} u_{tt} = u_{xx} \\ u(0,t) = u(4,t) = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = 0 \end{cases},$$

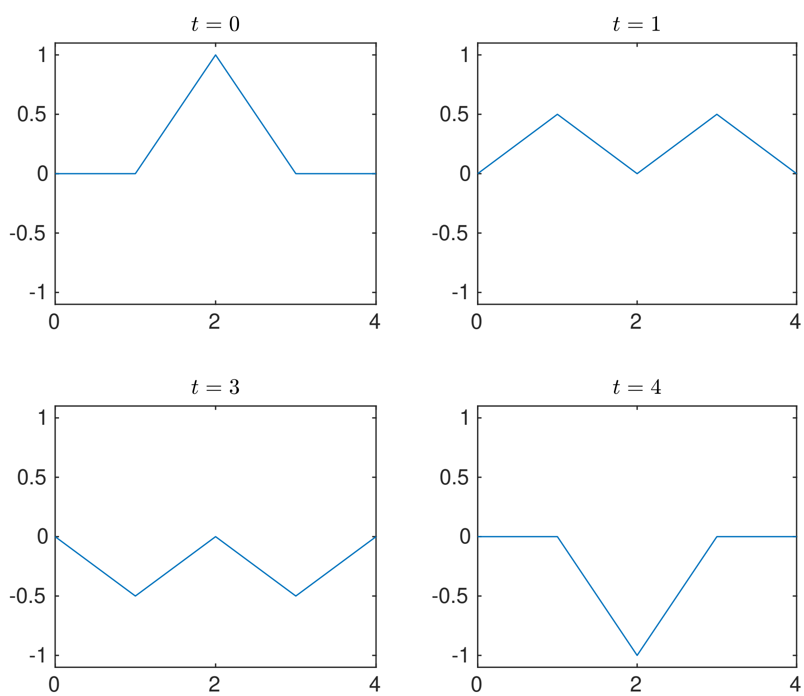
where

$$f(x) = \begin{cases} x-1 & 1 \le x \le 2 \\ 3-x & 2 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}.$$

The figure below illustrates the initial condition $f(x)$, and $u(x,1)$:



For $t > 1$, two branches of waves hit the boundary points $x = 0$ and $x = 4$. According to formula (3.1), the odd extension of the wave moves back after it hits the end point. The figure below depicts the solution $u(x,t)$ for $t = 0, 1, 3$, and 4.



Exercise 3.17. Verify the formula (3.1).

Exercise 3.18. Consider the following equation

$$\begin{cases} u_{tt} = u_{xx} - 1, \quad 0 < x < 1 \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = \frac{1}{2}x(x-1), \\ u_t(x,0) = \sin(\pi x) \end{cases}. \tag{3.2}$$

Verify that the following function is the solution of the above problem:

$$u(x,t) = \frac{1}{2}x(x-1) + \frac{1}{\pi}\sin(\pi x)\sin(\pi t).$$

Exercise 3.19. Consider the following wave problem

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}.$$

The energy of the wave function is defined by the following formula

$$E(t) := \int_0^L [|u_t(x,t)|^2 + c^2|u_x(x,t)|^2]\, dx.$$

a) Suppose that $f'(x)$ is integrable. Verify the following formula which is known as the conservation of energy of the wave function

$$E(t) = \int_0^L \{|g(x)|^2 + c^2|f'(x)|^2\}\, dx.$$

b) Conclude that the the given problem has a unique solution.

Exercise 3.20. Let $\Omega \subset \mathbb{R}^2$ be a bounded set with smooth boundary $\mathrm{bnd}(\Omega)$. Consider the following wave problem on Ω

$$\begin{cases} u_{tt} = c^2 \Delta u \\ u|_{\mathrm{bnd}(\Omega)} = 0 \\ u(x,y,0) = f(x,y) \\ u_t(x,y,0) = g(x,y) \end{cases},$$

and assume that

$$\iint_\Omega |g|^2 + c^2|\nabla f|^2\, dS < \infty.$$

a) The energy of the vibrating membrane is defined through the following integral

$$E(t) := \iint_\Omega [|u_t(x,y,t)|^2 + c^2|\nabla u(x,y,t)|^2]\, dS$$

Prove that $E(t)$ is constant and conclude the following relation

$$E(t) = \iint_\Omega [|g|^2 + c^2|\nabla f|^2]\, dS.$$

b) Prove that the above problem can not have multiple solutions.

Exercise 3.21. Consider the following wave problem

$$\begin{cases} u_{tt} = u_{xx} - u \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}.$$

i. Show the following conservation of energy for the above problem

$$\int_0^L (|u|^2 + |u_t|^2 + |u_x|^2)\, dx = \int_0^L (|f|^2 + |f'|^2)\, dx.$$

ii. If $f(0) = f(L)$, show that

$$\int_0^L (|u|^2 + |u_t|^2 + |u_x|^2)\, dx = \int_0^L |f + f'|^2\, dx.$$

Exercise 3.22. Consider the following damped equation

$$\begin{cases} u_{tt} + \xi u_t = u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases},$$

where $\xi k \geq 0$ is a constant. For $E(t)$, the energy defined as

$$E(t) := \int_0^L (|u_t|^2 + |u_x|^2)\, dx,$$

show the following relation

$$e^{-2\xi t} E(0) \leq E(t) \leq E(0).$$

Exercise 3.23. Consider the following wave problem

$$\begin{cases} u_{tt} = c^2 \Delta u \\ u|_{\mathrm{bnd}(\Omega)} = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases},$$

where $x \in \Omega \subset \mathbb{R}^n$, and Ω is a bounded domain with smooth boundary $\mathrm{bnd}(\Omega)$. Verify the following conservation of energy formula for the problem

$$\int_\Omega [|u_t(x,t)|^2 + c^2 |\nabla u(x,t)|^2]\, dV = \int_\Omega [|g(x)|^2 + c^2 |\nabla f(x)|^2]\, dV.$$

Exercise 3.24. Consider the following wave problem

$$\begin{cases} u_{tt} = \Delta u - u \\ u(t, \mathrm{bnd}(\Omega)) = 0, \\ u(0,p) = f(p), \partial_t u(0,p) = 0 \end{cases},$$

for $x \in \Omega \subset \mathbb{R}^n$, and Ω is a bounded domain with the smooth boundary $\mathrm{bnd}(\Omega)$. Show the following conservation of energy for the above problem

$$\int_\Omega (|u|^2 + |u_t|^2 + |\nabla u|^2)\, dV = \int_\Omega (|f|^2 + |\nabla f|^2)\, dV.$$

Exercise 3.25. Consider the following damped equation

$$\begin{cases} u_{tt} + \xi u_t = \Delta u \\ u|_{\mathrm{bnd}(\Omega)} = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases},$$

where $\xi > 0$ is a constant, and Ω is the unit cube $[0,1]^n$ in $\mathbb{R}^n$.

