# Chapter 2

# Linear First-Order PDEs

The general first-order partial differential equation (PDE) for a two-variable function, denoted as $u = u(x, y)$, can be expressed in the form:

$$F(x, y, u, u_x, u_y) = 0.$$

Here, $u_x$ and $u_y$ represent the partial derivatives of $u$ with respect to $x$ and $y$, respectively. The function $F$ establishes a functional relationship between the function $u$, its partial derivatives, and the independent variables $x$ and $y$. The general first-order partial differential equation for a function $u = u(x_1, \ldots, x_n)$ of $n$ independent variables, denoted as $x_1, \ldots, x_n$, can be represented as:

$$F(x, u, \nabla u) = 0,$$

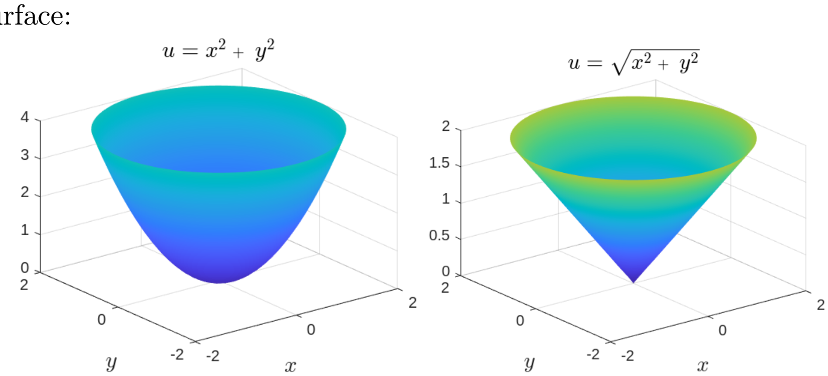
Here, $\nabla u$ is a vector denoted as $\nabla u = (\partial_1 u, \ldots, \partial_n u)$, which comprises the partial derivatives of $u$ with respect to each independent variable $x_1, \ldots, x_n$.

**Definition 2.1.** *A classical solution of the equation $F(x, u, \nabla u) = 0$, for $x \in \mathbb{R}^n$, $\nabla u = (\partial_1 u, ..., \partial_n u)$, is a smooth function $u = u(x)$ defined on an open set $\Omega \subset \mathbb{R}^n$ such that $F(x, u(x), \nabla u(x)) = 0$, is an identity for all $x \in \Omega$.*

For example, it is possible to verify that functions of the form $u = h(x^2 + y^2)$ for arbitrary smooth functions $h$ is the classical solution of the equation

$$y u_x - x u_y = 0.$$

For example, the function $u = x^2 + y^2$ is a classical solution to the equation for all $(x, y) \in \mathbb{R}^2$, while $u = \sqrt{x^2 + y^2}$ is a solution only on $\mathbb{R}^2 - \{(0, 0)\}$. The graph of these two solution shown below. Observe that the graph of a classical solution of a first-order PDE in two variable $x, y$ is a smooth surface:



## 2.1 Classification of first-order PDEs

In this book, we will exclusively focus on the study of first-order PDEs falling within the categories of linear, semi-linear, and quasi-linear. The subject of fully nonlinear equations will be introduced in a separate book dedicated to that topic.

**Linear equations.** The general form of a linear first-order PDE for a function $u=u(x)$, $x=(x_1,\ldots,x_n)$ is given by:

$$\sum_{j=1}^{n} v_j(x)\,\partial_j u(x) + v_0(x)\,u = r(x),$$

for some (usually) continuous functions $v_{\mathrm{j}}(x)$ and $r(x)$.

**Semi-linear equations.** A semi-linear equation is characterized by the general form:

$$\sum_{j=1}^{n} v_j(x)\,\partial_j u(x) = r(x,u),$$

The difference between a linear and semi-linear equation is that a semi-linear equation can be nonlinear with respect to $u$ (and not with the partial derivatives $u_x$ and $u_y$).

**Quasi-linear equations.** A quasi-linear equation assumes the general form:

$$\sum_{j=1}^{n} v_j(x,u)\,\partial_j u(x) = r(x,u),$$

The difference between a quasi-linear and semi-linear equation is that in the former case, the coefficients of partial derivatives are function of $u$ as well.

**Fully-nonlinear equations.** A fully nonlinear equation is an equation where one or all of the partial derivatives are nonlinear. For example, the equation:

$$|u_x|^2 + |u_y|^2 = 1,$$

is a fully nonlinear first-order equation for $u=u(x,y)$. We wont study this type of equation in this book.

**Exercise 2.1.** Classify the following first-order equations

a) $u_x + u_y = e^x\,u$

b) $x\,u_x + y\,u_y = e^u$

c) $u_x + (u^2)_x = 1 + u$

d) $u_x\,u_y + u\,u_z = 1$

**Exercise 2.2.** Consider the linear equation

$$u_x + u_y = -u.$$

Verify that every function of the forms $u = f(y-x)\,e^{-x}$, $u = f(y-x)\,e^{-y}$ satisfies the equation, where $f$ is a smooth arbitrary function.

**Exercise 2.3.** Consider the following quasi-linear equation

$$u_x + u\,u_y = 0.$$

a) Verify that every function of the implicit form $u = f(y - u\,x)$ solves the equation.

b) What is the explicit solution if we know $u$ satisfies the condition $u = 2y + 1$ along the $y$-axis?

**Exercise 2.4.** Verify that the function $u = 1 - \sqrt{x^2+y^2}$ is a solution to the fully nonlinear equation

$$|u_x|^2 + |u_y|^2 = 1.$$

What is the domain of $u$ if it considered as a classical solution? The solution satisfies the auxiliary condition $u=1$ on the unit circle $x^2+y^2=1$. Note that $u = \sqrt{x^2+y^2} - 1$ solves the PDE and the auxiliary condition too.

## 2.2 Characteristic method and ODEs along curves

The characteristic method is a powerful technique for solving first-order partial differential equations, and it is especially useful for semi-linear and quasi-linear equations. By using this method, one can derive the general solution to such equations. Furthermore, the characteristic method has a geometric interpretation that can be illustrated through the Cauchy problem.

### 2.2.1 Introductory remark: ODE along a curve

In our study of ordinary differential equations (ODEs), we explored equations of the form:

$$\frac{du}{dx} = f(x, u).$$

Here, $u = u(x)$ represents a single-variable function. Geometrically, we interpret the $x$-variable as the $x$-axis in the standard direction. The solution to this equation consists of a one-parameter family of functions $u = u(x; c)$, where $c \in \mathbb{R}$, such that for any $x$ within the domain of $u$, the following relation holds:

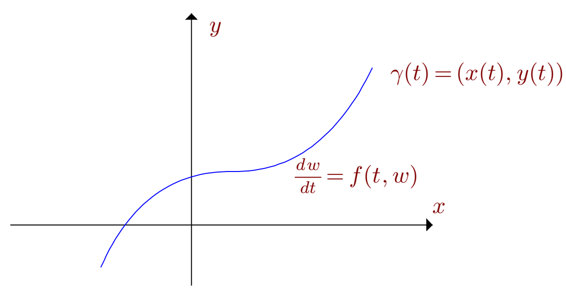
$$\frac{d}{dx} u(x; c) = f(x, u(x; c)).$$

Now, let's shift our focus to a parametric curve $\gamma(t)$ in the $xy$-plane. An ordinary differential equation (first-order) along $\gamma(t)$ takes the form:

$$\frac{du \circ \gamma}{dt} = f(t, u \circ \gamma).$$

Here, $u \circ \gamma$ is defined at any $t$ as $(u \circ \gamma)(t) = u(\gamma(t))$. If we denote $w$ as $u \circ \gamma$, we arrive at the equation:

$$\frac{dw}{dt} = f(t, w).$$

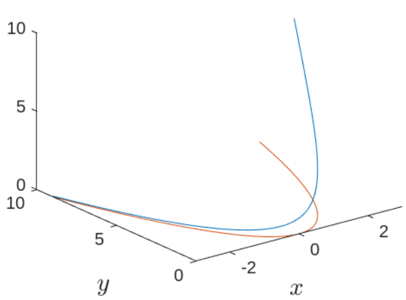



For instance, consider $\gamma(t)$ given by $\gamma = (t, t^2)$, a parabola in the $xy$-plane along which the differential equation:

$$\frac{dw}{dt} = w,$$

is defined. Suppose $u$ at the point $(1, 0)$ is $1$, corresponding to $\gamma(0)$. Then, we obtain $w$ as:

$$w(t) = u(\gamma(t)) = e^t.$$



**Remark 2.2.** Solving differential equations along curves can sometimes result in non-valid solutions. For instance, let's consider the circle $C$ represented by the parametric curve:

$$\gamma(t) = (\cos(t), \sin(t)),$$

for $t$ in the interval $[0, 2\pi)$, along with the initial value problem:

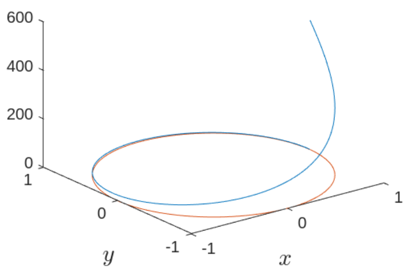
$$\frac{dw}{dt} = w, w(0) = 1,$$

defined for $w(t) = u(\gamma(t))$. The solution to this equation is $w(t) = e^t$ for $t \in [0, 2\pi)$. However, the function $u$ is not continuous on the circle because:

$$u(\gamma(0)) = u(1, 0) = 1,$$

and

$$u\left(\lim_{t \to 2\pi} \gamma(t)\right) = u(\gamma(0)) = 1 \neq \lim_{t \to 2\pi} u(\gamma(t))$$




**Exercise 2.5.** Consider the family of د $\{y = x^2 + c; c \in \mathbb{R}\}$ for parameter $c \in \mathbb{R}$. Consider the ordinary differential equation

$$\frac{du}{dx} = 0,$$

along all curves in this family. Show that $u$ on the $xy$-plane can be described by $u = h(y - x^2)$ where $h$ is an arbitrary smooth function.

### 2.2.2 A simple type of equations

Let's begin with the following simple equation:

$$u_x + v(x, y)\, u_y = 0, \tag{2.1}$$

where $u$ is a smooth two-variable function, $u = u(x, y)$. We'll relate the independent variable $y$ to $x$ through the equation:

$$\frac{dy}{dx} = v(x, y),$$

and assume that the solution to this equation is expressed as $y = Y(x; c)$, where $c$ is a parameter of the solution to this ordinary differential equation (ODE). This family of curves is known as the characteristic curves of the given partial differential equation (PDE). The reason is that along each curve $y = Y(x; c)$, the PDE reads as an ODE:

$$\frac{d}{dx} u(x, Y(x; c)) = 0. \tag{2.2}$$

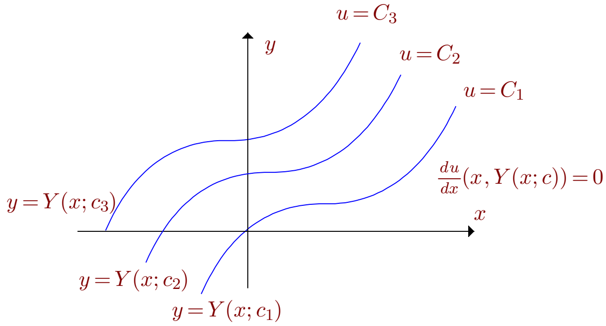
Note that, by the chain rule, we have:

$$\frac{d}{dx} u(x, Y(x; c)) = u_x(x, y) + \frac{dy}{dx} u_y(x, y) = u_x(x, y) + v(x, y)\, u_y(x, y).$$

Equation (2.2) can be simply solved for the constant function:

$$u(x, Y(x; c)) = C,$$

where $C$ is constant along the characteristic curve $y = Y(x; c)$ for a fixed $c$. Therefore, $C$ is a function of $c$, written as $C = h(c)$, where $h$ is an arbitrary function.




Let's assume that the equation $y = \boldsymbol{y}(x; c)$ can be solved for $c$ as $c = g(x, y)$. Then, we can express $u(x, y)$ as

$$u(x, y) = h(g(x, y)), \tag{2.3}$$

for an arbitrary smooth function h. Now, let's verify that the solution (2.3) satisfies equation (2.1):

$$\begin{cases} u_x = h'(g(x, y))\, g_x \\ u_y = h'(g(x, y))\, g_y \end{cases}.$$

This implies:

$$u_x + v(x, y)\, u_y = h'(g(x, y))(g_x + v(x, y)\, g_y)$$

Utilizing the relation $c = g(x, y)$, we have

$$0 = g_x\, dx + g_y\, dy,$$

which leads to:

$$g_x + \frac{dy}{dx} g_y = 0.$$

Substituting this into our previous equation:

$$(h(g(x, y)))_x + v(x, y)(h(g(x, y)))_y = 0.$$

This confirms that the solution (2.3) satisfies equation (2.1).

**Example 2.3.** Consider the partial differential equation:

$$u_x + u_y = 0.$$

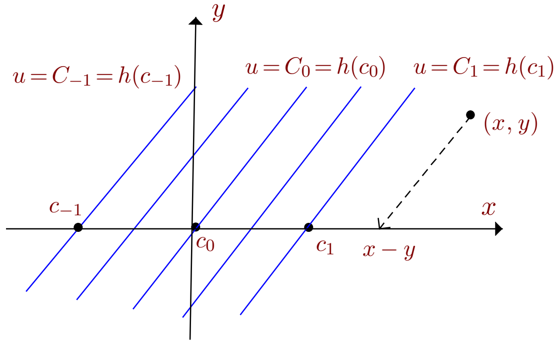
To apply the characteristic method, we begin by finding the characteristic equation:

$$\frac{dy}{dx} = 1.$$

This equation has the solution $y = x + c$, where $c$ is a parameter. Now, let's consider the characteristic family, denoted as $\{c = y - x; c \in \mathbb{R}\}$. Along each characteristic curve, $u$ remains constant due to the equation: $\frac{du}{dx} = 0$. Thus, we can express $u$ as $u = C$ along each line $c = y - x$, where $C$ depends on $c$. Consequently, we have:

$$u(x, y) = h(y - x).$$

Here, $h$ is an arbitrary smooth function. The characteristic curves in this case are straight lines with a slope of $1$.




**Remark 2.4.** In the example we solved earlier, we obtained the solution $u$ in terms of an arbitrary function $h(y-x)$. Consequently, any function of the form $u=\sin(y-x)$, $u=e^{-(y-x)^2}$, $u=(y-x)^3+x-y$, and so on, satisfies the given partial differential equation $u_x=u_y=0$. This type of solution is known as a general solution.

The concept of a general solution here is akin to the concept of the general solution for a first-order ordinary differential equation that typically contains a constant parameter rather than an arbitrary function. In subsequent discussions, we will explore how to determine the specific form of the arbitrary function $h$ with the help of auxiliary conditions for the problem.

**Exercise 2.6.** Find the general solution of the equation

$$u_x+xu_y=u.$$

### 2.2.3 Characteristic method for semi-linear PDEs

Let's consider the following equation:

$$v_1(x,y)u_x+v_2(x,y)u_y=v_3(x,y,u), \tag{2.4}$$

where $v_1$, $v_2$, and $v_3$ are smooth functions. The objective is to transform this partial differential equation into a set of first-order ordinary differential equations along characteristic curves.

Recall the differential of a two-variable function $u=u(x,y)$ as $du=u_x\,dx+u_y\,dy$. Comparing the expression of $du$ with equation (2.4) implies the following system

$$\frac{du}{v_3(x,y,u)}=\frac{dx}{v_1(x,y)}=\frac{dy}{v_2(x,y)}. \tag{2.5}$$

By relating $x$ to $y$ through the characteristic equation:

$$\frac{dy}{dx}=v(x,y), \tag{2.6}$$

where $v=\frac{v_2}{v_1}$, we obtain a family of curves $\gamma_c$: $c=g(x,y)$, where $c$ is an arbitrary constant. The given PDE reduces to the following equation along each $\gamma_c$:

$$\frac{du}{dx}=\frac{v_3}{v_2}. \tag{2.7}$$

Suppose this equation is solved for $u=U(x,c,C)$, where $C$ depends on $\gamma_c$ and thus can be expressed as $C=h(c)$ for an arbitrary smooth function $h$. Hence, the general solution can be expressed as:

$$u=U(x,g(x,y),h(g(x,y))).$$

**Example 2.5.** Let's solve the following partial differential equation:

$$x u_x + y u_y = x y.$$

The characteristic equation is given by:

$$\frac{dy}{dx} = \frac{y}{x},$$

which we can solve to obtain $y = cx$. The ordinary differential equation for $u$ becomes:

$$\frac{du}{dx} = y.$$

Substituting $y = cx$ into this equation, we have:

$$\frac{du}{dx} = cx.$$

This ordinary differential equation can be solved to find:

$$u = \frac{1}{2} c x^2 + C.$$

Here, $C$ can be expressed as an arbitrary smooth function in terms of $c$. Therefore, the general solution can be written as:

$$u(x, y) = \frac{x y}{2} + h\left(\frac{y}{x}\right).$$

**Theorem 2.6.** *The general solution of equation (2.4) is*

$$u = U(x, g(x, y), h(g(x, y))), \tag{2.8}$$

*where $h$ is an arbitrary smooth function, $c = g(x, y)$ is the equation of characteristic curves solution of the equation (2.6), and $U$ is the solution of the equation (2.7).*

**Proof.** We have

$$\begin{cases} u_x = U_x + U_y\, g_x + u_z\, h'(g)\, g_x \\ u_y = U_y\, g_y + u_z\, h'(g)\, g_y \end{cases}.$$

Multiplying the first equation by $v_1$ and the second one by $v_2$, we obtain

$$v_1 u_x + v_2 u_y = v_1 U_x + U_y (v_1 g_x + v_2\, g_y) + u_h\, h'(g)\, (v_1 g_x + v_2\, g_y).$$

By the equation $c = g(x, y)$, we have

$$0 = g_x\, dx + g_y\, dy,$$

and by the equality

$$\frac{dx}{v_1} = \frac{dy}{v_2},$$

we obtain:

$$v_1 g_x + v_2\, g_y = 0.$$

Hence, we obtain the equality:

$$v_1\, u_x + v_2\, u_y = v_1\, U_x.$$

On the other hand, from the equation $u = U(x, c, C)$, we have $du = U_x\, dx$. Utilizing the equation

$$\frac{du}{dx} = \frac{v_3}{v_1},$$

yields $v_1 U_x = v_3$, that proves the equation $v_1 u_x + v_2 u_y = v_3$. $\square$

**Definition 2.7.** *Given a fixed value of $c$ in the real numbers, the curve $\gamma_c$, which is the solution of equation (2.6), is referred to as a characteristic curve of the differential equation (2.4). Since equation (2.4) reduces to an ordinary differential equation when evaluated along $\gamma_c$ for any fixed $c$, we obtain an infinite system of ordinary differential equations for the family of characteristic curves $\{\gamma_c; c \in \mathbb{R}\}$. The system (2.5) is known as the characteristic system of the associated partial differential equation.*

**Example 2.8.** The existence of a general solution, even for linear first-order PDEs, is not always a trivial question. Consider the equation:

$$x u_x + y u_y = \alpha u,$$

where $\alpha$ is a constant. We will examine three cases: $\alpha = 0, -1$, and $1$.

