# Chapter 7
# Heat and Wave Equations

In this chapter we present an elementary discussion on partial differential equations including one dimensional heat and wave equations. The second volume of this book is dedicated entirely to partial differential equations and studies the first and second order equations in great detail. To solve simple heat equations, we need two important tools: 1) eigenvalue problems 2) FOURIER series. We discuss them in sequel.

## 7.1 Introduction

### 7.1.1 Heat equation

Consider a conductive rod of length $L$. We can model this rod as a one dimensional object running from $0$ to $L$. Assume that the rod is insulated along $(0, L)$ and it can possibly exchange heat from its boundary point $x=0$ and or $x=L$. Let $u(x,t)$ denotes the temperature of the point $x$ at time $t$. Note there here the function $u$ is a function of two independent variables $x,t$, where $x\in[0,L]$ denotes the position along the rod and time $t$ is measured for $t\geq 0$. Let the temperature distribution of the rod at time $t=0$ is $u(x,0)=f(x)$. The initial heat distribution makes generally a flow of heat for $t>0$, and thus the temperature changes with time. In the second volume of the book, we derive the equation that describes $u(x,t)$ as a partial differential equation which is

$$\frac{\partial u}{\partial t}=D\frac{\partial^2 u}{\partial x^2}. \tag{7.1}$$

The above equation is called a heat equation. Here $D>0$ is a positive constant that depends on the heat capacity, the density and other physical factors of the rod.

The physical interpretation of the equation is clear. Fix $x_0\in(0,L)$. The term $\frac{\partial u}{\partial t}(x_0,t)$ measures the rate of change of the temperature at $x_0$ with respect to $t$. This quantity is equal to $\frac{\partial^2 u}{\partial x^2}(x_0,t)$. But notice that for fixed $t$, the term $\frac{\partial u}{\partial x}(x_0,t)$ measures the rate of change of $u$ at $x_0$ with respect to the *adjacent points* to $x_0$ at the fixed $t$. The term $\frac{\partial u}{\partial x}$ is also called the *heat gradient* because it measures the difference between $x_0$ and its neighboring points. For example if $\frac{\partial u}{\partial x}(x_0,t_0)=0$, them the total heat that flows through $x_0$ at the fixed time $t_0$ is equal zero. The term $\frac{\partial^2 u}{\partial x^2}(x_0,t)$ then measures the acceleration of the heat flow passing through $x_0$ at the fixed time $t$.

The equation (7.1) is not complete without the conditions at the boundary points $x=0$, $L$. In this chapter we consider three important cases:

1. DIRICHLET boundary conditions
2. NEUMANN boundary conditions
3. Mixed or ROBIN's boundary conditions

### DIRICHLET B.Cs.

In a DIRICHLET problem, the value $u(x,t)$ are known at $x=0, L$, and thus the equation reads

$$\begin{cases} \frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}. \\ u(0,t) = a, u(L,t) = b \end{cases},$$

where we assume that $a, b$ are constant values independent of $t$. If values $a, b$ are zero, the heat problem is called a *homogeneous problem*. Note that in this case, there is no thermal source at the boundary *and* along the rod. If a thermal source present along the rod that generate heat with the rate $h(x)$, the DIRICHLET problem reads

$$\begin{cases} \frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + h(x) \\ u(0,t) = a, u(L,t) = b \end{cases}. \tag{7.2}$$

Note that here we assumed again that $h$ is independent of $t$; see the figure (7.1). For more general cases refer to the second volume of this book.

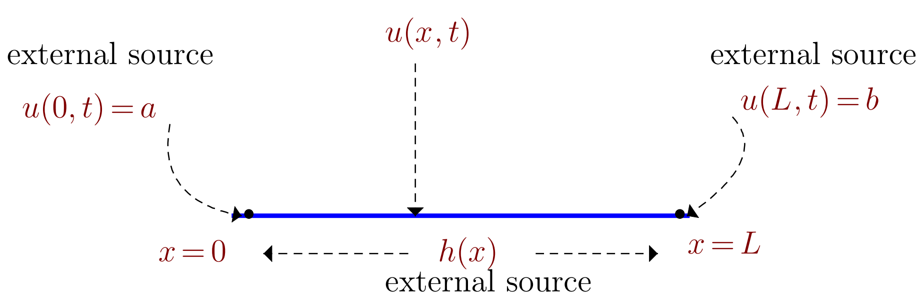


**Figure 7.1.**

### NEUMANN B.Cs .

In a NEUMANN problem, it is assumed that the derivative of $u$, that is, $\frac{\partial u}{\partial x}$ are $0$ at the boundary points $x=0, L$. Therefore, a homogeneous NEUMANN heat problem reads

$$\begin{cases} \frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + h(x) \\ \frac{\partial u}{\partial x}(0,t) = 0, \frac{\partial u}{\partial x}(L,t) = 0 \end{cases}. \tag{7.3}$$

The physical interpretation of the NEUMANN B.Cs is as follows. First note that $\frac{\partial u}{\partial x}(x, t)$ measures the net flow of heat passing through the point $x$ at time $t$. Therefore, the homogeneous NEUMANN boundary condition states that the net heat flow through the end point is zero. Equivalently, end points can be considered insulated so that it does not allow any heat escapes or enters through these points.

**Robin's BCs.**

The general form of mixed or Robin's boundary condition is

$$\begin{cases} a_1 u(0,t) + b_1 \frac{\partial u}{\partial x}(0,t) = 0 \\ a_2 u(L,t) + b_1 \frac{\partial u}{\partial x}(L,t) = 0 \end{cases}, \tag{7.4}$$

where $a_1, b_1$ or $a_2, b_2$ can not be simultaneously zero. Of important cases are when one side is in Dirichlet condition and another side is in the Neumann condition, like $u(x,t) = 0$ and $\frac{\partial u}{\partial x}(L,t) = 0$.

### 7.1.2 Steady state solution of the heat equation

A steady state solution to a heat equation is a solution that remain unchanged with respect to $t$. Therefor, the steady state is a function only of $x$. Let $v(x)$ be the steady state solution to the equation (7.2). Hence $v(x)$ satisfies the equation

$$\begin{cases} 0 = Dv''(x) + h(x) \\ v(0) = a, v(L) = b \end{cases}.$$

The above equation is simply solved by integration. We will see later on that

$$\lim_{t \to \infty} u(x,t) = v(x),$$

regardless of the initial heat distribution $u(x,0) = f(x)$. In particular, for a homogeneous Dirichlet problem, that is, when $a = b = 0$ and $h(x) = 0$, the steady state solution is the solution to the equation

$$\begin{cases} 0 = v''(x) \\ v(0) = v(L) = 0 \end{cases},$$

which is simply $v(x) \equiv 0$ for all $x$. In fact, we expected the latter case based on the common sense. If $u(0,t) = u(L,t) = 0$, then the end points are kept at $0$ degree for all $t > 0$. In absence of any other source $h(x)$ along the rod, we expect that all initial thermal energy associate to $u(x,0)$ disperses into these two boundary points and the initial temperature approaches $0$, the same temperature of the end points (the same thing when you put your food in a refrigerator).

The steady state solution for a Neumann problem is more interesting. First of all, we note that the term $\frac{\partial u}{\partial x}$ is proportional to the rate of heat exchange between adjacent points and for this it is called the *heat gradient*. If the tempera tire difference between adjacent points are high, the heat gradient is higher and vice versa Therefore, $\frac{\partial u}{\partial x}(0, t) = 0$, and $\frac{\partial u}{\partial x}(L,t) = 0$ mean physically that boundary points $x = 0, L$ are insulated, and thus no heat exchanges through these two points. In absence of any external source term $h(x)$ along the rod, we *expect* that the heat approaches to the value equal to the average of the initial heat distribution $u(x,0)$:

$$v(x) = \frac{1}{L} \int_0^L u(x,0)\, \mathrm{d}x. \tag{7.5}$$

For a proof of the equation see the problem set.

## 7.2 Wave equation

Consider an elastic string of length $L$. Again we can model this string as a one dimensional object running from $x=0$ to $x=L$ in the $(x,u)$-plane. Assume that the string are fastened at its boundaries $x=0,L$. Let $u(x,t)$ denote the position of the point $x$ at time $t$ in this plane. If $u(x,0)$ is the initial displacement of the string at time $t=0$, and $\frac{\partial u}{\partial t}(x,0)$ is the initial velocity of the point $x$ at time $t=0$, then the string we be generally in motion for $t>0$. The position function $u(x,t)$ satisfies the following equation which is called a *wave equation*

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2} \tag{7.6}$$

A DIRICHLET wave problem without any external force has the form

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2} \\ u(0,t)=u(L,t)=0 \\ u(x,0)=f(x), \frac{\partial u}{\partial t}(x,0)=g(x) \end{cases}. \tag{7.7}$$

We note that in a wave equation, the partial derivative with respect to $t$ is of order $2$, and for this reason the equation is accompanied by $2$ initial conditions, i.e., $u(x,0)$, and $\frac{\partial u}{\partial t}$.

