Chapter 1

Introduction

Partial differential equations, or PDEs for short, are an important type of differential equation that arise as natural mathematical models in many physical problems. They allow us to describe the behavior of a system in terms of functions that depend on multiple variables, such as time and space.

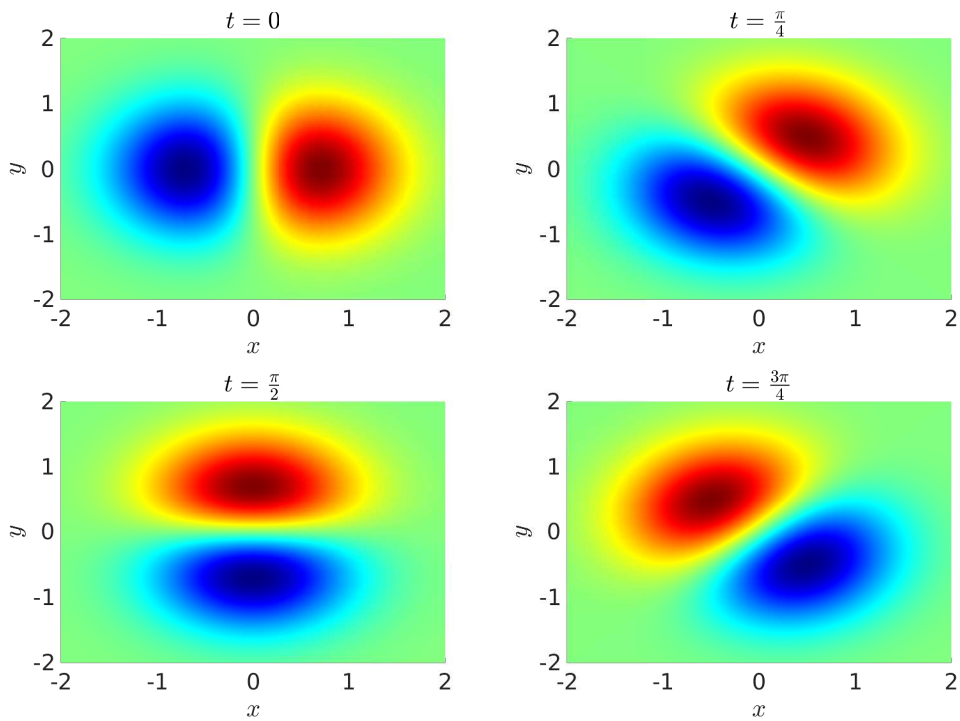
For example, consider the classical heat equation, which describes the distribution of heat in a conducting material over time. This equation can be derived from Fourier's law of heat conduction, which states that the rate of heat transfer through a material is proportional to the temperature gradient. By applying this law to an infinitesimal volume element in the material, one obtains a PDE for the temperature distribution.

Similarly, the wave equation can be derived from Newton's second law of motion, which relates the force acting on a body to its acceleration. By applying this law to a small element of a string or other vibrating object, one obtains a PDE for the displacement of the element as a function of time and position.

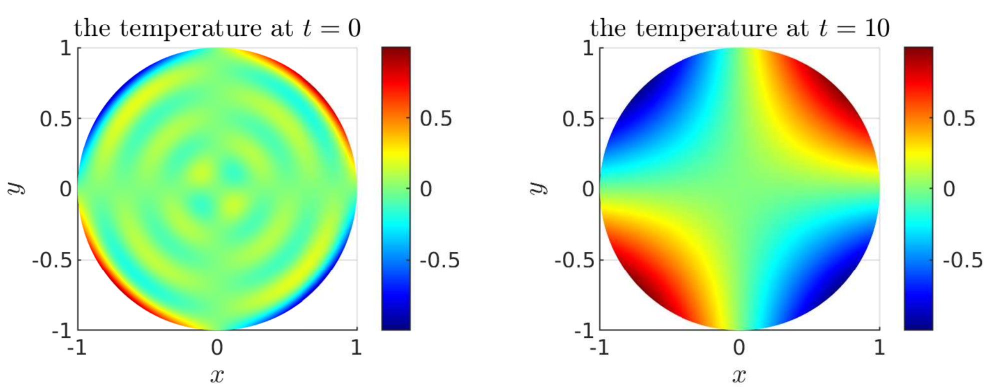
In both of these examples, the PDEs are derived from fundamental physical laws and provide a mathematical description of the underlying physical phenomenon. By solving these equations, we can make predictions about how the system will behave under different conditions, such as changes in temperature or initial conditions.

Our main focus is to introduce four important types of PDEs at an entry-level technicality, without delving into their theoretical aspects. These include:

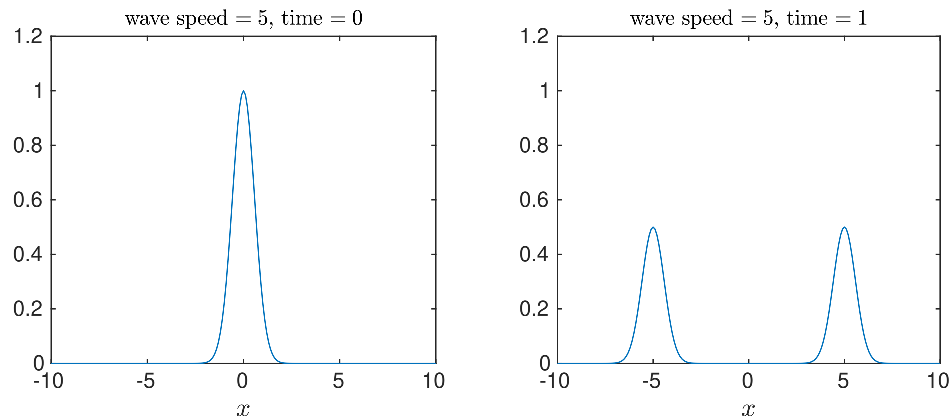
1. The transport equation, which describes how the density of a fluid flow moving in the plane or space changes according to a given velocity. This PDE is of first-order. The following figure illustrates the variation of mass density of a fluid flow in the xy-plane with the velocity field $V=(-y,x)$.



2. The heat problem, which involves determining how the temperature of a conductive medium changes in time, given an initial heat distribution along a conductive rod or continuum. This problem leads to a second-order PDE. The figure below shows the evolution of temperature over time in a unit disk.



3. The wave equation, which describes how an elastic string or a 2D membrane responds to an initial disturbance, leading to oscillations and the propagation of the disturbance. This PDE is also of second-order. The figure below illustrates how an initial disturbance propagates as a bilateral wave, splitting into two branches moving with a given velocity to the left and right.



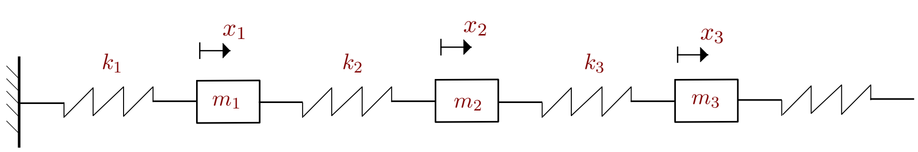
4. The Poisson equation, which arises in the context of electrostatics and involves finding the potential function generated in space by an electric charge distributed in a domain Ω in the plane or space. This problem is reduced to a second-order PDE.

1.1 From ODEs to PDEs

In the following discussion, we will explore the natural extension of well-known ordinary differential equations to partial differential equations. We will focus on the derivation of some simple PDEs and highlight how they arise from the ODE versions of the same problems. This will illustrate the process of extending mathematical models of physical phenomena from one dimension to multiple dimensions.

1.1.1 From mass-spring system to vibrating string

Remember that the dynamics of a single or finite set of quantities that depends on a single independent variable leads to a single or set of differential equations called ordinary differential equations (ODEs). An important example of this type of equation is the harmonic oscillator: consider a set of n point masses $m_1, \ldots, m_n$ that are connected in series to n springs with stiffnesses $k_1, \ldots, k_n$, as shown in the figure below.