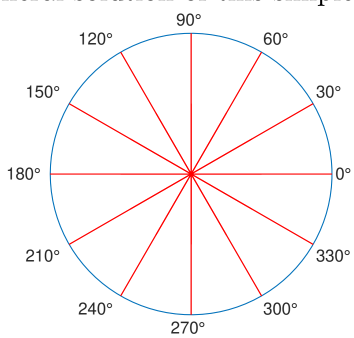
For $\alpha = 0$, the characteristic system is:

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{0},$$

and the equation for the characteristic curves in the $xy$-plane is:

$$y dx + x dy = 0.$$

The general solution of this simple ODE is $y = cx$, which is shown in the following figure:




Along the characteristic line $\gamma_c$, we have $\frac{du}{dx} = 0$, and thus $u$ is constant along $\gamma_c$. Furthermore, all characteristic lines intersect at the origin, so $\gamma_c$ carries the information of $u$ at the origin. This implies that $u(x, y) = u(0,0) = C$, a constant for all $(x, y)$. There is no other solution of the equation in this case.

Now, consider $\alpha = -1$. We will show that the only possible solution is $u \equiv 0$. In this case, the solution $u$ along $\gamma_c$ satisfies the ODE:

$$\frac{du}{dx} = -\frac{1}{x} u,$$

and therefore, $u = \frac{C}{x}$ along $\gamma_c$: $y = cx$ with respect to $x$. On the other hand, the PDE implies $u(0, 0) = 0$, and thus $C = 0$, implying that $u(x, y)$ is identically zero in this case.

For $\alpha = 1$, the solution u satisfies the ODE:

$$\frac{du}{dx} = \frac{1}{x} u,$$

with the solution $u = Cx$, for a constant $C$. The general solution in this case is:

$$u(x, y) = f\left(\frac{y}{x}\right) x.$$

The form of the solution imposes a restriction on the form of the function f if u is assumed to be smooth inside the unit disk. For example, for $f(z)=z^2$, the solution is not even continuous at the origin.

**Exercise 2.7.** Consider the following equation

$$2\sqrt{x}\,u_x+u_y=0.$$

a) Draw the characteristic curves of the equation.

b) Show that $u(4,3)=u(9,4)$.

c) Write down the general solution of the equation. Note that the solution is not differentiable with respect to $x$ at $x=0$.

**Exercise 2.8.** Consider the following equation

$$u_x+\sqrt{y}\,u_y=\alpha u,\, y>0$$

where $\alpha$ is a constant.

a) First assume $\alpha=0$. Draw the characteristic curve in the xy-plane and use the relation $\frac{du}{dx}=0$ to find $u(1,1)$. if $u=\frac{1}{1+x^2}$ on the $x$-axis.

b) Find the general solution of the equation for $\alpha=-1$. What is $u(1,1)$ if $u=\frac{1}{1+x^2}$ on the $x$-axis?

**Exercise 2.9.** Consider the equation

$$u_x+u_y=0.$$

Show that if $u$ is the solution of the equation, then it is impossible that $u=x$ on the unit circle $x^2+y^2=1$.

**Exercise 2.10.** Consider the following function

$$yu_x+xu_y=0.$$

a) What are the characteristic curves of the equation in the xy-plane?

b) Find the general solution of the equation and draw some integral surfaces of the equation.

**Exercise 2.11.** Find the general solution of the following equations

a) $xu_x+yu_y=xyu$.

b) $-yu_x+xu_y=2xyu$.

c) $u_x+u_y=x(y-x)\,u$

d) $xu_x+yu_y=x(y+u)$

**Exercise 2.12.** Consider the following equation

$$u_x+3x^2u_y=y^{\frac{1}{3}}a(x)+y^{\frac{2}{3}}b(x),$$

where $a,b$ are smooth functions. Show that the general solution of the equation is

$$u(x,y)=f(y-x^3)\,A(x)+g(y-x^3)\,B(x),$$

where $A(x),B(x)$ are respectively anti-derivatives of $x\,a(x)$ and $x^2\,b(x)$.

**Exercise 2.13.** Find the general solution of the following equation

$$u_x+u_y=u+xu^2.$$

**Exercise 2.14.** Consider the following equation

$$xu_x+yu_y=f(u).$$

If $f(0)\neq 0$, show that the equation can not have a smooth solution inside a unit disk.

**Exercise 2.15.** Consider the following equation

$$-yu_x+xu_y=u.$$

If $u(x,y)$ is a smooth solution in the closure of the unit ball in the xy-plane, show that $u=0$.

Hint: Since $u$ is smooth in the closure of the unit ball $B$, it has an absolute max and min in $\mathrm{cl}(B)$. If the max and min occurs inside $B$, then $u=0$. Let max and min occur on the boundary. Conclude the relation $-yu_x+xu_y=0$ on the boundary. Use this relation and conclude $u=0$.

**Exercise 2.16.** Consider the following equation

$$yu_x+u_y=u.$$

Take $y$ as the independent variable for the characteristic curves $x=x(y)$, and find the general solution of the equation.

### 2.2.4 Semi-linear equation in general dimension

Now, we generalize the method for semi-linear PDEs in general dimension. Consider the following equation:

$$\sum_{j=1}^{n} v_j(x)\,\partial_j u=r(x,u), x\in\mathbb{R}^n, \tag{2.9}$$

where $v_j(x)$ are smooth functions of $x=(x_1,\ldots,x_n)$. To solve this equation, we first consider the characteristic system:

$$\frac{dx_1}{v_1(x)}=\cdots=\frac{dx_n}{v_n(x)}=\frac{du}{r(x)}.$$

The characteristic equation in the space $(x_1,\ldots,x_n)$ is

$$\frac{dx_1}{v_1(x)}=\cdots=\frac{dx_n}{v_n(x)}.$$

By taking $x_1$ as the independent variable, we can write the system as:

$$\begin{cases}\frac{dx_2}{dx_1}=\frac{v_2(x)}{v_1(x)}\\ \vdots\\ \frac{dx_n}{dx_1}=\frac{v_n(x)}{v_1(x)}\end{cases}. \tag{2.10}$$

We can solve system (2.10) for a given parameter $c=(c_2,\ldots,c_n)$ as the implicit system:

$$c_2=g_2(x),\ \ldots, c_n=g_n(x).$$

We assume that the above implicit solutions are solvable for $x$ in terms of $x_1$ and $c$, so we can write:

$$x_2=x_2(x_1;c),\ \ldots, x_n=x_n(x_1;c).$$

The intersection of surfaces

$$c_2=g_2(x),\ \ldots, c_n=g_n(x),$$

reduces to a smooth curve $\gamma_c$, $c=(c_2,\ldots,c_n)$, which is the same as the parametric curves:

$$\gamma_c: \{x_2=x_2(x_1;c),\ \ldots, x_n=x_n(x_1;c)\}.$$

Next, we consider the derivative of u along the curve $\gamma_c$, which is given by:

$$\frac{du}{dx_1}=\frac{r(x,u)}{v_1(x)}. \tag{2.11}$$

By solving this equation, we obtain a function $u=U(x_1,c_2,\ldots,c_n,C)$. Therefore, taking $C=h(c)$, for an arbitrary smooth function $f$, the general solution to equation (2.9) can be expressed as:

$$u=U(x_1,g_2(x),\ldots,g_n(x),h(g_2(x),\ldots,g_n(x))).$$

**Example 2.9.** Let's solve the partial differential equation:

$$u_x + u_y + u_z = 0.$$

The characteristic equations in the space $(x, y, z)$ is given by:

$$\frac{dy}{dx} = 1, \frac{dz}{dx} = 1,$$

which is solved for $c_2 = y - x$, $c_3 = z - x$. The characteristic curves parameterized by $x$ is

$$\gamma_c = (x, x + c_2, x + c_3)$$

for $c = (c_2, c_3)$. The given PDE along $\gamma_c$ becomes: $\frac{du}{dx} = 0$, which implies $u = C$. Replacing $C$ by $h(c_1, c_2)$ for an arbitrary smooth function $h$, we obtain the general solution

$$u(x, y, z) = h(y - x, z - x).$$

**Exercise 2.17.** Consider the following equation

$$u_x - z u_y + y u_z = \alpha u,$$

where $\alpha$ is a constant.

a) Show that the characteristic curves of the equation have the following equations

$$\gamma_{\{c_1, c_2\}}(x) = (x, c_1 \cos(x) - c_2 \sin(x), c_1 \sin(x) + c_2 \cos(x)).$$

What is the shape of the characteristic curves?

b) The characteristic curves passes through the plane $x = 0$. Show that for $\alpha = 0$, we have

$$u(0, 1, 1) = u\left(\frac{\pi}{2}, -1, 1\right).$$

c) Show that the solution of the given PDE along $\gamma_{\{c_2, c_2\}}(x)$ is of the form

$$u(x; c_1, c_2) = h(c_1, c_2)\, e^{\alpha x},$$

for arbitrary function $h$. Find the general solution.

**Exercise 2.18.** Consider the following equation

$$u_x + 2\sqrt{y}\, u_y + 2\sqrt{z}\, u_z = 0,$$

where $y \geq 0$ and $z \geq 0$. Find the general solution of the equation. Verify the equality $u(2, 4, 4) = u(1, 1, 1)$.

**Exercise 2.19.** The singular equation

$$x u_x + y u_y + z u_z = 0.$$

If $u(x, y, z)$ is a smooth solution inside the unit ball in $\mathbb{R}^3$, show that $u(x, y, z) = C$ a constant for all $x, y, z$.

### 2.2.5 Characteristic method for quasi-linear equations

The method of characteristics for quasi-linear partial differential equations may lead to the emergence of new phenomena, such as shocks, which we will discuss later. Let's consider the following equation

$$v_1(x, y, u)\, u_x + v_2(x, y, u)\, u_y = v_3(x, y, u). \tag{2.12}$$

The coefficients of the quasi-linear equations depend on $u$. The characteristic system of the PDE is:

$$\frac{dx}{v_1} = \frac{dy}{v_2} = \frac{du}{v_3}. \tag{2.13}$$

Let $c_1 = \phi(x, y, u)$ and $c_2 = \psi(x, y, u)$ be implicit solutions of the characteristic system. We have the following theorem:

**Theorem 2.10.** *Let $v_1, v_2$, and $v_3$ be smooth functions in equation (2.12), and let $c_1 = \phi(x, y, u)$ and $c_2 = \psi(x, y, u)$ be implicit solutions of its associated characteristic system. Then, the general solution of the equation in implicit form is given by $f(\phi, \psi) = 0$ for any smooth function $f$ satisfying $f_u \neq 0$.*

**Example 2.11.** Let's consider the following equation called the Burger's equation:

$$u_x + u u_y = 0.$$

The characteristic system is given by:

$$\frac{dx}{1} = \frac{dy}{u} = \frac{du}{0},$$

By taking $x$ as the independent variable and setting $y = y(x)$, we obtain:

$$\frac{dy}{dx} = u, \frac{du}{dx} = 0.$$

This system is solved for $c_1 = u$ and $c_2 = y - x u$. The general implicit solution of the equation is derived by $f(c_1, c_2) = 0$, for arbitrary smooth function $f$, or equivalently $f(u, y - x u) = 0$. Alternatively, we can write the solution as $u = g(y - x u)$ for arbitrary smooth function $g$. Note that $u$ in both cases is in the implicit form.

**Example 2.12.** Consider the equation

$$u_x + y u u_y + z u u_z = 0.$$

To solve this equation, we need to determine the characteristic system, which is given by:

$$\frac{dy}{dx} = y u, \frac{dz}{dx} = z u, \frac{du}{dx} = 0.$$

Solving this system of differential equations, we obtain $e^{-x} y = c_1 u$, $e^{-x} z = c_2 u$, and $u = c$, where $c$, $c_1$ and $c_2$ are arbitrary constants. Substituting these values into the equation, we get the general implicit solution as

$$f(u, e^{-x} y u^{-1}, e^{-x} z u^{-1}) = 0.$$

Alternatively, the solution can be expressed as $u = g\left(\frac{y e^{-x}}{u}, \frac{z e^{-x}}{u}\right)$.

**Proof. (of the theorem)** Since $f(\phi(x, y, u(x, y)), \psi(x, y, u(x, y)))$ is identically zero, taking derivatives with respect to $x$ and $y$ gives:

$$\begin{cases} f_\phi(\phi_x + u_x \phi_u) + f_\psi(\psi_x + u_x \psi_u) = 0 \\ f_\phi(\phi_y + u_y \phi_y) + f_\psi(\psi_y + u_y \psi_u) = 0 \end{cases}.$$

Note that $f_u \neq 0$ implies that $f_\phi$ and $f_\psi$ cannot both be identically zero. This in turn implies that the following determinant is zero:

$$\begin{vmatrix} \phi_x + u_x \phi_u & \psi_x + u_x \psi_u \\ \phi_y + u_y \phi_y & \psi_y + u_y \psi_u \end{vmatrix} = 0.$$

Expanding the determinant and simplifying, we get:

$$(\phi_u \psi_y - \psi_u \phi_y) u_x + (\phi_x \psi_u - \phi_u \psi_x) u_y = \phi_y \psi_x - \phi_x \psi_y. \tag{2.14}$$

On the other hand, we have:

$$\begin{cases} d\phi = v_1\,\phi_x + v_2\,\phi_y + v_3\,\phi_u = 0 \\ d\psi = v_1\,\psi_x + v_2\,\psi_y + v_3\,\psi_u = 0 \end{cases},$$

that gives

$$\frac{v_1}{\phi_u\,\psi_y - \psi_u\,\phi_y} = \frac{v_2}{\phi_x\,\psi_u - \phi_u\,\psi_x} = \frac{v_3}{\phi_y\,\psi_x - \phi_x\,\psi_y}. \tag{2.15}$$

Matching relations (2.14) and (2.15) gives $v_1\,u_x + v_2\,u_y = v_3$, which completes the proof. □

**Exercise 2.20.** Find the general solution of the following equations

a) $u\,u_x + u_y = u^2$

b) $\sin u\,u_x + u_y = 1$ (hint: you can take $y$ as the independent variable just for the simplicity)

c) $x\,u_x + (1+u)\,u_y = u$

d) $u_x + u\,u_y - u\,u_z = 1$

**Exercise 2.21.** Consider the following equation

$$u_x + u\,u_y = 0.$$

a) Find the general solution in the implicit form. Then for the auxiliary condition $u(0,y) = y$, find an explicit solution. Determine the domain $(x,y)$ where the solution exists smoothly.

b) In general, the implicit solutions can not be transformed to an explicit form, although, the existence of such explicit forms is guaranteed by the *implicit function theorem*. Let $u(0,y) = e^y$. Obtain the implicit solution of the equation and use the theorem to show the existence of the explicit solution.

**Exercise 2.22.** Consider the following equation

$$\begin{cases} u_x + u\,u_y + u\,u_z = 0 \\ u(0,y,z) = f(y,z) \end{cases}.$$

a) Find the solution of the equation in the implicit form.

b) Find the explicit solution if $f(y,z) = y + z$.

**Exercise 2.23.** Consider the following quasi-linear equation

$$\partial_1 u + \sum_{j=2}^{n} v_j(x,u)\,\partial_j u = r(x,u), x \in \mathbb{R}^n$$

The characteristic systems is

$$\begin{cases} \dfrac{dx_j}{dx_1} = v_j, j = 2, \ldots, n \\ \dfrac{du}{dx_1} = r \end{cases}.$$

Let the system sis solved for

$$c = \psi(x,u), c_j = \phi_j(x,u), j = 2, \ldots, n.$$

Show that the function $f(\phi_j(x,u), \psi(x,u)) = 0$ solves the given partial differential equation.