### 7.2.1 D'ALEMBERT's formula

The wave equation (7.6) admits admits a closed form solution which is know as D'ALEMBERT SOLUTION. To see how the solution is derived, we use the change of variables $\xi=x-ct$, $\eta=x+ct$. The geometrical reason for using that change of variables will be discussed in the second volume of this book. We note that this change of variables also changes the form of the differential equation. In fact, we have

$$\frac{\partial}{\partial t}=\frac{\partial\xi}{\partial t}\frac{\partial}{\partial\xi}+\frac{\partial\eta}{\partial t}\frac{\partial}{\partial\eta}=-c\frac{\partial}{\partial\xi}+c\frac{\partial}{\partial\eta}=c\left(\frac{\partial}{\partial\eta}-\frac{\partial}{\partial\xi}\right).$$

Similarly,

$$\frac{\partial}{\partial x}=\frac{\partial\xi}{\partial x}\frac{\partial}{\partial\xi}+\frac{\partial\eta}{\partial x}\frac{\partial}{\partial\eta}=\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}.$$

Substituting above formula into the wave differential equation give

$$\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2}=c^2\left(\frac{\partial}{\partial\eta}-\frac{\partial}{\partial\xi}\right)\left(\frac{\partial}{\partial\eta}-\frac{\partial}{\partial\xi}\right)-c^2\left(\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)\left(\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)=-2c^2\frac{\partial^2 u}{\partial\xi\partial\eta}.$$

Therefore, $\frac{\partial^2 u}{\partial\xi\partial\eta}=0$, that implies $u=h_1(\xi)+h_2(\eta)$ for arbitrary twice differentiable functions $h_1,h_2$. Therefore, we can write the solution $u$ in terms of the original variables $x,t$ as

$$u(x,t)=h_1(x-ct)+h_2(x+ct).$$

The above solution is the general solution of the wave equation (7.6). In order to adjust it to the problem (7.7), we have to choose $h_1,h_2$ such that the solution satisfies the boundary conditions as well as the given initial conditions at $t=0$.

It is simply seen that the problem (7.7) has a closed form solution of the form

$$u(x,t)=\frac{1}{2}[f_{\text{odd}}(x+ct)+f_{\text{odd}}(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}g_{\text{odd}}(s)\,\mathrm{d}s,$$

where $f_{\text{odd}}$, $g_{\text{odd}}$ are the odd extension of functions $f, g$ respectively and periodic of period $2L$, that is, $f_{\text{odd}}(x+2L)=f_{\text{odd}}(x)$ and $g_{\text{odd}}(x+2L)=g_{\text{odd}}(x)$. The above formula is called D'Alembert's solution. Let us verify the formula. First for the boundary condition $x=0$. We have

$$u(0,t)=\frac{1}{2}[f_{\text{odd}}(ct)+f_{\text{odd}}(-ct)]+\frac{1}{c^2}\int_{-ct}^{ct}g_{\text{odd}}(s)\,\mathrm{d}s=0,$$

because $f(-ct)=f(ct)$ and the integral of the odd function $g_{\text{odd}}$ on a symmetric domain is equal to zero. At $x=L$, we have

$$f_{\text{odd}}(L-ct)=-f_{\text{odd}}(ct-L)=-f_{\text{odd}}(ct-L+2L)=-f_{\text{odd}}(L+ct).$$

Similarly, by taking $s=v+L$, we derive

$$\int_{L-ct}^{L+ct}g_{\text{odd}}(s)\,\mathrm{d}s=\int_{-ct}^{ct}g_{\text{odd}}(v+L)\,\mathrm{d}v.$$

Let $h(v)=g_{\text{odd}}(v+L)$. We show $h(-v)=-h(v)$. In fact, we have

$$h(-v)=g_{\text{odd}}(-v+L)=g_{\text{odd}}(-v+2L-L)=g_{\text{odd}}(-v-L)=-g_{\text{odd}}(v+L)=-h(v).$$

Therefore, we obtain

$$\int_{-ct}^{ct}g_{\text{odd}}(v+L)\,\mathrm{d}v=0.$$

So far, we have shown that the D'Alembert solution satisfies the boundary conditions at $x=0, L$. The verification that the solution satisfies given initial conditions as well as the differential equation is left as an exercise to the reader.

## Problems

**Problem 7.1.** Here we prove the formula (7.5). Consider the following Neumann problem

$$\begin{cases} \frac{\partial u}{\partial t}=D\frac{\partial^2 u}{\partial x^2} \\ \frac{\partial u}{\partial x}(0,t)=0, \frac{\partial u}{\partial x}(L,t)=0 \end{cases}$$

Define the *energy function*

$$E(t)=\int_0^L u(x,t)\,\mathrm{d}x.$$

Use the equation and show $E'(t)=0$. Conclude that $E(t)$ is independent of $t$ and thus (7.5).

**Problem 7.2.** Consider the equation (7.3). We show that there is no steady state solution to the equation.

a) Show the following relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L u(x,t)\mathrm{d}x=\int_0^L h(x)\mathrm{d}x.$$

b) Solve the equation derived in part a) and conclude

$$\int_0^L u(x,t)\,\mathrm{d}x = \left(\int_0^L h(x)\mathrm{d}x\right)t + \int_0^L f(x)\mathrm{d}x,$$

where $u(x,0)=f(x)$. Conclude that

$$\lim_{t\to\infty} u(x,t),$$

is unbounded.

**Problem 7.3.** Consider the equation (7.7). The energy of the function $u(x,t)$ is defined as follows

$$E(t) = \int_0^L \left|\frac{\partial u}{\partial t}(x,t)\right|^2 \mathrm{d}x + c^2\int_0^L \left|\frac{\partial u}{\partial x}(x,t)\right|^2 \mathrm{d}x.$$

Show the following formula which is known as the conservation of energy of wave function

$$E(t) = \int_0^L |g(x)|^2\,\mathrm{d}x + c^2\int_0^L |f'(x)|^2\,\mathrm{d}x.$$

**Problem 7.4.** Verify that the D'ALEMBERT solution to the homogeneous wave equation (7.7) satisfies the initial conditions and the differential equation itself.

**Problem 7.5.** Consider the following wave equation

$$\begin{cases} \partial_{tt}u = 4\partial_{xx}u \\ u(0,t)=u(\pi,t)=0 \\ u(x,0)=\sin(3x), \partial_t u(x,0)=0 \end{cases}.$$

a) Verify that the function $u(x,t)=\cos(6t)\sin(3x)$ solves the problem.

b) Show that the above solution coincides the D'ALEMBERT's solution.

## 7.3 Eigenvalue problem

Let us start off by considering the following boundary value problem

$$\begin{cases} \frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} & 0<x<L, t>0 \\ u(0,t)=0, u(L,t)=0 & t\ge 0 \end{cases}.$$

The main technique to solve such differential equation is the *separation of variables*, that is, to assume the solution $u$ as $u(x,t)=X(x)T(t)$. In fact, there is no *a priori* reason for that assumption, but we will see how it helps us to solve the equation. Substituting the *separated solution* into the differential equation yields

$$\frac{T'(t)}{DT(t)} = \frac{X''(x)}{X(x)}.$$

Evidently, the above equality can be possible only if both sides are a constant (the left hand side is a function of $t$ while the right hand side is a function of $x$). Let us denote this constant by $-\lambda$ (the negative sign appears only by historical reason and there is no real reason for that). The we obtain

$$\frac{T'(t)}{DT(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

and by that, we reach the two equations: $T' = -\lambda T$, and $X'' = -\lambda X$. But the solution $u$ must satisfies in addition to the equation, the given boundary conditions at $x = 0, L$, that is,

$$X(0)T(t) = 0, X(L)\,T(t) = 0.$$

If $T(t)$ is identically zero, then $u(x,t)$ is identically zero, that is in general is not acceptable (except the initial condition of the equation is zero and then the solution is trivial). Therefore, we obtain two boundary conditions for $X(x)$ as $X(0) = X(L) = 0$. Therefore, we look for function $X(x)$ that satisfies the following equation

$$\begin{cases} \frac{\mathrm{d}^2}{\mathrm{d}x^2} X = -\lambda X \\ X(0) = X(L) = 0 \end{cases}. \tag{7.8}$$

Equations of the above form are generally called *eigenvalue problems*. In fact, we would like to find non-trivial functions $X(x)$ called *eigenfunctions* that preserves their structure under the second order differentiation. The values $\lambda$ in the above equation are called *eigenvalues*. Therefore, solving this type of equations is crucial to solve a partial differential equation.

### 7.3.1 General Dirichlet eigenvalue problem

The most general form of the Dirichlet eigenvalue problem is

$$\begin{cases} a(x)\,X'' + b(x)\,X' + c(x)X = -\lambda X \\ X(x_0) = 0, X(x_1) = 0 \end{cases},$$

where $a(x) > 0$ in the interval $[x_0, x_1]$. As we learned in the previous chapters, we are unable, in general, to solve equations with the variable coefficients in closed form, and for this, there is no hope to determine the closed form solutions to the above eigenvalue problem. However, we can say several important things about the eigenvalues and eigenfunctions of the problem. The interested reader is referred to the second volume of this book for a detailed discussion on this subject. Here, we restrict ourselves to simple eigenvalue problem.