i. For $E(t)$, the energy defined by

$$E(t) := \int_\Omega |u_t|^2 + \int_\Omega |\nabla u|^2,$$

show that $E(t) \leq E(0)$ for $t > 0$.

ii. Show the following relation

$$E(t) \leq E(0)\, e^{-\xi t}$$

3.3 Poisson and Laplace equations

The Poisson and Laplace equations are two important partial differential equations that arise in many areas of science and engineering. The Poisson equation describes the steady-state behavior of a field, such as the electric potential or temperature, in the presence of a given source or charge distribution. The Laplace equation is a special case of the Poisson equation where the source is zero, and it describes the equilibrium state of the field. These equations have many applications in physics, engineering, and mathematics, and their solutions can provide valuable insights into the behavior of physical systems. In this chapter, we will discuss the Poisson and Laplace equations and their solutions in one, two, and three dimensions.

3.3.1 Derivation of Poisson and Laplace equations

Let Ω be an open subset of $\mathbb{R}^n$. The Poisson's equation has the general form $-\Delta u = f$. If Ω is bounded with boundary $\mathrm{bnd}(\Omega)$, the associated boundary condition can be

$$\alpha u + \beta \frac{\partial u}{\partial n}\bigg|_{\mathrm{bnd}(\Omega)} = g,$$

where α and β are constants. In particular, if f is identically zero, the partial differential equation is called a Laplace equation.

The Poisson equation can be considered as the steady-state solution of a heat problem. For example, the steady-state solution of the non-homogeneous heat equation

$$\begin{cases} u_t = \Delta u + f & \text{on } \Omega \\ \alpha u + \beta \frac{\partial u}{\partial n} = g & \text{on } \mathrm{bnd}(\Omega) \\ u(x,0) = u_0(x) \end{cases},$$

is expressed as the equation

$$\begin{cases} -\Delta u = f & \text{on } \Omega \\ \alpha u + \beta \frac{\partial u}{\partial n} = g & \text{on } \mathrm{bnd}(\Omega) \end{cases},$$

which is independent of the initial condition u_0.

In the electrostatic context, the Poisson and Laplace equations arise as the potential field of electric charges. Consider a distribution of electric charge with density $\rho(r)$ in $\mathbb{R}^3$. The electric field at $r\in\mathbb{R}^3$ generated by this distribution is determined by the following integral:

$$E(r)=\frac{1}{4\pi\varepsilon_0}\int_{\mathbb{R}^3}\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\rho(r')\,dr', \tag{3.3}$$

where $\varepsilon_0>0$ is the permittivity constant. We can apply Gauss's law, which relates the flux of electric field through a surface to the charge enclosed by that surface, to a bounded domain $\Omega\subset\mathbb{R}^3$:

$$\oiint_{\mathrm{bnd}(\Omega)} E(r)\cdot\nu\,dS=\frac{Q}{\varepsilon_0},$$

where Q is the total charge enclosed by $\mathrm{bnd}(\Omega)$:

$$Q=\iiint_\Omega \rho\,dV.$$

Using the divergence theorem, this can be rewritten in terms of the divergence of the electric field:

$$\iiint_W \mathrm{div}\,(E)\,dV=\frac{1}{\varepsilon_0}\iiint_\Omega \rho\,dV.$$

It turns out that the electric field E is potential in the sense that there is a scalar function ϕ such that $E=-\nabla\phi$. Accordingly, we obtain the following differential equation for ϕ:

$$-\Delta\phi(r)=\frac{\rho(r)}{\varepsilon_0}.$$

Exercise 3.26. Recall that if we put a pointwise electric charge q at the origin, it produces an electric field E that bu the Columb law is

$$E(r)=\frac{q}{4\pi\varepsilon_0\,|r|^2}\hat{r},$$

for any $r\in\mathbb{R}^3-\{0\}$, where $\hat{r}$ is the unit vector in the direction of r.

a) Verify that for $\phi=-\frac{q}{4\pi\varepsilon_0\,|r|}$, we have $E=-\nabla\phi$.

b) For any $r\in\mathbb{R}^3-\{0\}$, verify the relation $\Delta\phi(r)=0$.

Exercise 3.27. f the electrical field $E(r)$ is known, its associated potential ϕ with the relation $E=-\nabla\phi$ is derived by the relation

$$\phi(r)=\int_\infty^r E(r')\cdot\hat{T}\,dl,$$

where $\hat{T}$ is the unit tangent vector to the straight line connecting ∞ to r.

Exercise 3.28. The most well know conservative field in physics is the gravitation field. If a particle with the mass m is located at the origin in $\mathbb{R}^3$, then its potential $U(x,y,x)$ is

$$U=-\frac{m}{|r|},$$

for $r\in\mathbb{R}^3-\{0\}$. Now, suppose that in $\mathbb{R}^n, n\geq 3$, the potential function U has the form

$$U(x_1,\ldots,x_n)=-\frac{m}{(x_1^2+\cdots+x_n^2)^{\alpha/2}},$$

for some constant α. Find α such that $\Delta U = 0$.

Exercise 3.29. Despite the heat and wave equations, the Poisson and Laplace equations with Neumann boundary conditions can be tricky and the existence of a solution may fail. Consider the following equation:

$$\begin{cases} -\Delta u = f & \text{on } \Omega \\ \frac{\partial u}{\partial n} = g & \text{on } \mathrm{bnd}(\Omega) \end{cases}.$$

a) Use the divergence theorem to show that the necessary condition for the existence of a solution to the equation is:

$$\oint_{\mathrm{bnd}(\Omega)} g = \int_{\Omega} f.$$

b) Even if a solution u exists for the equation, show that the solution is not unique and there are infinitely many other solutions.

c) Let $B \subset \mathbb{R}^2$ be the unit disk centered at the origin. Find α such that the following equation is solvable.

$$\begin{cases} \Delta u = 1 + x^2 y \\ \frac{\partial u}{\partial n}\big|_S = \alpha \end{cases},$$

where $S = \mathrm{bnd}(B)$ is the unit circle.