## 2.3 Theoretical aspects

### 2.3.1 The geometrical interpretation of a first-order PDE

Consider the partial differential equation:

$$v_1(x,y,u)\,u_x + v_2(x,y,u)\,u_y = v_3(x,y,u). \tag{2.16}$$

Here, $v_1, v_2$, and $v_3$ are functions of $x, y$, and $u$. Let $z = u(x, y)$ is a solution to this equation. This function defines a surface on the space $(x, y, z)$:

$$S: (x, y, u(x, y)).$$

Recall that the vector $\vec{n}: (-u_x, -u_y, 1)$ is perpendicular on $S$ at any point $(x, y, z)$ on $S$. Now, assume that the vector field $\vec{V}: (x, y, z) \mapsto (v_1, v_2, v_3)$ is given on the space. The partial differential equation (2.16) then can be stated from the geometrical point of view as follow:

***Geometrical interpretation of a first-order PDE****: Given a vector field $\vec{V}: (x, y, z) \rightarrow (v_1, v_2, v_3)$, the equation of a tangent surface $z = u(x, y)$ to $\vec{V}$ is given by the partial differential equation*

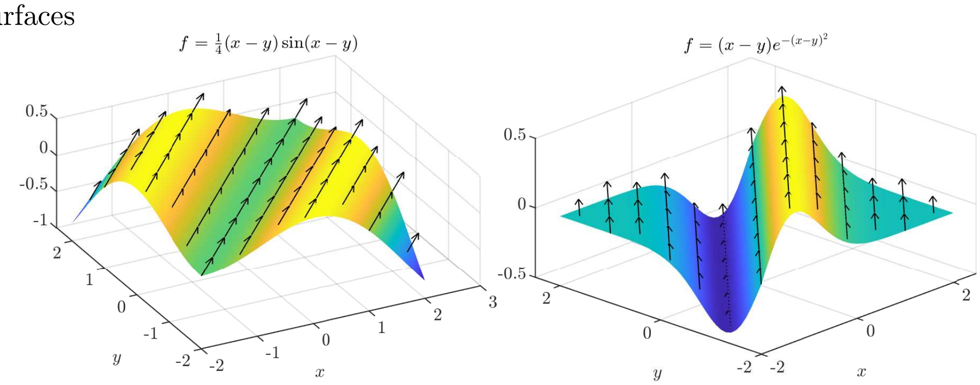
$$v_1(x, y, u)\, u_x + v_2(x, y, u)\, u_y = v_3(x, y, u).$$

*Conversely, the solution of the above partial differential equation defines a surface $z = u(x, y)$ that is locally tangent to the vector field $\vec{V}$.*

For example, consider the equation: $u_x + u_y = 0$. The vector field is

$$\vec{V}: (x, y, z) \rightarrow (1, 1, 0)$$

The figure below shows two different surfaces that are tangent to this vector field at all point of the surfaces




In fact, it can be seen that the all surface generated by the function $u = f(y - x)$ for arbitrary smooth $f$ has this tangent property as

$$\vec{V} \cdot \vec{n} = (1, 1, 0) \cdot (f'_x, -f'_y, 1) = 0.$$

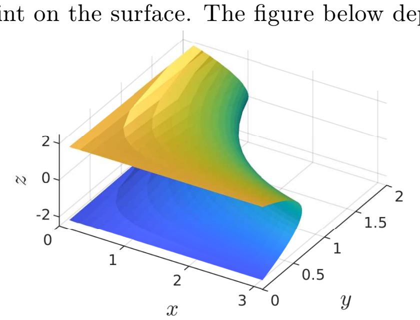
**Problem 2.1.** Let $V = (v_1(x, y, z), v_2(x, y, z), v_3(x, y, z))$ be a given vector field in space.

a) Show that the existence of a surface $\phi(x, y, z) = 0$ which is perpendicular to the vector fields at each point on the surface satisfies the following equation

$$v_1\, dx + v_2\, dy + v_3\, dz = 0.$$

b) The existence of the solution to the above ODE is not a trivial question. Show that if $\nabla \times V = 0$, then there is a surface $\phi(x, y, z) = 0$ that satisfies the above ODE. Hint: Note that if $\nabla \times V = 0$, then $V$ is potential.

c) As an example, consider the vector field $V = (y, x, z)$. Find a surface which is perpendicular to $V$ at each point on the surface. The figure below depicts one of such surfaces.



**Problem 2.2.** Consider the vector field $V = (-y, x, 0)$. This field does not satisfy the condition $\nabla \times V = 0$, however, a surface still exists which is perpendicular to $V$. Show that the surface expressed parametrically as $\Sigma(x, z) = (x, \alpha x, -xz)$ has such a property.

**Problem 2.3.** Show that there is no surface $\phi(x, y, z) = 0$ with the normal vectors $(x, y, xy)$.

## 2.3.2 Parametric solution surfaces

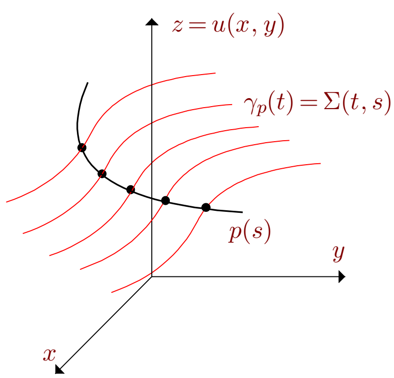
Now, consider a curve $\gamma(t)$ on the solution surface (yet unknown) $S$. This curve is tangent to the vector field $\vec{V}$ at any point on the curve. Therefore, the equation of $\gamma(t)$ is determined by the following ordinary differential equation

$$\begin{cases} \frac{dx}{dt} = v_1(x(t), y(t), z(t)) \\ \frac{dy}{dt} = v_2(x(t), y(t), z(t)) \\ \frac{dz}{dt} = v_3(x(t), y(t), z(t)) \end{cases}. \tag{2.17}$$

This first-order system of ordinary differential equation has a unique solution if an initial condition is set for the system. Let's assume that we know a point $p_0$ on $S$. We can set the initial condition as:

$$\gamma(0) = p_0 \colon \begin{cases} x(0) = x_0 \\ y(0) = y_0 \\ z(0) = z_0 \end{cases}.$$

With this initial condition, a curve $\gamma_{p_0}(t)$ is obtained on $S$. In this way, to determine $S$, we need to a family of curves $\{\gamma_p(t)\}$ where $p$ lies on a curve on $S$ as shown below:




For example, assume the curve $p(s) = (s, 0, s\sin(s))$ lies on $S$ of the solution of the equation $u_x + u_y = 0$. The solution surface is spanned by the curves determined by the following system of ODEs

$$\begin{cases} \frac{dx}{dt} = 1 \\ \frac{dy}{dt} = 1 \\ \frac{dz}{dt} = 0 \end{cases},$$

accompanied with the conditions $x(0) = s, y(0) = 0, z(0) = s\sin(s)$. This system is solved as

$$x(t, s) = t + s, y(t, s) = t, z(t, s) = s\sin(s).$$

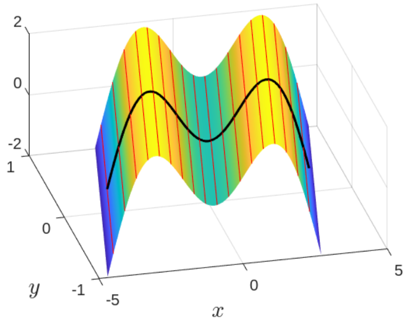
In this way, we obtain a parametric surface

$$\Sigma(t,s)=(t+s,t,s\sin(s)).$$

It is simply seen that this parametric surface is algebraically represented as

$$u(x,y)=(y-x)\sin(y-x).$$

The figure below depicts the "data line" $p(s)$ in black and the space characteristic curves $\gamma_p(t)$ in red.




**Remark.** The advantage of using parametric form for representing solutions is that it can represent very complicated surfaces, whereas explicit functions $z=u(x,y)$ represent only restricted classes of surfaces. The example below further clarifies this point.

**Example 2.13.** Consider the following problem

$$\begin{cases} 0.2x\,u_x-u\,u_y=y \\ u|_{\{x=y^2\}}=y \end{cases}.$$

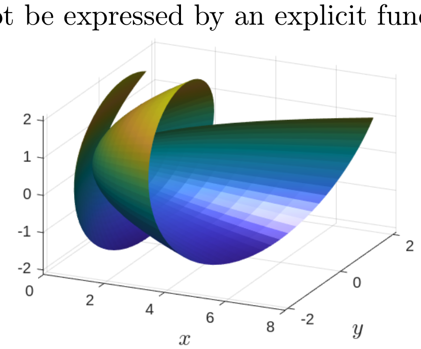
Here $u$ is given along the curve $x=y^2$ in the xy-plane. This data can be parameterized in terms of $s$ as: $p(s)=(s^2,s,s)$. The system of characteristic equations is:

$$\begin{cases} \frac{dx}{dt}=0.2x \\ \frac{dy}{dt}=-u \\ \frac{dz}{dt}=y \end{cases},$$

with the initial conditions $x_0=s^2$, $y_0=s$ and $z_0=s$. The parametric representation of the integral surface is obtained as:

$$\Sigma(t,s)=(s^2e^{0.2t},s(\cos t-\sin t),s(\sin t+\cos t)).$$

As shown in figure below, the solution of the equation represents a complicated surface which can not be expressed by an explicit function $z=u(x,y)$.



Even though this surface is not associated with an explicit function, it is possible to define an explicit function that locally coincides with this surface. Thus, classical solutions of partial differential equations can only be defined locally in a neighborhood of the data curve.

**Exercise 2.24.** Use the parametrization and find the solution of the following Cauchy problems. Use Matlab and draw the integral surface

a) $u u_x + y u_y = x$, $u = \sin(x)$ on $y = 1$.

b) $(y - u)\, u_x + (u - x)\, u_y = u$, $u = x^2$ on $y = 0$.

The following code generates the integral surfaces of the given equation in part b).

```
s = linspace(0, 2, 100); % define the range of s
t = linspace(0, 10, 100); % define the range of t
[S, T] = meshgrid(s, t); % create a grid of s and t values
X = S .* cos(T); % compute the x-coordinates
Y = S .* sin(T); % compute the y-coordinates
Z = S .^ 2 .* exp(0.2 * T);% compute the z-coordinates
surf(X, Y, Z) % plot the surface
shading interp; camlight
xlabel('x');
ylabel('y');
zlabel('z');
title('Surface plot of x = s*cos(t), y = s*sin(t), z = s^2*exp(0.2*t)');
```

### 2.3.3 Cauchy problem

In this section, we demonstrate how to obtain the particular solution of a given first-order PDE from the general solution, using an auxiliary condition or additional information about the solution. This process is similar to deriving particular solutions from the general solutions of ODEs, using initial conditions.

**Definition 2.14.** *A problem of the form*

$$\begin{cases} v_1(x,y,u)\, u_x + v_2(x,y,u)\, u_y = v_3(x,y,u) \\ u|_C = g \end{cases} \tag{2.18}$$

*where $C$ is a curve in an open set $\Omega \subset \mathbb{R}^2$ in the* xy*-plane, is called a Cauchy problem.*

Let's consider the following problem:

$$\begin{cases} u_x + u_y = 0 \\ u|_{\{y=0\}} = \frac{1}{1+2x^2} \end{cases}.$$

The general solution to the equation is $u = h(y - x)$ for an arbitrary smooth function $h$. Utilizing the initial condition $u(x,0) = \frac{1}{1+2x^2}$ is

$$\frac{1}{1+2x^2} = h(-x),$$

and thus $h(y - x) = \frac{1}{1+2(y-x)^2}$, and thus the particular solution to the given Cauchy problem is given by:

$$u(x,y) = \frac{1}{1+2(y-x)^2}.$$

**Exercise 2.25.** Find the particular solution of the following Cauchy problem

$$\begin{cases} u_x + (x+y)u_y = u \\ u|_{x=0} = \sin(y) \end{cases}.$$

**Exercise 2.26.** Find the particular solution of the following Cauchy problem

$$\begin{cases} u_x + u_y = 2x(y-x)u \\ u|_{x=0} = e^y \end{cases}.$$

**Exercise 2.27.** The Cauchy problem can be generalized to higher dimensions. Find the solution of the following equation for $u = u(x, y, z)$:

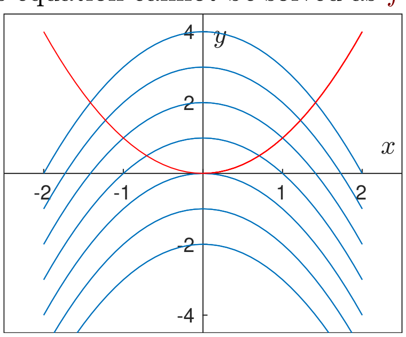
$$\begin{cases} x u_x + y u_y + z u_z = 0 \\ u|_{\{z=1\}} = xy \end{cases}$$

Examples below highlight some of the issues that can arise when attempting to extract a particular solution from a general solution.

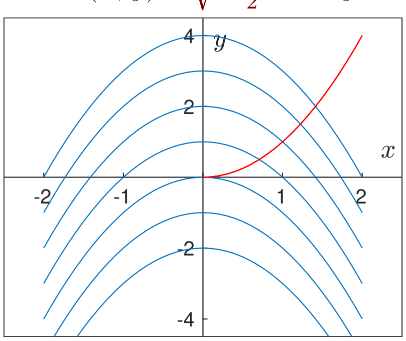
**Example 2.15.** Consider the Cauchy problem:

$$\begin{cases} u_x - 2x\, u_y = 0 \\ u|_{\{y=x^2\}} = x \end{cases}.$$

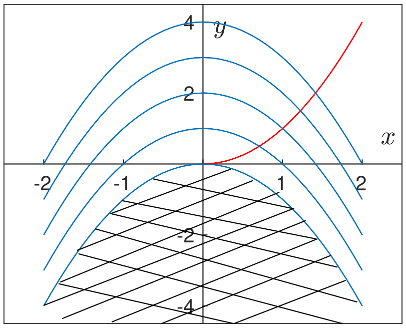
The general solution of the PDE is $u = f(y + x^2)$ for an arbitrary smooth function $f$. We can use the auxiliary condition $u = x$ on the line $C: y = x^2$ to determine $f$, which gives us $x = f(2x^2)$. However, this equation cannot be solved as $f(2)$ can have two possible values, $\pm 1$, leading to a contradiction.



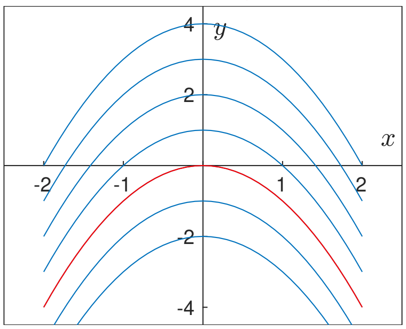

This problem occurs because the characteristic curves of the PDE, $y = -x^2 + c$, intersect the data curve $C: y = x^2$ at more than one point, resulting in multiple possible values of $u$. To resolve this issue, we can consider only one branch of the data curve, such as $y = x^2, x \geq 0$, to obtain a unique solution $u(x, y) = \sqrt{\frac{y + x^2}{2}}$ for $y \geq -x^2$.




However, this solution is still not unique in the region $y < -x^2$, where the characteristic curves do not intersect the data curve. The value of $u$ along these curves can be chosen arbitrarily.




If we change the data curve to $y=-x^2$, which is also a characteristic curve, there is no unique solution in the region $y>-x^2$ and $y<-x^2$. In this case, $x=f(0)$, which is not solvable from the general solution $u=f(y+x^2)$ and the data curve $y=-x^2$.




**Example 2.16.** Let us consider the following equation:

$$\begin{cases} -yu_x+xu_y=u \\ u|_C=x \end{cases},$$

where $C$ is the $x$-axis for $x\geq 0$. To apply the method of characteristics, we first need to find the equation of the characteristic curves. Using the characteristic equation

$$\frac{dx}{-y}=\frac{dy}{x},$$

we get $x^2+y^2=c,$ where $c$ is a positive constant. Therefore, the characteristic curves are circles centered at the origin. In one of previous exercise, we asked the read to show that this equation does not have any smooth solution inside a disk. We can see this fact by trying to solve this equation explicitly. Taking $x$ as the independent variable, we can rewrite the PDE as

$$\frac{du}{dx}=\frac{-1}{y}u.$$

To solve this equation, we need to express $y$ as a function of $x$. However, this cannot be done through the obtained implicit function $x^2+y^2=c$. One way to overcome this difficulty is to use the parametric representation of the characteristic curves. Let $t$ be a parameter and consider the characteristic system:

$$\begin{cases} \frac{dx}{dt}=-y \\ \frac{dy}{dt}=x \end{cases}.$$

The solution to this system is

$$\gamma_p(t)=(s\cos(t),s\sin(t)),$$

where $s$ is a non-negative parameter. Note that we used the data curve $C$ to write the initial point of the characteristic curve as $\gamma_s(0)=(s,0)$. Now, we can express $u$ in terms of $t$ as follows:

$$\frac{du}{dt}=u,$$

which is a separable ODE with solution

$$u(\gamma_s(t))=u(\gamma_s(0))\,e^t=u(s)\,e^t=s\,e^t.$$

To determine the domain of $t$, note that $\gamma_s(0)=\gamma_s(2\pi)$, which implies

$$u(\gamma_s(2\pi))=u(\gamma_s(0))=s.$$

However, we also have

$$u(\gamma_s(2\pi))=s\,e^{2\pi},$$

so we conclude that the domain of $t$ can not be $[0,2\pi]$. Note that for $x=s\cos t$, $y=s\sin t$ and $u=s\,e^t$, we obtain the integral surface

$$\Sigma(t,s)=(s\cos t, s\sin t, s\,e^t).$$

This solution can be put in the algebraic form as

$$u(x,y)=\sqrt{x^2+y^2}\,e^{\operatorname{atan}\left(\frac{y}{x}\right)}.$$

**Exercise 2.28.** Find the particular solution of the following Cauchy problem

$$\begin{cases} u_x+(x+y)u_y=u \\ u|_{x=0}=\sin(y) \end{cases}.$$

**Exercise 2.29.** Find the particular solution of the following Cauchy problem

$$\begin{cases} u_x+u_y=2x(y-x)u \\ u|_{x=0}=e^y \end{cases}.$$

**Exercise 2.30.** The Cauchy problem can be generalized to higher dimensions. Find the solution of the following equation for $u=u(x,y,z)$:

$$\begin{cases} x u_x+y u_y+z u_z=0 \\ u|_{\{z=1\}}=xy \end{cases}$$

**Exercise 2.31.** Consider the following Cauchy problem

$$\begin{cases} -y u_x+\omega_0^2 u_y=\alpha u \\ u|_C=x \end{cases},$$

where $C$ is the line $y=x$ for $x\geq 0$, and $\omega_0$ is a constant. Use parametric representation of characteristic curves in terms of the parameter $t$ and find the solution for $\alpha=0$. If $\alpha\neq 0$, find the domain of $t$ and indicate the domain of the solution in the xy-plane.