Of most important are the following simple problem (7.8). Let us solve the equation. The characteristic polynomial is $r^2 = -\lambda$ and thus $r_{1,2} = \pm\sqrt{-\lambda}$. There are three possible cases for $\lambda$, that are $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$. We first show that there is no eigenfunction for $\lambda \le 0$. Multiply the equation by $X(x)$ and integrate in the interval $[0, L]$. We obtain

$$\int_0^L X''(x)\,X(x) = -\lambda \int_0^L |X(x)|^2.$$

Integration by parts formula, simplifies the left hand side of above relation to

$$\int_0^L X''(x)\,X(x) = X'(x)X(x)|_0^L - \int_0^L |X'(x)|^2 = -\int_0^L |X'(x)|^2.$$

Note that the boundary term is zero in the above relation due to the boundary conditions at $x = 0, L$. We obtain finally

$$\int_0^L |X'(x)|^2 = \lambda \int_0^L |X(x)|^2,$$

and thus $\lambda \geq 0$. Again if $\lambda = 0$, we obtain $X'(x) = 0$ that gives in turn $X(x) = \text{constant}$. But $X(0) = 0$ that gives $X(x) = 0$ identically and this contradicts the fact that eigenfunctions are non-trivial. Therefore, $\lambda > 0$ and thus $r_{1,2} = \pm i\sqrt{\lambda}$. The solution to the equation $X'' = -\lambda X$ for $\lambda > 0$ is

$$X(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x).$$

Applying the boundary condition $X(0) = 0$ gives $c_1 = 0$. The second boundary condition $X(L) = 0$ gives $c_2 \sin(\sqrt{\lambda}\ L) = 0$. Since $c_2$ must be non-zero (why?) we obtain $\sqrt{\lambda}\ L = n\pi$, and therefore eigenvalues of the problems are obtained as $\lambda_n = \frac{n^2\pi^2}{L^2}$ associated to the eigenfunctions $X_n(x) = \sin\left(\frac{n\pi}{L}x\right)$ for $n = 1, 2, \cdots$. Note that $n$ starts from $1$ and not $0$. Also, there is no linearly independent eigenfunction for negative integers $n$. The pair $(\lambda_n, X_n)$ is called *eigenpair* of the eigenvalue problem.

### 7.3.2 General NEUMANN eigenvalue problem

For NEUMANN eigenvalue problems, the boundary conditions are in the form of NEUMANN, and accordingly the problem reads

$$\begin{cases} a(x)\, X'' + b(x)\, X' + c(x) X = -\lambda X \\ X'(x_0) = 0, X'(x_1) = 0 \end{cases},$$

where $a(x) > 0$ in $[x_0, x_1]$ as before. Again, there is no general method to solve the above eigenvalue problem (with variable coefficients, and for this, we restrict our discussion mainly to equation with constant coefficients. The simplest eigenvalue problems of the type NEUMANN is

$$\begin{cases} \frac{\mathrm{d}^2}{\mathrm{d}x^2} X = -\lambda X \\ X'(0) = X'(L) = 0 \end{cases}. \tag{7.9}$$

It is left to the reader as a simple exercise to verify that in this case $\lambda \geq 0$ and there is no eigenfunction for $\lambda < 0$. For $\lambda = 0$, we have $X''(x) = 0$, that gives $X(x) = Ax + B$ and applying the boundary conditions implies $A = 0$ and thus $X(x) = B$ a nonzero constant. Without loss of generality, we can assume $X_0(x) = 1$ (multiplication by arbitrary constant is also an eigenfunction). For $\lambda > 0$, we obtain $X_n(x) = \cos\left(\frac{n\pi}{L}x\right)$ and $\lambda_n = \frac{n^2\pi^2}{L^2}$, $n = 1, 2, \cdots$. Without loss of generality, we can write $\left(\frac{n^2\pi^2}{L^2}, \cos\left(\frac{n\pi}{L}x\right)\right)$ for $n = 0, 1, \cdots$, as the eigenpair of the NEUMANN eigenvalue problem.

### 7.3.3 ROBIN's eigenvalue problem

As we saw before, the mixed or ROBIN's boundary conditions is of the form (7.4) and thus the general form of a ROBIN's eigenvalue problem is

$$\begin{cases} a(x)\, X'' + b(x)\, X' + c(x) X = -\lambda X \\ a_1 X(x_0) + b_1 X'(x_0) = 0 \\ a_2\, X(x_1) + b_2\, X'(x_1) = 0 \end{cases},$$

where $a_1^2+b_1^2\neq 0$ and $a_2^2+b_2^2\neq 0$. The simplest case of mixed problem is when one side of the boundary is DIRICHLET and the other side is NEUMANN, for example $X(x_0)=0$, $X'(x_1)=0$. Let us solve the following eigenvalue problem

$$\begin{cases} \frac{\mathrm{d}^2}{\mathrm{d}x^2}X=-\lambda X \\ X(0)=X'(L)=0 \end{cases}. \tag{7.10}$$

It is left as a simple exercise to the reader to show that there is no non-trivial eigenfunction for $\lambda\leq 0$. The eigenfunctions of the problem are $X_n(x)=\sin\left(\frac{(2n-1)\pi}{2L}x\right)$ and $\lambda_n=\frac{(2n-1)^2\pi^2}{4L^2}$ for $n=1,2,\cdots$.

## Problems

**Problem 7.6.** Find eigenvalues an eigenfunctions of the given DIRICHLET problems

a)

$$\begin{cases} X''+2X'=-\lambda X \\ X(0)=X(1)=0 \end{cases}.$$

b)

$$\begin{cases} x^2X''+xX'+4X=-\lambda X \\ X(1)=X(e)=0 \end{cases}$$

(Note that the substituting $x=e^t$ transforms the equation to an equation with constant coefficients)

c)

$$\begin{cases} x^2X''+3xX'+X=-\lambda X \\ X(1)=X(e)=0 \end{cases}$$

**Problem 7.7.** Show that in the eigenvalue problem (7.9), $\lambda\geq 0$ and there is no non-trivial eigenfunction for $\lambda<0$.

**Problem 7.8.** Find eigenvalues an eigenfunctions of the given DIRICHLET problems

a)

$$\begin{cases} X''+2X'=-\lambda X \\ X'(0)=X'(1)=0 \end{cases}.$$

b)

$$\begin{cases} x^2X''+xX'+4X=-\lambda X \\ X'(1)=X'(e)=0 \end{cases}$$

(Note that the substituting $x=e^t$ transforms the equation to an equation with constant coefficients) c)

$$\begin{cases} x^2X'' + 3xX' + X = -\lambda X \\ X'(1) = X'(e) = 0 \end{cases}$$

**Problem 7.9.** Show that in the eigenvalue problem (7.10), $\lambda > 0$ and there is no non-trivial eigenfunction for $\lambda \leq 0$.

**Problem 7.10.** Show that the eigenvalue problem

$$\begin{cases} \frac{\mathrm{d}^2}{\mathrm{d}x^2}X = -\lambda X \\ X(0) = 0, X(L) + X'(L) = 0 \end{cases},$$

has non-trivial eigenfunctions only for $\lambda > 0$.

**Problem 7.11.** Find the eigenvalues and eigenfunctions of the following problem

a)

$$\begin{cases} X'' = -\lambda X \\ X'(0) = X(1) = 0 \end{cases}$$

b)

$$\begin{cases} x^2X'' + xX' = -\lambda X \\ X(1) = X'(e) = 0 \end{cases}$$

**Problem 7.12.** Consider the following eigenvalue problem

$$\begin{cases} X'' + 2X' = -\lambda X \\ X(0) = 0, X(1) + X'(1) = 0 \end{cases}.$$

Show that eigenfunctions are of the form $X_n(x) = e^{-x}\sin(\omega_n x)$, where $\omega_n$ are the solutions to the algebraic equation $\tan(\omega) = -\omega$.

**Problem 7.13.** Consider the following ordinary boundary value problem

$$\begin{cases} (1+x^2)X'' - e^x X = -\lambda X \\ X(0) = X(1) = 0 \end{cases}.$$

Use the energy method employed in this section to show that the condition $\lambda > 1$ is necessary the problem has a solution.

## 7.4 FOURIER series

We start off by a definition.

**Definition 7.1.** *A function $f(x)$, $x \in (a,b)$ is called piecewise continuous if it is continuous for all points $x \in (a,b)$ except possibly at finitely many points. In addition, if $z \in (a,b)$ is a discontinuity point of $f(x)$ then both left and right limit exist*

$$f(z^+) = \lim_{x \to z^+} f(x),\ f(z^-) = \lim_{x \to z^-} f(x).$$

*Furthermore, the following limits exist*

$$f(a^+) = \lim_{x \to a^+} f(x),\ f(a^-) = \lim_{x \to b^-} f(x).$$

*A function $f(x)$, $a\in(a,b)$ is called piecewise continuously differentiable if it is continuously differentiable everywhere in $(a,b)$ except possibly at finitely many points. If $z\in(a,b)$ is a point where $f$ is not continuously differentiable, then right and left derivatives of $f$ at $z$ must exist. In addition, the right derivative of $f(x)$ at $x=a$ and the left derivative of $f(x)$ at $x=b$ must exist.*

In this section, we always assume that a function is piecewise continuously differentiable (we call it admissible by now) in the *finite interval* $(a,b)$. That is important we note $a\neq-\infty$, $b\neq\infty$. It is celebrated work of J. Fourier that a function defined on an finite interval $(a,b)$ can be represented by a series of trigonometric functions as

$$f(x)\sim\sum_{n=0}^{\infty}a_n\cos(n\omega x)+\sum_{n=1}^{\infty}b_n\sin(n\omega x), \tag{7.11}$$

for some constants $a_n,b_n$ where $\omega=\frac{2\pi}{b-a}$, and the coefficient $a_a,a_n$, and $b_n$ are

$$a_0=\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x,\, a_n=\frac{2}{b-a}\int_a^b f(x)\cos(n\omega x)\mathrm{d}x,\, b_n=\frac{2}{b-a}\int_a^b f(x)\sin(n\omega x)\mathrm{d}x. \tag{7.12}$$

But what is the meaning of notation $\sim$ in the formula? We have the following theorem.

