Chapter 6
Systems of Differential Equations

6.1 Introduction

6.1.1 Modeling with the system of equations

We start with a motivational example. Let $x(t)$, $y(t)$ be respectively the population of some kinds of prey and predator at time t. We would like to model the dynamic of the populations in terms of differential equations. Our model would be based on some assumptions. For example, in the absence of any predator, we assume that the population of prey increases according to the formula

$$\frac{dx}{dt} = r_1 x,$$

where $r > 0$ is the offspring rate of the prey. The hunting rate that we assume of the form $k_1 x y$ causes the decrease in $x(t)$ and thus we can write

$$\frac{dx}{dt} = r_1 x - k_1 x y,$$

for some constant $k_1 > 0$. Similarly, in the absence of any prey, the population of predators decreases according to the formula

$$\frac{dy}{dt} = -r_2 y,$$

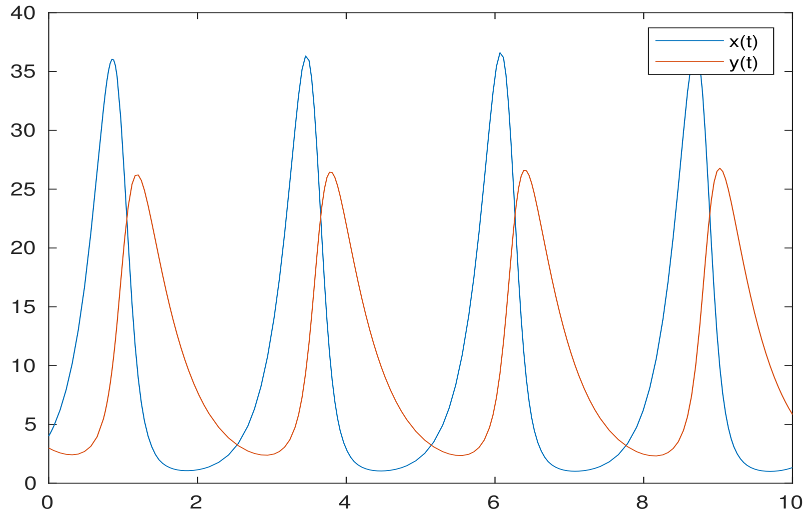
for some death rate $r_2 > 0$. The hunting rate $k_2\, x y$ contributes to the increasing of $y(t)$ and then

$$\frac{dy}{dt} = -r_2 y + k_2 x y,$$

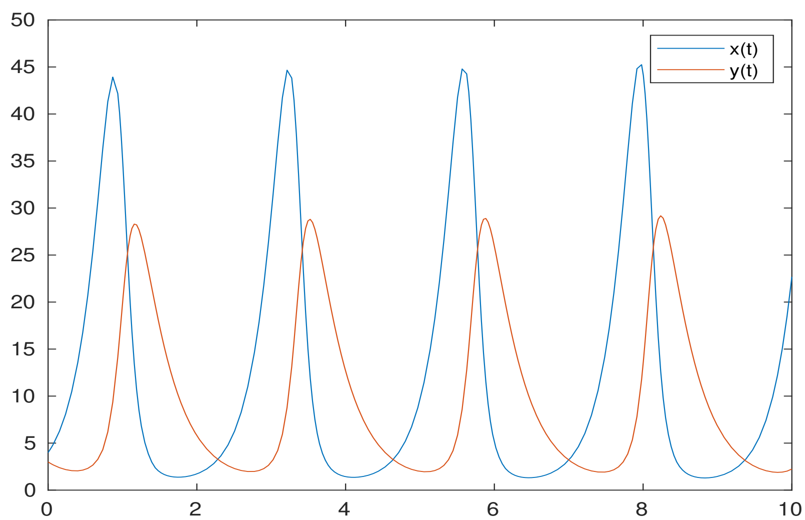
for some $k_2 > 0$. Finally, we can write the mathematical model of prey-predator as follows

$$\begin{cases} \frac{dx}{dt} = r_1 x - k_1 x y \\ \frac{dy}{dt} = -r_2 y + k_2 x y \end{cases}. \tag{6.1}$$

Note that the above equations constitute a *system* in the following sense: to solve the first equation for x, one needs $y(t)$ which is derived by solving the second equation for $y(t)$, while solving the second equation for $y(t)$ needs $x(t)$ which needs the first equation to be solved for $x(t)$. Therefore, the above system must be solve *simultaneously* for $x(t)$, $y(t)$. The above system can not be solved by standard methods we have learned in previous sections for *scalar equations*. However, the *numerical methods* can be employed to solve the equation. The following figure shows the solution for $r_1 = 4, k_1 = 0.4$ and $r_2 = 2, k_2 = 0.2$ in $0 \le t \le 10$ and initial condition $(x(0),\ y(0)) = (4,\ 3)$. Observe how the population of predator follows the population of prey with a specific delay time.