The equation that describes the dynamics of m_1 is derived by Newton's second law $f = ma$ as

$$m_1 \frac{d^2 x_1}{dt^2} = -k_1 x_1 + k_2(x_2 - x_1),$$

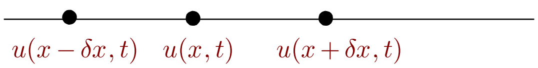
where $-k_1 x_1$ is the Hook's force exerted by the spring k_1, and $k_2(x_2 - x_1)$ is the force exerted by k_2 on m_1. Similarly, the equation that describes the dynamics of m_2 is:

$$m_2 \frac{d^2 x_2}{dt^2} = -k_2(x_2 - x_1) + k_3(x_3 - x_2).$$

The equation for $m_3, \ldots, m_n$ are derived in a similar manner. In this way, we obtain a system of n second-order ODEs

$$\begin{cases} m_1\, \ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1) \\ m_2\, \ddot{x}_2 = -k_2(x_2 - x_1) + k_3(x_3 - x_2) \\ \vdots \\ m_n\, \ddot{x}_n = -k_n(x_n - x_{n-1}) \end{cases}.$$

Now, consider a vibrating string along the x-axis, which can be modeled as an infinite mass-spring system connected in series. For a control length δx with density ρ, the control mass is $\rho\delta x$. Let $u(x,t)$ denote the position of this control length. This piece is under the forces of masses $\rho(x-\delta x)\delta x$, which is posited at $u(x+\delta x, t)$, and $\rho(x+\delta x)\delta x$ at the position $u(x+\delta x, t)$.



The force on the mass at x is

$$F = -\frac{k}{\delta x}[u(x,t) - u(x-\delta x, t)] + \frac{k}{\delta x}[u(x+\delta x, t) - u(x,t)],$$

where k is the stiffness of the string, and according to the control length δx, the ratio $\frac{k}{\delta x}$ reflects the impact of this length on the motion of mass in δx. Therefore, we can write

$$\rho\,\delta x\, u_{tt}(x,t) = -k\frac{u(x,t) - u(x-\delta x, t)}{\delta x} + k\frac{u(x+\delta x, t) - u(x,t)}{\delta x},$$

where $u_{tt}(x,t) := \frac{\partial^2 u}{\partial t^2}$ is the acceleration of the control length δx. By using the relations

$$\frac{u(x,t) - u(x-\delta x, t)}{\delta x} \approx u_x(x-\delta x, t), \frac{u(x+\delta x, t) - u(x,t)}{\delta x} \approx u_x(x,t),$$

we obtain

$$\rho\,\delta x\, u_{tt}(x,t) \approx -k u_x(x-\delta x, t) + k u_x(x,t),$$

and by dividing by δx, we obtain

$$\rho\, u_{tt}(x,t) \approx k\frac{u_x(x,t) - u_x(x-\delta x, t)}{\delta x},$$

where the right-hand side approaches $k u_{xx}$ when $\delta x \to 0$. Finally, we derive the following differential equation for the oscillation of the vibrating string:

$$u_{tt}(x,t) = \frac{k}{\rho}\, u_{xx}(x,t).$$

Exercise 1.1. For a mass-spring system consists of n masses $m_1, \ldots, m_n$ connected to n springs $k_1, \ldots, k_n$ in series, the total energy is defined as

$$E(t) = \sum_{j=1}^{n} \frac{1}{2} m_j\, |\dot{x}_j(t)|^2 + \frac{1}{2} k_1 |x_1(t)|^2 + \sum_{j=2}^{n} \frac{1}{2} k_j\, |x_j(t) - x_{j-1}(t)|^2.$$

Verify that the system conserves the energy, that is, $\frac{dE}{dt} = 0$, and conclude $E(t) = E(0)$.

Exercise 1.2. The conservation of energy also applies to the wave equation of a string. Let us consider a string of length L, fixed at two endpoints $x=0$ and $x=L$. We denote the position of the point x at time t by $u(x,t)$. The string forms a dynamic system if either its initial kinetic or potential energy is nonzero. The initial kinetic energy of the string can be defined as follows:

$$K(t) := \frac{1}{2} m v^2 = \frac{1}{2}\int_0^L \rho\, |u_t(x,t)|^2\, dx.$$

The Hook's stretch potential energy is

$$U(t) = \frac{k}{2}\int_0^L |u_x(x,t)|^2\, dx,$$

Verify that the total energy $E(t) = K(t) + U(t)$ of the string is constant, i.e., $\frac{dE}{dt} = 0$, and conclude that $E(t) = E(0)$. Thus, in order to solve a wave equation, we require the initial disturbance $u(x,0)$ and initial velocity $u_t(x,0)$.

1.1.2 General remarks

As we saw from the above example, ODEs and PDEs differ in some significant aspects. We can summarize the differences as follows.

- The solution to an ODE is a function or a set of functions of only one independent variable, whereas for a PDE, the solution depends on two or more independent variables. For the wave equation, the solution u depends on two variables, x and t, and represents the position of point x at time t. For this reason, the derivatives in a partial differential equation are partial derivatives instead of ordinary ones.

- While ODEs are related to pointwise quantities, PDEs are mathematical models of distributed systems or continua. For this reason, PDEs are sometimes considered as an infinite set of ODEs.

- The general solution of an ODE depends on one or more constant parameters. For example, the solution of the harmonic oscillator $\ddot{x} + \frac{k}{m}x = 0$ can be expressed as a linear combination of two fundamental solutions $\{\cos(\omega_0 t), \sin(\omega_0 t)\}$, where $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural frequency of the single mass-spring system. The solution is given by

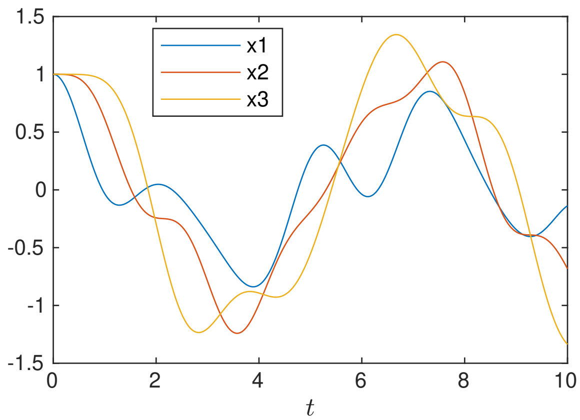
  $$x(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t),$$

  where c_1 and c_2 are arbitrary constant parameters. In contrast, the solution of PDEs usually depends on arbitrary functions. For example, the solution of the wave equation $u_{tt} = \frac{k}{\rho} u_{xx}$ for constant k and ρ can be of the form $u(x,t) = f(x - ct) + g(x + ct)$, where $c = \sqrt{\frac{K}{\rho}}$, and f and g are arbitrary smooth functions.

- The geometry of the solution to an ODE is a curve, while for a PDE it is a surface in space or generally a hypersurface. For example, the graph of the solution $u(x,t)$ of a wave equation in the space (x,t,u) is a surface and not a curve.

- The solution of the wave equation can be interpreted from a physical perspective as a traveling wave moving to the right as $f(x - ct)$ and left as $g(x + ct)$ with velocity $c = \sqrt{\frac{k}{\rho}}$. In contrast, the ordinary differential equations for a coupled mass-spring system do not reflect such a traveling speed. For example, consider a system of three masses connected to three springs in series, where the initial displacement of the first mass is $x_1(0) = 1$ and the other two are at rest without stretch. Assuming zero initial velocity for all three masses ($\dot{x}_1(0) = \dot{x}_2(0) = \dot{x}_3(0)$), the initial kinetic energy of the system is zero, and the system evolves its dynamics based on its initial potential energy $U(0) = \frac{1}{2}|x_1(0)|^2$.

  The following figure shows the motion of all three masses. As we observe, the motion of m_1 affects m_3 immediately, i.e., the propagation speed of the disturbance in the system is infinite.