**Theorem 7.1.** *Assume that $f(x)$ is an admissible function (piecewise continuously differentiable) in an finite interval $(a,b)$. If $x\in(a,b)$ is a continuity point of $f(x)$, then*

$$f(x)=\lim_{n\to\infty}\sum_{k=0}^{n}a_n\cos(n\omega x)+\sum_{k=1}^{n}b_k\sin(n\omega x),$$

*where $\omega=\frac{2\pi}{b-a}$, and $a_0$, $a_n$, and $b_n$ are determined by the formula (7.12). If $x\in(a,b)$ in a discontinuity point of $f(x)$, then*

$$\frac{1}{2}[f(x^+)+f(x^-)]=\lim_{n\to\infty}\sum_{k=0}^{n}a_n\cos(n\omega x)+\sum_{k=1}^{n}b_k\sin(n\omega x).$$

*Furthermore, at $x=a,b$ we have*

$$\begin{aligned}\frac{1}{2}[f(a^+)+f(b^-)]&=\lim_{n\to\infty}\sum_{k=0}^{n}a_n\cos(n\omega a)+\sum_{k=1}^{n}b_k\sin(n\omega a)=\\&=\lim_{n\to\infty}\sum_{k=0}^{n}a_n\cos(n\omega b)+\sum_{k=1}^{n}b_k\sin(n\omega b).\end{aligned}$$

In this chapter, we mainly consider the Fourier series of functions defined on a symmetric domain $(-L,L)$. In this setting, the Fourier series is written

$$f(x)\sim\sum_{n=0}^{\infty}a_n\cos\Big(\frac{n\pi}{L}x\Big)+\sum_{n=1}^{\infty}b_n\sin\Big(\frac{n\pi}{L}x\Big),$$

and

$$a_0=\frac{1}{2L}\int_{-L}^{L}f(x),\, a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\Big(\frac{n\pi}{L}x\Big),\, b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\Big(\frac{n\pi}{L}x\Big).$$

**Remark 7.1.** Notice that $a_0$ is equal to $\bar{f}$, the average of $f$ on its domain $(-L,L)$.

**Remark 7.2.** If $f(x)$ is an odd function in $(-L,L)$, then $a_n=0$, $n=0,1,\cdots$, and the series contains only sine terms. This fact agrees our expectation since sine is an odd function. Similarly, if $f(x)$ is an even function then $b_n=0$, $n=1,2,\cdots$, and the series contains only cosine terms (including the constant function for $n=0$).

**Example 7.1.** Consider the function

$$f(x)=\begin{cases} 1 & 0<x<1 \\ -1 & -1<x<0 \end{cases}.$$

Here $a=-1, b=1$ and $\omega=\pi$. The function is odd and it is simply seen that $f_0=f_n^c=0$ for all $n\geq 1$. The sine coefficients are $f_n^s=\frac{2}{n\pi}(1-\cos n\pi)$ and thus

$$f(x)\sim \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos n\pi}{n}\sin(n\pi x). \tag{7.13}$$

The figure (7.2) shows partial sums $S_{15}, S_{150}$ of the series in the interval $-1<x<1$.

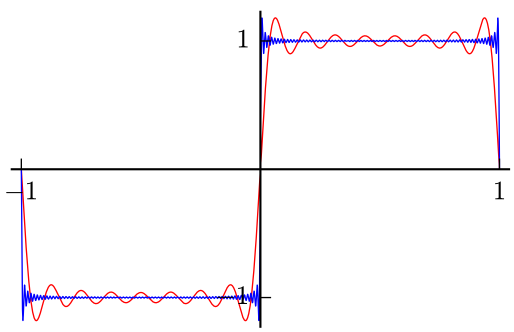


**Figure 7.2.**

We make following observations from the figure

i. $S_n(x)$ approximates the original function more accurately for larger $n$.

ii. $S_n(0)=0$ for all $n$. In fact, this is the average value of $f(0^+)$ and $f(0^-)$.

iii. $S_n(1)=S_n(-1)=0$, the average of $f(1^+), f(1^-)$.

iv. $S_n(x)$ shows jumps near discontinuity points $x=-1,0,1$ regardless of the values of $n$. This phenomena is called GIBBS *phenomena* after the American physicist J. W. GIBBS. See the appendix for a detailed discussion of this phenomena.

**Example 7.2.** The series representation of the function $f(x)=x^2$ in $-1<x<1$ is

$$x^2\sim f_0+\sum_{n=1}^{\infty} f_n^c\cos(n\pi x),$$

where $f_0 = \frac{1}{3}$ and $f_n^c = \frac{4(-1)^n}{n^2\pi^2}$. Note that the function is even and $f_n^s = 0$ for all $n$. The figure (7.3) shows the partial sums $S_2, S_{20}$.

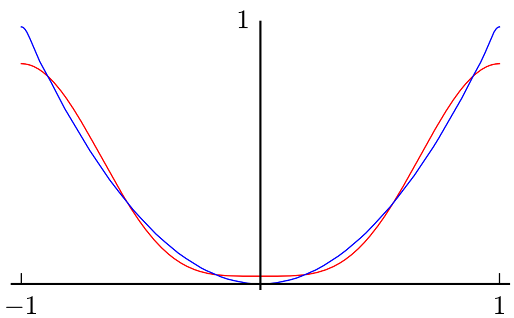


**Figure 7.3.**

**Example 7.3.** The function

$$f(x) = \begin{cases} x+1 & -1 \leq x \leq 0 \\ 1 & 0 \leq x \leq 1 \end{cases},$$

has the following series representation

$$f(x) \sim \frac{3}{4} + \frac{1}{\pi^2}\sum_{n=1}^{\infty} \frac{1-\cos(n\pi)}{n^2}\cos(n\pi x) - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n}\sin(n\pi x).$$

The plot of $S_{10}, S_{50}$ are shown in the figure (7.4). Note that $S_n(-1) = S_n(1) = 0.5$ for all $n$ which is the average limits $f(-1^+)$ and $f(1^-)$.

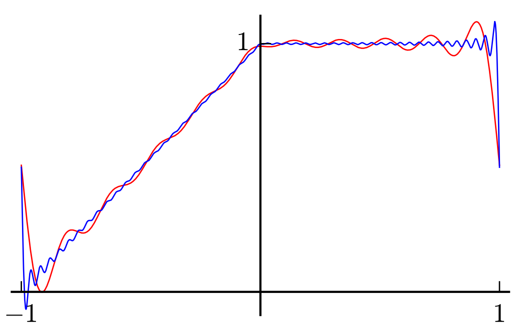


**Figure 7.4.**

Let us clarify the convergence of the series at the end points. In the first example, the end points are $x = \pm 1$ and the series converges to the value $0$ that is equal to the average value $\frac{f(1^-)+f(-1^+)}{2}$. Note that the series (7.13) is periodic with the period $T = 2$; see the figure (7.5). Therefore, the series at $x = 1$ converges to the average of the left and right limits at this point. A similar argument holds at $x = -1$. Geometrically, we can imagine that $f$ is defined on a circle and thus the end points $x = \pm 1$ coincides on the circle.

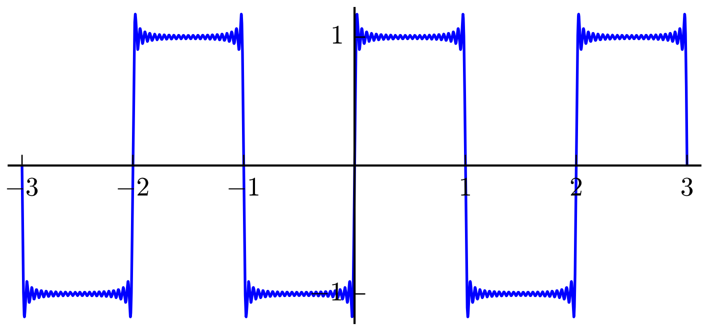


**Figure 7.5.**

### 7.4.1 Sine and cosine FOURIER series

If an admissible function is given on $(0, L)$, then we can extend it to an odd function $f_{\text{odd}}(x)$, or alternatively to an even function $f_{\text{ev}}(x)$ on $(-L,\ L)$. The extensions $f_{\text{odd}}$ and $f_{\text{ev}}$ are respectively defined by the relations

$$f_{\text{odd}}(x)=\begin{cases} f(x) & 0<x<L \\ -f(-x) & -L<x<0 \end{cases},\ f_{\text{ev}}(x)=\begin{cases} f(x) & 0<x<L \\ f(-x) & -L<x<0 \end{cases}.$$

Since $f_{\text{odd}}$ is an odd function, its associated FOURIER series contains only sine functions. This resulted series is called the *sine* FOURIER series of $f(x)$. Similarly, the associated FOURIER series for $f_{\text{ev}}$ contains only cosine terms and it is called the *cosine* FOURIER series of $f(x)$. Since $f_{\text{odd}}(x)$ is defined on $(-L, L)$, its FOURIER series is

$$f_{\text{odd}}(x)\sim\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right),$$

where

$$b_n=\frac{1}{L}\int_{-L}^{L} f_{\text{odd}}(x)\sin\left(\frac{n\pi}{L}x\right)=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right).$$

On the other hand, since the series represents $f_{\text{odd}}$ on $(-L, L)$, then definitely it represents $f(x)$ on $(0, L)$ since on $(0, L)$ two function $f_{\text{odd}}$ and $f$ are the same. Therefore, we can write

$$f(x)\sim\sum_{n=1}^{\infty}\left[\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\right]\sin\left(\frac{n\pi}{L}x\right).$$

Similarly, the cosine FOURIER series of $f(x)$ in $(0, L)$ is

$$f(x)=\bar{f}+\sum_{n=1}^{\infty}\left[\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)\right]\cos\left(\frac{n\pi}{L}x\right),$$

where

$$\bar{f}=\frac{1}{L}\int_0^L f(x).$$

**Example 7.4.** Consider the function $f(x)=x$ in $(0,1)$. The original FOURIER series of this function according to the formula (7.11) is

$$f(x)\sim\sum_{n=0}^{\infty}a_n\cos(2n\pi x)+\sum_{n=1}^{\infty}b_n\sin(2n\pi x),$$

where

$$a_0=\bar{f}=\int_0^1 x=\frac{1}{2}, a_n=2\int_0^L x\cos(2n\pi x)=0, b_n=2\int_0^1 x\sin(2n\pi x)=-\frac{1}{n\pi},$$

and thus

$$f(x)\sim\frac{1}{2}-\sum_{n=1}^{\infty}\frac{1}{n\pi}\sin(2n\pi x).$$

The graph is shown in the figure (7.6).