It may seem that if the death rate r_2 increases then $y(t)$ will decrease. The following figure show the solution for $r_2 = 2.5$ instead of $r_2 = 2$ for the above figure. While the maximum of $y(t)$ in the previous case was about 27, the increase of the death rate r_2 increases $y(t)$ to more than 29. The reason is clear. The decreasing of $y(t)$ will increase $x(t)$ that means more food and thus the increase in $y(t)$. This mutual affection is a characteristic of *systems*.

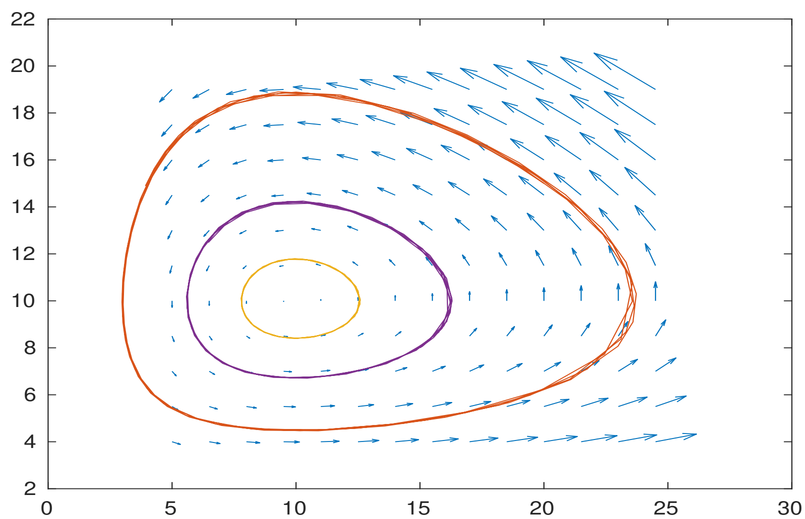


6.1.2 Phase plane

In the prey-predator example, the plane $(x,\ y)$ is called the *phase plane*. Note that in the final analysis, the solution of the system is $(x(t),\ y(t))$ that can be considered as a *parametric curve* $\gamma(t) := (x(t),\ y(t))$ in the phase plane. This curve is called a *trajectory* of the system. On the other hand, the right hand side of the system (6.1) can be considered as a *vector field* in the phase plane, that is,

$$F(x,y) = \begin{pmatrix} r_1 x - k_1 x y \\ -r_2 y + k_2 x y \end{pmatrix}.$$

The following figure show the vector field for parameters $r_1 = 4$, $k_1 = 0.4$, $r_2 = 2$, $k_2 = 0.2$ and a few of trajectories.



It is observed that the trajectories are *tangent* to the vector field in the phase plane. This fact is evident from the system (6.1) as well. If $\gamma(t)$ is the solution of the system, then

$$\frac{d}{dt}\gamma(t) = F(\gamma(t)).$$

Remember that $\frac{d}{dt}\gamma(t)$ is the tangent vector to $\gamma(t)$ at any time t and thus $F(\gamma(t))$ is tangent to the trajectories $\gamma(t)$.

Observe also that the trajectories are closed curves. A closed curve is called an *orbit* of the system and shows that the solutions of the system is *periodic* with respect to time t. In other word, if $\gamma(t)$ is closed, then there is some $\omega > 0$ such that $\gamma(t+\omega) = \gamma(t)$ and thus $x(t), y(t)$ are periodic functions with the period ω. In addition, it seems that the trajectories orbits around a specific point. This point is called an *equilibrium* of the system. Let us find the equilibrium of (6.1). Since the time derivative at the equilibrium is zero, we have

$$\begin{pmatrix} r_1 x - k_1 x y \\ -r_2 y + k_2 x y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

that gives $p_1 = (0,0)$, and $p_2 = \left(\frac{r_2}{k_2}, \frac{r_1}{k_1}\right)$. For the chosen parameters of the figure, the nontrivial equilibrium is $(10,10)$ which is clear from the figure as well.

Problem 6.1. The prey-predator system in the phase plane reduces to a scalar differential equation as

$$\frac{dy}{dx} = \frac{-y(r_2 - k_2 x)}{x(r_1 - k_1 y)}.$$

Solve the above equation for $y = y(x)$.