Exercise 1.3. For each of the following partial differential equation, verify that the given solution satisfies the equation

a) $y\partial_x u - x\partial_y u = 0$, $u = f(x^2 + y^2)$

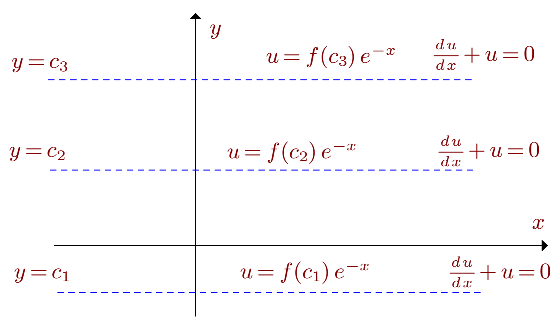
b) $x\partial_x u - y\partial_y u = 0$, $u = f(x, y)$

c) $\partial_x u - \partial_y u = -u$, $u = f(x + y)\, e^{-x}$

Exercise 1.4. For a multivariable function $u = u(x, y)$, equations that involve only the partial derivatives of one variable are generally referred to as defective equations. One example is the equation $u_x + u = 0$, where $u = u(x, y)$. This equation can be solved by the methods outlined in ODEs, resulting in the solution $u(x, y) = f(y)\, e^{-x}$, where f is an arbitrary function. The function f serves as a constant parameter for integrating the ODE

$$\frac{du}{dx} + u = 0.$$

The following figure show the geometry of the above argument



Using the appropriate method, solve the following equations, where u is a two-variable function $u = u(x, y)$:

a) $u_{xx} + c^2 u = 0$, $c > 0$ a constant.

b) $u_x + u = y$

c) $\partial_y u + u = u^2$

d) $u_{xy} + u_x = 0$

e) $x^2 u_{xx} + x u_x + u = y$.

1.1.3 From Newton cooling law to heat equation

Remember from ODEs that the temperature of a quantity located in an ambient space of temperature T_e follows the Newton's cooling rule:

$$\frac{dT}{dt} = \alpha(T_e - T),$$

where $\alpha > 0$ is a proportionality factor that depends on the conductivity of the material, and $T = T(t)$ is the temperature function of the material as a function of time t. The function $T_e - T$ is the temperature gradient of the material compared to the temperature of the ambient space, T_e. The solution of the above ODE is an exponential function:

$$T(t) = T_e + (T_0 - T_e)\, e^{-\alpha t},$$

where $T_0 = T(0)$ is the initial temperature of the material.

Now, consider a conductive rod that is insulated from the surrounding environment, with an initial temperature profile given by the function $f(x)$. The temperature at any point x of the rod at time t is denoted by $u(x, t)$. Consider the interval between points a and b on this rod. If $q(x, t)$ is the density of the thermal energy, then the total thermal energy in this interval is:

$$Q_{[a,b]}(t) = \int_a^b c\rho q(x, t)\, dx,$$

where ρ is the mass density and c is the specific heat capacity of the material. The change in energy from t to $t + \delta t$ is:

$$Q_{[a,b]}(t + \delta t) - Q_{[a,b]}(t) = \int_a^b c\rho\, [q(x, t + \delta t) - q(x, t)]\, dx,$$

and by thermodynamics:

$$q(x, t + \delta t) - q(x, t) \approx u(x, t + \delta t) - u(x, t).$$

In the limiting case as $\delta t \to 0$, we obtain:

$$\frac{dQ_{[a,b]}}{dt}(t) = \int_a^b c\rho u_t(x, t)\, dx.$$

On the other hand, if no thermal energy is produced or lost in the interval, the only factor that contributes to the change of thermal energy in this interval is the energy escaping through the endpoints $x = a$ and $x = b$. The Fourier law states that the heat gradient is proportional to u_x, that is:

$$\frac{dQ_{[a,b]}}{dt}(t) = \alpha u_x(b, t) - \alpha u_x(a, t),$$

where α is the conductivity factor of the material. Assuming α is a constant, the above relation can be written as:

$$\alpha u_x(b, t) - \alpha u_x(a, t) = \int_a^b \alpha u_{xx}(x, t)\, dx,$$

and finally:

$$\int_a^b [c\rho u_t(x, t) + \alpha u_{xx}(x, t)]\, dx = 0.$$

This integral equation holds for arbitrary interval $[a,\ b]$, and if the integrand is continuous, we conclude the following partial for $u(x, t)$:

$$u_t(x, t) = k u_{xx}(x, t),$$

for $k = \frac{\alpha}{\rho c}$. This is the one-dimensional heat equation, which governs the evolution of temperature in a homogeneous material. The equation states that the time rate of change of the thermal energy in any interval is proportional to the rate of change of temperature and to the second spatial derivative of temperature.

The heat equation has important applications in various fields such as physics, engineering, and finance. In physics, it is used to model the diffusion of heat in a medium, the propagation of electromagnetic waves, and the behavior of quantum systems. In engineering, it is used to design and analyze heat transfer systems, and to model the behavior of materials subjected to thermal stress. In finance, it is used to model the pricing of financial derivatives, such as options and futures, where the underlying asset price follows a random diffusion process.

Exercise 1.5. Consider the heat equation $u_t = k u_{xx}$ where $k > 0$ is a constant.

a) Verify that the following function called the fundamental solution satisfies the equation for $t > 0$

$$\Phi(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-\frac{x^2}{4kt}}.$$

b) Assume that $f(x)$ is a smooth function. Verify the following relation

$$\lim_{t\to 0} \int_{-\infty}^{\infty} \Phi(x,t)\, f(x)\, dx = f(0).$$

as long as the integral exists. The above relation implies that $\Phi(x,t)$ behaves like a Dirac delta function when $t \to 0$.

Exercise 1.6. By Newton's cooling law, the heat flows from hot spots to cold spot.

a) Explain how this fact is embedded in the derivation of the heat equation. In particular, explain why k must be a positive constant.

b) If $k < 0$, the equation is called a reverse heat equation. Explain why in this case heat flows from cold spots to hot spots.

Exercise 1.7. Consider a conductive rod of length L which is insulated along its length and are insulated at the end points $x = 0$ and $x = L$. The insulation can be expressed mathematically as $u_x(0,t) = 0$ and $u_x(L,t) = 0$, where u_x denotes the temperature gradient. If so, show that the thermal energy in the system is conserved, that is, $\frac{dQ}{dt} = 0$, where Q is

$$Q_{[0,L]}(t) = \int_0^L u(x,t)\, dt.$$

1.1.4 From population dynamics to migration

The exponential equation, formulated in the eighteenth century, was the first mathematical model for population dynamics. It describes the growth of a living species in a certain region through the equation $\frac{dP}{dt} = \alpha P$, where $P = P(t)$ denotes the population and α is the growth rate. However, this model has proven to be inaccurate, and more sophisticated models, such as the logistic model, have been suggested.

In this context, we consider a possible extension of population dynamics for a living species that migrates along a long strip. Specifically, we examine a living zone in the shape of a strip populated with animals that have a fertility rate of α. We assume that these animals migrate along the strip with a constant velocity of c. Our objective is to determine the population in an arbitrary segment of the strip at any given time.

To achieve this, we need to define the population density function $\rho(x,t)$ such that the population in a segment $[a,b]$ at time t is equal to

$$P_{[a,b]}(t) = \int_a^b \rho(x,t)\, dx.$$

Here, we present the derivation through the following exercise.

Exercise 1.8. Based on the above settings

a) Suppose the offspring rate is zero, $\alpha=0$. The rate of change of $P_{[a,b]}$ in time is

$$\frac{dP_{[a,b]}}{dt}=\frac{d}{dt}\int_a^b \rho(x,t)\,dx=\int_a^b \rho_t(x,t)\,dx.$$

By conservation of population, this rate of change is equal to the rate of moving in this segment minus the rate of animals leaving this segment. Show that

$$\frac{dP_{[a,b]}}{dt}=c\rho(a,t)-c\rho(b,t),$$

and conclude

$$\int_a^b [\rho_t(x,t)+c\rho_x(x,t)]\,dx=0.$$

The above relation holds for any segment $[a,b]$, and thus we conclude the following differential equation

$$\rho_t(x,t)+c\rho_x(x,t)=0.$$

b) Verify that the function $\rho=f(x-ct)$ solves the above PDE for arbitrary smooth function f. Conclude that if $\rho(x,0)=\rho_0\,e^{-x^2}$, then $\rho(x,t)=\rho_0\,e^{-(x-ct)^2}$. If $c=1$, find the population in the segment $[-1,1]$ at time $t=3$.

c) Now, let us assume that $\alpha>0$ and $\frac{d\rho}{dt}=\alpha\rho$. Verify that the PDE in this case is

$$\rho_t+c\rho_x=\alpha\rho.$$

Verify that the function $\rho=f(x-ct)\,e^{\alpha t}$ solves the equation. For $\rho(x,0)=100e^{-x^2}$, $c=1$ and $\alpha=0.3$, draw the solution $\rho(x,t)$ at $t=0,1,2,3$.