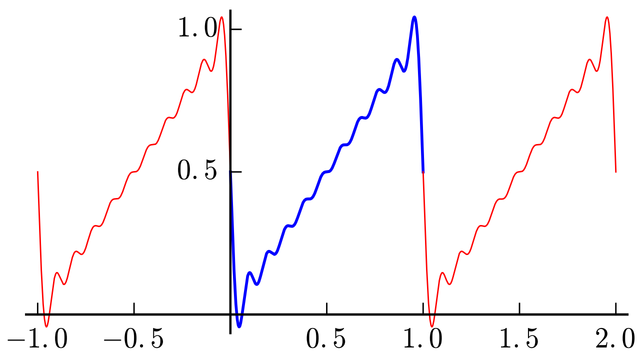


**Figure 7.6.**

Notice that the series is periodic with the period $T=1$, that is, $f(x+1)=f(x)$. The graph is sketched for $n=10$. For higher values of $n$, the series represent the original function more accurately. The figure (7.7) is sketched for $n=20$.

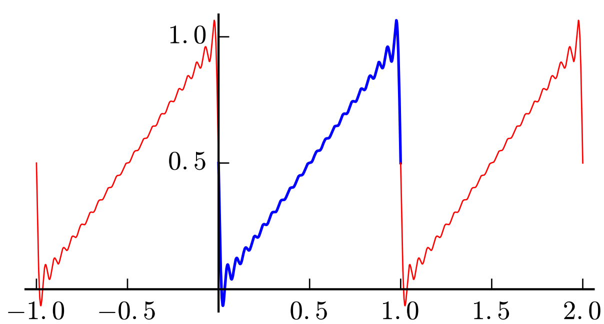


**Figure 7.7.**

Note that regardless of how many terms are used to draw the FOURIER series, there is always an overshoot on the discontinuity points $x=0,1$. This phenomena known as GIBBS phenomena that we discuss in greater detail in the second volume of the book. Also note that at the discontinuity points, the series converges to the average left and right limits that here is equal to $\frac{1}{2}$.

Now let us obtain the sine FOURIER series of the function. The odd extension of the function $f(x)$ in $(0,1)$ is $f_{\text{odd}}=x$ in $(-1,1)$. Therefore we can write

$$f(x)\sim\sum_{n=1}^{\infty}\left[2\int_0^1 x\sin(n\pi x)\right]\sin(n\pi x)=\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n\pi}\sin(n\pi x).$$

The figure (7.8) shows the graph of the series.

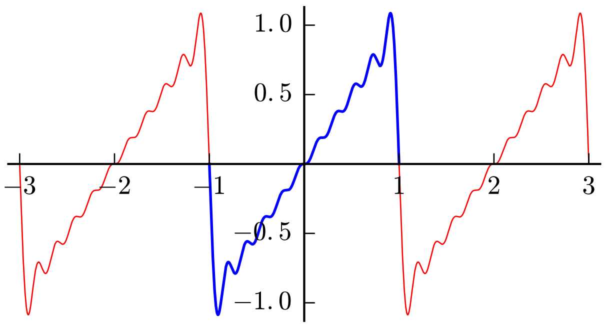


**Figure 7.8.**

Note that the period of the series in this case is $T=2$ instead of $T=1$ for the original FOURIER series. The sine FOURIER series represents the odd extension of the function in $(-1,1)$ and thus the function $f(x)$ in $(0,1)$. Similarly the cosine FOURIER series of $f(x)$ is done by the even extension $f_{\text{ev}}(x)=|x|$ that is

$$f(x)\sim\bar{f}+\sum_{n=1}^{\infty}\left[2\int_0^1 x\cos(n\pi x)\right]\cos(n\pi x)=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{2((-1)^n-1)}{n^2\pi^2}\cos(n\pi x).$$

The figure (7.9) shows the series.

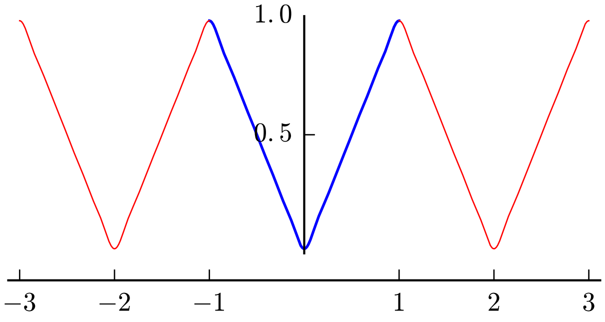


**Figure 7.9.**

## Problems

**Problem 7.14.** Find the FOURIER series of the following functions

a) $f(x)=\begin{cases}1 & 0<x<1\\ -1 & -1<x<0\end{cases}$

b) $f(x)=\begin{cases}1 & -1<x<0\\ x-1 & 0<x<1\end{cases}$

c) $f(x) = \cos(x)$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

**Problem 7.15.** write the original and the cosine FOURIER series of the function $f(x) = \sin(x)$, $0 < x < \pi$.

**Problem 7.16.** Write the sine FOURIER series of the function $f(x) = \cos(x)$, $0 < x < \frac{\pi}{2}$.

**Problem 7.17.**

a) Find coefficients $b_n$ such that
$$x^2 = \sum_{n=1}^{\infty} b_n \sin(nx), 0 < x < \pi$$

b) Find coefficients $a_n$ such that
$$x^2 = \sum_{n=0}^{\infty} a_n \cos(nx), 0 < x < \pi$$

**Problem 7.18.** Consider the function $f(x) = x$ on $0 < x < 1$.

a) Find the cosine FOURIER series of $f$.

b) Find the sine FOURIER series of $f$.

c) Find the complex FOURIER series of $f$.

d) If $S_n(x)$ is the partial sum of the Fourier series of $f$ on $(0,1)$, what is the value of $\lim_{n\to\infty} S_n(0)$?

e) Prove the following identity
$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

**Problem 7.19.** Let $f(x) = x^2$ on $(0,1)$.

a) Write down the FOURIER series of $f$ and draw $S_3(x), S_5(x), S_6(x)$ on $(-2,2)$. What is the value of the series at $x = 0, 1$. How does the series behave near $x = 0$ and $x = 1$?

b) Write down the complex form of the FOURIER series of $f$.

c) Write down the sine FOURIER series of $f$ and draw $S_3(x), S_5(x), S_6(x)$ on $(-2,2)$.

d) Write down the cosine FOURIER series of $f$ and draw $S_3(x), S_5(x), S_6(x)$ on $(-2,2)$.

e) Use the FOURIER series of $x^2$ on $(-1,1)$ to show:
$$\frac{\pi^2}{12} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \cdots$$

**Problem 7.20.** Write down the FOURIER series of the function $f(x) = x$ for $x \in (0,2)$ and $f(x+2) = f(x)$ and deduce the following identity called LEIBNIZ's formula
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

**Problem 7.21.** Find the FOURIER series of the function $f(x) = x^3$ in $x \in (-\pi, \pi)$ and deduce the following identity
$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots.$$

**Problem 7.22.** Let $f(x) = e^{-x}$ on $(0,1)$.

a) Write down the FOURIER series of $f$ and draw $S_5(x), S_{10}(x)$ on $(-2,2)$.

b) Write down the complex form of the FOURIER series.

c) Write down the sine FOURIER series of $f$ and draw $S_5(x), S_{10}(x)$ on $(-3,3)$.

d) Write down the cosine FOURIER series of $f$ and draw $S_2(x), S_{10}(x)$ on $(-3,3)$.

**Problem 7.23.** Let $f(x)=\cos(x)$ on $\left(0,\frac{\pi}{2}\right)$.

a) Write down the FOURIER series of $f$ and draw $S_3(x)$, $S_5(x)$, $S_6(x)$ on $(-\pi,\pi)$. How the series behave near $x=0$ and $x=\pi/2$?

b) Write down the sine FOURIER series of $f$ and draw $S_3(x), S_5(x), S_6(x)$ on $(-\pi,\pi)$. How the series behave near $x=0$ and $x=\pi/2$?

c) Write down the cosine FOURIER series of $f$ and draw $S_3(x), S_5(x), S_6(x)$ on $(-\pi,\pi)$.

d) For the partial sum $S_n$ of the part $(a)$, find the the square error for $n=3,5,6$.

**Problem 7.24.** Find the cosine FOURIER series for the function $f(x)=\sin(x)$ on $(0,\pi)$.

**Problem 7.25.** Let us solve the following boundary problem by FOURIER series method

$$\begin{cases} y''+y=1 \\ y(0)=y(1)=0 \end{cases}.$$

i. Assume that the solution to the equation is written as

$$y(x)=\sum_{n=1}^{\infty} Y_n \sin(n\pi x).$$

Substitute $y(x)$ into the equation and find coefficients $Y_n$.

ii. Show that the obtained series is absolutely convergent.

iii. Why is this series a true solution to the given problem?