Exercise 3.30. The MAXWELL's equations are as follows

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0}, \nabla \cdot B = 0, \nabla \times E = -B_t, \nabla \times B = \mu\sigma E + \mu\varepsilon_0 E_t,$$

where E, B are respectively the electric and magnetic fields in space, ρ is the density of the electric charge, μ is the magnetic inductive capacity and ε is the electric inductive capacity and σ is the electrical conductivity. Use the above equations and derive the following equation

$$\Delta E = \mu\varepsilon_0 E_{tt} + \mu\sigma E_t + \frac{1}{\varepsilon_0}\nabla\rho$$

3.3.2 Fundamental solution of Poisson's equation

Consider the equation $-\Delta u = f(r)$, where $r \in \mathbb{R}^3$. We can write f as a convolution integral using the Dirac delta function δ:

$$f(r) = \int_{\mathbb{R}^3} f(r')\, \delta(r - r')\, dV.$$

This suggests that the solution to the above equation is related to the solution of the equation $-\Delta u = \delta(r)$. Recall that the potential function $\phi = -\frac{1}{4\pi\varepsilon_0 |r|}$ describes the electric field of a pointwise unit charge at the origin. We can take $u(r) = \Phi(r) := \frac{1}{4\pi\, |r|}$ for $r \in \mathbb{R}^3 - \{0\}$, and although a rigorous proof is beyond the scope of this book, we can verify the following:

$$-\Delta\Phi = 0 = \delta(r),$$

for $r \in \mathbb{R}^3 - \{0\}$. Moreover, for any ball B_a of radius a centered at 0, we have:

$$\iiint_{B_a} \delta(r)\, dV = 1.$$

Using the divergence theorem, we can rewrite the left-hand side of the above equation:

$$-\iiint_{B_a} \Delta\Phi \, dV = -\oiint_{\mathrm{bnd}(B_a)} \nabla\Phi \cdot \nu dS = -\oiint_{\mathrm{bnd}(B_a)} \nabla\frac{1}{4\pi\,|r|} \cdot \hat{r}\, dS,$$

where we used the fact that $\nu = \hat{r}$ on the surface of the ball B_a. Using the relation

$$\nabla\frac{1}{4\pi\,|r|} = -\frac{\hat{r}}{4\pi\,|r|^2},$$

and the equality $|r|^2 = a^2$ on $\mathrm{bnd}(B_a)$, we obtain

$$-\oiint_{\mathrm{bnd}(B_a)} \nabla\frac{1}{4\pi\,|r|} \cdot \hat{r}\, dS = \frac{1}{4\pi a^2} \oiint_{\mathrm{bnd}(B_a)} dS = 1.$$

Therefore,

$$-\iiint_{B_a} \Delta\Phi(r)\, dV = \iiint_{B_a} \delta(r)\, dV,$$

for any $a > 0$. We can now obtain the solution to the equation $-\Delta u = f$ using the convolution integral:

$$u(r) = \iiint_{\mathbb{R}^3} f(r')\, \Phi(r - r')\, dV.$$

Example 3.4. Consider the equation $-\Delta u(r) = |r|\, e^{-|r|}$. The solution of the equation is

$$u(r) = \frac{1}{4\pi} \iiint_{\mathbb{R}^3} \frac{|r'|\, e^{-|r'|}}{|r - r'|}\, dV.$$

The value of u at the origin $r = 0$ is equal to

$$u(0) = \frac{1}{4\pi} \iiint_{\mathbb{R}^3} e^{-|r'|}\, dV = \frac{1}{4\pi} \lim_{R\to\infty} \int_{-\pi}^{\pi} \int_0^{\pi} \int_0^R e^{-r'} r'^2 \sin(\theta)\, dr'\, d\theta\, d\phi = 2.$$

For $r \neq 0$, we can use a numerical integrator to find the value of $u(r)$ by the above integral.

Remark 3.2. While the convolution solution for the Poisson equation works well on $\mathbb{R}^n$, it does not always satisfy the desired boundary conditions when applied to bounded domains Ω. In fact, finding a closed-form solution for the Poisson equation on a general bounded domain with arbitrary boundary conditions is typically impossible. However, in later chapters, we will develop tools to solve this equation on specific types of domains such as rectangles, disks, cylinders, and spheres.

Exercise 3.31. Let $f = e^{-|r|^2}$. Use a numerical integrator and find the solution of equation $-\Delta u = f$ on the sphere $|r| = 1$.

Exercise 3.32. Let us find the fundamental solution of the POISSON equation in $\mathbb{R}^2$

$$-\Delta\Phi = \delta(r).$$

Assume $\Phi = \Phi(|r|)$, where $|r| = \sqrt{x^2 + y^2}$. Calculate $\Delta\Phi$ and obtain the following equation for $r \neq 0$:

$$\frac{d^2\Phi}{d|r|^2} + \frac{1}{|r|}\frac{d\Phi}{d|r|} = 0,$$

Solve the equation and obtain $\Phi(r) = C \ln|p|$ for some constant C. Find C.

3.3.3 Harmonic functions

Harmonic functions play an important role in many areas of mathematics and physics. A smooth function u defined on an open set $\Omega \subset \mathbb{R}^n$ is called harmonic if it satisfies the Laplace equation $\Delta u(x) = 0$ for all $x \in \Omega$. There are many examples of harmonic functions in $\mathbb{R}^n$, including some simple ones that can be obtained using basic calculus. The following exercise gives a few such examples.

Exercise 3.33. Verify that the following functions are solutions to the LAPLACE equation in $\mathbb{R}^2$:

i. $f(x, y) = \text{constant}$

ii. $f(x, y) = ax + by$

iii. $f(x, y) = x^2 - y^2$

iv. $f(x, y) = \cos(nx)\cosh(ny)$.

Harmonic functions possess several interesting properties. One of these properties is that the net flux of the gradient vector field $F = \nabla u$ through an arbitrary closed surface S is zero. Specifically, for any closed surface S that encloses the open set Ω, we have

$$0 = \int_\Omega \Delta u = \oint_S \nabla u \cdot \nu, \tag{3.4}$$

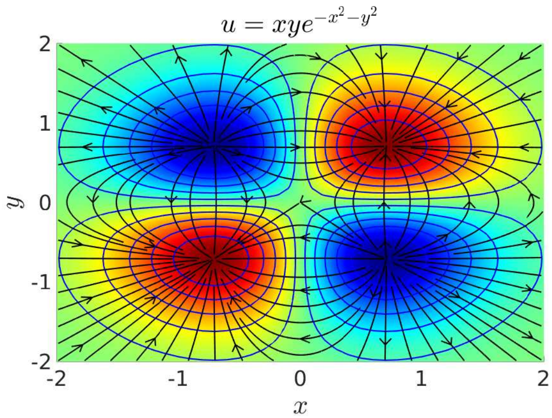
where ν is the unit outward normal vector to S. This property also explains why the Laplace equation $\Delta u = 0$ corresponds to the steady state of a heat equation $u_t = \Delta u$: if we interpret u as the temperature function, then the gradient ∇u represents the heat flux, and since u is steady, the net heat flux through any closed surface S must be zero, otherwise the temperature of points inside S would change in time. This observation leads to the following important theorem

Theorem 3.2. *A harmonic function u in an open set Ω can not have a proper maximum and minimum inside Ω.*

The theorem expresses a fundamental result in the theory of harmonic functions. It states that a smooth function u, which is harmonic in an open set Ω, cannot have a proper maximum or minimum inside Ω. By proper maximum or minimum, we mean a point x_0 and a neighborhood D containing x_0 such that $u(x_0) > u(x)$ or $u(x_0) < u(x)$ for all $x \in D$.