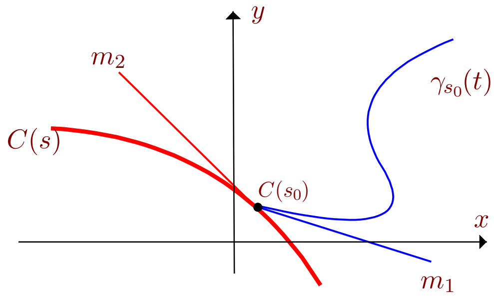
**Exercise 2.32.** Consider the following Cauchy problem

$$\begin{cases} (-y-\alpha x)\,u_x+(x-\alpha y)\,u_y=u \\ u|_C=f(x) \end{cases},$$

where $C$ is the $x$-axis for $x\geq 0$, and $\alpha>0$. Show that the problem is solvable in classical sense only if $f(x)=0$.

### 2.3.4 Well-posedness and existence of integral surfaces

As we observed in previous examples, if the data curve of a Cauchy problem is not a characteristic curve, then there is a solution that can be extended locally. The following figure shows this situation schematically.




Here, the data curve $C$ is parametrized by $s$ as $C = C(s)$, and the planar characteristic curves $\gamma$ are parameterized by $t$. Note that $\gamma_s(t)$ is a characteristic curve passing through $C(s)$ at $t = 0$. Let $s_0$ be a point on $C(s)$ in its domain. If $\gamma'_{s_0}(0)$ and $C'(s_0)$ are non-parallel, then there is a $t_0 > 0$ such that $\gamma_{s_0}(t)$ exists for $0 \le t < t_0$.

**Theorem 2.17.** *Consider the Cauchy problem*

$$\begin{cases} v_1(x,y)\,u_x + v_2(x,y)\,u_y = v_3(x,y,u) \\ u|_C = f \end{cases}$$

*where $v_1, v_2$, and $v_3$ are smooth functions, and $C$ is a smooth curve in the* xy*-plane. Assume there exists $(x_0, y_0) \in C$ such that*

$$C'(x_0, y_0) \nparallel (v_1(x_0, y_0), v_2(x_0, y_0)). \tag{2.19}$$

*Then there exists an open neighborhood $\Omega$ of $(x_0, y_0)$ and a smooth function $u = u(x, y)$ on $\Omega$ that solves the given Cauchy problem.*

The proof of the theorem is based on a standard theorem on the existence and uniqueness of the solution to ordinary differential equations. Note that if condition (2.19) holds, then due to the continuity of $v_1, v_2$, and $C'$ at $(x_0, y_0)$, the condition holds for an open neighborhood of $(x_0, y_0)$. Then, the existence of a domain $\Omega$ for the solution $u(x, y)$ is reduced to the existence and uniqueness of the ordinary differential equation

$$\frac{du}{dt} = v_3(x(t), y(t), u).$$

However, it is important to note that the theorem only provides a sufficient condition for the existence of a solution, and there may be cases where the condition is not satisfied but a solution still exists. For instance, consider the problem:

$$\begin{cases} u_x + \sqrt{y}\,u_y = 0 \\ u(x,0) = f(x) \end{cases}$$

This problem has the solution $u(x, y) = f(x - 2\sqrt{y})$ which is defined for $y \ge 0$, even though it is not generally differentiable on the $x$-axis.

The existence of a parametric surface for a quasi-linear first-order PDE is established in a similar manner. We have the following theorem.

**Theorem 2.18.** *Consider the Cauchy problem*

$$\begin{cases} v_1(x,y,u)\,u_x + v_2(x,y,u)\,u_y = v_3(x,y,u) \\ u|_C = f \end{cases},$$

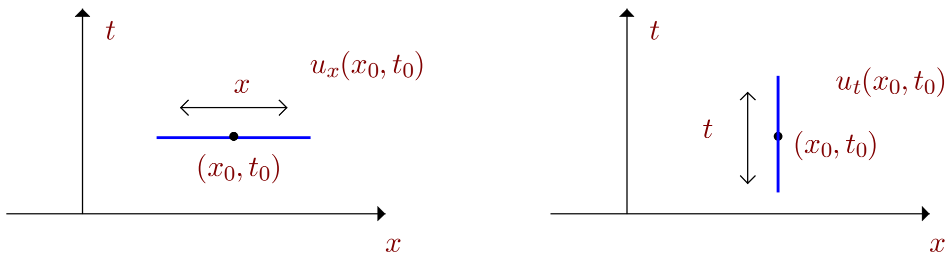
*where $v_1, v_2, v_3$ are smooth functions and $C$ is a smooth and non-characteristic curve in* xy*-plane. Let $\Gamma(s)$ be the parametrized space data curve $(C(s), f(s))$. Fix $s_0$ in the domain of $\Gamma(s)$ and let $(x_0, y_0, u_0) = \Gamma(s_0)$. If*

$$\Gamma'(s_0) \times V(x_0, y_0, u_0) \neq 0,$$

*where $V = (v_1, v_2, v_3)$, then there exist $\alpha, \beta > 0$ such that the given problem has an integral surface $\Sigma(t, s)$ for $t \in (-\beta, \beta)$ and $s \in (s_0 - \alpha, s_0 + \alpha)$.*

## 2.4 Time dependent functions and fluid flow

Consider a fluid distributed along the $x$-axis at time $t$, with $u(x,t)$ representing its density function. Let $x_0$ be a fixed point on the $x$-axis. The rate of change in $u(x_0, t)$ as time progresses is given by the partial derivative $u_t(x_0, t)$. Meanwhile, $u_x(x, t_0)$ measures the rate of difference in density between $x$ and its neighboring points.




Now, consider a fluid with density function $u(x,t)$ flowing with velocity function $V(x)$. This velocity function can be thought of as a one-dimensional vector field: $x \to V(x)$. The properties of this field affect the density function $u$. For example, suppose we have a control volume $[-1, 1]$ moving with velocity $V(x) = x$. The total mass within this volume at time $t = 0$ is given by the integral

$$M = \int_{-1}^{1} u(x, 0)\, dx.$$

What can be said about the mass of this packet at time $t = 1$? Let $x_0$ be a fixed point on the $x$-axis. According to the velocity field, the position of $x_0$ at time $t$ can be determined by the differential equation $\frac{dx}{dt} = V$. For $V(x) = x$, we obtain $x(t) = x_0 e^t$. Therefore, the control volume $[-1, 1]$ expands to the volume $[-e, e]$ at time $t = 1$. Assuming there is no sink or source of mass production, the total mass is conserved, and thus

$$M = \int_{-e}^{e} u(x, 1)\, dx.$$

Since the volume of the control volume has increased, the density $u(x, 1)$ must have decreased to maintain the same mass. The main objective of this section is to address how the density function $u$ of a $n$-dimensional fluid flow changes over time.

**Exercise 2.33.** Consider the following density distribution

$$u(x,t) = e^{-\frac{(x-t)^2}{2}}.$$

It is simply seen that the total mass along the $x$-axis for any $t$ is equal to

$$\int_{-\infty}^{\infty} u(x,t)\, dx = \sqrt{2\pi},$$

and thus we can write that

$$\lim_{t\to\infty}\int_{-\infty}^{\infty} u(x,t)\,dx=\sqrt{2\pi}.$$

Find the *steady state* distribution when $t\to\infty$ and calculate

$$\int_{-\infty}^{\infty}\lim_{t\to\infty} u(x,t)\,dx.$$

This shows that the following equality does not hold always

$$\lim_{t\to\infty}\int_{-\infty}^{\infty} u(x,t)\,dx=\int_{-\infty}^{\infty}\lim_{t\to\infty} u(x,t)\,dx.$$

### 2.4.1 Change of density along flow lines

Consider a velocity field $V=(v_1(x),\ldots,v_n(x))$ of a fluid flow, where $x=(x_1,\ldots,x_n)$. Let $x_0$ be a fixed point in $\mathbb{R}^n$, and let $\gamma_{x_0}(t)$ be the flow line of $x_0$, determined by the system of ODEs

$$\frac{dx_j}{dt}=v_j(x),\ j=1,\ldots,n,$$

with initial condition $x(0)=x_0$. Our goal is to find the rate of change of the density function $u$ along the flow line $\gamma_{x_0}(t)$. This rate of change is given by the time derivative:

$$\frac{du}{dt}(\gamma_{x_0},t)=u_t(\gamma_{x_0},t)+\nabla u(\gamma_{x_0},t)\cdot\frac{d\gamma_{x_0}}{dt},$$

where $\nabla u(t,x)$ represents the gradient of $u$. This can be expressed in coordinate form as:

$$\frac{du}{dt}(x,t)=u_t(x,t)+\sum_{j=1}^{n} v_j(x)\,\partial_j u(x,t).$$

The derived formula provides us with the change of $u$ along the flow lines. Therefore, the question arises: can we determine $u(x,t)$ if we know the initial density $u(x,0)$?

**Exercise 2.34.** Verify that the above formula is a straightforward application of the chain rule.

**Example 2.19.** Consider a fluid flow in the $\mathrm{xy}$-plane, where the density at $t=0$ is given by $u(x,y,0)=f(x,y)=x^2e^{-x^2-y^2}$, and the velocity field $V$ is

$$(x,y)\mapsto(-\alpha x-y,x-\alpha y),$$

with $\alpha\geq 0$ being a constant. The trajectories of the particles are determined by the following system of differential equations:

$$\begin{cases}\frac{dx}{dt}=-\alpha x-y\\ \frac{dy}{dt}=x-\alpha y\end{cases}.$$

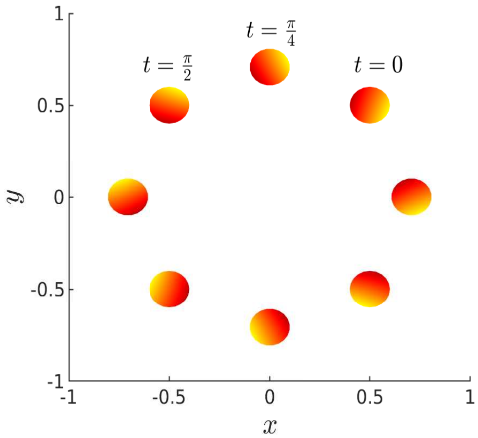
For the first case, let $\alpha=0$, that leads to the trajectory

$$\gamma_{(x_0,y_0)}(t)=(x_0\cos(t)-y_0\sin(t),x_0\sin(t)+y_0\cos(t)),$$

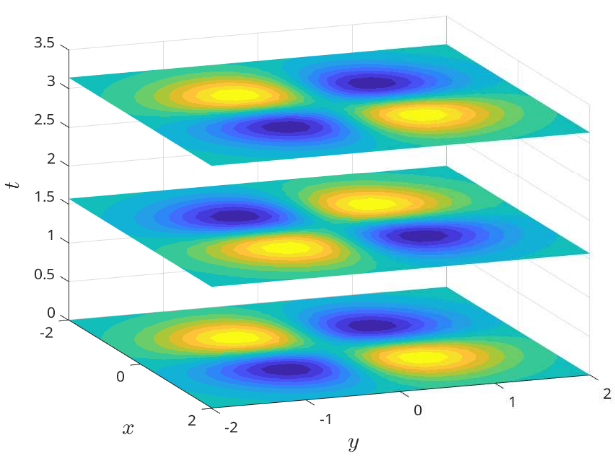
where $(x_0,y_0)$ is the initial position at $t=0$. In matrix form, the flow line passing through $(x_0,y_0)$ is given by:

$$\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}=\begin{pmatrix}\cos(t)&-\sin(t)\\ \sin(t)&\cos(t)\end{pmatrix}\begin{pmatrix}x_0\\ y_0\end{pmatrix}.$$

Note that the coefficient matrix is the rotation matrix $R_t$. Consider a control volume $D$ centered at $(0.5, 0.5)$. The following graph illustrates $u(x(t), y(t), t)$ for $t = 0 : \frac{\pi}{4} : 2\pi$. Observe how the control volume centered at $(0.5, 0.5)$ rotates around the origin while also rotating around itself. The velocity field is responsible for this rotation, as it is a rotational field with a non-zero curl, denoted as $\nabla \times V = 2$.



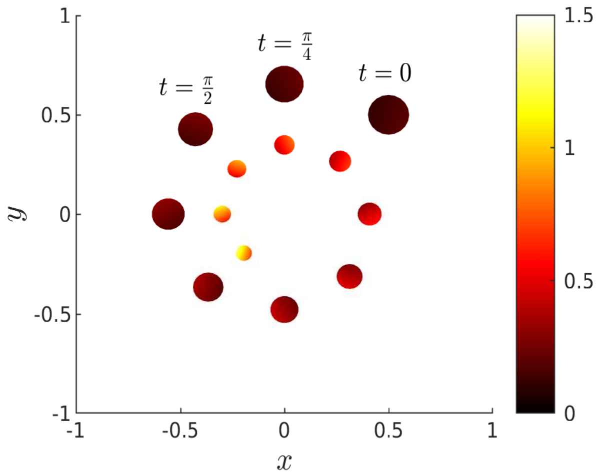

The following figure shows the density at different time slices, specifically for times $t = 0, \frac{\pi}{2}$, and $\pi$. As shown in the figure, the density rotates in a circular pattern around the origin as it is clear from the differential equations of the trajectory $\frac{dx}{dt} = -y, \frac{dy}{dt} = x$. Furthermore, we observe that the density along the trajectory of the fluid particles remains constant, which means the density remains constant along the trajectory. This type of fluid flow, where the density remains constant along the trajectory of the fluid particles, is called an incompressible flow.




Now, let $\alpha = 0.1$. The trajectory of the control volume is an inward spiral towards the origin, given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-0.1t} \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

The following graph illustrates this scenario, showing how the control volume centered at $(0.5, 0.5)$ forms an inward spiral while also rotating around itself. Observe how the control volume gradually becomes smaller and by the conservation law, the density in $D$ increases to balance the decrease in the volume.




The above example illustrate how the velocity filed $V(x)$ affects the density of particles along the flow lines. The following theorem make a relationship between the divergence $\nabla\cdot V(x)$ and the change of density along the flow line.

**Theorem 2.20.** *Consider a control volume $D$ in a fluid flow that moves according to the velocity field $V$. If the divergence of $V$, denoted by $\nabla\cdot V(\gamma_D(t))$, is positive, then the volume of $D$ increases along the trajectory $\gamma_D(t)$. On the other hand, if $\nabla\cdot V(\gamma_D(t))$ is negative, the volume of $D$ decreases along the trajectory, and if $\nabla\cdot V(\gamma_D(t))$ is zero, the volume of $D$ remains constant.*

**Corollary 2.21.** *If $\nabla\cdot V=0$, the density of a fluid-flow remains constant along its trajectory in the absence of any material sink or source.*

In the first scenario of the example, the velocity field $V=(-y,x)$ had a divergence of $\nabla\cdot V=0$, which implies that the volume of the control volume remained constant along $\gamma_D(t)$. As a result, the density $u$ remained unchanged along the flow lines. In the second scenario, the velocity field was $V=(-y-\alpha x, x-\alpha y)$, with a divergence of $\nabla\cdot V=-2\alpha<0$ for $\alpha>0$. This indicates that the volume of the control volume decreased with time, leading to an increase in the density $u$ to balance the volume decrease.

### 2.4.2 Continuity equation

Given a region $\Omega$ in a fluid flow, the total mass of the fluid in $\Omega$ at time $t$, denoted by $M_\Omega(t)$, changes due to two factors. The first factor is the net flow rate of fluid passing through the boundary $\mathrm{bnd}(\Omega)$ per unit time. This can be expressed mathematically as the surface integral

$$\Phi(t):=-\oiint_{\mathrm{bnd}(\Omega)} u(x,t)V(x)\cdot\nu dV.$$

where $V$ is the velocity vector of the fluid and $\nu$ is the unit outward normal vector to the boundary. The second factor is the presence of a source or sink of material in the fluid, which can be represented mathematically as the volume integral

$$Q(t)=\iiint_\Omega f(x,t)\,dV,$$

where $f(x,t)$ is the rate of material added or removed per unit volume at point $x$ at time $t$. Therefore, we can write the rate of change of $M_\Omega(t)$ as:

$$\frac{dM_\Omega(t)}{dt} = -\oiint_{\mathrm{bnd}(\Omega)} u(x,t)V(x)\cdot\nu dV + \iiint_\Omega f(x,t)\,dV.$$

This equation is known as the mass conservation equation, or the continuity equation, and it plays a fundamental role in fluid dynamics.