**Problem 7.26.** Show that the trivial solution $y(x)=0$, is the unique solution to the problem

$$\begin{cases} y''+\sin(\pi x)y=0 \\ y(0)=y(1)=0 \end{cases}.$$

# 7.5 Solution to heat and wave equations

## 7.5.1 Homogeneous DIRICHLET heat problems

Consider the following DIRICHLET problem

$$\begin{cases} \partial_t u = D\partial_{xx} u & \text{for } 0 < x < L, t > 0 \\ u(0,t) = u(L,t) = 0 & \text{for } t \geq 0 \\ u(x,0) = f(x) & \text{for } 0 < x < L \end{cases}. \tag{7.14}$$

Let us *assume* that the solution $u(x,t)$ is of the form $u(x,t) = T(t)\,X(x)$, for some unknown functions $T(t)$ and $X(x)$. We do not know in advance if the assumption is true. This solution is called a *separated solution*. Substituting $u$ into the equation leads to the following equation

$$\frac{T'(t)}{DT(t)} = \frac{X''(x)}{X(x)}. \tag{7.15}$$

Since $t$ and $x$ are independent variables, the equality (7.16) holds if and only if we have

$$\frac{T'(t)}{DT(t)} = \frac{X''(x)}{X(x)} = -\lambda, \tag{7.16}$$

for some constant $\lambda$. The negative sign in the front of $\lambda$ is just for historical convention. Therefore, we obtain two equations for $T$, and $X$

$$T' = -\lambda DT, X'' = -\lambda X.$$

Moreover, according to the condition $u(0,t) = 0$, we derive $X(0)T(t) = 0$. Since $T(t)$ can not be identically zero, we obtain $X(0) = 0$. For the boundary condition at $x = L$, we have $X(L) = 0$, and therefore, we reach the following DIRICHLET eigenvalue problem for $X(x)$

$$\begin{cases} X'' = -\lambda X \\ X(0) = X(L) = 0 \end{cases}. \tag{7.17}$$

As we saw in above the obtained eigenvalue problem has the eigenvalues $\lambda_n = \frac{n^2\pi^2}{L^2}$, and eigenfunctions $X_n(x) = \sin\left(\frac{n\pi}{L}x\right)$ for $n = 1, 2, \cdots$. In this way, we obtain infinitely many solutions

$$u_n(x,t) = e^{-\frac{n^2\pi^2}{L^2}Dt} \sin\left(\frac{n\pi}{L}x\right).$$

It is simply verified that $u_n(x,t)$ satisfies both the differential equation and the boundary conditions (it is left as a simple exercise to the reader). However, $u_n(x,t)$ does not satisfies the initial condition in general. Is the obtained solution wrong? Here we see hoe the *superposition principle* helps us to write the correct solution. Thanks to the linearity of the differential equation, we can write the solution to the problem as a linear combination in the *series form*

$$u\,(x,t) = \sum_{n=1}^{\infty} b_n u_n(x,t) = \sum_{n=1}^{\infty} b_n\, e^{-\frac{n^2\pi^2}{L^2}Dt} \sin\left(\frac{n\pi}{L}x\right).$$

(the technicality that we are using the superposition of infinitely many terms is resolved in the second volume of this book). Now, let us see for what values of $b_n$ the solution satisfies the initial condition as well. At $t=0$, we have

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\Big(\frac{n\pi}{L}x\Big), 0<x<L$$

and thus $b_n$ must be the coefficients of the sine FOURIER series of the function $f(x)$, that is,

$$b_n=\frac{2}{L}\int_0^L f(x)\sin\Big(\frac{n\pi}{L}x\Big)\,\mathrm{d}x.$$

Therefore, the true solution to the homogeneous DIRICHLET heat problem is

$$u(x,t)=\sum_{n=1}^{\infty}\left[\frac{2}{L}\int_0^L f(x)\sin\Big(\frac{n\pi}{L}x\Big)\,\mathrm{d}x\right]e^{-\frac{n^2\pi^2}{L^2}Dt}\sin\Big(\frac{n\pi}{L}x\Big). \tag{7.18}$$

**Example 7.5.** Consider the following heat problem

$$\begin{cases}\partial_t u=D\partial_{xx}u & 0<x<\pi, t>0\\ u(0,t)=u(\pi,t)=0 & t\geq 0\\ u(x,0)=2\sin(3x)-3\sin(4x) & 0<x<\pi\end{cases}. \tag{7.19}$$

Here $L=\pi$ and $D=1$, and thus the solution is

$$u(t,x)=\sum_{n=1}^{\infty} b_n\, e^{-Dn^2t}\sin(nx). \tag{7.20}$$

To determine $b_n$, we use the initial condition, that is,

$$2\sin(3x)-3\sin(4x)=\sum_{n=1}^{\infty} b_n\sin(nx),$$

and thus $b_3=2$, $b_4=-3$ and $b_n=0$ for $n\neq 3,4$. The solution is then

$$u(x,t)=2e^{-9Dt}\sin(3x)-3e^{-16Dt}\sin(4x).$$

Here we could determine $b_n$ by the simple match. However, if the initial condition is not an eigenfunction of the associated eigenvalue, we have to use the FOURIER series as the formula (7.18). Consider the following problem

$$\begin{cases}\partial_t u=D\partial_{xx}u & 0<x<\pi, t>0\\ u(0,t)=u(\pi,t)=0 & t\geq 0\\ u(x,0)=1 & 0<x<\pi\end{cases}.$$

Here the coefficients $b_n$ are

$$b_n=\frac{2}{\pi}\int_0^{\pi}\sin(nx)\,\mathrm{d}x=\frac{2(1-\cos(n\pi))}{n\pi}.$$

The figure (7.10) at the left shows $u(x,t)$ for some values of $t$ and $D=1$. The right figure shows the solution for $D=2$.

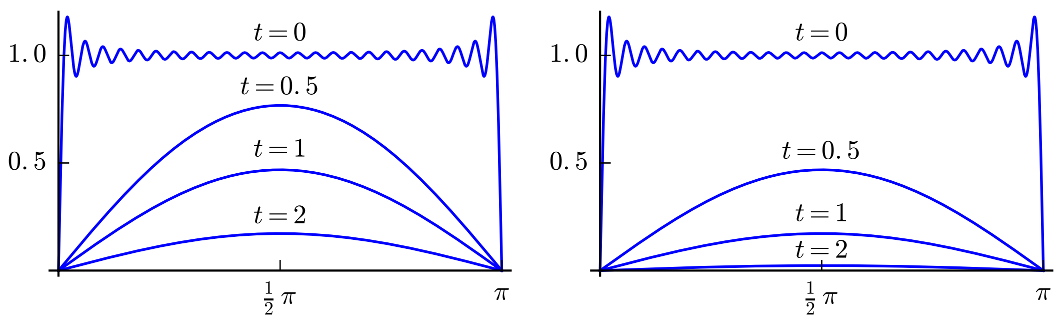


**Figure 7.10.**

We have the following observations from the graph of solution:

a) The solution is smooth for $t>0$ even the initial data is discontinuous. In the above example the initial data is discontinuous at $x=0,\pi$, however, for any $t>0$, regardless of how small $t$ is, the solution is smooth.

b) The solution approaches zero in long term. This is due to the exponential factor $e^{-Dn^2t}$ in the solution. Note that the zero solution is the steady state solution $u_e$ to the problem. Here the positive factor $D>0$ determines the speed that the solution $u(x,t)$ approaches zero.

c) It takes longer time for the mid point $x=\frac{\pi}{2}$ to lose its initial heat than points adjacent to the end points.

d) More terms of the series needed to catch the initial data more accurately because of the discontinuity at the end points.

### 7.5.2 Homogeneous Neumann heat problem

Fir the problem

$$\begin{cases} \partial_t u = D\partial_{xx} u & \text{for } 0<x<L, t>0 \\ \partial_x u(0,t) = \partial_x u(L,t) = 0 & \text{for } t \geq 0 \\ u(x,0) = f(x) & \text{for } 0<x<L \end{cases}.$$

The separated solution $u(x,t) = X(x)\,T(t)$ results to the following eigenvalue problem for $X(x)$

$$\begin{cases} X'' = -\lambda X \\ X'(0) = 0, X'(L) = 0 \end{cases}.$$

As we saw before, the above eigenvalue problem has the eigenvalues $\lambda_n = \frac{n^2\pi^2}{L^2}$, and the eigenfunctions $X_n(x) = \cos\left(\frac{n\pi}{L}x\right)$ for $n = 0, 1, 2, \cdots$. The associate equation for $T(t)$, that is $T' = -\lambda DT$ has the solution $T_n(t) = e^{-\frac{n^2\pi^2}{L^2}Dt}$ and thus the solution can be written in the series form

$$u(x,t) = \sum_{n=0}^{\infty} a_n\, e^{-\frac{n^2\pi^2}{L^2}Dt} \cos\left(\frac{n\pi}{L}x\right).$$

Notice that in this case $n$ starts form $0$ not $1$. The coefficients $a_n$ are determined by the aid of the initial condition $u(x,0)$, i.e.,

$$f(x)=\sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right).$$

The above series implies that $a_n$ must be the cosine FOURIER series of $f(x)$, that is,

$$a_0=\bar{f}, a_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right), n=1,2,\cdots.$$

Therefore, we obtain

$$u(x,t)=\bar{f}+\sum_{n=1}^{\infty}\left[\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)\right]e^{-\frac{n^2\pi^2}{L^2}Dt}\cos\left(\frac{n\pi}{L}x\right).$$

Note that as we expected for NEUMANN heat problem

$$\lim_{n\to\infty} u(x,t)=\bar{f}.$$

**Example 7.6.** Let us solve the following NEUMANN heat problem

$$\begin{cases}\partial_t u=\partial_{xx}u & 0<x<\pi, t>0\\ \partial_x u(0,t)=\partial_x u(\pi,t)=0 & t\geq 0\\ u(x,0)=x+5\cos(3x) & 0<x<\pi\end{cases}. \tag{7.21}$$