1.2 Differential operators in higher dimensions

In order to extend the study of partial differential equations to higher dimensional spaces, we require the use of appropriate differential operators. Among the most crucial of these operators are the gradient, divergence, and Laplacian. These operators enable us to express and manipulate partial derivatives in these spaces, which is essential for solving many important problems in fields such as physics, engineering, and mathematics.

Before we begin, let's establish some general notation used in this book. The set of real numbers is denoted by $\mathbb{R}$, and the set of n-tuples by $\mathbb{R}^n$. In $\mathbb{R}^2$, a point is written as (x,y); in $\mathbb{R}^3$, as (x,y,z); and in $\mathbb{R}^n$, as $x=(x_1,\ldots,x_n)$. The standard unit vectors in $\mathbb{R}^n$ are denoted by

$$\hat{e}_1,\ldots,\hat{e}_n,$$

where $\hat{e}_j$ is a vector with zeros in all coordinates except the j-th coordinate, which is 1. Thus, a point x can be considered as a vector $x=x_1\,\hat{e}_1+\cdots+x_n\,\hat{e}_n$. The set of all vectors in $\mathbb{R}^n$ forms a vector space with scalar multiplication given by

$$\lambda\,(x_1,\ldots,x_n)=(\lambda x_1,\ldots,\lambda x_n),\lambda\in\mathbb{R}$$

and vector addition defined as

$$(x_1,\ldots,x_n)+(y_1,\ldots,y_n)=(x_1+y_1,\ldots,x_n+y_n).$$

The magnitude of a vector $x=(x_1,\ldots,x_n)$ is denoted as $|x|$ and defined as

$$|x|=\left(\sum_{j=1}^n x_j^2\right)^{\frac{1}{2}}$$

The standard dot product between two vectors x and y is denoted as $x \cdot y$ and defined as:

$$x \cdot y = \sum_{j=1}^{n} x_j y_j.$$

Alternatively, we may use the more general notation $\langle , \rangle$ for the dot product, i.e., $\langle x, y \rangle = x \cdot y$. The distance between two points x and y in $\mathbb{R}^n$ is defined as

$$|x - y| = \sqrt{\langle x - y, x - y \rangle} = \sqrt{(x_1 - y_1)^2 + \cdots + (x_\mathrm{n} - y_\mathrm{n})^2}.$$

For a point $a \in \mathbb{R}^\mathrm{n}$ and $r > 0$, $B_\mathrm{r}(a)$ denotes the r-ball centered at a. The unit ball with the center at the origin is denoted by B. A set $\Omega \subset \mathbb{R}^\mathrm{n}$ is open if for any $a \in \Omega$, there exists an $r > 0$ such that $B_r(a) \subset \Omega$. Equivalently, a set is open if that every point in the set has a neighborhood that is contained entirely within the set. The boundary of a set Ω is denoted by $\mathrm{bnd}(\Omega)$, and its closure by $\bar{\Omega}$. For example, the set $\Omega = \{(x, y), r_1 < |x - y| < r_2\}$ is open, with the closure $\bar{\Omega} = \{(x, y), r_1 \leq |x - y| \leq r_2\}$. The set $\Omega_1 = \{(x, y), r_1 < |x - y| \leq r_2\}$ is neither open nor closed.

We use the notation u_x, u_y for the partial derivatives of a two-variable function $u = u(x, y)$:

$$u_x := \frac{\partial u}{\partial x}, u_y := \frac{\partial u}{\partial x}.$$

On occasion, we also use the notation $\partial_x u$ and $\partial_y u$ for the partial derivatives. For a function u of n variables, $u = u(x_1, \ldots, x_\mathrm{n})$, the partial derivatives are denoted by $\partial_\mathrm{j} u$ for $\frac{\partial u}{\partial x_j}$. The second-order partial derivatives are similarly denoted by $\partial_{ij} u = \frac{\partial^2 u}{\partial x_i \partial x_j}$.

1.2.1 Differential operators in Cartesian coordinate

The differential operator ∇, called *nabla*, in $\mathbb{R}^n$ is defined as:

$$\nabla := \hat{e}_1 \partial_1 + \cdots + \hat{e}_n \partial_n$$

Let $u = u(x_1, \ldots, x_n)$ be a smooth scalar function defined on an open set $\Omega \subset \mathbb{R}^n$. The gradient of u, denoted by ∇u, is defined as:

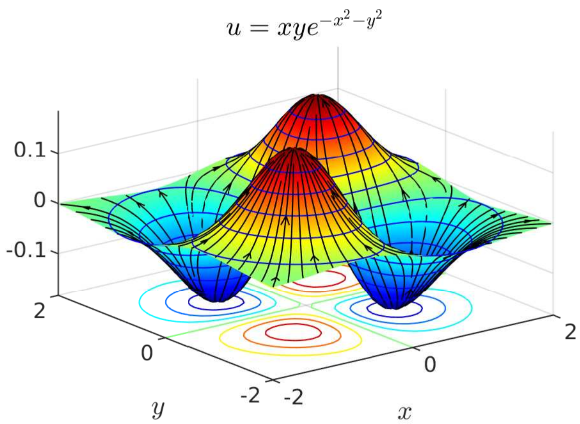
$$\nabla u = \partial_1 u \, \hat{e}_1 + \cdots + \partial_n u \, \hat{e}_n.$$

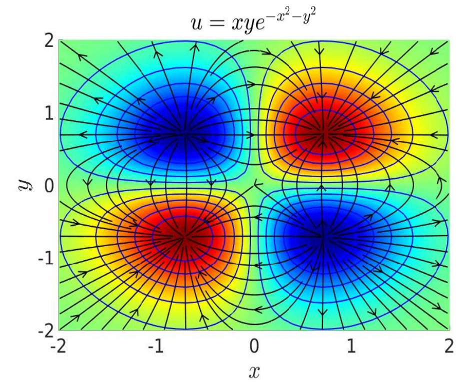
For a fixed point $x_0 \in \Omega$, $\nabla u(x_0)$ is a vector, and $\nabla u(x)$ for $x \in \Omega$ defines a vector field over Ω which assigns a vector to each point in Ω. This vector field is called the gradient vector field. The gradient of u, is a vector field that points in the direction of maximum increase of u at each point in Ω. The following figures illustrate the graph of the function $u = xy \exp(-x^2 - y^2)$ and the gradient trajectories which are the solutions of the following system:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \nabla u(x, y),$$

or equivalently

$$\begin{cases} \frac{dx}{dt} = (1 - 2x^2) y e^{-x^2 - y^2} \\ \frac{dy}{dt} = (1 - 2y^2) x e^{-x^2 - y^2} \end{cases}.$$





The direction of the gradient flow lines in heat flow or fluid dynamics is opposite to that of heat or fluid flow. This is because heat and fluid tend to move from regions with higher temperatures or pressures to those with lower temperatures or pressures, whereas the gradient vector field points in the direction of maximum increase of the scalar function at each point.

In the context of a two-dimensional elastic membrane, the function $u(x,\ y)$ represents the vertical displacement of the membrane at point (x, y). The tension $T(x, y)$ on the boundary of the membrane, $\mathrm{bnd}(\Omega)$, is defined as the product of a constant τ and the gradient of $u(x, y)$, $\nabla u(x, y)$. This means that the tension at a point on the boundary is proportional to the rate of change of the displacement at that point.

The vertical Hook's force exerted at a point (x, y) in the interior of the membrane is defined as the limit of the average tension along the boundary of a shrinking region Ω around the point, as the size of Ω approaches zero. The force is given by the line integral of the tension over the boundary $\mathrm{bnd}(\Omega)$, with respect to the outward unit normal vector ν, divided by the area of Ω

$$F(x, y) = \lim_{|\Omega| \to 0} \frac{1}{A(\Omega)} \oint_{\mathrm{bnd}(\Omega)} \tau \nabla u \cdot \nu dS.$$

In other words, the Hook's force measures the rate of change of the tension with respect to the area of the membrane. This force is important in studying the behavior of elastic membranes, such as in the design of structures that use membranes as a supporting element.