6.1.3 Linearization of nonlinear systems

As we saw above, the prey-predator model is a nonlinear system due to the multiplicative term xy. Despite linear systems, nonlinear ones sometimes show very complicated behaviors, and thus the linearized version of a nonlinear system is sometimes used instead of the original one. With the linearization model of single variable functions as follows

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0),$$

we can write the linearization of a smooth function $\begin{pmatrix} f \\ g \end{pmatrix}: D \subset \mathbb{R}^2 \to \mathbb{R}^2$ at a given point (x_0, y_0) as follows

$$\begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} \approx \begin{pmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{pmatrix} + J_f(x_0, y_0) \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix},$$

where J_f is the Jacobi matrix

$$J_f = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}.$$

The most common point of a dynamical system for the linearization is the equilibrium point. For the prey-predator system (6.1) with the equilibrium $\left(\frac{r_2}{k_2}, \frac{r_1}{k_1}\right)$, we have

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} \approx F\left(\frac{r_2}{k_2}, \frac{r_1}{k_1}\right) + \begin{bmatrix} 0 & -\frac{k_1 r_2}{k_2} \\ \frac{k_2 r_1}{k_1} & 0 \end{bmatrix} \begin{pmatrix} x - \frac{r_2}{k_2} \\ y - \frac{r_1}{k_1} \end{pmatrix},$$

and since $F\left(\frac{r_2}{k_2}, \frac{r_1}{k_1}\right) = 0$, we derive the linearized system as follows

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} \approx \begin{bmatrix} 0 & -\frac{k_1 r_2}{k_2} \\ \frac{k_2 r_1}{k_1} & 0 \end{bmatrix} \begin{pmatrix} x - \frac{r_2}{k_2} \\ y - \frac{r_1}{k_1} \end{pmatrix}.$$

Furthermore, if we take $X = x - \frac{r_2}{k_2}$, $Y = y - \frac{r_1}{k_1}$, we obtain the following simple linear system

$$\frac{d}{dt}\begin{pmatrix} X \\ Y \end{pmatrix} \approx \begin{bmatrix} 0 & -\frac{k_1 r_2}{k_2} \\ \frac{k_2 r_1}{k_1} & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \end{pmatrix},$$

or equivalently

$$\begin{cases} \frac{dX}{dt} = -\frac{k_1 r_2}{k_2} Y \\ \frac{dY}{dt} = \frac{k_2 r_1}{k_1} X \end{cases}.$$

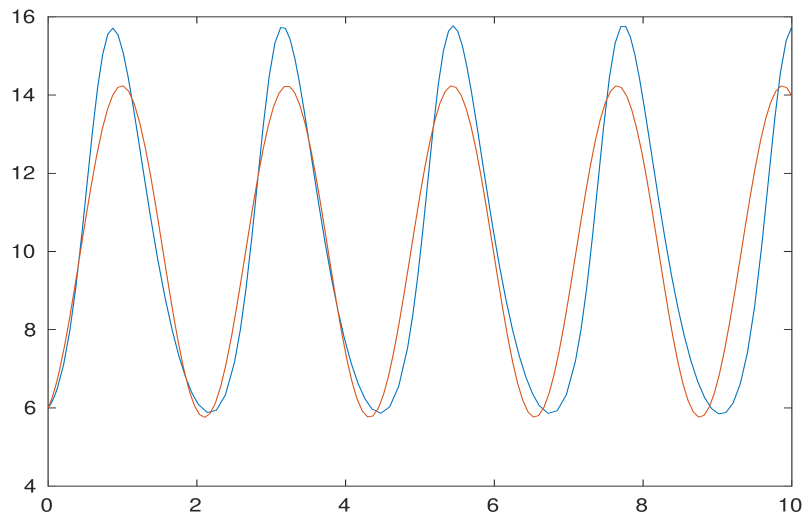
Note that the above system yields the following equation for $X(t)$ by taking derivative of the first equation and substituting $\frac{d}{dt}Y$ from the second equation