For $L=\pi$, we obtain the solution

$$u(t,x)=\bar{f}+\sum_{n=1}^{\infty} a_n e^{-n^2t}\cos(nx).$$

We have $\bar{f}=\frac{\pi}{2}$ and

$$a_n=\frac{2}{\pi}\int_0^\pi (x+5\cos(3x))\cos(nx)\,\mathrm{d}x=\frac{2(\cos(n\pi)-1)}{\pi n^2}+\begin{cases}5 & n=3\\ 0 & n\neq 3\end{cases},$$

Therefore, the solution is

$$u(x,t)=\frac{\pi}{2}+5e^{-9t}\cos(3x)+\sum_{n=1}^{\infty}\frac{2(\cos(n\pi)-1)}{\pi n^2}e^{-n^2t}\cos(nx).$$

As we expect, we have the following steady state solution

$$\lim_{t\to\infty} u(t,x)=\bar{f}=\frac{\pi}{2}.$$

### 7.5.3 Homogeneous DIRICHLET wave problems

In previous section, we saw that a wave problem has a closed form solution called D'ALEMBERT solution. Here we apply the separation of variable technique to write the solution in series form. Consider the following problem

$$\begin{cases}\partial_{tt}u=c^2\,\partial_{xx}u\\ u(0,t)=u(L,t)=0\\ u(x,0)=f(x), \partial_t u(x,0)=g(x)\end{cases}. \tag{7.22}$$

The separation of variables leads to the following equations for $T(t)$ and $X(x)$

$$\left\{ T'' = -c^2\lambda^2 T \,, \left\{ \begin{array}{l} X'' = -\lambda X \\ X(0) = X(L) = 0 \end{array} \right. . \right.$$

As before, the associated eigenvalue problem has eigenvalues $\lambda_n = \frac{n^2\pi^2}{L^2}$, and the eigenfunctions $X_n(x) = \sin\left(\frac{n\pi}{L}x\right)$. Then the associated time equation has the solution

$$T_n(t) = A_n \cos\left(\frac{n\pi}{L}ct\right) + B_n \sin\left(\frac{n\pi}{L}ct\right),$$

and thus the solution to the wave problem is

$$u(t,x) = \sum_{n=1}^{\infty} \left[ A_n \cos\left(\frac{n\pi}{L}ct\right) + B_n \sin\left(\frac{n\pi}{L}ct\right) \right] \sin\left(\frac{n\pi}{L}x\right).$$

In order that the above series to be the true solution to the problem, the coefficients $A_n$, $B_n$ must be determined such that $u(x,t)$ satisfies the given initial conditions. Applying the first initial condition leads to the relation

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right),$$

and thus

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) \mathrm{d}x.$$

Similarly, from the relation

$$g(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \sin\left(\frac{n\pi}{L}x\right),$$

the coefficients $B_n$ are determined by the formula

$$B_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right) \mathrm{d}x.$$

**Example 7.7.** Consider the following wave problem

$$\begin{cases} \partial_{tt}u = c^2 \partial_{xx}u & 0 < x < \pi \\ u(0,t) = u(\pi,t) = 0 & \\ u(x,0) = x(\pi - x), \partial_t u(x,0) = 0 & 0 < x < \pi \end{cases} . \tag{7.23}$$

For $L = \pi$, we obtain

$$A_n = \frac{4(1 - \cos(n\pi))}{n^3\pi}, B_n(0) = 0,$$

and thus

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4(1 - \cos(n\pi))}{n^3\pi} \cos(cnt) \sin(nx). \tag{7.24}$$

The figure (7.11) shows the solution for some values of $t$ and for $c = 1$ (the left) and $c = \sqrt{2}$ (the right). As it is observed from the figure, the factor $c$ determines the speed of the traveling wave. Moreover, the solution is $\frac{2\pi}{c}$ periodic, that is, $u\left(x, t + \frac{2\pi}{c}\right) = u(x,t)$.

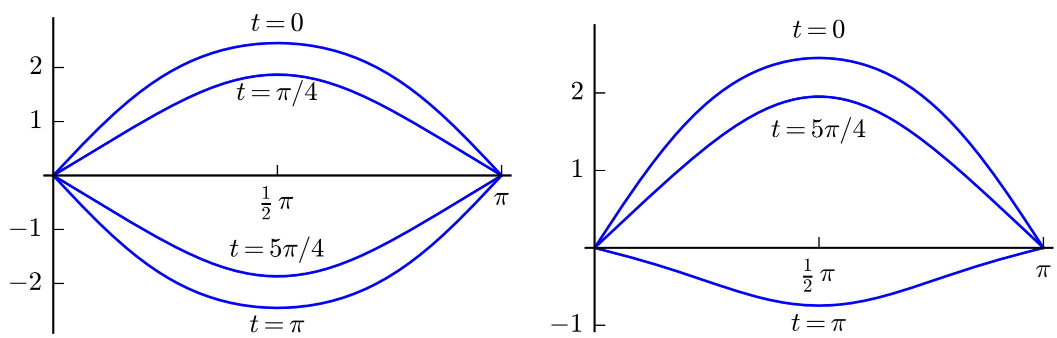


**Figure 7.11.**

What is the relation between the D'ALEMBERT solution and the series solution found above? In order to show that the series solution is the same as the D'ALEMBERT solution, we use the trigonometric identities to write

$$\sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}ct\right)\sin\left(\frac{n\pi}{L}x\right)=\frac{1}{2}\sum_{n=1}^{\infty} A_n\left[\sin\left(\frac{n\pi}{L}(x+ct)\right)+\sin\left(\frac{n\pi}{L}(x-ct)\right)\right]. \tag{7.25}$$

Note that

$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)=f_{\text{odd}}(x),$$

where $f_{\text{odd}}$ is the odd extension of $f$ in $[-L,L]$, and thus

$$\sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}ct\right)\sin\left(\frac{n\pi}{L}x\right)=\frac{1}{2}[f_{\text{odd}}(x+ct)+f_{\text{odd}}(x-ct)].$$

Similarly, we have

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}ct\right)\sin\left(\frac{n\pi}{L}x\right)=\frac{1}{2}\sum_{n=1}^{\infty} B_n\left[\cos\left(\frac{n\pi}{L}(x+ct)\right)-\cos\left(\frac{n\pi}{L}(x-ct)\right)\right], \tag{7.26}$$

On the other hand, we have

$$\int_{x-ct}^{x+ct} g_{\text{odd}}(s)\mathrm{d}s=\sum_{n=1}^{\infty} B_n\frac{n\pi c}{L}\int_{x-ct}^{x+ct}\sin\left(\frac{n\pi}{L}s\right)=$$
$$=\sum_{n=1}^{\infty} B_n\, c\left[\cos\left(\frac{n\pi}{L}(x-ct)\right)-\cos\left(\frac{n\pi}{L}(x+ct)\right)\right]$$

Therefore

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}ct\right)\sin\left(\frac{n\pi}{L}x\right)=\frac{1}{2c}\int_{x-ct}^{x+ct} g_{\text{odd}}(s)\,\mathrm{d}s.$$

Putting together the obtained formula, we reach

$$u(t,x)=\frac{1}{2}[f_{\text{odd}}(x+ct)+f_{\text{odd}}(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct} g_{\text{odd}}(s)\,\mathrm{d}s, \tag{7.27}$$

that is the D'ALEMBERT formula.

### 7.5.4 Non-homogeneous problems

Consider a simple heat equation with the nonzero DIRICHLET boundary condition, that is,

$$\begin{cases} \partial_t u = D\partial_{xx} u + h(x) \\ u(0,t) = a, u(L,t) = b \\ u(x,0) = f(x) \end{cases}.$$

In the above equation, we have three source terms, one heat source (or sink) at $x=0$, one at $x=L$ and another one which is distributed along the rod is $h(x)$. Like ordinary differential equations, the logic to solve linear non-homogeneous equations is to add a *particular solution* to the homogeneous solution. The associated homogeneous solution which is solution to the equation

$$\begin{cases} \partial_t u = D\partial_{xx} u \\ u(0,t) = u(L,t) = 0 \end{cases},$$

is

$$u_h(x,t) = \sum_{n=1}^{\infty} b_n \, e^{-\frac{n^2\pi^2}{L^2}Dt} \sin\!\left(\frac{n\pi}{L}x\right).$$

For a particular solution, we notice that all source terms are independent of time $t$. Then we can *assume* a particular solution of the form $u_p = v(x)$. Substituting this into the equation results to

$$\begin{cases} 0 = Dv'' + h(x) \\ v(0) = a, v(L) = b \end{cases}.$$

The above is a simple ordinary equation for $v(x)$ and is solved by double integration. The general solution is then

$$u(x,t) = u_h(x,t) + v(x) = \sum_{n=1}^{\infty} b_n \, e^{-\frac{n^2\pi^2}{L^2}Dt} \sin\!\left(\frac{n\pi}{L}x\right) + v(x).$$

To determine coefficients $b_n$, we use the initial condition and obtain

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\!\left(\frac{n\pi}{L}x\right) + v(x),$$

and thus

$$b_n = \frac{2}{L}\int_0^L [f(x) - v(x)] \sin\!\left(\frac{n\pi}{L}x\right) \mathrm{d}x.$$

**Example 7.8.** Consider the following problem

$$\begin{cases} \partial_t u = \partial_{xx} u + e^{-x} & 0 < x < 1 \\ u(0,t) = 0, u(1,t) = 1 - e^{-1} & t \geq 0 \\ u(x,0) = 1 & 0 < x < 1 \end{cases}. \tag{7.28}$$

The equation of the particular solution is

$$\begin{cases} -v'' = e^{-x} \\ v(0) = 0, v(1) = 1 - e^{-1} \end{cases}, \tag{7.29}$$

and therefore $v(x) = 1 - e^{-x}$. Therefore, the general solution to the equation is

$$u(x,t) = \sum_{n=1}^{\infty} b_n\, e^{-n^2\pi^2 Dt} \sin(n\pi x) + 1 - e^{-x}.$$

The coefficients $b_n$ are determined by the formula

$$b_n = 2\int_0^1 e^{-x} \sin(n\pi x)\, \mathrm{d}x = \frac{2\pi n}{\pi^2 n^2 + 1}(1 - e^{-1}\cos(n\pi)).$$

Note that $u(t,\, x) \xrightarrow{t\to\infty} v(x)$, and thus $v(x)$ is the steady state solution $u_e$; see the figure (7.12).