To understand why this is true, consider the associated vector field $F = \nabla u$. As we saw in the first chapter, proper maximum points behave like sinks for the vector field, while minimum points behave like sources. In either case, the net flux passing through the level curves around the maximum or minimum points is not zero, which violates the relation (3.4).

For example, the function $u = xye^{-x^2-y^2}$ can not be a harmonic function. The figure below illustrates the associated vector field of the function. Observe that the net flux passing through the level curves around the maximum and minimum points are non-zero.



The following fact is an immediate result of the above argument:

Theorem 3.3. (Maximum principle) *If u is harmonic function in an bounded open set $\Omega \subset \mathbb{R}^n$ and if u is continuous on $\mathrm{cl}(\Omega)$, then u attains its maximum and minimum on $\mathrm{bnd}(\Omega)$.*

Exercise 3.34. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary $\mathrm{bnd}(\Omega)$. Use the maximum principle and prove that the unique solution of the Laplace equation

$$\begin{cases} \Delta u = 0 & \text{on } \Omega \\ u = a & \text{on } \mathrm{bnd}(\Omega) \end{cases},$$

for a a constant, is the constant solution $u = a$.

Exercise 3.35. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary $\mathrm{bnd}(\Omega)$. Use the maximum principle and show that if there is a solution to the following problem

$$\begin{cases} \Delta u = f & \text{on } \Omega \\ u = g & \text{on } \mathrm{bnd}(\Omega) \end{cases},$$

then it is unique.

Exercise 3.36. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary $\mathrm{bnd}(\Omega)$. Use the maximum principle and show that if there is a solution to the following problem

$$\begin{cases} -\Delta u = f & \text{on } \Omega \\ u + \frac{\partial u}{\partial n} = g & \text{on } \mathrm{bnd}(\Omega) \end{cases},$$

then it is unique.

Exercise 3.37. Let Ω be an open bounded set with smooth boundary $\mathrm{bnd}(\Omega)$. Prove that the necessary condition that the following problem has a non-trivial solution is $\lambda > 0$

$$\begin{cases} \Delta u = -\lambda u & \text{on } \Omega \\ u + \frac{\partial u}{\partial n} = 0 & \text{on } \mathrm{bnd}(\Omega) \end{cases}.$$

Exercise 3.38. Let us write the wave equation on a bounded domain Ω as follows

$$\frac{1}{c^2} u_{tt} = \Delta u.$$

When $c \to \infty$, the equation approached to the Laplace equation $\Delta u = 0$. Therefore, we can consider the Laplace equation as a wave propagation equation with the wave speed infinity. This justifies that if the condition on $\text{bnd}(\Omega)$ changes, this change affects immediately on the value of u inside Ω.

a) Verify that the function $u = 1 + \varepsilon r^2 \sin(2\theta)$ satisfies the Laplace equation

$$\begin{cases} \Delta u = 0 & \text{on } B \\ u = 1 + \varepsilon \sin(2\theta) & \text{on bnd(B)} \end{cases},$$

where B is the unit disk in $\mathbb{R}^2$.

b) For $\varepsilon = 0$, the unique solution is $u = 1$. For the boundary condition $u|_{\text{bnd}(B)} = 1 + \varepsilon \sin(2\theta)$, the solution inside B changes immediately to $u = 1 + \varepsilon r^2 \sin(2\theta)$. Run the following code in Matlab and draw the solution for $\varepsilon = 0, 0.5, 1$.

```
[th,r]=meshgrid(-pi:pi/20:pi,0:0.2:1);
up=@(e) 1+e*r.^2.*sin(2*th);
[x,y]=pol2cart(th,r,up(0.5));
axis([-1 1 -1 1 0 2]);
subplot(2,2,1)
surf(x,y,up(0));
axis([-1 1 -1 1 0 2]);
title('$\epsilon=0$','interpreter','latex','fontsize',12);
subplot(2,2,2)
surf(x,y,up(0.5));
axis([-1 1 -1 1 0 2]);
title('$\epsilon=0.5$','interpreter','latex','fontsize',12);
subplot(2,2,3)
surf(x,y,up(1));
axis([-1 1 -1 1 0 2]);
title('$\epsilon=1$','interpreter','latex','fontsize',12)
```

Another consequence of the relation (3.4) is the Mean Value Property of harmonic function. In its rigorous form, it states that if u is a harmonic function on an open set Ω in $\mathbb{R}^n$, and $B_r(x)$ denotes the open ball of radius r centered at x, and $S_r(x)$ denotes the sphere $\text{bnd}(B_r(x))$, then for any $x \in \Omega$ and $r > 0$ such that $B_r(x) \subseteq \Omega$, we have:

$$u(x) = \frac{1}{|S_r|} \oint_{S_r(x)} u(z)\, dS,$$

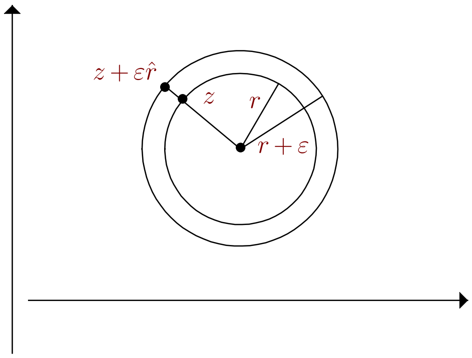
where $|S_r|$ denotes the volume of the ball sphere S_r. In other words, the value of a harmonic function at any point is the average of its values over any sphere centered at that point.

In the case of harmonic functions on $\Omega \subset \mathbb{R}^2$, the Mean Value Property can be expressed as follows: if u is a harmonic function on an open set Ω, and $B_r(z_0)$ denotes the open disk of radius r centered at $z_0 = (x_0, y_0)$ and contained in Ω, then

$$u(z_0) = \frac{1}{2\pi r} \oint_{C_r(z_0)} u(z)\, dz,$$

where $C_r(z_0)$ denotes the circle of radius r centered at z_0, oriented counterclockwise. In other words, the value of a harmonic function at any point is the average of its values over any circle centered at that point.

To see why this is true, suppose u is harmonic in an open set $\Omega \subset \mathbb{R}^2$ and consider two circles centered at $z_0 = (x_0, y_0)$ with radii r and $r + \varepsilon$ inside Ω. For an arbitrary point z on C_r, the circle of radius r, consider the associated point $z + \varepsilon\, \hat{r}$ on $C_{r+\varepsilon}$.