$$\frac{d^2 X}{dt} + r_1 r_2 X = 0,$$

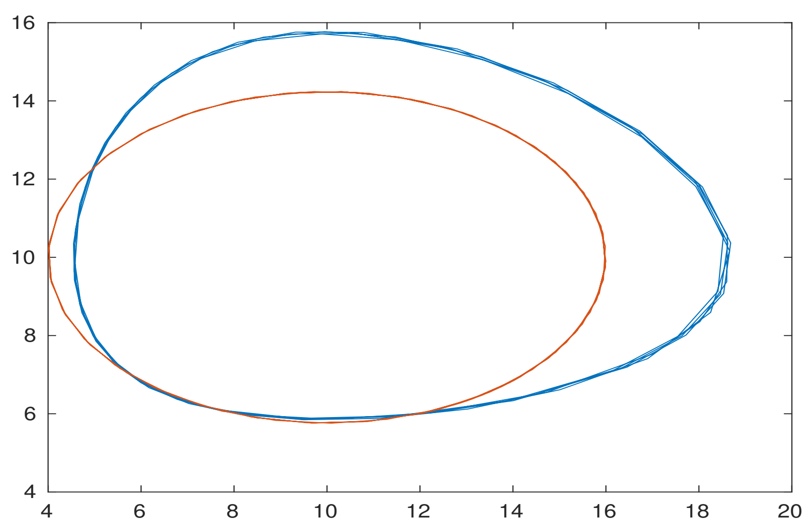
which is solved for

$$X(t) = C_1 \cos(\sqrt{r_1 r_2}\, t) + C_2 \sin(\sqrt{r_1 r_2}\, t).$$

A similar solution is derived simply for $Y(t)$. The following figure shows $y(t)$ for the original system and the linearized system with the same parameters r_1, k_1, r_2, k_2 as before and the initial condition $x_0 = 12$, $y_0 = 6$. The population of prey shows similar behavior. Note that (x_0, y_0) is not very far from the equilibrium point $(10, 10)$.



The trajectory in the phase plane shows the differences and similarities better:

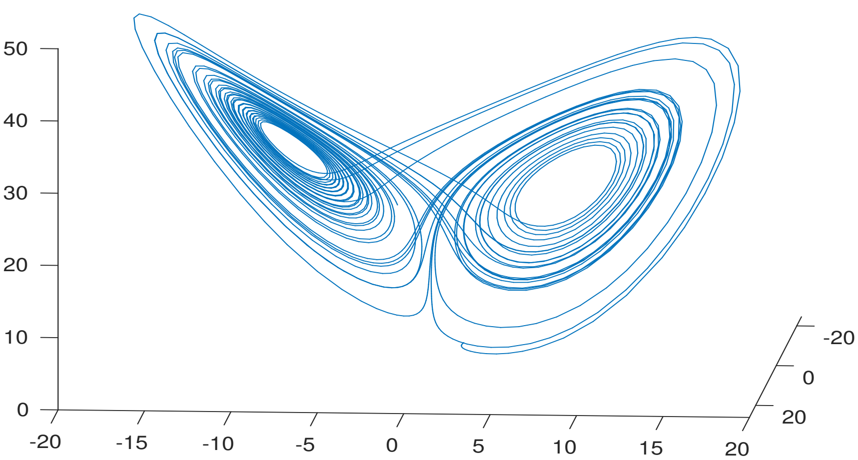


Remark 6.1. Nonlinear systems sometimes show sometimes extremely complicated or eve *chaotic* dynamics. Therefore, the linearization of such system does not show the intrinsic dynamics of them completely. Of the most well-known such system is the Lorenz system studied by E. Lorenz in 1963 in his work in atmospheric convection of

the following form

$$\begin{cases} x' = \alpha(y - x) \\ y' = r x - y - x z \ , \\ z' = x y - \beta z \end{cases}$$

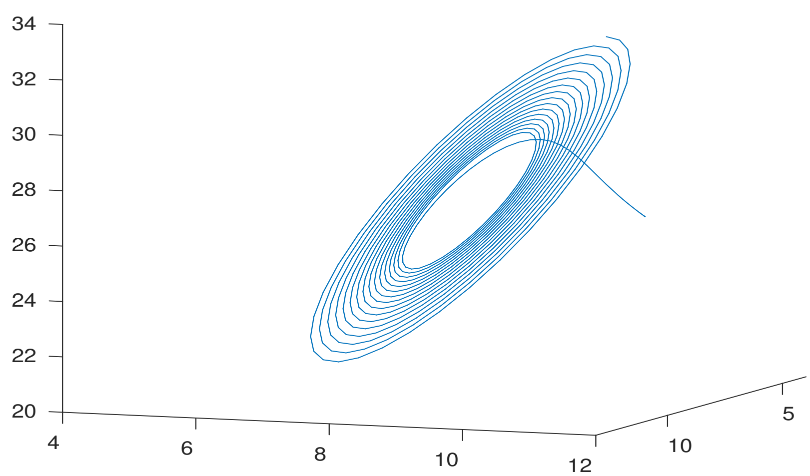
for some constants α, r, β. The following figure shows the solution in the *phase space* (x, y, z) for specific value $\alpha = 10, r = 28, \beta = \frac{8}{3}$.



It is simply seen that the system has a non-trivial equilibrium at $(\sqrt{72}, \sqrt{72}, 27)$, and thus the linearized system around this equilibrium is

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{bmatrix} -10 & 10 & 0 \\ 1 & -1 & -\sqrt{72} \\ \sqrt{72} & \sqrt{72} & -\frac{8}{3} \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The solution of the above linear system is spiral and never chaotic.