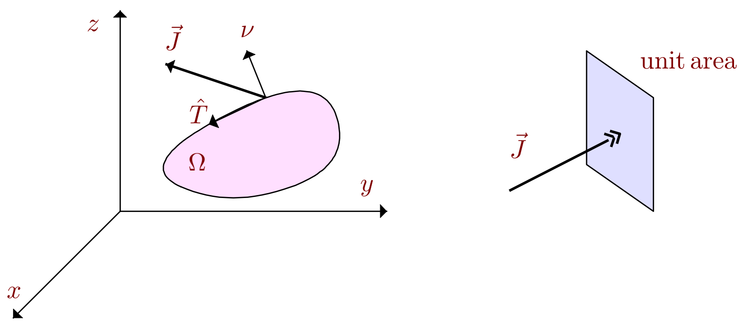
On the other hand, the left hand side of the above equation is

$$\frac{dM_\Omega(t)}{dt} = \frac{d}{dt}\iiint_\Omega \rho(x,t)\,dV = \iiint_\Omega \rho_t(x,t)\,dV,$$

and by equating the two expressions for $\frac{dM_\Omega}{dt}$, we arrive at the continuity equation for the fluid flow in $\Omega$:

$$\iiint_\Omega \rho_t(x,t)\,dV = -\oiint_{\mathrm{bnd}(\Omega)} u(x,t)V(x)\cdot\nu dS + \iiint_\Omega f(x,t)\,dV.$$

The material flux $J(x,t)=u(x,t)V(x)$ measures the rate at which fluid passes through a unit area per unit time. When considering the flux at a particular point $x$ on the boundary $\mathrm{bnd}(\Omega)$ of the bounded, open set $\Omega$ in $\mathbb{R}^3$, the unit normal vector $\nu(x)$ is needed to determine the flux through the surface. This is because the flux $J(x,t)$ can be decomposed into two components: 1) a component in the direction of the unit normal vector $\nu(x)$, and 2) a tangential component that does not leave the closure of $\Omega$, denoted by $\bar{\Omega}$. Only the component of $J(x,t)$ that is perpendicular to the boundary, i.e., $J(x,t)\cdot\nu(x)$, contributes to the flow leaving $\Omega$ through $\mathrm{bnd}(\Omega)$. The figure below illustrates this concept.




The differential form of the continuity equation can be derived by applying Gauss's theorem to the divergence of the material flux density. Specifically, the theorem yields:

$$\iiint_\Omega u_t(x,t)\,dV = -\iiint_\Omega \nabla\cdot[u(x,t)V(x)]dV + \iiint_\Omega f(x,t)\,dV,$$

which simplifies to:

$$\iiint_\Omega \{u_t(x,t) + \nabla\cdot[u(x,t)\,V(x)] - f(x,t)\}\,dV = 0.$$

Since the integral holds for every arbitrary subdomain of $\Omega$, we conclude that the integrand must be identically zero, yielding the continuity equation:

$$u_t(x,t) + \nabla\cdot[u(x,t)\,V(x)] = f(x,t). \tag{2.20}$$

This equation expresses the conservation of mass, where $f(x,t)$ represents the source or sink of material in the fluid.

**Exercise 2.35.** Show the relation

$$\operatorname{div}[u(x,t)\,V(x)] = \nabla u \cdot V(x) + u(x,t)\operatorname{div}[V(x)],$$

and conclude that the conservation equation can be written as follows

$$u_t + V \cdot \nabla u = -u \operatorname{div}[V].$$

Assuming $f$ is identically zero in equation (2.20), we can rewrite the equation using the result from the previous exercise as follows:

$$u_t + V \cdot \nabla u = -u \nabla \cdot V. \tag{2.21}$$

Let $\gamma_z(t)$ be the trajectory of a particle at $z$ which is determined by the equation $\frac{dz}{dt} = V(z)$. In this case, the left hand side of the above equation reads $\frac{du}{dt}(t, \gamma_x(t))$, and then we have

$$\frac{du}{dt}(\gamma_z(t), t) = -u \nabla \cdot V(\gamma_z(t)).$$

If $\nabla \cdot V(\gamma_z(t)) = 0$, then $\frac{du}{dt}(\gamma_z(t), t)$ and this implies that the density of $x$ remains the same at its trajectory, an therefore, equation (2.21) leads to the solution

$$u(\gamma_z(t), t) = u(\gamma_z(0), 0),$$

and by the fact $\gamma_z(0) = z$, we obtain

$$u(\gamma_z(t), t) = u(z, 0).$$

If we denote $\gamma_z(t)$ by $x$ and then $z = \gamma_x(-t)$, we derive the solution

$$u(x,t) = u(\gamma_x(-t), t).$$

**Definition 2.22.** *A flow moving along the velocity filed $V(x)$ is called incompressible if $\operatorname{div}(V)$.*

**Exercise 2.36.** We aim to show that the total mass within a moving control volume $D$, defined by the trajectory $\gamma_D(t)$, remains constant when there are no external sources or sinks. To begin, we express the total mass $M_{D(t)}$ as the integral of the fluid density $u(x,t)$ over the volume $D(t)$ at time $t$:

$$M_{D(t)} = \iiint_{D(t)} u(x,t)\,dx.$$

Taking the time derivative of the above equation, we get:

$$\frac{dM_{D(t)}}{dt} = \frac{d}{dt}\iiint_{D(t)} u(x,t)\,dx.$$

Using the Leibniz integral rule, we can write:

$$\frac{d}{dt}\iiint_{D(t)} u(x,t)\,dx = \iiint_{D(t)} u_t(x,t)\,dx + \oiint_{\operatorname{bnd}(D(t))} u V(x) \cdot \nu\, dS.$$

Use the divergence theorem (Gauss theorem) and show

$$\frac{d}{dt}\iiint_{D(t)} u(x,t)\,dx = 0,$$

and conclude that $M_{D(t)}$ remains constant along $\gamma_D(t)$.

## 2.5 Linear and semi-linear transport equations

The general form of a semi-linear transport equation in $\mathbb{R}^n$ is given by

$$u_t + V(x) \cdot \nabla u = f(x,t),$$

where $V(x)$ is the velocity field. However, in some cases, the velocity field may also depend on the density $u$ in addition to $x$, leading to the quasi-linear equation:

$$u_t + V(x,u)\cdot\nabla u = f(x,t).$$

In general, the independent variable for transport equations is time $t$. For semi-linear equations, the characteristic system is given by

$$\frac{dx}{dt} = V(x),$$

while for quasi-linear equations, the characteristic system becomes:

$$\begin{cases} \frac{dx}{dt} = V(x,u) \\ \frac{du}{dt} = f(x,t) \end{cases}.$$

**Example 2.23.** The simplest form of the transport equation is:

$$\begin{cases} u_t + c\,u_x = 0 \\ u(x,0) = u_0(x) \end{cases}.$$

This equation is also known as the unilateral wave equation due to the directionality of the wave propagation. Since $V(x) = c$ is constant, the flow is incompressible, and the trajectory $x(t) = ct + x_0$ yields the ordinary differential equation $\frac{du}{dt} = 0$, which has the solution:

$$u(x(t),t) = u(x_0,0) = u_0(x_0) = u_0(x - ct).$$

This solution corresponds to a traveling wave propagating to the right (if $c>0$) or to the left (if $c<0$).

The solution to the transport equation in higher dimensions is similar. Consider the equation:

$$\begin{cases} u_t + C\cdot\nabla u = 0 \\ u(x,0) = u_0(x) \end{cases},$$

where $C = (c_1, ..., c_n)$ and $x = (x_1, \ldots, x_n)$. The flow lines are:

$$x_j(t) = c_i\,t + x_0^j,$$

where $j = 1, ..., n$ and $x_0^j = x_j(0)$. Thus, we have $u(t, x(t)) = f(x_0)$, and for $x_0 = x - Ct$, the solution is:

$$u(x,t) = u_0(x_1 - c_1 t, \ldots, x_n - c_n t).$$

**Example 2.24.** Consider the following equation

$$\begin{cases} u_t + c x u_x = \alpha u \\ u(x,0) = \frac{1}{1+x^2} \end{cases},$$

Here, $c$ is a constant. When $\alpha = -c$, the equation is conservative, and can be written as:

$$u_t + \nabla\cdot[c x u] = 0.$$

Let's first solve the equation for $\alpha = -c$, where $c > 0$. We can derive the flow lines by solving:

$$\frac{dx}{dt} = cx,$$

which gives us $\gamma_{x_0}$: $x(t) = x_0\, e^{ct}$. The PDE along $\gamma_{x_0}$ can then be expressed as:

$$\frac{du}{dt} = -cu,$$

which can be solved to give:

$$u(x(t), t) = u(x_0, 0)\, e^{-ct} = \frac{e^{-ct}}{1 + x_0^2}.$$

Replacing $x_0$ by $x\, e^{-ct}$, we obtain

$$u(x, t) = \frac{e^{-ct}}{1 + x^2\, e^{-2ct}}.$$

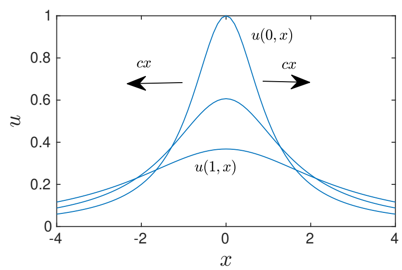
As expected, the total mass at $t = 0$ is conserved for $t > 0$. In fact, we have:

$$\int_{-\infty}^{\infty} u(x, t)\, dx = \int_{-\infty}^{\infty} \frac{e^{-ct}}{1 + x^2\, e^{-2ct}}\, dx \overset{z = xe^{-ct}}{=\!=\!=\!=} \int_{-\infty}^{\infty} \frac{dz}{1 + z^2} = \pi.$$

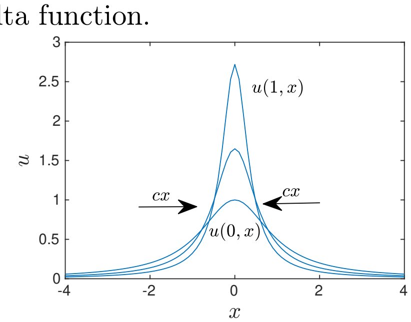
This is the same as the total mass at $t = 0$. Therefore, we can write:

$$\int_{-\infty}^{\infty} u(x, t)\, dx = \int_{-\infty}^{\infty} u(x, 0)\, dx.$$

Note that since $V(x) = cx$, the particles move with velocity proportional to $x$. Consequently, the initial profile becomes fatter in time as the total mass remains the same.




For negative constant $c$, $c < 0$, the solution has the same structure. However, $u(x, t)$ becomes more concentrated around $x = 0$ in this case. It is seen that for $t \to \infty$, $u(x, t)$ approaches the Dirac delta function.




If $\alpha = 0$, the equation is no longer conservative, and the solution is:

$$u(x, t) = \frac{1}{1 + x^2\, e^{-2ct}}.$$

This this case, we have

$$\int_{-\infty}^{\infty} u(x, t)\, dx = \int_{-\infty}^{\infty} \frac{dx}{1 + x^2\, e^{-2ct}} = \pi\, e^{ct} > \int_{-\infty}^{\infty} u(x, 0)\, dx.$$

**Exercise 2.37.** Consider the following equation

$$u_t + e^x\, u_x = 0,$$

for $t \geq 0$. Draw the characteristic curves and find the general solution of the equation.

**Exercise 2.38.** Consider the following equation

$$\begin{cases} u_t + x u_x - y u_y = \alpha u \\ u(x, y, 0) = \begin{cases} 1 & x^2 + y^2 \leq 1 \\ 0 & \text{otherwise} \end{cases} \end{cases}.$$

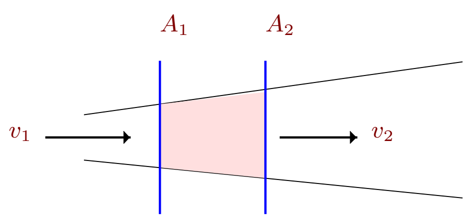
a) For what value of $\alpha$ the equation is conservative?

b) For this $\alpha$, find $u(x, y, t)$.

c) Generalize the above results in $\mathbb{R}^n$

$$\begin{cases} u_t + \sum_{j=1}^{2n} (-1)^j\, \partial_j u = \alpha u \\ u(x, y, 0) = \begin{cases} 1 & \sum_{j=1}^{2n} x_j^2 \leq 1 \\ 0 & \text{otherwise} \end{cases} \end{cases}.$$

**Problem 2.4.** In the flow of a *incompressible fluid* in a long pipeline, the velocity depends to the area intersection; see the following.




Use the conservation of mass principle, and show that

$$\frac{v_1}{v_2} = \frac{A_2}{A_1}.$$

**Problem 2.5.** Suppose the density function of a fluid is given by $u(t, x, y) = e^{-t}(x^2 + y^2)$. Find the total mass in the unit ball $B = \{(x, y); x^2 + y^2 < 1\}$. What is the rate of change of the total mass in $B$ at time $t = 1$?

**Problem 2.6.** Suppose the initial density function for a 1D flow is $u_0(x) = 1 + x$. If the velocity field for the flow is $v = t + x$, we know that $u(x, t) = e^{-2t}(1 + t + x)$ for $t > 0$. Verify the conservation law for this flow in an arbitrary segment $[x_1, x_2]$.

**Problem 2.7.** For a 2D flow, suppose $V = (x, y)$. If the initial density function is $u_0(x, y) = x^2 + y^2$, then the density function at $t > 0$ is $u(t, x, y) = (x^2 + y^2)\, e^{-4t}$. Verify directly the conservation law for this flow.

**Problem 2.8.** Assume that the initial density of a matter is given by $u_0(x, y) = e^{-(x^2+y^2)}\,\mathrm{kg}/m^2$. If the mass moves with the constant speed $V = 1 m/s$ in the direction of $x$-axis, find the total mass in the unite disk $(x-1)^2 + y^2 < 1$ the time $t = 1$.

**Problem 2.9.** Verify that the function $u(x, y, t) = xy$ is a solution to the problem

$$\begin{cases} u_t + x u_x - y u_y = 0 \\ u(x, y, 0) = xy \end{cases}.$$

Verify the conservation law

$$\frac{d}{dt}\int_B u(x, y, t)\, dS = \int_{\mathrm{bnd}(B)} \vec{J} \cdot \nu\, d\ell,$$

where $B$ is the unit ball in $\mathbb{R}^2$.

**Problem 2.10.** Write down the continuity equation for function $u$ for the velocities given below, and determine if they are incompressible

a) $V = -x$

b) $V = u$

c) $V=(y,x)$

d) $V=(1,0,u)$

**Problem 2.11.** Consider the velocity field $V=(-y,x)$ in $\mathbb{R}^2$. If the initial density is $u_0(x,y)=\frac{1}{1+x^2+y^2}$, find the density of the point $(-3,0)$ at time $t=\pi$ without solving the transport equation. Find the total mass in the unit disk at time $t=1$.

**Problem 2.12.** Suppose the initial density function of a 2D fluid is given by $u_0(x,y)=e^{-x^2-y^2}$. If the fluid moves with the velocity $V=(-y,10x-2y)$, find the parametric equations of a particle initially located at $(x_0,y_0)=(0,3)$. Find the density of the point $(x,y)=(0,3e^{-\pi/3})$ at time $t=\frac{\pi}{3}$.

**Problem 2.13.** Suppose the velocity filed of a 2D fluid is given by $V=(1+x^2,1)$. Find the flow line passing through the origin and show that the density is decreasing along this line.

**Problem 2.14.** Write the continuity equation for a fluid moving with the velocity $V=(xy,y-x)$. Is the flow incompressible? Repeat the problem for the velocity filed $V=(\alpha x+y,e^x-y)$ and find $\alpha$ such that the flow is incompressible.

**Problem 2.15.** Assume that the density function of a fluid (in absence of any source term) is $u(x,y,t)=e^{-t}(x^2+y^2)$. If we know the divergence of the velocity field $V=(v_1,v_2)$ is equal to $1$, show the relation $xv_1+yv_2=0$.

**Problem 2.16.** Assume that the initial density of a matter is given by $u_0(x)=\frac{2}{\pi(1+x^2)}$ kg/m. If the mass moves with constant velocity $V=1m/s$ along $x$-axis, find the total mass in the segment $[0,2]$ at $t=1$.

**Problem 2.17.** Assume that the velocity of some fluid is given by $V=x$. Write down the continuity equation for the flow and obtain the density at $t=1$ if the initial density is $u_0(x)=\frac{e^{-2}}{1+x^2}$.

**Problem 2.18.** Determine the characteristic line along which the solution to the problem

$$\begin{cases} u_t+xu_x=0 \\ u(x,0)=e^{-|x|} \end{cases},$$

is $u=e^{-1}$.

**Problem 2.19.** Draw the characteristic lines of the equation

$$u_t-xu_x=0.$$

If the initial data is $u(x,0)=\tan^{-1}x+\frac{\pi}{2}$, find $u(x,t)$.

**Problem 2.20.** Consider the equation

$$u_t+v(x)u_x=0,$$

where $v(x)$ is the following function

$$v(x)=\begin{cases} -x+1 & 0<x\le 1 \\ -x-1 & -1<x<0 \\ 0 & |x|\ge 1 \end{cases}.$$

If the initial data $f$ is the function

$$f(x)=\begin{cases} |x| & |x|\le 1 \\ 1 & |x|\ge 1 \end{cases}.$$

**Problem 2.21.** Solve the following linear problem

$$\begin{cases} u_t+(1+x^2)\,u_x=t \\ u(x,0)=x \end{cases}.$$

**Problem 2.22.** Find the solution to the following

$$\begin{cases} u_t+xu_x=u \\ u(x,0)=\sin(x) \end{cases}.$$

**Problem 2.23.** Solve the following equation

$$\begin{cases} u_t+xu_x=xe^{-t}u \\ u(x,0)=\sin(x) \end{cases}.$$

**Problem 2.24.** Consider the equation

$$3t^{2/3}u_t + u_x = 0,$$

for the function $u = u(x,t)$.

a) Write the equation of the characteristic line passing through the point $t=1, x=2$.

b) Find the value $u(1,2) - u(4,3)$ (you should provide the reason for the value)

c) Find $u(1,2)$ if $u(x,0) = e^{-x^2}$

**Problem 2.25.** Consider the following semi-linear problem

$$\begin{cases} u_t + u_x = 2te^{-u} \\ u(x,0) = \ln(1+x^2) \end{cases}.$$

Find region in the plane $(x,t)$ where $u < 1$.