We can calculate the limit:

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left\{\frac{1}{r+\varepsilon}\oint_{C_{r+\varepsilon}(z_0)}u(z)\,dl-\frac{1}{r}\oint_{C_r(z_0)}u(z)\,dl\right\}.$$

Using polar coordinates for both integrals, we obtain the expression in the bracket as:

$$\frac{1}{r+\varepsilon}\oint_{C_{r+\varepsilon}(z_0)}u(z)\,dl-\frac{1}{r}\oint_{C_r(z_0)}u(z)\,dl=\varepsilon\int_{-\pi}^{\pi}\nabla u(|\xi|,\theta)\cdot\hat{r}\,d\theta,$$

where ξ is a point in the line connecting z to $z+\varepsilon\,\hat{r}$. Therefore, we obtain:

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left\{\frac{1}{r+\varepsilon}\oint_{C_{r+\varepsilon}(z_0)}u(z)\,dl-\frac{1}{r}\oint_{C_r(z_0)}u(z)\,dl\right\}=\lim_{\varepsilon\to 0}\int_{-\pi}^{\pi}\nabla u(|\xi|,\theta)\cdot\hat{r}\,d\theta,$$

and if we pass the limit inside the integral, we obtain

$$\lim_{\varepsilon\to 0}\int_{-\pi}^{\pi}\nabla u(|\xi|,\theta)\cdot\hat{r}\,d\theta=\int_{-\pi}^{\pi}\nabla u(r,\theta)\cdot\hat{r}\,d\theta=0$$

where the last equality is justified by the relation (3.4). This calculation justifies the fact that:

$$\frac{d}{dr}\frac{1}{r}\oint_{C_r(z_0)}u(z)\,dl=0,$$

and then the integral

$$\frac{1}{r}\oint_{C_r(z_0)}u(z)\,dl$$

is a constant. In particular, if $r\to 0$, we obtain

$$\lim_{r\to 0}\frac{1}{r}\oint_{C_r(z_0)}u(z)\,dl=u(z_0)\lim_{r\to 0}\frac{1}{r}\oint_{C_r(z_0)}dl=2\pi u(z_0),$$

and then

$$u(x_0,y_0)=\frac{1}{2\pi r}\oint_{C_r(z_0)}u(z)dl.$$

This is the first version of the mean value property of harmonic functions in 2D.

Example 3.5. Let us verify the mean value property for the harmonic function $u=x^2-y^2$. We have

$$\frac{1}{2\pi r}\int_{-\pi}^{\pi}\delta^2(\cos^2\theta-\sin^2\theta)\,d\theta=\frac{r}{2\pi}\int_{-\pi}^{\pi}\cos(2\theta)d\theta=0,$$

and the obtained value is equal to $u(0,0)$. Now, consider the arbitrary point z_0: (x_0, y_0). The circle C of radius r centered at z_0 has the following representation in the polar coordinate

$$x = x_0 + r\cos\theta,\ y = y_0 + r\sin\theta.$$

Hence,

$$\frac{1}{2\pi r}\oint_{C_r}(x^2 - y^2)dl = \frac{1}{2\pi r}\int_{-\pi}^{\pi}[(x_0 + r\cos\theta)^2 - (y_0 + r\sin\theta)^2]d\theta = x_0^2 - y_0^2 = u(z_0).$$

Exercise 3.39. Verify that the function $u = x^3 + y^3 - 3x^2y - 3xy^2$ is harmonic and calculate the following integral

$$\oint_C u(x, y)\, dl,$$

where C is the circle of radius 1 centered at the point $(1,1)$.

Exercise 3.40. We prove another version of Mean Value Property for harmonic functions in 2D

$$u(x_0, y_0) = \frac{1}{\pi r^2}\iint_{B_r(x_0, y_0)} u(x, y)\, dx\, dy.$$

For this, use the polar coordinate and write

$$\iint_{B_r(x_0, y_0)} u(x, y)\, dx\, dy = \int_0^r \int_{-\pi}^{\pi} u(r, \theta)\, r\, dr\, d\theta.$$

Then, use the first version of the mean value property and conclude the claim.

Exercise 3.41. Use the spherical coordinate and prove the first and second versions of the Mean Value Property for harmonic functions in 3D.

3.4 Classification of linear second-order PDEs

The differential equations we have explored thus far are instances of *linear* second-order equations. Understanding the behavior of their solutions is a significant aspect of studying these equations, both qualitatively and quantitatively. In order to analyze them effectively, it is helpful to classify linear second-order partial differential equations into three main categories: *elliptic*, *parabolic*, and *hyperbolic* equations. This classification is based on the properties of the *principal part* of the differential operator near a specific point, which in turn determines the characteristic behavior of the solution in the vicinity of that point.

3.4.1 Operator form, linearity and superposition

A useful perspective on linear differential equations is to view them as operator equations. When dealing with a real-valued function $u(x)$ defined on a domain $\Omega \subset \mathbb{R}^n$, the partial derivative $\frac{\partial u}{\partial x_j}$ can be seen as a mapping ∂_j that takes functions from the domain of continuously differentiable functions and returns continuous functions:

$$\partial_j \colon C^1(\Omega) \to C(\Omega).$$

Here, C^1 represents the set of continuously differentiable functions, and C represents the set of continuous functions.

By considering $\frac{\partial u}{\partial x_j}$ as an operator ∂_j, we can analyze how it acts on functions and how it relates to other operators in the differential equation. This viewpoint allows us to study the properties of the operator, such as linearity, composition, and its effect on the solutions of the differential equation. Similarly, we can extend the notion of differential operators to higher order derivatives. For instance, the second partial derivative operator can be denoted as ∂_{ij} and maps functions from the domain of twice continuously differentiable functions:

$$\partial_{ij}: C^2(\Omega) \to C(\Omega).$$

Furthermore, we can define a complete linear second-order differential operator as:

$$L := \sum_{i,j=1}^{n} a_{ij}(x)\, \partial_{ij} + \sum_{j=1}^{n} b_j(x)\, \partial_j + c(x), \tag{3.5}$$

where $a_{i,j}(x), b_j(x)$ and $c(x)$ are some continuous or smooth functions defined on the domain Ω. This general form allows us to represent a wide range of second-order linear differential equations.

Recall from linear algebra that a mapping T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is called linear if it satisfies the following relation for any $\alpha_1, \alpha_2 \in \mathbb{R}$ and $u_1, u_2 \in \mathbb{U}$:

$$T(\alpha_1\, \vec{v}_1 + \alpha_2\, \vec{v}_2) = \alpha_1\, T(\vec{v}_1) + \alpha_2\, T(\vec{v}_2).$$

Similarly, we can apply this concept to the differential operator L. The operator L is linear because it satisfies the following property:

$$L(\alpha_1\, u_1(x) + \alpha_2\, u_2(x)) = \alpha_1\, T(u_1(x)) + \alpha_2\, T(u_2(x)),$$

where $u_1(x)$ and $u_2(x)$ are functions in the domain of the operator L, and α_1 and α_2 are constants.