Another important fact that we will use extensively in the next chapter of this book is the time or mass derivative of a scalar function along a smooth curve. For a smooth parametrized curve $\gamma(t) = (x(t), y(t))$, the derivative of $u(x, y)$ along γ is defined as:

$$\frac{du}{dt}(\gamma(t)) = \nabla u(\gamma(t)) \cdot \gamma'(t),$$

where $\gamma'(t)$ is the tangent vector to γ at time t. This expression represents the instantaneous rate of change of $u(x, y)$ at the point $\gamma(t)$ along the curve $\gamma(t)$, and it takes into account both the direction and magnitude of the tangent vector $\gamma'(t)$. In other words, it tells us how quickly $u(x, y)$ is changing as we move along the curve $\gamma(t)$.

In the context of fluid mechanics, if $\gamma(t)$ denotes the trajectory of a fluid particle in the xy-plane, then $\gamma'(t)$ represents its velocity vector, and $\frac{du}{dt}$ represents the instantaneous rate of change of $u(x, y)$ at each point on the trajectory of the particle as it moves through the fluid.

1.1. It appears that the natural world operates more justly than our modern economic system. In the context of money, wealth tends to flow from poor individuals to those who already possess more, following the gradient of wealth inequality. However, the laws of physics dictate the opposite behavior: heat, pressure, mass, and other quantities flow from regions of high density to those of lower density.

Exercise 1.9. Consider the temperature distribution in the xy-plane given by $T(x,y)=10\exp(x^2-y^2)$. Suppose a runner is moving along the curve $\gamma(t)=(\cos(t),\sin(t))$, and wearing a hand clock marked from 0 to 60. What is the angular velocity of the hand of the runner's watch as they move along the curve? Find the angular velocity of the hand clock if the runner runs given by the following function $\gamma(t)=(\cos(\omega_0 t),\sin(\omega_0 t))$ for some $\omega_0>0$.

The divergence of a vector field $F=(f_1,\dots,f_n)$ on a domain $\Omega\subset\mathbb{R}^n$ is a scalar function denoted by $\nabla\cdot F$, defined by

$$\nabla\cdot F=\sum_{j=1}^{n}\frac{\partial f_j}{\partial x_j}.$$

Geometrically, the divergence of a vector field measures the extent to which the field "flows out" from a given point in space. If $\nabla\cdot F(x_0)>0$, then the point x_0 behaves like a source for the field; that is, the field flows outward from x_0. If $\nabla\cdot F(x_0)>0$, then x_0 behaves like a sink, and the field flows inward toward x_0. If $\nabla\cdot F(x_0)=0$, then x_0 is a neutral point for the field. The concept of divergence is fundamental in fluid dynamics, where it is used to study the behavior of fluids in motion.

For example, the vector field $F=\nabla u$, for $u=xy\exp(-x^2-y^2)$, is equal to

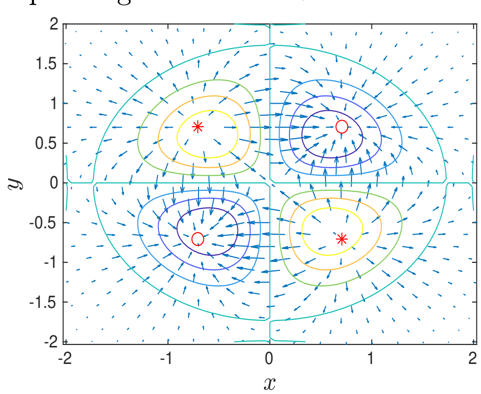
$$\nabla\cdot F(x,y)=4xy(x^2+y^2-3)\exp(-x^2-y^2).$$

For a vector field $x\mapsto F(x)=(f_1(x),\dots,f_n(x))$ for $x\in\Omega\subset\mathbb{R}^n$, the divergence denoted by $\nabla\cdot F$ is defined as the scalar function

$$\nabla\cdot F=\sum_{j=1}^{n}\partial_j f_j.$$

The divergence of a vector field measures how much the vector field "flows out" from a given point in space. If $\nabla\cdot F(x_0)$ is positive, then x_0 behaves like a source for the given field. If $\nabla\cdot F(x_0)$ is negative, it behaves like a sink and if $\nabla\cdot F(x_0)$ is zero, the point is a neutral point for the field.

The following figure depicts the vector field $F=\nabla u$, where $u=xy\exp(-x^2-y^2)$, and the corresponding contours of $\nabla\cdot F$.



One of the important results related to the divergence of a vector field is the Gauss or divergence theorem. The theorem relates the integral of the divergence of a vector field over a volume to the flux of the vector field through the surface enclosing the volume. It is a fundamental theorem in vector calculus and has many applications in physics and engineering.

Theorem 1.1. (Gauss) *Assume that F is a smooth vector field in an open bounded set $\Omega\subset\mathbb{R}^n$. The following equality holds:*

$$\int_\Omega\nabla\cdot F dV=\oint_{\mathrm{bnd}(\Omega)}F\cdot\nu dS,$$

where dV is the volume element, dS is the surface element, ν is the unit normal vector to the surface, and the dot product $F \cdot \nu$ is the flux of the vector field through the surface.

Exercise 1.10. Let B_R^n denote the ball of radius R in $\mathbb{R}^n$ and S_R^{n-1} the surface of the ball. Use divergence theorem to show that

$$V(B_R^n) = \frac{A(S_R^{n-1})}{n} R,$$

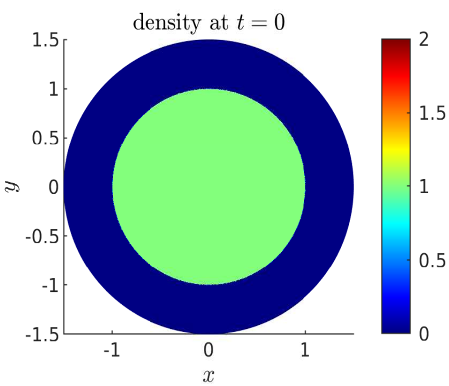
where $V(B_R^n)$ and $A(S_R^{n-1})$ are the volume and surface area of B_R^n and S_R^{n-1} respectively. Verify your formula for $n = 2, 3$. (Hint: consider the function $u = x_1^2 + \cdots + x_n^2$).

Remark 1.1. The divergence of a vector field at a point x can be defined using the divergence theorem. Specifically, let F be a smooth vector field in an open set containing x, and let Ω be a small, open, three-dimensional ball centered at x. Then, the divergence of F at x is given by:

$$\nabla \cdot f(x) = \lim_{|\Omega| \to 0} \frac{1}{\mathrm{vol}(\Omega)} \oint_{\mathrm{bnd}(\Omega)} F \cdot \nu dS,$$

where $\mathrm{vol}(\Omega)$ is the volume of Ω, and the limit is taken as the size of Ω approaches zero. In other words, the divergence of F at x is equal to the flux of F across the boundary of a small ball centered at x, divided by the volume of the ball.

Consider a homogeneous fluid with a constant density distributed in the region $x^2 + y^2 < 1$, which is shown in the figure below:



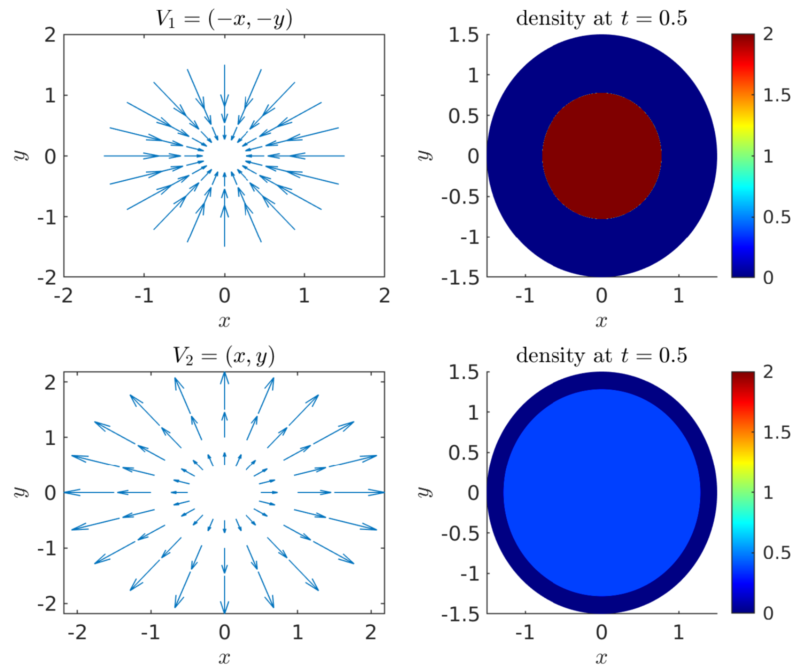
Suppose the fluid moves according to the velocity vector field $V_1 = (-x, -y)$. As expected from the shape of the vector field, the fluid will concentrate at the origin at later times, and the density of the fluid will increase around the origin. In fact, the flow lines of this vector field are the solutions of the following system of ODEs

$$\begin{cases} \frac{dx}{dt} = -x \\ \frac{dy}{dt} = -y \end{cases},$$

which is solved for $\gamma(t) = (x_0 e^{-t}, y_0 e^{-t})$. The top figure in the following image depicts this scenario. If the velocity field changes to $V_2 = (x, y)$, the fluid will spread out, according to the system

$$\begin{cases} \frac{dx}{dt} = x \\ \frac{dy}{dt} = y \end{cases},$$

and the density will decrease around the origin, as shown in the bottom figure.