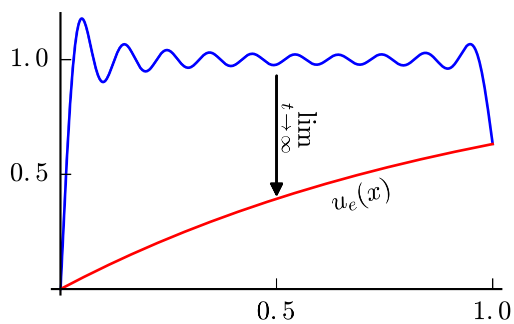


**Figure 7.12.**

**Example 7.9.** Consider the following problem

$$\begin{cases} \partial_{tt} u = \partial_{xx} u + \pi^2 \sin(2\pi x) \quad 0 < x < 1 \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = 0, \partial_t u(x,0) = 0 \end{cases}. \tag{7.30}$$

The particular solution to the problem is obtained by solving the equation

$$\begin{cases} -v'' = \pi^2 \sin(2\pi x) \\ v(0) = v(1) = 0 \end{cases}. \tag{7.31}$$

Clearly we have $v(x) = \frac{1}{4}\sin(2\pi x)$ and thus

$$u(x,t) = \frac{1}{4}\sin(2\pi x) + \sum_{n=1}^{\infty} [A_n \cos(n\pi t) + B_n \sin(n\pi t)] \sin(n\pi x). \tag{7.32}$$

Applying initial conditions gives $B_n = 0$ and $A_n = -\frac{1}{4}\left\{\begin{smallmatrix} 1 & n=2 \\ 0 & n\neq 2 \end{smallmatrix}\right.$, and finally

$$u(x,t) = \frac{1}{4}\sin(2\pi x)(1 - \cos(2\pi t)). \tag{7.33}$$

The non-homogeneous NEUMANN problem is not as trivial as DIRICHLET problem. In the problem set, we solve an example of it.

### 7.5.5 Convergence of solutions

The following theorems states the main result about the convergence of the series solutions derived in this chapter. The detailed discussion is given in the second volume of this book.

**Theorem 7.2.** *Let $f(x)$ be an admissible function as defined in the section of* FOURIER *series, and let $(f_n)$ be the sine* FOURIER *coefficients of $f(x)$. Then the sequence of partial sums*

$$S_n(x,t)=\sum_{j=1}^{n} f_j\, e^{-n^2\pi^2 t/L^2}\sin\Big(\frac{n\pi}{L}x\Big), \tag{7.34}$$

*converges pointwise in $\Omega=(0,\infty)\times(0,L)$. Furthermore, the limit function $u(t,x)$ is $C^1$ with respect to $t$ and $C^2$ with respect to $x$.*

**Theorem 7.3.** *Assume that $u(t,x)$ is defined by the series*

$$u(x,t)=\sum_{n=1}^{\infty} f_n\, e^{-n^2\pi^2 t/L^2}\sin\Big(\frac{n\pi}{L}x\Big) \tag{7.35}$$

*Fix $t\in[0,\infty)$, then the following relations hold*

$$\lim_{x\downarrow 0} u(x,t)=\lim_{x\uparrow L} u(x,t)=0. \tag{7.36}$$

**Theorem.** *Assume that $x\in(0,L)$ is a continuity point for the initial data $f(x)$. Then the series solution (7.35) satisfies the relation*

$$\lim_{t\to 0} u(x,t)=f(x). \tag{7.37}$$

## Problems

**Problem 7.27.** Consider the equation

$$\partial_t u=10^{-4}\,\partial_{xx}u,\, 0<x<1, t>0.$$

a) Assuming that the end points are kept at zero degree and the initial data $u(x,0)$ is $100x(1-x)$. Find the solution $u(x,t)$ for $t>0$ and $x\in(0,1)$. Draw the temperature of the point $x=0.5$ in $t\geq 0$.

b) Solve the above problem if the boundary points $x=0,1$ are insulated. How many terms of the series of $u(x,t)$ are needed to guarantee that the partial sum approximate the true solution within $10^{-6}$ error.

**Problem 7.28.** Consider an elastic string fastened at its boundary $x=0,1$. The displacement of the point $x$ (in the vertical direction) at time $t$ follows the equation

$$\partial_{tt}u=4\partial_{xx}u.$$

If the initial displacement $u(x,0)$ is $0$ and the initial velocity is $\partial_t u(x,0)=10^{-2}(1-\cos(4\pi x))$, draw the displacement function of the point $x=\frac{1}{3}$ for $t>0$.

**Problem 7.29.** The following equation is called *damped wave* equation

$$\partial_{tt}u + 2\xi\partial_t u = c^2\partial_{xx}u,$$

where $\xi > 0$ is the damping factor. Assume that boundary points $x = 0, 1$ are fastened. Find $u(t, x)$ if the initial displacement is $u(x, 0) = x(1-x)$, and the initial velocity is $\partial_t u(x, 0) = 0$ in $0 < x < 1$. What is the limit $\lim_{t\to\infty} u(t, x)$.

**Problem 7.30.** Find the solution to the following damped wave equation

$$\begin{cases} \partial_{tt}u + \partial_t u = \partial_{xx}u \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = 2\sin(2\pi x), \partial_t u(x,0) = \sin(\pi x) \end{cases}$$

**Problem 7.31.** Solve the following homogeneous problems

i.

$$\begin{cases} \partial_t u = \partial_{xx}u \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = \cos(2\pi x) \end{cases}$$

ii.

$$\begin{cases} \partial_t u = 3\partial_{xx}u \\ \partial_x u(0,t) = \partial_x u(2,t) = 0 \\ u(x,0) = x\sin(\pi x) \end{cases}$$

iii.

$$\begin{cases} \partial_t u = \partial_{xx}u \\ u(1,t) = u(2,t) = 0 \\ u(x,1) = x - 1 \end{cases}$$

iv.

$$\begin{cases} \partial_{tt}u = 4\partial_{xx}u, \qquad 0 < x < 1 \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = 1, \partial_t u(x,0) = x \end{cases}$$

v.

$$\begin{cases} \partial_{tt}u = \partial_{xx}u, \qquad \pi < x < 2\pi \\ u(\pi,t) = u(2\pi,t) = 0 \\ u(x,0) = x\sin(x), \partial_t u(x,0) = 0 \end{cases}$$

vi.

$$\begin{cases} \partial_{tt}u = \partial_{xx}u, \qquad 0 < x < \pi \\ u(0,0) = u(\pi,0) = 0 \\ u(x,1) = 0, \partial_t u(x,1) = 2 \end{cases}$$

**Problem 7.32.** Solve the following non-homogeneous problems

i.

$$\begin{cases} \partial_t u = \partial_{xx}u + \cos(x) - 3\cos(3x) \\ \partial_x u(0,t) = \partial_x u(\pi,t) = 0 \\ u(x,0) = 0 \end{cases}$$

ii.

$$\begin{cases} \partial_t u = \partial_{xx}u \\ u(0,t) = 1, u(1,t) = 1 \\ u(x,0) = \sin(\pi x) \end{cases}$$

iii.

$$\begin{cases} \partial_t u = \partial_{xx}u - e^{-x} \\ u(0,t) = 0, u(\pi,t) = 1 \\ u(x,0) = e^{-x} \end{cases}$$

iv.

$$\begin{cases} \partial_{tt}u = \partial_{xx}u + \sin(2\pi x), & 0 < x < 1 \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = 0, \partial_t u(x,0) = 0 \end{cases}$$

v.

$$\begin{cases} \partial_{tt}u = c^2\,\partial_{xx}u \\ u(0,t) = -1, u(1,t) = 1 \\ u(x,0) = 2x - 1, \partial_t u(x,0) = \sin(2\pi x) - \sin(3\pi x) \end{cases}$$

**Problem 7.33.** Consider the following heat problem

$$\begin{cases} \partial_t u = \partial_{xx}u - u \\ u(0,t) = 0, u(1,t) = \sinh(1) \\ u(0,x) = x + \sinh(x) \end{cases},$$

a) Find the steady state solution $u_e(x)$ to the problem.

b) Write the solution to the problem as $u(x,t) = v(x) + u_h(x,t)$ and obtain the equation for $w$.

c) Solve the problem to find $u(x,t)$.

**Problem 7.34.** Consider the following problem

$$\begin{cases} \partial_t u = \partial_{xx}u + 2\partial_x u \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = 1 \end{cases}.$$

Solve the problem by the separation of variable technique.

**Problem 7.35.** Consider the following non-homogeneous NEUMANN problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial u}{\partial x}(0,t) = \alpha, \frac{\partial u}{\partial x}(L,t) = \beta \end{cases}.$$

Verify that the solution to the equation is of the form

$$u(x,t) = \alpha x + \frac{\beta - \alpha}{2L}x^2 + \frac{\beta - \alpha}{L}t + c_0 + \sum_{n=1}^{\infty} c_n\, e^{-n^2\pi^2 t/L^2} \cos\left(\frac{n\pi}{L}x\right).$$