Although we won't delve into the full vector space formulation of partial differential operators in this book, representing partial derivatives in operator notation can help us understand the classification of second-order linear PDEs. In general, a linear second-order partial differential equation can be written as:

$$L[u] = f,$$

where L is the operator defined in (3.5). Here, L represents a linear mapping that takes a function u as input and produces the corresponding function f as output. The goal of such an equation is to determine the function u, given a known function f.

Interpreting a linear partial differential equation as an operator equation has several advantages. It allows us to make a connection between linear algebra, which studies the properties of linear mappings in finite-dimensional vector spaces, and differential equations. One benefit of this approach is the superposition principle, which applies to homogeneous equations of the form $L[u] = 0$.

Proposition 3.1. *If $u_1, \ldots, u_N$ are homogeneous solutions of linear equation $L[u] = 0$, then the function*

$$u = \sum_{j=1}^{N} c_j u_j,$$

for arbitrary $c_j \in \mathbb{R}$ solves the equation. Moreover, if u_p is a particular solution for the equation $L[u] = f$, then the function

$$u = u_p + \sum_{j=1}^{N} c_j u_j,$$

solves the equation if $u_1, \ldots, u_N$ are in the kernel or null space of L.

Exercise 3.42. Let L be the linear operator $L = \partial_x + \partial_y$. Consider the equation $L[u] = x + y$.

a) Find the general homogeneous solution of the equation.

b) Verify that $u_p = xy$ is a particular solution of the equation. Therefore, $u = u_h + xy$ is the general solution of the equation. It is simply verified that $u_p = \frac{1}{2}x^2 + \frac{1}{2}y^2$ is also a particular solution. Show that the latter particular solution can be derived from the general solution $u = u_h + xy$.

Exercise 3.43. Find the general solution of the linear equation $L[u] = x$ for $u = u(x, y)$, where $L = \partial_t - c\partial_x$ for a constant c.

Exercise 3.44. Consider the following equation

$$u_{tt} - c^2 u_{xx} = 0,$$

and let L be the operator $L := \partial_{tt} - c^2 \partial_{xx}$. Show that

$$L = L_1 \circ L_2,$$

where $L_1 := \partial_t - c\partial_x$, $L_2 := \partial_t + c\partial_x$. With the aid of the the decomposition, find two linearly independent solution of the given equation.

By taking some precautions, we can also extend the superposition principle to the infinite or continuous versions. For example, if $u_1, u_2, \ldots$ are in the null space of L, then $L[u] = 0$ for the function

$$u = \sum_{j=1}^{\infty} c_j u_j.$$

Similarly, if $u(x; k)$ solves the equation $L[u] = 0$ for k in some interval I, then the following function also solves the equation:

$$u = \int_I c(k)\, u(x; k)\, dk.$$

Example 3.6. Consider the partial differential equation $\Delta u = 0$ defined on the half-plane $x > 0$. An example of a solution to this equation is given by the functions $u_k(x, y) = e^{-kx}\sin(ky)$ for any $k \in (-\infty, \infty)$. Now, suppose we define $c(k)$ as follows:

$$c(k) = \begin{cases} 0 & k < 0 \\ 1 & k > 0 \end{cases}.$$

Then, the function

$$u(x, y) := \int_{-\infty}^{\infty} c(k)\, u_k(x, y)\, dk = \frac{y}{x^2 + y^2},$$

solves the equation $\Delta u = 0$ on $x > 0$. However, as x approaches 0, the function $u(x, y)$ becomes unbounded and cannot be extended to the boundary of the domain. To remedy this, we can choose $c(k)$ as

$$c(k) = \begin{cases} 0 & k < 0 \\ e^{-k} & k > 0 \end{cases}.$$

This transforms the superposition integral to

$$u(x, y) = \frac{y}{(x+1)^2 + y^2}.$$

This solution can be extended to the boundary $x = 0$ in a proper manner.

Remark 3.3. The version of infinite superposition should be used with caution. For instance, let us consider the equation $L = \frac{d^2}{dx^2} + 1$ and the functions $u_n(x) = \frac{1}{n^2}\cos(nx)$ for any odd n, which solve the equation. The function

$$u = \sum_{n=1, n:\mathrm{odd}}^{\infty} u_n(x) = \cos(x) + \frac{1}{4}\cos(2x) + \frac{1}{9}\cos(3x) + \cdots +$$

converges to $f(x) = \frac{\pi^2}{8} - \frac{\pi}{4}|x|$ for $x \in (-\pi, \pi)$. However, this function is discontinuous and does not have a derivative at $x = 0$. Therefore, we cannot define $L[u]$ as equal to zero at $x = 0$, and we must be careful when using the infinite version of superposition.

Exercise 3.45. Find a solution to equation $\Delta u = 0$ on the half-plane $x > 0$ such that u on the boundary $x = 0$ is equal to $\frac{y}{4 + y^2}$.

Exercise 3.46. Consider linear equation $\partial_{xx} u = xy$, for $u = u(x, y)$.

a) Show that $u_p = \frac{1}{6}x^3 y$ solves the equation.

b) Find the homogeneous solution of the equation and write down the general solution.

Exercise 3.47. Find the general solution of the following equation

$$L[u] = xy,$$

for a function $u = u(x, y)$, where $L = \partial_{xy}$.

Exercise 3.48. We know that the linear equations enjoy superposition principle. This property does not necessarily hold for non-linear equations. Verify that functions $u_1 = x + c_1$ and $u_2 = y + c_2$ are the solutions to the nonlinear equation

$$|u_x|^2 + |u_y|^2 = 1;$$

however, $u_1 + u_2$ is not a solution.

3.4.2 Classification: constant coefficient case

Linear second-order partial differential equations can be classified based on their principal part, which is defined by the sum:

$$L_p := \sum_{i,j} a_{ij}(x)\, \partial_{ij}.$$

This classification is based on the idea that the behavior of a PDE near a given point is determined by the highest-order derivatives appearing in the equation at that point. To illustrate the logic behind the classification, we first consider the case where a_{ij} are constants, and the domain of the problem is open subsets of the plane.