Note that the behavior of the fluid is determined by the divergence of the velocity field. In the first scenario, the divergence of V_1 is negative, indicating that the fluid is flowing inward toward the origin, while in the second scenario, the divergence of V_2 is positive, indicating that the fluid is flowing outward from the origin. This observation is consistent with the divergence theorem presented earlier, which shows the relationship between the flow of a vector field and its divergence.

Exercise 1.11. Let the initial density of a fluid flow is given by the function

$$f(x,y) = \begin{cases} x\,y\exp(-x^2-y^2) & x^2+y^2<1 \\ 0 & \text{otherwise} \end{cases}.$$

Find the flow lines of the fluid if the velocity vector field is given by $V = (-y, x)$. Round the following code in Matlab and explain the change of density function in time.

```
[thet,r]=meshgrid(-pi:pi/1000:pi,0:0.005:1);
[x,y]=pol2cart(thet,r);
xt=@(t) x*cos(t)+y*sin(t);
yt=@(t) -x*sin(t)+y*cos(t);
zt=@(t) (xt(t).*yt(t).*exp(-x.^2-y.^2)).*(x.^2+y.^2<1)+0.*(x.^2+y.^2>=1);
for i=0:1:2
 subplot(2,2,i+1)
 surf(x,y,zt(i*pi/4)); shading interp; view(2); axis equal tight; grid off; colormap jet
 clim([-0.25,0.25]); colorbar;
 title(sprintf('density at $t=%.2f$',i*pi/4),'interpreter','latex','fontsize',10);
end
```

Exercise 1.12. A gradient vector field of the form $F = -\nabla u$ for a scalar filed is also called conservative fields. The reason is due to the following fact. Consider mass particle m moving along an arbitrary path $\gamma(t)$ in this field. The total energy of the mass is

$$E = \frac{1}{2}m\,|v|^2 + u(x).$$

Use the NEWTON's second law and show that the derivative of E along $\gamma(t)$ is zero. Therefore, a conservative force field *conserves* the total energy of a mass particle.

Exercise 1.13. The line integral of a vector field F along an arbitrary smooth path $\gamma(t)$ for $t\in(a,b)$ is defined as follows

$$\int_\gamma F := \int_a^b F(\gamma(t))\cdot\gamma'(t)\,dt.$$

Show that if $F=-\nabla u$, then the integral is independent of γ and is equal to $u(\gamma(b))-u(\gamma(a))$. This integral shows the amount of work or kinetic energy spend to move a particle from point $\gamma(a)$ to $\gamma(b)$ and as it is seen, this amount is equal to the difference between the potential at these two points.

Exercise 1.14. Show that the line integral of a vector field is independent of the parametrization, that is, if γ_1: $(c,d)\to C$ then

$$\int_\gamma F(\gamma(t))\cdot\gamma'(t)\,dt = \int_{\gamma_1} F(\gamma_1(t))\cdot\gamma_1'(t)\,dt.$$

Exercise 1.15. The divergence of a two dimensional smooth vector filed $F=(f,g)$ is

$$\nabla\cdot F = f_x + g_y.$$

Let R be the rectangle $R:=[-a,a]\times[-b,b]$. We prove the relation

$$\nabla\cdot F(0,0) = \lim_{a,b\to 0}\lim_{a,b\to 0}\frac{1}{A(R)}\oint_{\mathrm{bnd}(R)} F\cdot\nu\,dl$$

a) Show that the net flux passing through the boundary of R is

$$\int_{-b}^{b}\{f(a,y)-f(-a,y)\}dy + \int_{-a}^{a}\{g(x,b)-g(x,-b)\}dx.$$

b) Use the relations

$$f(a,y)-f(-a,y) = \int_{-a}^{a} f_x(x,y)dx, \qquad g(x,b)-g(x,-b) = \int_{-b}^{b} g_y(x,y)dy,$$

and show the relation

$$\int_{-b}^{b}\{f(a,y)-f(-a,y)\}dy + \int_{-a}^{a}\{g(x,b)-g(x,-b)\}dx = 4ab\{f_x(\xi_1,\eta_1)+g_y(\xi_2,\eta_2)\}$$

for some $-a<\xi_1,\xi_2<a$, $-b<\eta_1,\eta_2<b$ and conclude

$$\lim_{a,b\to 0}\frac{1}{A(R)}\oint_{\mathrm{bnd}(R)} F\cdot\nu dl = f_x(0,0)+g_y(0,0) = \nabla\cdot F(0,0).$$

c) Use the same argument and show that for the cube $C:=[-a,a]\times[-b,b]\times[-c,c]$ and the smooth field $F=(f,g,h)$, the following relation holds

$$\lim_{a,b,c\to 0}\frac{1}{\mathrm{vol}(C)}\oiint_{\mathrm{bnd}(C)} F\cdot\nu dS = f_x+g_y+h_z\Big|_{(0,0,0)}.$$

Exercise 1.16. Let $B_a\subset\mathbb{R}^n$ be a ball of radius a in n-dimensional space. Consider the vector field $F(x)=x$, and use the divergence theorem to conclude that the volume of B_a to its surface area is equal to $\frac{a}{n}$.

Exercise 1.17. Let $\Omega\subset\mathbb{R}^3$ be a domain with smooth boundary surface $\mathrm{bnd}(\Omega)$. Use the divergence theorem and show the following relations

a) Let F be a smooth vector field u a smooth scalar function, then

$$\mathrm{div}\,(uF) = u\,\mathrm{div}\,(F) + F\cdot\nabla u,$$

where div stands for $\nabla\cdot$.

b) Use the above result and show the following relation

$$\iiint_\Omega u\,\mathrm{div}\,(F)\,dV = \oiint_{\mathrm{bnd}(\Omega)} u\,F\cdot\nu\,dS - \iiint_\Omega F\cdot\nabla u\,dV.$$

c) If ϕ,ψ are smooth functions defined on Ω, then

$$\iiint_\Omega \psi\Delta\phi dV = \oiint_{\mathrm{bnd}(\Omega)}\psi\partial_n\phi dS - \iiint_\Omega \nabla\phi\cdot\nabla\psi dV,$$

where $\partial_n\phi$ stands for $\nabla\phi\cdot\nu$.

d) Use the above result and show the following relation called the GREEN's formula:

$$\iiint_{\Omega}(\phi\Delta\psi-\psi\Delta\phi)\,dV=\oint\!\!\!\oint_{\mathrm{bnd}(\Omega)}[\phi\,\partial_n\psi-\psi\,\partial_n\phi]\,dS.$$

The Laplacian operator $\Delta:=\nabla\cdot\nabla$ is a differential operator that is commonly used in mathematics and physics. As it is evident from its definition, the Laplacian of a scalar function u is equal to the divergence of the gradient of u, and in the coordinate form is equal to

$$\Delta u=\sum_{j=1}^{n}\partial_{jj}u,$$

for a smooth function $u=u(x_1,\ldots,x_n)$.

The Laplacian operator has important applications in physics, such as in the study of the diffusion equation, wave equation, and Schrödinger equation. In these contexts, the Laplacian operator describes the behavior of physical quantities such as temperature, pressure, electric potential, and wave functions. In the following discussion, we present an example of how the Laplacian operator is applied in the field of electrostatics.