**Problem 2.26.** Assume that the velocity field of a matter is give as $V = (-y, x)$. Find the density of the point $(1,0)$ at time $t=1$ if the plan $(x,y)$ is initially filled with a matter of density

i. $u_0 = \sqrt{x^2+y^2}$,

ii. $u_0 = xy$

**Problem 2.27.** Solve the following problem

$$\begin{cases} u_t + u_x + yu_y = xy \\ u(x,y,0) = \sin(xy) \end{cases}.$$

**Problem 2.28.** Solve the following problem

$$\begin{cases} u_t + u_x + u_y = u \\ u(x,y,0) = u_0(x,y) \end{cases}.$$

**Problem 2.29.** Consider the equation

$$u_t + yu_x - xu_y = 0.$$

a) Show that the equations of characteristic curves are

$$\begin{cases} x = x_0\cos(t) + y_0\sin(t) \\ y = -x_0\sin(t) + y_0\cos(t) \end{cases}.$$

b) Find the solution if $u(x,y,0) = xy$.

c) Now solve the following equation

$$\begin{cases} u_t + yu_x - xu_y = u \\ u(x,y,0) = xy \end{cases}.$$

## 2.6 Quasi-linear equations and shockwave

In general, $V$ in the continuity equation $u_t + \nabla \cdot [uV] = 0$ can be a function of $x$ and $u$, such that $V = V(x,u)$. One example of this is in modeling traffic flow on highways, where the velocity of cars depends inversely on the density of cars using the highway, described by the function

$$V(u) = V_{\max}\left(1 - \frac{u}{u_\infty}\right),$$

where $u_\infty$ is the density of cars at maximum capacity, $V_{\max}$ is the maximum speed of cars, and $0 \le u \le u_\infty$ represents the density of traffic on the highway. In this case, the continuity equation becomes

$$u_t + V_{\max}\left(1 - \frac{2u}{u_\infty}\right)u_x = 0. \tag{2.22}$$

Another important equation with $V$ as a function of $u$ is the Burgers' equation, where $V=\frac{1}{2}u$, giving the PDE

$$u_t+\left(\frac{1}{2}u^2\right)_x=0,$$

or equivalently, $u_t+u\,u_x=0$.

More generally, we can consider problems of the form:

$$\begin{cases} u_t+\operatorname{div}[g(u)]=0 \\ u(x,0)=u_0(x) \end{cases},$$

which can be written equivalently as:

$$\begin{cases} u_t+g'(u)\,u_x=0 \\ u(x,0)=u_0(x) \end{cases}.$$

The characteristic system for this problem is given by:

$$\begin{cases} \frac{dx}{dt}=g'(u) \\ \frac{du}{dt}=0 \end{cases}.$$

which leads to the following implicit solution: $u=u_0(x-g(u)t)$.

**Example 2.25.** Let's consider the continuity equation problem given by

$$u_t+u\,u_x=0,$$

with the initial condition $u(0,x)=x$. Although the initial condition $u(0,x)$ for $x<0$ is not physically meaningful as it leads to negative density, we retain this example to highlight an important feature of the continuity equation. By solving the characteristic system of equations

$$\frac{dx}{dt}=u,\frac{du}{dt}=0,$$

we obtain the solution $x=ut+x_0$ and $u(t,x(t))=u(0,x(0))=x_0$, where $x_0$ is replaced by $x-ut$. This yields the solution

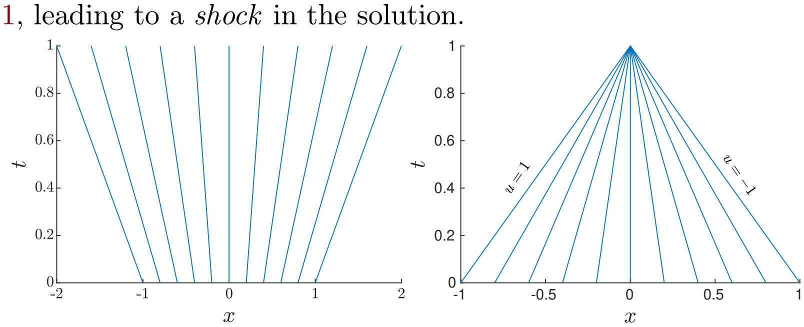
$$u(x,t)=\frac{x}{1+t},$$

which has a domain of $[0,\infty)$ for $t>0$.

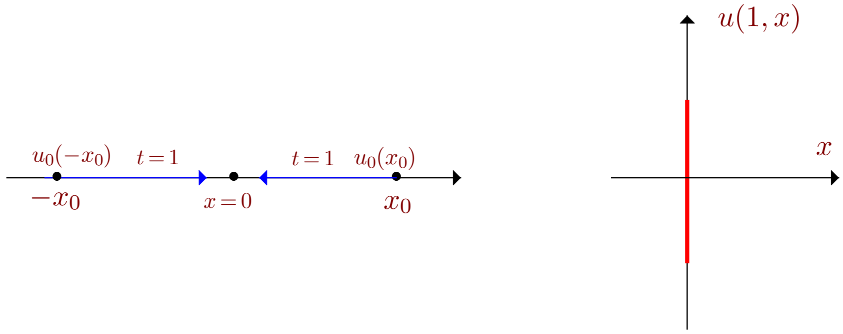
Suppose we now modify the initial condition to $-x$. The resulting solution is

$$u(x,t)=\frac{x}{t-1},$$

which has a domain of $[0,1)$. As shown in the figure, the characteristic lines in this case collide at $t=1$, leading to a *shock* in the solution.



The value of $u(1,0)$ can take any value in the range $(-\infty,\infty)$, as particles with different densities converge at $x=0$ at $t=1$ and the density becomes discontinuous at this point. The figure below shows this phenomena schematically. This concentration changes the initial density of $x=0$ to a discontinuous value.




**Exercise 2.39.** Solve the following quasi-linear equation and determine the domain of the solution

$$\begin{cases} u_t + u u_x = 0 \\ u(x,0) = |x| \end{cases}.$$

**Remark 2.26.** What can we say about the solution $u(x,t)$ for $t>1$? As we have seen, the solution is discontinuous at $t=1$, which means that the differential equation fails to hold for $t\geq 1$ as well. However, this does not mean that we cannot study the propagation of the shock beyond $t=1$ from a physical point of view. In fact, the development of a shock is a physical phenomenon that can be observed and studied experimentally.

One possible mathematical way to extend the discontinuous or shock solutions beyond the shock time $t=1$ is by using the concept of shockwave solutions. In this approach, we assume that the discontinuous solution propagates like a wave in time. Shockwave solutions are a common way of extending the solutions of partial differential equations beyond the point of discontinuity.

### 2.6.1 Riemann problem

In this section, we consider the initial value problem given by the scalar conservation law:

$$\begin{cases} u_t + u u_x = 0 \\ u(x,0) = u_0(x) := \begin{cases} 1 & x<0 \\ 1-x & 0<x<1 \\ 0 & x>1 \end{cases} \end{cases}. \tag{2.23}$$

To solve this problem, we first determine the characteristic system of equations:

$$\begin{cases} \frac{dx}{dt} = u \\ \frac{du}{dt} = 0 \end{cases},$$

which leads to the characteristic lines $x(t) = ut + x_0$. Along each characteristic line, the solution $u$ remains constant. Thus, we can write:

$$u(t,x(t)) = u_0(x_0),$$

where $x_0$ is the initial position of the characteristic line. Using the initial condition $u_0(x)$, we can find the value of $u$ along each characteristic line.

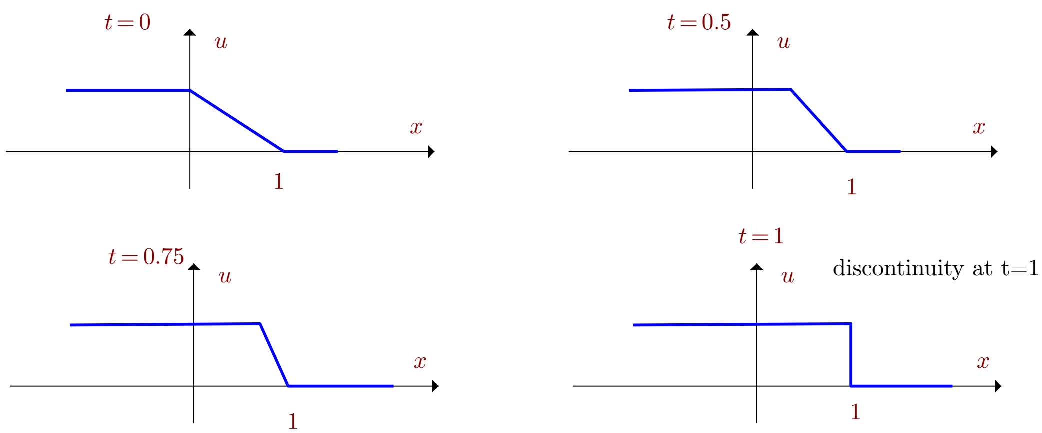
Next, we express $x_0$ in terms of $x$ and $t$ as $x_0 = x - ut$. Substituting this into the expression for $u(t, x(t))$, we obtain the solution $u$ as a function of $x, u$ and $t$:

$$u(x,t) = \begin{cases} 1 & x - ut < 0 \\ 1 - (x - ut) & 0 < x - ut < 1 \\ 0 & x - ut > 1 \end{cases}.$$

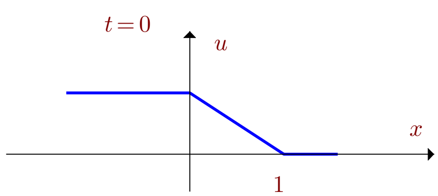
Simplifying the expression algebraically, we get:

$$u(x,t) = \begin{cases} 1 & x < t \\ \frac{1-x}{1-t} & t < x < 1 \\ 0 & x > 1 \end{cases}. \tag{2.24}$$

The figure below shows the solution $u(x,t)$ at several instances of time. Note how a shock (discontinuity) is developed at $t=1$.




Let's analyze the Riemann problem again using equation (2.23). In this equation, the function $u$ has a dual role, serving both as the density function and the material velocity $V$, where $V = \frac{dx}{dt} = u$. Now, let's examine the initial condition shown below at $t = 0$:




- Particles initially located at $x_0 \leq 0$ move to the right with velocity $V = 1$, as their density $u = 1$ is conserved. For example, a particle initially located at $x_0 = 0$ will reach $x = 1$ at $t = 1$, as evident from the characteristic equation $x(t) = ut + x_0$. When $x_0 = 0$ and $u = 1$, $x(1) = 1$.

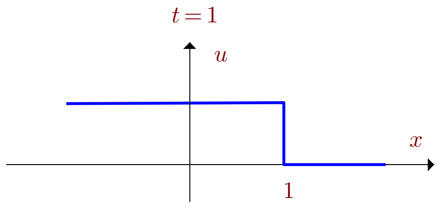
- Particles located initially at $x \in (0, 1)$ move with the velocity $V = 1 - x$ while they preserving their densities. For example, a particle at $x_0 = \frac{1}{2}$ has density $u = \frac{1}{2}$ and moves with speed $V = \frac{1}{2}$ to the right, reaching $x = 0$ at $t = 1$.

- All particles initially located at $x \geq 1$ have zero velocity and remain stationary on the $x$-axis for $t > 0$.

The above observations imply that the density of point $x=1$ at time $t=1$ can take any value between $0$ and $1$, as particles with different densities between $0$ and $1$ reach this point at $t=1$. Therefore, we can write the density $u(x,1)$ as

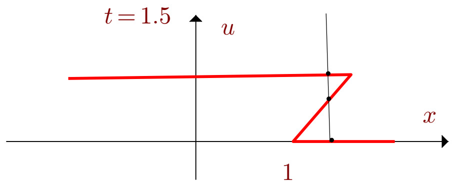
$$u(x,1)=\begin{cases} 1 & x<1 \\ 0 & x>1 \end{cases},$$

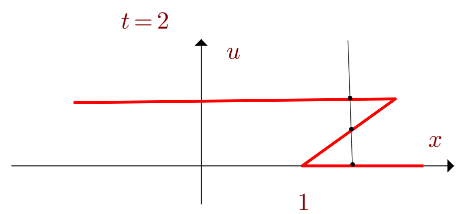
while $u(1,1)$ is not uniquely defined.




## 2.6.2 Extension of the solution

We will delve into the details of this problem in the second volume of the book. However, in this context, we aim to understand the elements of shock propagation through the conservation law. Our primary concern is to determine the density function $u(x,t)$ for $t>1$. It should be noted that the partial differential equation is not valid at $t=1$, and hence, we cannot use the solution (2.24) for $t>1$. In fact, using this solution for $t>1$ results in the existence of multiple non-physical solutions, as shown in the figure below.




To understand the propagation of shocks better, let us first consider the following one-dimensional wave equation:

$$\begin{cases} u_t+cu_x=0 \\ u(x,0)=u_0(x) \end{cases},$$

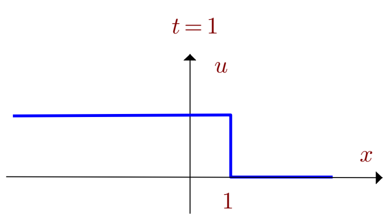
where $u_0$ is the following discontinuous function

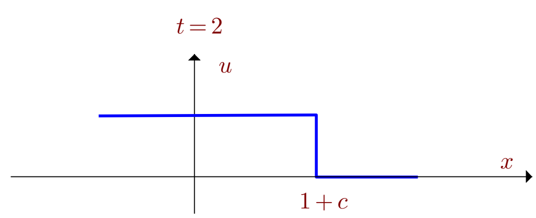
$$u_0(x)=\begin{cases} 1 & x<0 \\ 0 & x>0 \end{cases}.$$

As expected, the wave equation carries the initial condition with speed $c$ to the right or left depending on the sign of $c$ in the equation. Hence, the solution can be expressed as:

$$u(x,t)=u_0(x-ct).$$

The following figure shows the solution of the wave equation for $t=1$ and $t=2$. Note that the extension of the initial condition is plausible, and the solution is discontinuous only at one point for each time instance.



Let's return to the Riemann problem:

$$\begin{cases} u_t + u u_x = 0 \\ u(x,1) = \left\{ \begin{smallmatrix} 1 & x<1 \\ 0 & x>1 \end{smallmatrix} \right. \end{cases}.$$

Since $u$ can only take the values $0$ or $1$, except at the discontinuity point, it's reasonable to assume that this equation behaves like a wave equation. If we make this assumption, then the question becomes: at what speed will the initial condition propagate?

To answer this, we can use the conservation law and write the PDE as an integral equation:

$$\int_{x_0}^{x_1} \left\{ u_t + \left( \frac{1}{2} u^2 \right)_x \right\} dx = 0.$$

for any interval $(x_0, x_1)$ containing the shock position $x$ at time $t$. Integrating this equation over the interval $(x-\varepsilon, x+\varepsilon)$, where $\varepsilon$ is a small positive constant, we obtain

$$\int_{x-\varepsilon}^{x+\varepsilon} u_t(x,t)\, dx + \frac{1}{2} u_r^2 - \frac{1}{2} u_l^2 = 0, \tag{2.25}$$

where $u_l$ and $u_r$ are the values of $u$ to the left and right of the shock, respectively. On the other hand, we can write the integral in the above equation as

$$\int_{x-\varepsilon}^{x+\varepsilon} u_t(x,t)\, dx = \lim_{\delta t \to 0} \frac{1}{\delta t} \left\{ \int_{x-\varepsilon}^{x+\varepsilon} u(x, t+\delta t)\, dx - \int_{x-\varepsilon}^{x+\varepsilon} u(x,t)\, dx \right\}.$$

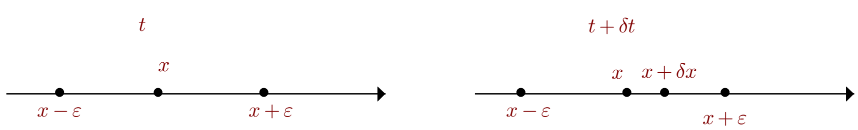
The first integral in the right-hand side of the above equation is equal to

$$\int_{x-\varepsilon}^{x+\varepsilon} u(x, t+\delta t)\, dx = u_l(\varepsilon + \delta x) + u_r(\varepsilon - \delta x),$$

and the second one is

$$\int_{x-\varepsilon}^{x+\varepsilon} u(x,t)\, dx = u_l\, \varepsilon + u_r\, \varepsilon.$$

The scenario is shown in the figure below.