It is simply seen that L_p can be rewritten in the matrix form as follows:

$$L_p := \begin{pmatrix} \partial_1 & \partial_2 \end{pmatrix} A \begin{pmatrix} \partial_1 \\ \partial_2 \end{pmatrix},$$

where A is the symmetric coefficient matrix given by

$$A = \begin{pmatrix} a_{11} & \frac{a_{12}+a_{21}}{2} \\ \frac{a_{12}+a_{21}}{2} & a_{22} \end{pmatrix}.$$

From linear algebra, the eigenvalues of A are real. Let λ_1 and λ_2 be the eigenvalues of A. Based on these eigenvalues, we can classify the PDE as follows:

- Elliptic: $\lambda_1 \lambda_2 > 0$.
  - Solutions to elliptic PDEs are smooth and have no singularities. This type of PDE typically arises in problems involving steady-state phenomena, such as in electrostatics or fluid mechanics. The most familiar example is the Poisson equation $-\Delta u = f$, where Δ operator can be written as

    $$\Delta = \begin{pmatrix} \partial_1 & \partial_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_1 \\ \partial_2 \end{pmatrix},$$

    with the coefficient matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ having eigenvalues $\lambda_1 = \lambda_2 = 1$.
- Parabolic: $\lambda_1 \lambda_2 = 0$.
  - Solutions to parabolic PDEs have a smoothing effect that can regularize initial data. This type of PDE often arises in problems involving diffusion, heat flow, or time-dependent phenomena that evolve in a certain direction. The most familiar equation is the heat equation $u_t = k u_{xx}$ in the plane (x, t). The coefficient matrix is

    $$A = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix},$$

    withe the eigenvalues $\lambda_1 = 0$, $\lambda_2 = k$.
- Hyperbolic: $\lambda_1 \lambda_2 < 0$.
  - Solutions to hyperbolic PDEs have wave-like behavior and can exhibit shock waves and other discontinuities. This type of PDE typically arises in problems involving wave propagation or vibrations. The most familiar example is a wave equation $u_{tt} = c^2 u_{xx}$ in the (x, t)-plane. The coefficient matrix of this equation is

    $$A = \begin{pmatrix} -1 & 0 \\ 0 & c^2 \end{pmatrix},$$

with eigenvalues $\lambda_1 = -1$, $\lambda_2 = c^2$.

Remark 3.4. It is known from linear algebra that $\lambda_1\lambda_2 = \det(A)$, and thus the classification can be done based on the sign of $\det(A)$ too. Therefore, we can classify the PDE based on the sign of $\det(A)$, as follows:

- Elliptic: $\det(A) > 0$
- Parabolic: $\det(A) = 0$
- Hyperbolic: $\det(A) < 0$

Example 3.7. Consider the following equation

$$2u_{xx} - \alpha u_{xy} + 2u_{yy} + 3u_x = u + x.$$

We classify the type of the equation based on the value of α. The coefficient matrix of the principal part of the equation is

$$A = \begin{pmatrix} 2 & \frac{\alpha}{2} \\ \frac{\alpha}{2} & 2 \end{pmatrix}.$$

We have $\det(A) = 4 - \frac{\alpha^2}{4}$, and thus the equation is elliptic if $\alpha^2 < 16$ or $-4 < \alpha < 4$. The equation is hyperbolic if $\alpha > 4$ or $\alpha < -4$. The equation is parabolic if $\alpha = 4$ or -4.

Exercise 3.49. Classify following equations

a) $u_{xx} + u_{yy} + 2u_{xy} + u_y = 1 - u$

b) $u_{xx} + 2u_{yy} - 3u_{xy} = u$.

Exercise 3.50. Determine the type of the following equation based on the values of α

$$u_{xx} + u_{yy} + 2\alpha u_{xy} = 0.$$

Definition 3.1. *Let L_p be the principal part of a linear second-order partial differential equation (PDE), given by*

$$L_p = \sum_{i,j=1}^{n} a_{ij}\,\partial_{ij},$$

where a_{ij} are constant coefficients. The associated coefficient matrix of the operator is defined by $A = [\tilde{a}_{ij}]$ where $\tilde{a}_{ij} = \frac{a_{ij} + a_{ji}}{2}$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A.

- *The PDE is called elliptic if all eigenvalues $\lambda_1, \ldots, \lambda_n$ have the same sign.*
- *The PDE is called hyperbolic if one eigenvalue has the opposite sign of the other eigenvalues.*
- *The PDE is called parabolic if one eigenvalue is zero, and the other eigenvalues have the same sign.*

Exercise 3.51. For function $u = u(x, y, t)$, classify the equations

a) $u_t = k\,\Delta u$,

b) $u_{tt} = c^2 \Delta u$.

c) $u_{xt} + u_{yy} + u_{xx} + 3u_x = 0$.

Exercise 3.52. Find α such that the following equation is parabolic in the coordinate (t, x, y):

$$u_t = 2u_{xx} + \alpha u_{xy} + 2u_{yy}.$$

3.4.3 Canonical form

For the principal part L_p, the terms ∂_{ij} for $i \neq j$ are called multiplicative terms. We can remove these terms by performing a linear transformation of the coordinate. Before introducing the transformation, let's explain the terminology of ellipticity, parabolicity, and hyberbolicity by comparing them to the types of conic sections.

Recall that the equation of a conic section is given by:

$$ax^2 + 2bxy + cy^2 + dx + ey = \text{const}.$$

The principal part of this section is $ax^2 + 2bxy + cy^2$, which can be represented in matrix form as

$$ax^2 + 2bxy + cy^2 = \begin{pmatrix} x & y \end{pmatrix} A \begin{pmatrix} x \\ y \end{pmatrix},$$

where the coefficient matrix A is symmetric and has real eigenvalues λ_1, λ_2, and two orthogonal eigenvectors. Let $Q = [\vec{v}_1 | \vec{v}_2]$ be the eigenvector matrix of A with $|\vec{v}_1| = |\vec{v}_2| = 1$.

By linear algebra, we know that:

$$Q^{-1} A Q = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Thus, the transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} = Q \begin{pmatrix} X \\ Y \end{pmatrix},$$

maps the old coordinate system to the new coordinate system (X, Y), and transforms the conic section to the following form:

$$\lambda_1 X^2 + \lambda_2 Y^2 = \text{first and zero order terms.}$$

If λ_1 and λ_2 have the same sign, the conic section is an ellipse. If one of λ_1 or λ_2 is zero, the conic section is a parabola. If λ_1 and λ_2 have opposite signs, then the conic section is a hyperbola.

This argument shows how we can remove the multiplicative terms and why the terminology is justified based on geometrical observation.