Let's consider an electric charge Q located at the origin of a three-dimensional space. The potential field generated by this charge at any point $r=(x,y,z)\neq(0,0,0)$ is given by

$$\Phi(r)=\frac{Q}{4\pi\varepsilon_0\,|r|},$$

where ε_0 is the permittivity of the air and $|r|$ is the distance of $\boldsymbol{x}$ to the origin

$$|r|=\sqrt{x^2+y^2+z^2}.$$

This potential field generates an electric field

$$E(r):=-\nabla\Phi(r)=\frac{Q\,r}{4\pi\varepsilon_0\,|r|^3},$$

with a magnitude of $|E(r)|=\frac{Q}{4\pi\varepsilon_0\,|r|^2}$ and direction of $\hat{r}=\frac{r}{|r|}$. Let B_R be the ball with radius R centered at the origin. We can calculate the net amount of electric field passing through the boundary of this ball, denoted by S_R, as follows:

$$\oint\!\!\!\oint_{S_R}E(r)\cdot\nu(r)\,dS,$$

where $\nu=\hat{r}$ for $r\neq 0$. Hence,

$$\oint\!\!\!\oint_{S_R}E(r)\cdot\nu(r)\,dS=\oint\!\!\!\oint_{S_R}\frac{Q}{4\pi\varepsilon_0\,r^2}\,dS=\frac{Q}{4\pi\varepsilon_0\,R^2}\oint\!\!\!\oint_{S_R}dS=\frac{Q}{\varepsilon_0}.$$

As observed, the net amount of electric field passing through S_R is independent of R and is equal to $\frac{Q}{\varepsilon_0}$.

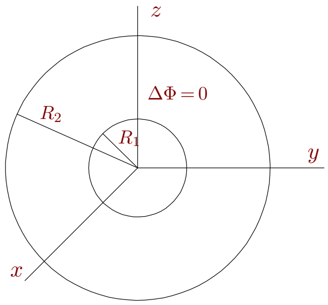
Now, let's consider the divergence of the electric field $E(r)$ for $r\neq 0$. By direct calculation, we have

$$\begin{aligned}\Delta\Phi=\nabla\cdot E(r)&=\frac{\partial}{\partial x}\left(\frac{Q\,x}{4\pi\varepsilon_0|r|^3}\right)+\frac{\partial}{\partial y}\left(\frac{Q\,y}{4\pi\varepsilon_0|r|^3}\right)+\frac{\partial}{\partial z}\left(\frac{Q\,z}{4\pi\varepsilon_0|r|^3}\right)\\&=\frac{Q}{4\pi\varepsilon_0}\left(\frac{|r|^3-3x^2|r|}{|r|^5}+\frac{|r|^3-3y^2|r|}{|r|^5}+\frac{|r|^3-3z^2|r|}{|r|^5}\right)=0\end{aligned}.$$

Therefore, $\Delta\Phi = 0$ everywhere except at $r = 0$.

Note that we can apply the divergence theorem to the integral of $\Delta\Phi$ over the region between two concentric balls B_{R_1} and B_{R_2}, which gives

$$\iiint_{B_{R_2}} \Delta\Phi \, dV = \iiint_{B_{R_1}} \Delta\Phi \, dV + \iiint_{B_{R_2} - B_{R_1}} \Delta\Phi \, dV.$$



By the result $\Delta\Phi = 0$ for $r \neq 0$, we conclude

$$\iiint_{B_{R_2}} \Delta\Phi \, dV = \iiint_{B_{R_1}} \Delta\Phi \, dV,$$

and by the Gauss theorem, we obtain

$$\oiint_{S_{R_2}} E \cdot \nu dS = \oiint_{S_{R_1}} E \cdot \nu dS.$$

One can use the above results and solve the following problems.

Exercise 1.18. Let Ω be any open bounded domain with the smooth boundary $\mathrm{bnd}(\Omega)$, and let and Φ be the potential field of a pointwise Q-charge. Show that if Q is located inside Ω, then

$$\iiint_{\Omega} \Delta\Phi \, dV = \frac{Q}{\varepsilon},$$

and if Q is located outside $\bar{\Omega}$, then

$$\iiint_{\Omega} \Delta\Phi \, dV = 0.$$

Exercise 1.19. Let two electric charges $Q, -Q$ are located inside an open bounded domain Ω. Show

$$\iiint_{\Omega} \Delta\Phi \, dV = 0.$$

Exercise 1.20. Suppose there is an open bounded set Ω in $\mathbb{R}^3$, and it contains an electric charge density q. The total electric charge inside Ω can be expressed as

$$Q = \iiint_{\Omega} q(r) \, dV.$$

Show the following relation

$$\iiint_{\Omega} \Delta\Phi \, dV = \frac{Q}{\varepsilon_0}.$$

Hint: consider a differential element qdV and let $E_q(r)$ be the electric field generated by qdV at point r.

1.2.2 Differential operators in polar and spherical coordinates

Remember that the polar coordinate is defined through the transformation

$$T(r, \theta) = (x, y) = (r\cos(\theta), r\sin(\theta)),$$

where $r \in [0, \infty)$ and $\theta \in [-\pi, \pi]$. In order to find the form of the operator nabla ∇ in this coordinate, wen need first to determine the vectors $\hat{r}, \hat{\theta}$, the unit vector along r and θ directions. For fixed $\theta = \theta_0$ The transformation $T(r, \theta_0)$ is a parameterized curve and then $T_r(r, \theta')$ is tangent to this curve. The unit vector $\hat{r}$ is then obtained as

$$\hat{r} = \frac{T_r(r, \theta_0)}{|T_r(r, \theta)|} = (\cos(\theta_0), \sin(\theta_0)).$$

As it is observed, the unit vector $\hat{r}$ depends on the chosen angle θ_0, and then

$$\hat{r} = \hat{r}(\theta) = (\cos(\theta), \sin(\theta)).$$

Now, fix $r = r_0$, and determine $\hat{\theta}$ by the relation

$$\hat{\theta} = \frac{T_\theta(r_0, \theta)}{|T_\theta(r_0, \theta)|} = (-\sin(\theta), \cos(\theta)).$$

As it is seen again, $\hat{\theta} = \hat{\theta}(\theta)$, is a function of θ and independent of r.

By the above calculations, it is evident that $\hat{r} \cdot \hat{\theta} = 0$, and thus they are orthogonal. Transformations for which this property holds are said orthogonal transformations. Moreover, we have

$$\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta}, \frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r},$$

and $\frac{\partial \hat{r}}{\partial r} = \frac{\partial \hat{\theta}}{\partial r} = 0$. Note that $\frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r}$ contributes to the centrifugal force experienced by a particle moving in a circular path around the origin in the polar coordinate system. The vector $\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta}$, on the other hand, points in the tangential direction to the curve traced out by a particle moving in a circular path around the origin in the polar coordinate system.

Now, let $u(r, \theta)$ be a given scalar function in polar coordinates. We can write ∇u in polar coordinates as:

$$\nabla u = \alpha(r, \theta)\ \hat{r} + \beta(r, \theta)\ \hat{\theta},$$

where we need to determine the appropriate functions α and β. To find α, we use the orthogonality condition, which yields:

$$\alpha(r, \theta) = \nabla u \cdot \hat{r} = (u_x, u_y) \cdot (\cos(\theta), \sin(\theta)) = u_x \cos(\theta) + u_y \sin(\theta).$$

Using the chain rule, we can also write:

$$\frac{\partial u}{\partial r} = u_x \frac{\partial x}{\partial r} + u_y \frac{\partial y}{\partial r} = u_x \cos(\theta) + u_y \sin(\theta),$$

and so we see that $\alpha(r, \theta) = u_r$. Similarly, we can find β as:

$$\beta(r, \theta) = \nabla u \cdot \hat{\theta} = -u_x \sin(\theta) + u_y \cos(\theta) = \frac{1}{r} u_\theta.$$

Therefore, we obtain:

$$\nabla_{(r,\theta)} u(r, \theta) = u_r \hat{r} + \frac{1}{r} u_\theta \hat{\theta},$$

and we can express the differential operator $\nabla_{(r,\theta)}$ as:

$$\nabla_{(r,\theta)} = \hat{r}\, \partial_r + \frac{1}{r} \hat{\theta}\, \partial_\theta,$$

where $\partial_r, \partial_\theta$ denote partial derivatives with respect to r and θ, respectively.