6.2 General first-order systems

6.2.1 Linear and nonlinear systems

The general form of a first-order system is

$$\frac{d}{dt}\boldsymbol{y} = \boldsymbol{f}(t, \boldsymbol{y}), \tag{6.2}$$

where $\boldsymbol{y} = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}$ and $\boldsymbol{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$. If $\boldsymbol{f}$ is independent of t, the system is called *autonomous*. The general form of a first-order autonomous system is

$$\frac{d}{dt}\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{y}). \tag{6.3}$$

A value $\bar{\boldsymbol{y}}$ is called an *equilibrium* for the above autonomous system if $\boldsymbol{f}(\bar{\boldsymbol{y}}) = 0$. In this case, $\boldsymbol{y}(t)$ is constant and $\boldsymbol{y}(t) = \bar{\boldsymbol{y}}$ for all t. If all f_i is linear with respect to y_1, ..., y_n, then the system is *linear*. The general form of a linear system is of the following form

$$\frac{d}{dt}\boldsymbol{y} = [a_{ij}(t)]\boldsymbol{y} + \boldsymbol{b}(t),$$

where $\boldsymbol{b} = \begin{pmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{pmatrix}$. If $\boldsymbol{b}$ is identically zero then the linear system is called *linear homogeneous*. If the *coefficient matrix* $A = [a_{ij}(t)]$ is independent of t, the system is called the linear system with *constant coefficients*.

It is interesting to note that every second and higher-order scalar differential equations can be rewritten as a system of first-order equations. The standard method to convert a higher order equation to a system is as follows. For example, consider the following second order equation

$$y'' = f(t, y, y').$$

Let us rename y by y_1 and write

$$\begin{cases} y_1' = y_2 \\ y_2' = f(t, y_1, y_2) \end{cases}.$$

The process for higher order equation is completely similar. For example, the equation

$$y''' = f(t, y, y', y''),$$

can be written as

$$\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = f(t, y_1, y_2, y_3) \end{cases}.$$

Example 6.1. Let us rewrite the following equation into a system

$$y''' + y'\,y'' + yy' = 1 + x.$$

We take $y_1 = y$, and write

$$\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = -y_2\, y_3 - y_1 y_2 + 1 + x \end{cases}.$$

The following linear equation

$$y'' + 3y' + 2y = e^t,$$

can be written in the following first-order system

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ e^t \end{pmatrix}$$

Definition 6.1. *A vector function $\boldsymbol{y} = \phi(t) \in \mathbb{R}^n$ is called a solution to the system (6.2) is there is an open interval I such that*

$$\phi'(t) = \boldsymbol{f}(t, \phi(t)), \forall t \in I.$$

Theorem 6.1. (existence & uniqueness) *Consider the following initial value problem*

$$\begin{cases} \frac{d}{dt}\boldsymbol{y} = \boldsymbol{f}(t, \boldsymbol{y}) \\ \boldsymbol{y}(t_0) = \boldsymbol{y}_0 \end{cases}.$$

If there is a cube D centered at $(t_0, \boldsymbol{y}_0) \in \mathbb{R} \times \mathbb{R}^n$ such that $\boldsymbol{f}$ is continuous on D, then the above initial value problem has at least one solution. In addition, if $\frac{\partial f_i}{\partial y_j}$ is continuous on D for all $f_i \in \boldsymbol{f}$ and $y_j \in \boldsymbol{y}$, then the problem has a unique solution.

Corollary 6.1. *A linear homogeneous system with constant coefficient matrix has exactly n linearly independent solution vector.*

Proof. Consider the following system

$$\frac{d}{dt}\boldsymbol{y} = [a_{ij}]\, \boldsymbol{y},$$

and the associated initial value problem

$$\begin{cases} \frac{d}{dt}\boldsymbol{y} = [a_{ij}]\, \boldsymbol{y} \\ \boldsymbol{y}(0) = \hat{e}_i \end{cases},$$

where $\hat{e}_i$ is the i^{th} unit vector in the direction of y_i. By the existence and uniqueness theorem, there is a unique solution $\phi_i(t)$ for each $i = 1, \ldots, n$. In fact, $\boldsymbol{f}(t, \boldsymbol{y}) = [a_{ij}]$ is continuous and $\frac{\partial f_i}{\partial y_j} = a_{ij}$ are continuous as well. Simply, the set $\{\phi_i(t)\}_{i=1}^n$ is a set of linearly independent vector because

$$\det[\phi_1(0)|\phi_2(0)|\cdots|\phi_n(0)] = \det[\hat{e}_1|\hat{e}_2|\cdots|\hat{e}_n] = 1 \neq 0$$

We leave the last part of the proof to the reader, that is, to prove that $\{\phi_i\}_{i=1}^n$ spans the solution set of the system; see the problem set. □

6.2.2 Vector fields and the geometry of trajectories

Consider a particle moving in the the plane (y_1, y_2) following the rule

$$\begin{cases} y_1' = f(y_1, y_2) \\ y_2' = g(y_1, y_2) \end{cases}. \tag{6.4}$$

The location function of the particle in the plane at time t is $\gamma(t) = (y_1(t),\, y_2(t))$. The path or trace of the particle in this plane is called the *trajectory* of the particle; see Fig.6.1. From the physics point of view, $\gamma'(t)$ is the velocity vector $\vec{v}$ of the trajectory which is the *tangent* vector to the trajectory at the point $\gamma(t)$:

$$\gamma'(t) = F(\gamma(t)),$$

for $F = \begin{pmatrix} f \\ g \end{pmatrix}$.