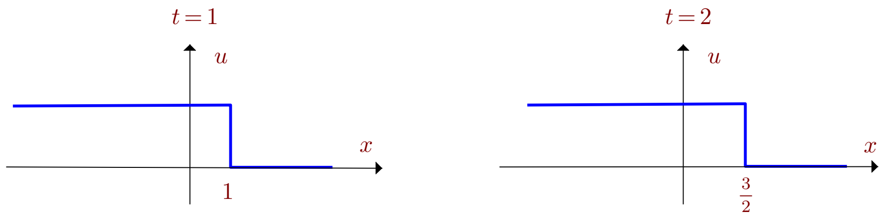

Hence, we can write

$$\int_{x-\varepsilon}^{x+\varepsilon} u_t(x,t)\, dx = v(u_l + u_r),$$

where $v$ is the velocity of the shock point $x$. Substituting the result into equation (2.25) yields

$$v = \frac{\frac{1}{2} v_l^2 - \frac{1}{2} u_r^2}{u_l + u_r} = \frac{u_l + u_r}{2}.$$

For our example with $u_l = 1$ and $u_r = 0$, we obtain the shock velocity $v = \frac{1}{2}$. The solution for $t = 1$, $2$ is illustrated in the following figure:

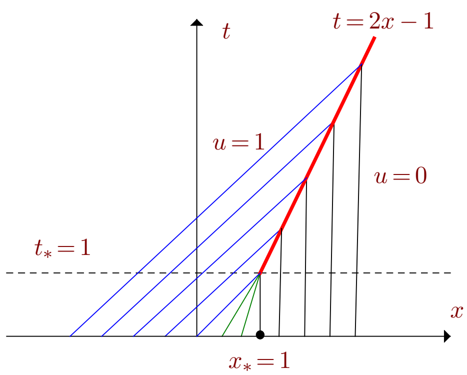



The solution $u(x,t)$ can be obtained using the shockwave solution of the Riemann problem. The formula for the solution is given as:

$$u(x,t) = \begin{cases} 1 & x < 1 + v(t-1) \\ 0 & x > 1 + v(t-1) \end{cases},$$

where $v = \frac{1}{2}$ is the shock velocity for the given example.

This solution is called the shockwave solution because it has a sharp transition between the two states, which moves with constant velocity v. The following figure illustrates the characteristic lines with the shock line.




**Example 2.27.** This example illustrate the fact that the graph of shockwave in $xt$-plane can be a curve. This means that the velocity of shock propagation can vary with time. Let us solve the following problem

$$\begin{cases} u_t + u u_x = 0 \\ u(x,0) = \begin{cases} 1+x & -1 < x < 0 \\ 1-x & 0 < x < 1 \end{cases} \end{cases}.$$

The equation of characteristic lines are $x(t) = u_0(x_0)\,t + x_0$ where $u_0(x_0) = \begin{cases} 1+x_0 & -1 < x_0 < 0 \\ 1-x_0 & 0 < x_0 < 1 \end{cases}$. Substituting $u_0(x_0)$ into the equation of characteristic line yields

$$x(t) = x_0 + \begin{cases} (1+x_0)t & -1 < x_0 < 0 \\ (1-x_0)t & 0 < x_0 < 1 \end{cases}.$$

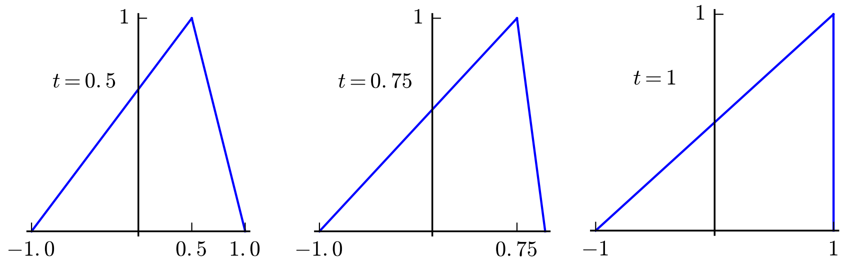
A simple algebraic calculation helps to solve $x_0$ in terms of $x, t$ as follows

$$x_0 = \begin{cases} \frac{x-t}{1+t} & -1 < x < t \\ \frac{x-t}{1-t} & t < x < 1 \end{cases}.$$

Accordingly, the solution $u(x,t)$ is obtained as

$$u(x,t) = \begin{cases} \frac{1+x}{1+t} & -1 < x < t \\ \frac{1-x}{1-t} & t < x < 1 \end{cases}.$$

The solution for $t=\frac{1}{2}, \frac{3}{4}$ and $t=1$ are shown below.

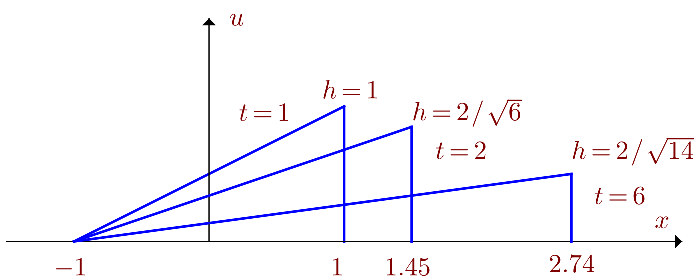



t is observed that at $t=1$, a discontinuity is formed at $x_*=1$. For $t>1$, the solution extends as a shockwave with speed $v$ that needs to be determined. Similar to the argument above, it can be justified that the shockwave propagates with speed $v=\frac{1}{2}$ at $t=1$. However, what about $v$ for $t>1$?

It is important to note that according to the conservation law, the total mass of the system remains conserved. This can be expressed as:

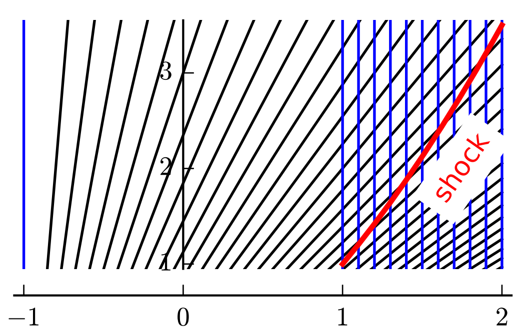
$$\int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty} u(x,0)\,dx = 1.$$

If the shockwave were to move with a constant velocity $v=\frac{1}{2}$ for all $t>1$, then the area under the curve $u(x,t)$ would increase with time, which would violate the conservation of mass. In order to maintain the constant mass equal to $1$, the height of the triangle, $h$, must decrease with time. This is depicted in the following figure.




It is important to note that the velocity of the shockwave is not constant in the case where the initial data has a jump discontinuity. Since the left value of the solution is $h$ and the right value is $0$, we can derive the shock velocity as $v=\frac{1}{2}h$, which in turn gives $v=\frac{1}{1+x}$, where $x$ is the distance from the discontinuity at $x=1$. Therefore, the shock velocity varies with respect to $x$.

To find the shock velocity in terms of $t$, we can integrate the relation $v=\frac{dx}{dt}=\frac{1}{1+x}$ with respect to $t$. This gives us $x(t)=\sqrt{2t+2}-1$, which implies that the shock velocity at time $t$ is $v=\frac{1}{\sqrt{2t+2}}$ for $t\geq 1$. The characteristic lines in the $(x,t)$-plane can be plotted to visualize the behavior of the solution.



**Exercise 2.40.** An alternative method to find the shockwave velocity for the above problem is to use the following formula

$$u_l(x,t) = \frac{x+1}{t+1},$$

and to solve the equation

$$v = \frac{dx}{dt} = \frac{x+1}{2(t+1)}.$$

Carry out the calculation and conclude $v = \frac{1}{\sqrt{2t+2}}$.

**Exercise 2.41.** Let us solve the following problem

$$\begin{cases} u_t + u u_x = 0 \\ u(x,0) = u_0(x) \end{cases},$$

where $u_0$ is as follows

$$u_0(x) = \begin{cases} 3 & x < 0 \\ 1 & x > 0 \end{cases}.$$

Draw the characteristic lines in the $(x,t)$-plane and find the shockwave solution.

**Exercise 2.42.** Consider the following problem

$$\begin{cases} u_t + u u_x = 0 \\ u(x,0) = u_0(x) \end{cases},$$

where $u_0$ is a continuously differentiable function.

a) Show that if $u' < 0$, the equation will develop a shock.

b) show that the first time of the appearance of the shock is derived by the following formula

$$t_* = \frac{-1}{\min_\xi u_0'(\xi)}$$

Hint: The shock is formed if two characteristic lines collide

$$x_0 + u_0(x_0)t = x_1 + u_0(x_1)t.$$

This gives $t = \frac{x_0 - x_1}{u_0(x_1) - u_0(x_0)}$. Minimize this and conclude the above relation.

**Exercise 2.43.** Consider the following equation

$$\begin{cases} u_t + g'(u)\, u_x = 0 \\ u(x,0) = u_0(x) \end{cases}.$$

If the problem develops a shock, show that the velocity of the shockwave is

$$v = \frac{g(u_l) - g(u_r)}{u_l + u_r},$$

where $u_l, u_r$ are the left and right limit of $u$ at the shock point $x$ respectively.

**Exercise 2.44.** Consider the following problem

$$\begin{cases} u_t + u u_x = 0 \\ u(x,0) = \begin{cases} 0 & x \le 0 \\ \frac{x}{\varepsilon} & 0 \le x \le \varepsilon \\ 1 & x \ge \varepsilon \end{cases} \end{cases},$$

for $\varepsilon > 0$. The problem does not develop shock.

a) Draw the initial condition and obtain the solution $u_\varepsilon(x,t)$.

b) Let $\varepsilon \to 0$ and find the solution of the following equation

$$\begin{cases} u_t + u u_x = 0 \\ u(x,0) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \end{cases}.$$

The solution of this equation is called the *rarefaction solution*.

**Problem 2.30.** Find an explicit solution of the following equation

$$\begin{cases} u_t + u u_x = x \\ u(x,0) = 0 \end{cases}.$$

**Problem 2.31.** Find the explicit solution to the following problem

$$\begin{cases} u_t + e^{u-x} u_x = 0 \\ u(x,0) = x \end{cases}.$$

**Problem 2.32.** Solve the following problem

$$\begin{cases} u_t + u u_x = u \\ u(x,0) = 1 + x \end{cases}.$$

**Problem 2.33.** Find an implicit solution to the following problem

$$\begin{cases} u_t + x u u_x = 0 \\ u(x,0) = x \end{cases}.$$

Use the implicit function theorem to justify that the solution exists in some interval of $t \in [0, T)$.

**Problem 2.34.** Find an implicit solution to the following problem

$$\begin{cases} \partial_t u + \sin(u)\, \partial_x u = 1 \\ u(x,0) = x \end{cases}.$$

Use the implicit function theorem to justify that the solution exists in some interval of $t$.

**Problem 2.35.** Draw the solution to the following problems for $t = 0, 1, 2$

$$\begin{cases} u + u u_x = 0 \\ u(x,0) = u_0(x) \end{cases},$$

where

$$u_0(x) = \begin{cases} 0 & x \le 0 \\ x & 0 \le x \le 1 \\ 1 & x \ge 1 \end{cases}.$$

**Problem 2.36.** Consider the following problem

$$\begin{cases} u_t + u u_x = 0 \\ u(x,0) = \begin{cases} 2 & x \le 0 \\ 2 - x & 0 \le x \le 1 \\ 1 & x \ge 1 \end{cases} \end{cases}.$$

i. Draw the characteristic lines and determine $t^*$.

ii. Find the shock wave speed.

iii. Draw the solution $u(x, 0.5)$, $u(x, 1)$, $u(x, 2)$ and $u(x, 4)$.

**Problem 2.37.** Draw the solution $u(1, x)$ and $u(2, x)$ to the following problem

$$\begin{cases} u + u u_x = 0 \\ u(x,0) = \begin{cases} 2 & x \le 0 \\ 1 & x \ge 0 \end{cases} \end{cases}.$$

**Problem 2.38.** Solve the following problem for $t < t_*$ and draw the solution for few values of $t$.

$$\begin{cases} u_t + u u_x = 0 \\ u(x,0) = \begin{cases} 1 + x & -1 < x < 0 \\ e^{-x} & 0 \le x \end{cases} \end{cases}.$$

**Problem 2.39.** For the following equation, draw the characteristics and find the shock wave speed. Draw the solution $u(1, x)$ and $u(2, x)$:

$$\begin{cases} u_t + u u_x = 0 \\ u(x,0) = \begin{cases} 2 - |x| & -1 < x \le 1 \\ 0 & \text{otherwise} \end{cases} \end{cases}.$$

**Problem 2.40.** The traffic flow in highways is usually modeled by the following equation

$$u_t + (1-u)u_x = 0,$$

where $u$ is the density of cars (number of cars in a unit length). Assume that the initial profile is as follows

$$u(x,0) = \begin{cases} 1 & x \le 0 \\ x+1 & 0 \le x \le 1 \\ 2 & x \ge 1 \end{cases}.$$

a) Draw the characteristic lines of the equation and find the collision time $t_\star$.

b) Draw $u(x,t)$ where $T = 2$.

**Problem 2.41.** For the following equation, find the collision time $t^*$. Use a software to fine $u(x,0.5)$ and $u(x,1)$.

$$\begin{cases} u_t + 2uu_x = 0 \\ u(x,0) = e^{-|x|} \end{cases}.$$

**Problem 2.42.** Solve the following problem

$$\begin{cases} u_t + uu_x = 0 \\ u(x,0) = \left\{\begin{smallmatrix} 1 & x<0 \\ 2 & x>0 \end{smallmatrix}\right. \end{cases}.$$

**Problem 2.43.** Solve the following problem

$$\begin{cases} u_t + uu_x = 0 \\ u(x,0) = \left\{\begin{smallmatrix} 0 & x<0 \\ 1 & 0<x<1 \\ 0 & x>1 \end{smallmatrix}\right. \end{cases}.$$

# Appendix A
# Curves and Surface

## A.1 Curves and surfaces

The techniques outlined in this chapter for solving first-order PDEs are primarily based on the concept of the derivative of a scalar function along a smooth curve. To apply these techniques, it is necessary to have an understanding of how smooth curves and surfaces are represented in the plane or in $\mathbb{R}^n$ for $n>2$.

### A.1.1 Curves and derivative along a curve

Curves in the plane $\mathbb{R}^2$ can be represented in several ways: through an explicit function as the graph $\{(x, f(x)); x\in D_f\}$ where $f$ is a smooth function, an implicit function $\phi(x,y)=c$, or a parametric form $\gamma(t)=(x(t), y(t))$, where $t$ ranges over an open interval $I\subset\mathbb{R}$. An implicit representation $\phi(x, y)=c$ of a curve can be transformed into an explicit one, $y=f(x)$, under certain conditions using the implicit function theorem; see the appendix of the book. Note that the graph of an explicit function $y=f(x)$ is a special case of a parameterized curve where the parameter is chosen to be $x$, so that $\gamma(x)=(x, f(x))$.

While explicit and implicit representations of curves focus on the curves as a geometrical object, or a set of points in the xy-plane, parametrized curves can be considered as the path or trajectory of a moving particle in the plane. Accordingly, the parameter $t$ represents time, and the function $\gamma(t)$ gives the position of the particle at each time $t$. This interpretation enables us to determine the velocity and acceleration vectors of the particle along its path. Moreover, the concept of parametrization can be generalized to higher dimensions, allowing us to represent more complicated curves and surfaces in $\mathbb{R}^n$.

A curve in $\mathbb{R}^n$ parametrized by a parameter $t$ is defined as a map $\gamma: I\subset\mathbb{R}\to\mathbb{R}^n$ given by $\gamma(t)=(x_1(t), x_2(t), \ldots, x_n(t))$. Here, $x_1, x_2, \ldots, x_n$ are scalar functions of $t$ that describe the coordinates of points on the curve, and $t$ is a parameter that varies over an open interval $I\subset\mathbb{R}$. The tangent vector at a specific point $\gamma(t)$ is defined as the derivative of $\gamma(t)$ with respect to $t$, written as $\gamma'(t)=(x_1'(t), x_2'(t), \ldots, x_n'(t))$. The magnitude of this vector is denoted by $|\gamma'(t)|$, and in the context of physics, it represents the speed of a moving particle along $\gamma(t)$.

The arc length $s(t)$ of a curve is defined as the integral of the speed $|\gamma'(t)|$ over the interval $[0,t]$, that is,

$$s(t)=\int_0^t |\gamma'(\tau)|\, d\tau.$$

By the fundamental theorem of calculus, the derivative of $s(t)$ with respect to $t$ is given by

$$\frac{ds}{dt}=|\gamma'(t)|,$$

which is the speed of a moving particle along the curve $\gamma(t)$.

The equation of a straight line $\gamma = (x_1(t), \ldots, x_n(t))$ in $\mathbb{R}^n$ passing through $c = (c_1, \ldots, c_n)$ is simply derived by the differential equations

$$\frac{d\gamma}{dt} = a,$$

where $a = (a_1, \ldots, a_n)$ is a constant. In the coordinate system, we have $\frac{dx_j}{dt} = a_j$ for $j = 1, \ldots, n$, which is solve for $x_j = a_j t + c_j$, or in the algebraic form:

$$\frac{x_1 - c_1}{a_1} = \cdots = \frac{x_n - c_n}{a_n}.$$

The vector $a$ determines the direction of the line and is tangent to $\gamma$ as $\gamma'(t) = a$.

For a general curve $\gamma(t)$ whose direction changes continuously according to the vector function $V(x) = (v_1(x), \ldots, v_n(x))$, where $x = (x_1, ..., x_n)$, the equation is derived by solving the following differential equation:

$$\frac{d\gamma}{dt} = V(\gamma(t)),$$

or in the coordinate system, $\frac{dx_j}{dt} = V(x_1(t), \ldots, x_n(t))$, for $j = 1, ..., n$, where $\gamma(t) = (x_1(t), ..., x_n(t))$. If $\gamma(t)$ is considered as the trajectory of a particle in $\mathbb{R}^n$, the vector $V(\gamma(t))$ is the velocity of the particle at the point $\gamma(t)$, and the vector function $V = (v_1, ..., v_n)$ is said to be the velocity vector field.