Example 3.8. Consider the equation

$$2x^2 + 2y^2 - 2xy = 1.$$

In the matrix form, we can represent the equation as

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1.$$

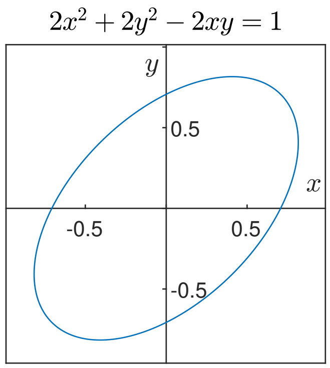
The coefficient matrix $A=\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ has two eigenvalues $\lambda=1,3$ with associated eigenvectors $\vec{v}_1=\begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$, $\vec{v}_2=\begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$. Consider the coordinate transformation

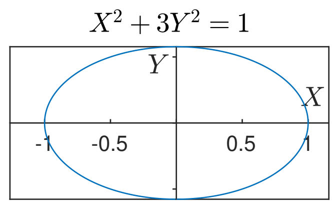
$$Q\begin{pmatrix} X \\ Y \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}, \tag{3.6}$$

where Q is the matrix of eigenvectors. By performing this transformation to the equation, we obtain

$$\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} X & Y \end{pmatrix}Q^t\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}Q\begin{pmatrix} X \\ Y \end{pmatrix}=\begin{pmatrix} X & Y \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}.$$

Thus, the given equation in the new coordinate system has the representation $X^2+3Y^2=1$ which is a ellipse. Note that Q is a 45 degree rotation matrix, and thus the coordinate change is just a rotation of the old coordinate system.





We can employ the above technique for linear partial differential equation.

Example 3.9. Let us rewrite the following equation in the canonical form

$$2u_{xx}+2u_{yy}-2u_{xy}+3u_x=u+x.$$

The coefficient matrix of the principal part is $A=\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, with the eigenvalues $\lambda_1=1$, and $\lambda_2=3$. The eigenvectors of the coefficient matrix A are $\vec{v}_1=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{v}_2=\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, and thus $Q=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ is the transformation matrix. Define the new coordinate system ∂_X,∂_Y through the following transformation

$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}=Q\begin{pmatrix} \partial_X \\ \partial_Y \end{pmatrix}. \tag{3.7}$$

By performing this transformation to the principal part of the equation, we reach

$$2u_{xx}+2u_{yy}-2u_{xy}=u_{XX}+3u_{YY}.$$

We need also to transform the first and zero terms of the equation as well. From (3.7), we have

$$\partial_x=\frac{\sqrt{2}}{2}\partial_X-\frac{\sqrt{2}}{2}\partial_Y,$$

and $x = \frac{\sqrt{2}}{2}X - \frac{\sqrt{2}}{2}Y$. Finally, the given PDE reduces to the following one in the new coordinate system

$$u_{XX} + 3u_{YY} + \frac{3\sqrt{2}}{2}u_X - \frac{3\sqrt{2}}{2}u_Y = u + \frac{\sqrt{2}}{2}X - \frac{\sqrt{2}}{2}Y.$$

Exercise 3.53. Write down the canonical form of the following equation

$$u_{xx} + u_{yy} + 4u_{xy} + \sqrt{2}\,u_y = u$$

Exercise 3.54. Find values of α such that the equation

$$u_{xx} + 3u_{yy} + 2\alpha\,u_{xy} + 2u_x - 3u_y = 0,$$

is elliptic, parabolic or hyperbolic. For $\alpha = 1$ write down the equation in the new coordinate and determine the associated linear transformation.

Exercise 3.55. Write the following equations in the canonical form and classify them into elliptic, parabolic and hyperbolic equations

a) $u_{xy} = 0$

b) $u_{xx} + 2u_{xy} = u$

c) $u_{xx} - u_{yy} - 2u_{xy} = u_x + u$

d) $3u_{xx} + 4u_{xy} + 6u_{yy} = u_x - u$

e) $u_{xx} - 4u_{xy} + 4u_{yy} - u_x = 1$

f) $-2u_{xx} - 6u_{xy} + 6u_{yy} - u_x = 0$

g) $u_{xx} + 4u_{xy} + u_{yy} + \sqrt{2}\,u_y = u$

Exercise 3.56. Consider the following equation

$$u_{tt} + u_{xx} + 4\,u_{tx} = \sqrt{2}x.$$

a) Determine the type of equation in terms of elliptic, parabolic or hyperbolic equation. Your answer should be based on a calculation.

b) Use a change of coordinate from (t, x) to (T, X) and rewrite the equation in the normal form without any multiplicative partial differentiation.

c) For the new equation in the coordinate (T, X), find a solution in the form $u(T, X) = \alpha X^a + \beta T^b$ for suitable constants α, a, β, b.

Exercise 3.57. Consider the following equation

$$2u_{xx} + 3u_{yy} + 2\sqrt{2}\,u_{xy} = 0$$

a) Determine the type of the equation in terms of ellipticity, parabolicity and hyberbolicity.

b) Write the canonical form of the equation with the aid of a transformation of the type

$$Q\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \begin{pmatrix} \partial_X \\ \partial_Y \end{pmatrix},$$

for an appropriate matrix Q.

c) In the new coordinate (X, Y), use the transformation $\xi = Y + i2X$, $\eta = Y - i2X$, and derive the following equation in (ξ, η)

$$u_{\xi\eta} = 0.$$

Find the general solution of the equation in (X, Y).

3.4.4 Classification: variable coefficients

The classification of linear second-order PDEs with variable coefficients is similar to that of equations with constant coefficients. However, for variable coefficient equations, the classification is local, meaning it holds only in an open neighborhood of the focal point, and not for the entire domain. The following example illustrates this fact.

Example 3.10. Consider the differential operator

$$L := (1 + x^2 y)\partial_{xx} - 2xy\partial_{xy} + y\partial_{yy}.$$

The coefficient matrix of L_p is

$$A(x, y) = \begin{pmatrix} 1 + x^2 y & -xy \\ -xy & y \end{pmatrix}.$$

Note that $\det(A) = y$. Thus, L is elliptic in the half plane $y > 0$ and hyperbolic in the half plane $y < 0$. For any point on the x-axis, the equation exhibits degenerate behavior.

Exercise 3.58. Consider the differential operator

$$L = x\,\partial_{xx} + x\,y\,\partial_{xy} + y\partial_{yy}.$$

Rewrite L in the form

$$L = \mathrm{div}\,([a_{ij}(x, y)]\,\nabla) + \text{first term derivatives}$$

where $[a_{ij}]$ is a symmetric matrix.

Problem 3.1. Find domains in which the following operators can be classified into elliptic, parabolic or hyperbolic type

i. $L := y\partial_{xx} - y\partial_{yy} + 2\,(x - 1)\partial_{xy}$

ii. $L := \cos(x\,y)\partial_{xx} + \cos(x\,y)\partial_{yy} + 2\sin(x\,y)\partial_{xy}$.

iii. $L := x\,\partial_{xx} + y\partial_{yy} + 2xy\partial_{xy}$

iv. $L := \sqrt{x^2 + y^2}\partial_{xx} + \sqrt{x^2 + y^2}\partial_{yy} + 2\,\partial_{xy}$.