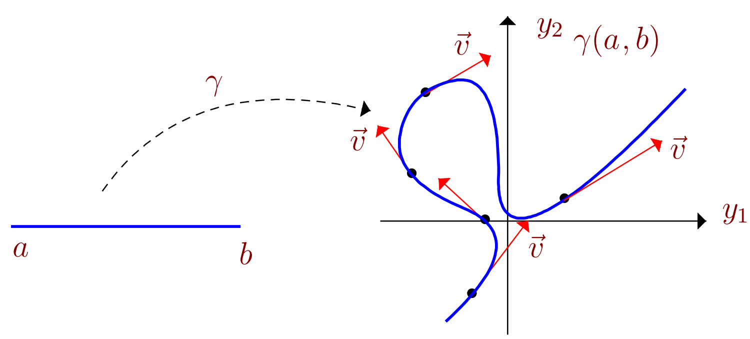
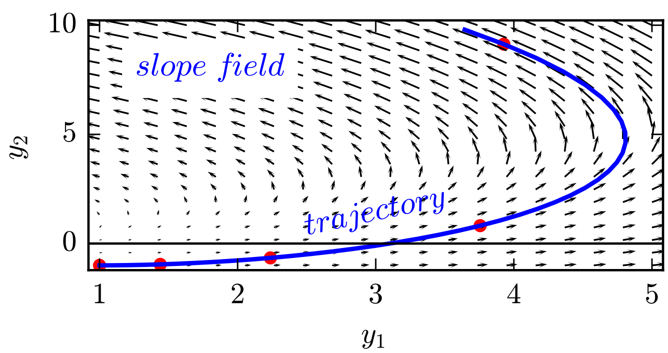


Figure 6.1.

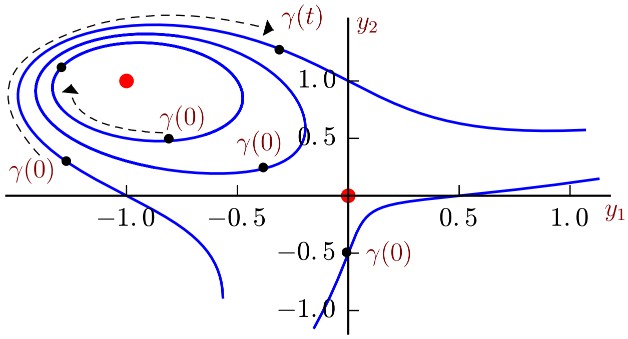
From the geometry point of view, $F = \begin{pmatrix} f \\ g \end{pmatrix}$ defines a vector field on the phase plane (y_1, y_2). The following figure shows how the particle moves in a velocity vector field. The geometry of this vector field provides us with some information about the trajectories of the particle in the plane.



For example, let us consider the following velocity vector field

$$F = \begin{pmatrix} y_1^2 + y_2^2 + 2y_1 \\ y_1^2 + y_2^2 - 2y_2 \end{pmatrix}$$

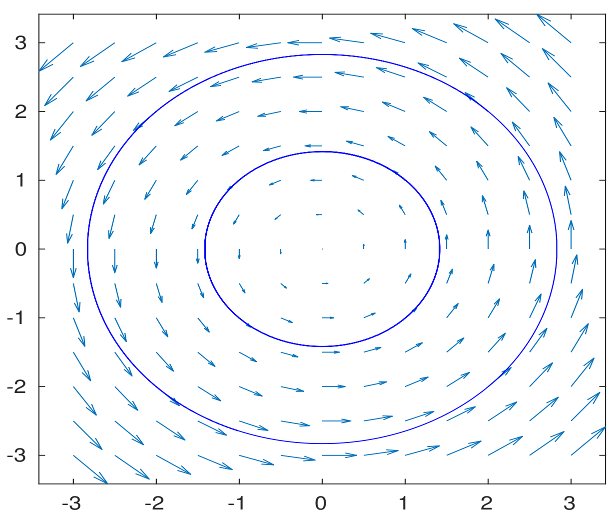
Obviously points $(0,0)$ and $(-1,1)$ are critical points or equilibria of the filed. The trajectory of the particle that moves in this field depends on the initial position. The following figure shows a few trajectories of the system based on the initial location of the particle.