**Example A.1.** Consider a particle moving in the xy-plane with the directional vector function $V(x, y) = (-y, x)$. The trajectory $\gamma(t)$ of the particle can be derived by solving the system of differential equations:

$$\frac{dx}{dt} = -y, \frac{dy}{dx} = x.$$

This system is equivalent to the exact equation $x dx + y dy = 0$ in the xy-plane, which has the implicit solution $x^2 + y^2 = c$. Alternatively, the system can be converted to a second-order equation for $x(t)$ as $x'' + x = 0$, with solution

$$x(t) = x_0 \cos(t) - y_0 \sin(t),$$

where $x_0$ is the initial position of the particle (i.e., $x(0) = x_0$), and $y_0 = -x'(0)$. Using $x(t)$ and the second differential equation, we can find

$$y(t) = x_0 \sin(t) + y_0 \cos(t),$$

and thus the equation of the trajectory is obtained as

$$\gamma(t) = (x_0 \cos(t) - y_0 \sin(t), x_0 \sin(t) + y_0 \cos(t)).$$

In particular, if $x_0 = 1$ and $y_0 = 0$, we obtain the trajectory

$$\gamma_{(1,0)}(t) = (\cos(t), -\sin(t)).$$

As we can see, the trajectory of the particle is the unit circle in the xy-plane. We can alternatively solve the system by putting it in the matrix form

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

with the fundamental matrix $\Phi(t)$ given by:

$$\Phi(t) = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}.$$

Note that:

$$\Phi(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \gamma_{(1,0)}(t).$$

The velocity vector of the particle with the trajectory $\gamma_{(1,0)}(t)$ is

$$\gamma'_{(1,0)}(t) = (-\sin(t), \cos(t)),$$

and through the relation

$$\gamma_{(1,0)}(t) \cdot \gamma'_{(1,0)}(t) = 0,$$

we observe that $\gamma'_{(1,0)}(t)$ is perpendicular to the trajectory of the particle.

Let $\gamma(t)$ be a parametric curve in $\mathbb{R}^n$, and let $u = u(x_1, \ldots, x_n)$ be a continuously differentiable function. The derivative of $u$ along $\gamma(t)$ is defined as:

$$\frac{du}{dt} := \sum_{j=1}^{n} \partial_j u(\gamma(t)) \frac{dx_j}{dt} = \nabla u(\gamma(t)) \cdot \gamma'(t),$$

where $\nabla u$ denotes the gradient of $u$, given by the vector:

$$\nabla u = \begin{pmatrix} \partial_1 u \\ \vdots \\ \partial_n u \end{pmatrix}.$$

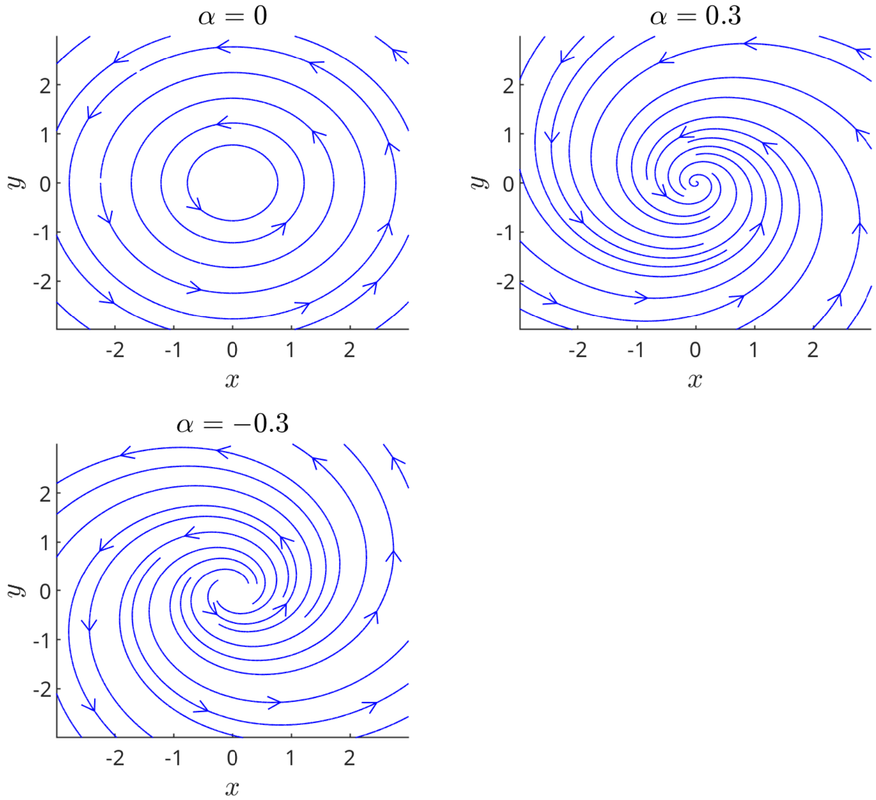
The operator $\nabla$ is also known as nabla and plays an important role in the context of multivariable functions. For example, if $u$ represents the density function in $\mathbb{R}^n$, and $\gamma(t)$ is a smooth curve, then $\frac{du}{dt}(\gamma(t))$ represents the rate of change of the density when a control volume of mass is moving along $\gamma$.

**Example A.2.** The above explanation highlights the relationship between curves in $\mathbb{R}^n$ and systems of first-order ordinary differential equations. Specifically, a curve in $\mathbb{R}^n$ can be obtained by solving a system of first-order ordinary differential equations, and conversely, the solution of a system of ordinary differential equations corresponds to a curve. This connection between curves and differential equations is fundamental to many areas of mathematics and has wide-ranging applications in physics, engineering, and other fields.

Consider the system

$$\begin{cases} \frac{dx}{dt} = -y - \alpha x \\ \frac{dy}{dt} = x - \alpha y \end{cases},$$

where $\alpha$ is a constant. The trajectory associated with $\alpha = 0, 0.3, -0.3$, are shown below




In the xy-plane, the equation the equation can be written as

$$(x-\alpha y)\, dx + (y+\alpha x) dy = 0,$$

which is not exact if $\alpha \neq 0$, meaning that energy is not conserved along the paths of motion. The equivalent second-order equation of the system is:

$$x'' + 2\alpha x' + x = 0,$$

which can which can be further transformed to:

$$\frac{d}{dt}\left(\frac{1}{2}|x'|^2 + \frac{1}{2}x^2\right) = -2\alpha\, |x'|^2.$$

Integrating this equation yields:

$$E(t) = E(0) - 2\alpha \int_0^t |x'(s)|^2\, ds.$$

If $\alpha > 0$, the particles move down spirally to the origin along $V(x, y)$, while if $\alpha < 0$, they gain energy and move outward spirally.

**Exercise A.1.** Find the force field associated to the velocity filed given in the example above using the relation $F = m\gamma''(t)$, and show that this force field is centrifugal. Consider the trajectory starting at $(1, 0)$ for simplicity. Additionally, demonstrate that the energy of the particle is conserved along its trajectory for the energy given by

$$E(t) = m|\gamma(t)|^2 + \frac{1}{2} m|\gamma'(t)|^2.$$

**Exercise A.2.** Consider the vector field $V = (-y, x, 1)$ and suppose a particle located initially at $(1, 0, 0)$ moves according to this vector field. Find the trajectory of the particle and show the relation $\gamma'(t) \cdot \gamma''(t) = 0$ for all $t$. Thus, conclude that the frame $\left(\frac{\gamma'(t)}{|\gamma'(t)|}, \frac{\gamma''(t)}{|\gamma''(t)|}, \frac{\gamma'(t) \times \gamma''(t)}{|\gamma'(t) \times \gamma''(t)|}\right)$ defines a local coordinate system for the particle.

**Exercise A.3.** With a curve mapping, we can determine the *length* of a curve. For a given explicit curve $y=f(x)$ or $(x, f(x))$ for $x\in(x_0, x_1)$ the differential length is

$$ds=\sqrt{1+|f'|^2}\,dx,$$

and thus the arc length is

$$s=\int_{x_0}^{x_1}\sqrt{1+|f'|^2}\,dx.$$

Similarly, for a general parametrization $\gamma(t)$, the differential length is

$$ds=|\gamma'(t)|\,dt=\sqrt{|x'(t)|^2+|y'(t)|^2}\,dt,$$

and thus

$$s=\int_{t_0}^{t_1}\sqrt{|x'(t)|^2+|y'(t)|^2}\,dt.$$

The interesting point is that the arc length formula is independent of the parametrization. Let $C\subset\mathbb{R}^2$ be a geometric curve and let $\gamma_1\colon(a,b)\to C$ and $\gamma_2\colon(c,d)\to C$ be two smooth parametrization of $C$. Show

$$\int_a^b|\gamma_1'(t)|\,dt=\int_c^d|\gamma_2'(t)|\,dt.$$

**Exercise A.4.** Consider the temperature distribution in the xy-plane given by $T(x,y)=10\exp(x^2-y^2)$. Suppose a runner is moving along the curve $\gamma(t)=(\cos(t),\sin(t))$, and wearing a hand clock marked from $0$ to $60$. What is the angular velocity of the hand of the runner's watch as they move along the curve? Find the angular velocity of the hand clock if the runner runs given by the following function $\gamma(t)=(\cos(\omega_0\,t),\sin(\omega_0\,t))$ for some $\omega_0>0$.

### A.1.2 Surfaces in space

There are several ways to represent a surface in three-dimensional space. One common approach is to express the surface as the graph of an explicit function of the form $z=f(x,y)$, where $z$ is the dependent variable and $x,y$ are the independent variables. Alternatively, the surface can be defined implicitly through an equation of the form $\phi(x,y,z)=c$.

The first approach is a convenient way to visualize and manipulate the surface. However, this representation is only possible for surfaces that can be written as a function of $x$ and $y$, such as paraboloids or planes. The second approach is more general and can represent surfaces that cannot be written explicitly as a function of $x$ and $y$, such as spheres or tori. The implicit function theorem provides conditions for the existence and differentiability of a smooth function $z=f(x,y)$ that satisfies the equation $\phi(x,y,f(x,y))=c$. In particular, the theorem requires the existence of a point $(x_0,y_0,z_0)$ such that $\phi(x_0,y_0,z_0)=c$ and the partial derivative $\frac{\partial\phi}{\partial z}(x_0,y_0,z_0)$ is nonzero. This condition guarantees that the equation $\phi(x,y,z)=c$ defines the surface locally as a graph over the xy-plane, and the function $f(x,y)$ can be obtained by solving for $z$ in terms of $x$ and $y$.

Finally, a surface can be parameterized using a set of equations that describe how $x$, $y$, and $z$ vary with two independent variables, typically denoted by $t$ and $s$. In particular, a parameterized surface in three-dimensional space is given by the equation

$$\Sigma(t,s)=(x(t,s),y(t,s),z(t,s)),$$

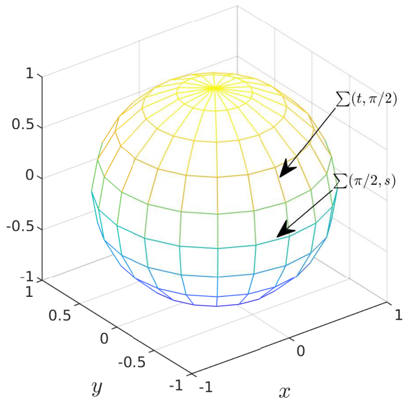
where $x$, $y$, and $z$ are functions of the independent variables $t$ and $s$, and $(t,s)$ belongs to an open set $D$ in the plane. This method allows for the representation of surfaces with complex shapes and topologies, such as the torus or the Mobius strip, and can also be used to represent higher-dimensional surfaces. For instance, an m-dimensional hypersurface in $\mathbb{R}^n$ where $m<n$ can be represented by a vector function of the form

$$\Sigma(t_1,\ldots,t_m)=(x_1(t_1,\ldots,t_m),\ldots,x_n(t_1,\ldots,t_m)).$$

Moreover, the parametric representation of a surface reveals the relationship between curves and surfaces. For example, a surface $\Sigma(t,s)$ in $\mathbb{R}^3$ contains the lines $\gamma_{s_0}(t)=(x(t,s_0), y(t,s_0), z(t,s_0))$, where $t$ ranges over an interval for each $s_0$, and the curves $\gamma_{t_0}(s)=(x(t_0,s), y(t_0,s), z(t_0,s))$, where $s$ ranges over an interval for each $t_0$. For instance, let $D=[0,\pi]\times[-\pi,\pi]$, and consider the parametrization given by:

$$\Sigma(t,s)=(\sin t\cos s, \sin t\sin s, \cos t),$$

for $(t,s)$ in $D$. This parametrization maps the rectangle $D$ onto the surface of the unit sphere. Each line $\{s_0\}\times[0,\pi]$, for $s\in[-\pi,\pi]$, is mapped to the vertical curve $\Sigma(t,s_0)$, and each line $[-\pi,\pi]\times\{t_0\}$ to the horizontal curves $\Sigma(s,t_0)$. In the spherical coordinate system, $t$ is usually denoted by $\theta$, the angle with the $z$-axis, and $s$ by $\phi$, the angle with the $x$-axis.




It is worth noting that the graph of an explicit function $z=f(x,y)$ is actually a special case of a parameterized surface, where the surface is defined by the equation $\Sigma(x,y)=(x,y,f(x,y))$. The tangent and normal vectors to a surface $\Sigma$ can be found as follows. Let $(t_0,s_0)$ be a fixed point on $\Sigma$. The curve map $\gamma_{s_0}(t)=(x(t,s_0), y(t,s_0), z(t,s_0))$ lies on $\Sigma$, and its tangent vector $\gamma'_{s_0}(t_0)$ is equal to $\Sigma_t(t_0,s_0)$. Similarly, the curve map $\gamma_{t_0}(s)=(x(t_0,s), y(t_0,s), z(t_0,s))$ lies on $\Sigma$, and its tangent vector is $\Sigma_s(t_0,s_0)$. The space spanned by $\Sigma_t(t_0,s_0)$ and $\Sigma_s(t_0,s_0)$ is the tangent space to the surface $\Sigma$ at $(t_0,s_0)$, which is just the tangent plane to $\Sigma$ at that point. A surface $\Sigma$ is said to be smooth if $\Sigma_t(t,s)$ and $\Sigma_s(t,s)$ exist and are linearly independent for all $(t,s)$ in the domain of $\Sigma$. The normal vector to $\Sigma$ is then given by the cross product

$$\nu=\frac{\Sigma_t\times\Sigma_s}{|\Sigma_t\times\Sigma_s|}.$$

Note that if $\Sigma_t(t_0,s_0)$ and $\Sigma_s(t_0,s_0)$ are linearly independent, then $\Sigma_t(t_0,s_0)\times\Sigma_s(t_0,s_0)\neq 0$, and the normal vector is well-defined. For example, the normal vector to the unit sphere at the point $\Sigma(\pi/2,0)=(1,0,0)$ with the parametrization given above can be calculated as follows: $\Sigma_t(\pi/2,0)=(0,0,-1)$ and $\Sigma_s(\pi/2,0)=(0,1,0)$. Therefore, $\Sigma_t(\pi/2,0)\times\Sigma_s(\pi/2,0)=(1,0,0)$ and $|\Sigma_t(\pi/2,0)\times\Sigma_s(\pi/2,0)|=1$, so the normal vector is $\nu=\frac{\Sigma_t\times\Sigma_s}{|\Sigma_t\times\Sigma_s|}=(1,0,0)$, which is clearly perpendicular to the tangent plane of the sphere at the point $\Sigma(\pi/2,0)$.

**Exercise A.5.** The area differential $dS$ is equal to $|\Sigma_t\times\Sigma_s|\,dt\,ds$, and thus the area of the surface $\Sigma$ is equal to

$$S=\iint_D|\Sigma_t\times\Sigma_s|\,dt\,ds.$$

In particular, $dS=\sqrt{1+|\nabla u|^2}dx\,dy$ is the differential area of an explicit surface $z=u(x,y)$. For example, if $\Sigma$ represents a material surface of density $\rho(x,y,z)$, then the total mass on $\Sigma$ is equal to

$$M=\iint_D\rho(\Sigma(t,s))|\Sigma_t\times\Sigma_s|\,dt\,ds.$$

Use the spherical coordinate and and calculate the area of the potion $0 \leq \theta \leq \theta_0$ of the unit sphere for $\theta_0 < \frac{\pi}{2}$. Calculate the total mass on the surface if $\rho = (x^2 + y^2)z$.

**Exercise A.6.** Write down the equation of the tangent plane to the smooth surface $\phi(x, y, z) = 0$ at $(x_0, y_0, z_0)$ on the surface.

**Exercise A.7.** Let $\phi(x, y, z) = 0$ denote a surface $\Sigma$. Show that $\nabla\phi$ is perpendicular to all tangent vectors on the surface $\Sigma$. Hint: Let $\gamma = (x(t), y(t), z(t))$ be arbitrary smooth curve lying on $\Sigma$.

Let's now consider the equations of surfaces in $\mathbb{R}^3$. Suppose we have a plane passing through a point $p_0 = (x_0, y_0, z_0)$ and perpendicular to a constant vector $\vec{n} = (a, b, c)$. Its equation can be expressed as a

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$

which is the expansion of the dot product $(p - p_0) \cdot \vec{n} = 0$, for $p = (x, y, z)$ on the plane. Equivalently, if we are given two linearly independent vectors $V_1 = (a_1, b_1, c_1)$ and $V_2 = (a_2, b_2, c_2)$, we can write the parametric equation of the plane containing these two vectors as

$$\Sigma(t, s) = (t a_1 + s a_2, t b_1 + s b_2, t c_1 + s c_2).$$

Note both vectors lie in the obtained plane. If we were provided with only one vector, the equation of the plane could not be uniquely determined.