With the nabla operator in polar coordinates, we can determine the Laplacian $\Delta := \nabla \cdot \nabla$, which is crucial for our discussion of second-order partial differential equations. By using the Laplacian in polar coordinates, we can solve a variety of problems involving circular or cylindrical domains. For example, we can model the behavior of electric or magnetic fields inside a cylindrical conductor, or the flow of fluids inside a circular pipe. The Laplacian also plays an important role in the study of harmonic functions, which are solutions to the Laplace equation $\Delta u = 0$.

Based on the relation $\Delta := \nabla \cdot \nabla$ and the derived $\nabla_{(r,\theta)}$, we can obtain the Laplacian in polar coordinates as follows:

$$\Delta = \nabla \cdot \nabla = \left(\hat{r}\,\partial_r + \frac{1}{r}\hat{\theta}\,\partial_\theta\right) \cdot \left(\hat{r}\,\partial_r + \frac{1}{r}\hat{\theta}\,\partial_\theta\right).$$

Using the orthogonality relation $\hat{r} \cdot \hat{\theta} = 0$, and the partial derivative relations:

$$\begin{cases} \partial_r \hat{r} = \partial_r \hat{\theta} = 0 \\ \partial_\theta \hat{r} = \hat{\theta} \\ \partial_\theta \hat{\theta} = -\hat{r} \end{cases},$$

we can simplify the Laplacian to the following form in polar coordinates:

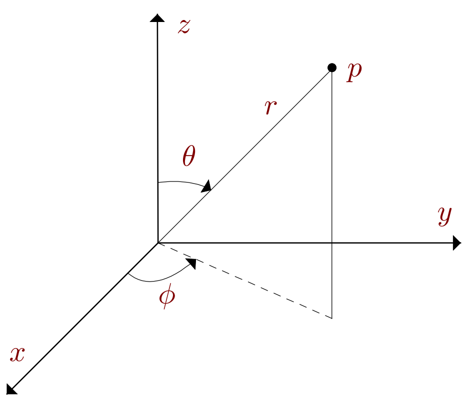
$$\Delta_{(r,\theta)} = \partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta}.$$

Exercise 1.21. Find the form of nabla ∇ in the cylindrical coordinate and then find the Laplacian in this coordinate system.

Let's now turn to the spherical coordinate system. In this system, the transformation is defined through the following relations:

$$T(\rho, \theta, \phi) = (x, y, z) = (r \sin(\theta)\cos(\phi), r\sin(\theta)\sin(\phi), r\cos(\theta)).$$

Here, $r \in [0, \infty)$ is the distance from the origin, $\theta \in [0, \pi]$ is the angle with respect to the z-axis, and $\phi \in [-\pi, \pi]$ is the angle with respect to the x-axis.



The unit vector $\hat{r}$ is defined as

$$\hat{r} = \frac{T_r(r,\theta,\phi)}{|T_r(r,\theta,\phi)|} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)).$$

Similarly, we have the following expressions for $\hat{\theta}$ and $\hat{\phi}$:

$$\hat{\theta} = \frac{T_\theta(r,\theta,\phi)}{|T_\theta(r,\theta,\phi)|} = (\cos(\theta)\cos(\phi), \cos(\theta)\sin(\phi), -\sin(\theta)),$$

$$\hat{\phi} = \frac{T_\phi(r,\theta,\phi)}{|T_\phi(r,\theta,\phi)|} = (-\sin(\phi), \cos(\phi), 0).$$

It can be easily verified that $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ are mutually orthogonal, and thus the transformation T defines an orthogonal transformation.

Given a scalar function $u = u(r,\ \theta,\ \phi)$, we can determine its gradient using the following expression:

$$\nabla u = \alpha(r, \theta, \phi)\,\hat{r} + \beta(r, \theta, \phi)\,\hat{\theta} + \gamma(r, \theta, \phi)\,\hat{\phi},$$

where functions α, β, and γ need to be determined. To find these functions, we can use the orthogonality conditions:

$$\alpha = \nabla u \cdot \hat{r} = u_x \sin(\theta)\cos(\phi) + u_y \sin(\theta)\sin(\phi) + u_z \cos(\theta) = u_r,$$

$$\beta = \nabla u \cdot \hat{\theta} = u_x \cos(\theta)\cos(\phi) + u_y \cos(\theta)\sin(\phi) - u_z \sin(\theta) = \frac{1}{r} u_\theta,$$

$$\gamma = \nabla u \cdot \hat{\phi} = -u_x \sin(\phi) + u_y \cos(\phi) = \frac{1}{r\sin(\theta)} u_\phi.$$

Using these expressions, we can derive the form of $\nabla_{(r,\theta,\phi)}$:

$$\nabla_{(r,\theta,\phi)} = \hat{r}\,\partial_r + \frac{1}{r}\hat{\theta}\,\partial_\theta + \frac{1}{r\sin(\theta)}\hat{\phi}\,\partial_\phi.$$

This can be used to determine the Laplacian in this coordinate system.

Exercise 1.22. To derive the Laplacian in the spherical coordinate:

a) Show the following relations for the spherical transformation

$$\partial_r \hat{r} = 0, \partial_\theta \hat{r} = \hat{\theta}, \partial_\phi \hat{r} = \sin(\theta)\,\hat{\phi}$$

$$\partial_r \hat{\theta} = 0, \partial_\theta \hat{\theta} = -\hat{r}, \partial_\phi \hat{\theta} = \cos(\theta)\hat{\phi}$$

$$\partial_r \hat{\phi} = 0, \partial_\theta \hat{\phi} = 0, \partial_\phi \hat{\phi} = -\sin(\theta)\,\hat{r} - \cos(\theta)\,\hat{\theta}.$$

b) Find the divergence of a vector filed $F = \alpha(r, \theta, \phi)\hat{r} + \beta(r, \theta, \phi)\,\hat{\theta} + \gamma(r, \theta, \phi)\hat{\phi}$ in this coordinate system.

c) Use $\nabla_{(r,\theta,\phi)}$ and the result of the above problem, and derive $\Delta_{(r,\theta,\phi)}$ which is

$$\Delta_{(r,\theta,\phi)} := \frac{1}{r^2}\partial_r(r^2\,\partial_r) + \frac{1}{r^2\sin(\theta)}\partial_\theta(\sin(\theta)\,\partial_\theta) + \frac{1}{r^2\sin^2(\theta)}\,\partial_{\phi\phi}.$$

Exercise 1.23. In this exercise, we will derive the form of the Laplacian for a general orthogonal curvilinear coordinate system defined by the transformation:

$$T(q_1, q_2, q_3) = (x(q_1, q_2, q_3), y(q_1, q_2, q_3), z(q_1, q_2, q_3)),$$

where q_1, q_2, and q_3 are the coordinates of a point in the curvilinear system, and x, y, and z are their Cartesian counterparts.

a) Find the unit vectors $\hat{q}_1$, $\hat{q}_2$, and $\hat{q}_3$ of the coordinate system.

b) Use the orthogonality condition and show the following relation for the nabla operator $\nabla_{(q_1,q_2,q_3)}$

$$\nabla = \frac{1}{|T_1|}\,\hat{q}_1\,\partial_1 + \frac{1}{|T_2|}\hat{q}_2\,\partial_2 + \frac{1}{|T_3|}\,\hat{q}_3\,\partial_3,$$

where $\partial_j = \frac{\partial}{\partial q_j}$, and $T_j = \frac{\partial T}{\partial x_j}$.

c) We use symbols $\gamma_{i,j}^k$ for the coordinates of the partial derivatives of $\dfrac{\partial\,\hat{q}_i}{\partial q_j}$ as follows

$$\frac{\partial\,\hat{q}_i}{\partial q_j} = \gamma_{ij}^1\,\hat{q}_1 + \gamma_{ij}^2\,\hat{q}_2 + \gamma_{ij}^3\,\hat{q}_3.$$

Show that $\gamma_{ij}^i = 0$.

d) Use the above notation and conclude

$$\Delta := \nabla \cdot \nabla = \sum_{i=1}^{3} \frac{1}{|T_i|^2}\partial_{ii} + \sum_{i=1}^{3} \frac{1}{|T_i|}\partial_i\left(\frac{1}{|T_i|}\right)\partial_i + \sum_{i,j=1}^{3} \frac{1}{|T_i|\,|T_j|}\gamma_{i,j}^j\,\partial_i$$

e) For a given vector field $F = P\,\hat{q}_1 + Q\,\hat{q}_2 + R\,\hat{q}_3$, find $\operatorname{div}(F)$.