Example 6.2. Consider the following system

$$\begin{cases} y_1' = -y_2 \\ y_2' = y_1 \end{cases}.$$

The vector field is rotational as it is seen from the curl of the filed $\nabla \times \begin{pmatrix} -y_2 \\ y_1 \\ 0 \end{pmatrix} = 2\hat{k}$. The two trajectories are shown in the following figure.



Let $V = |y_1(t)|^2 + |y_2(t)|^2$ be the scalar function of the magnitude of the solution. Then it is simply seen that

$$\frac{d}{dt}V = 2\,y_1(t)\frac{dy_1}{dt} + 2y_2\frac{dy_2}{dt} = -2\,y_1(t)\,y_2'(t) + 2\,y_2(t)\,y_1(t) = 0,$$

and thus $V(y_1, y_2) = \text{const.}$, that is, the shape of the curve are circle. Now consider the following system

$$\begin{cases} y_1' = -y_2 - \varepsilon y_1 \\ y_2' = y_1 \end{cases},$$

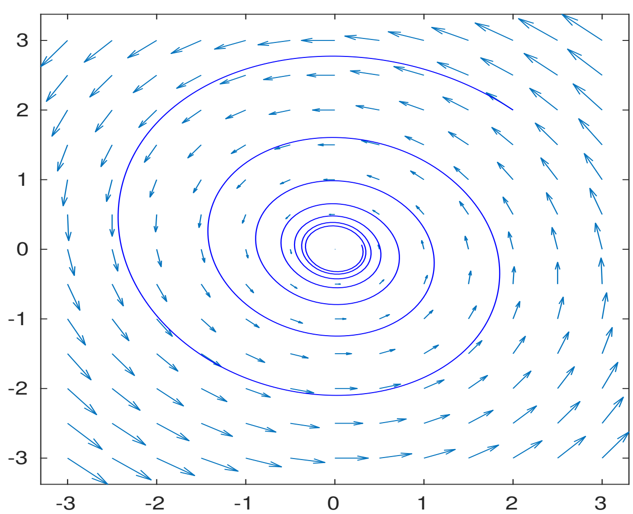
for $\varepsilon > 0$, and let $V(y_1, y_2)$ be the same function as previous. We have

$$\frac{d}{dt}V = -2\varepsilon y_1^2(t),$$

and thus $V(y_1(t), y_2(t))$ decreases with respect to time. It turns out that

$$\lim_{t\to\infty} V(y_1(t), y_2(t)) = 0,$$

and $y_1(t), y_2(t)$ approach zero in long term as the following figure shows.



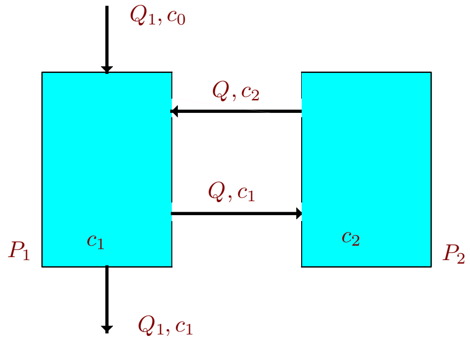
It is simply seen that the trajectory of the following system is outward spiral

$$\begin{cases} y_1' = -y_2 + \varepsilon y_1 \\ y_2' = y_1 \end{cases},$$

for $\varepsilon > 0$.

Problems

Problem 6.2. We would like to model the water pollution diffusion of two connected ponds. Consider two ponds P_1 and P_2 connected by two channels as shown in the following figure:



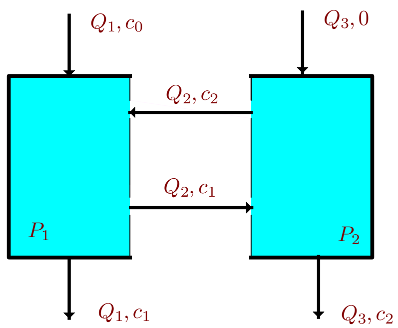
We make the following assumptions: i) channels carry constant streams $Q\frac{m^3}{s}$ between two ponds, ii) the pond P_1 contains $V_1\, m^3$ and the pond P_2 contains $V_2\, m^3$ of pure water at time $t=0$, iii) a pollutant stream $Q_1\,\frac{m^3}{s}$ of the concentration $c_0\,\frac{\text{gr}}{m^3}$ runs into P_1 and simultaneously same amount is discharged out of the pond. Ponds are gradually contaminated by the polluted streams. We would like to determine functions $c_1(t)\,\frac{\text{gr}}{m^3}$ and $c_2(t)\,\frac{\text{gr}}{m^3}$, the pollution of ponds P_1, P_2 respectively.

a) Write down the conservation law for the quantity $c_1\,V_1$, and $c_2\,V_2$. Justify that the relations are as follows

$$\begin{cases} \frac{d(c_1 V_1)}{dt} = c_0\,Q_1 + c_2\,Q - c_1(Q_1+Q) \\ \frac{d(c_2 V_2)}{dt} = c_1\,Q - c_2 Q \end{cases}.$$

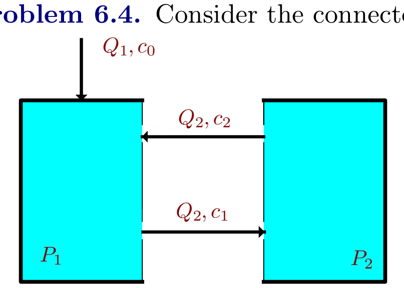
b) Verify that V_1, V_2 are constants, and write the system of differential equations for c_1, c_2 with given initial conditions.

Problem 6.3. Consider the connected ponds shown below.



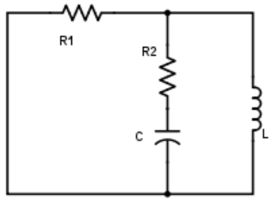
A constant flow $Q_1 m^3/s$ of polluted water $c_0\,\text{gr}/m^3$ runs into the pond P_1 and a constant flow Q_3 of pure water runs into P_2. Write down a system of differential equations describing the concentration $c_1(t)$, $c_2(t)$, the pollution concentration of P_1 and P_2. Suppose P_1 and P_2 contain respectively $V_1\,m^3$ and $V_2\,m^3$ water initially.

Problem 6.4. Consider the connected ponds shown below.



A constant flow $Q_1 m^3/s$ of polluted water $c_0\,\text{gr}/m^3$ runs into the pond P_1. Write down a system of differential equations describing the concentration $c_1(t)$, $c_2(t)$, the pollution concentration of P_1 and P_2. Suppose P_1 and P_2 contain respectively $V_1\,m^3$ and $V_2\,m^3$ water initially.

Problem 6.5. Consider the circuit shown below.



We would like to obtain an equation for the $V_c(t)$, the voltage across the capacitance C, and $i(t)$, the electrical current in the inductor L.

a) Let $i_1(t)$ and $i_2(t)$ denote respectively the electrical current in the resistors $R1$ and $R2$. Use KIRSCHOFF's law and conclude

$$L\frac{di}{dt} = -R_1 i_1, \tag{6.5}$$

$$R_1 i_1 + R_2 i_2 + V_c = 0. \tag{6.6}$$

b) Substitute $i_1 = i_2 + i$ into (6.6) and conclude

$$i_2 = -\frac{1}{R_1 + R_2}V_c - \frac{R_1}{R_1 + R_2}i. \tag{6.7}$$

c) Use the relation $i_2 = C\frac{dV_c}{dt}$, and rewrite (6.7) as

$$\frac{dV_c}{dt} = -\frac{1}{(R_1 + R_2)C}V_c - \frac{R_1}{(R_1 + R_2)C}i. \tag{6.8}$$

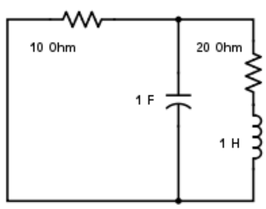
d) Use the relation $L\frac{di}{dt} = -R_1 i_2 - R_1 i$ and substitute i_2 from (6.7) and conclude

$$\frac{di}{dt} = \frac{R_1}{(R_1 + R_2)L}V_c - \frac{R_1 R_2}{(R_1 + R_2)L}i. \tag{6.9}$$

e) Rewrite the system for the vector $\begin{pmatrix} V_c \\ i \end{pmatrix}$. It should has the form

$$\begin{pmatrix} V_c \\ i \end{pmatrix}' = \begin{pmatrix} -\frac{1}{(R_1+R_2)C} & -\frac{R_1}{R_1+R_2} \\ \frac{R_1}{(R_1+R_2)L} & -\frac{R_1 R_2}{(R_1+R_2)L} \end{pmatrix} \begin{pmatrix} V_c \\ i \end{pmatrix}.$$

Problem 6.6. For the circuit shown below write down a system of first order equations for $i(t)$ the electric current in the inductor and V_c the voltage across the capacitance.



Problem 6.7. Verify that vector functions $\vec{\phi}_1(x) = \begin{pmatrix} \cos(x) \\ -\sin(x) \end{pmatrix}$ and $\vec{\phi}_2(x) = \begin{pmatrix} \sin(x) \\ \cos(x) \end{pmatrix}$ are two core solutions to the system

$$\begin{cases} y_1' = y_2 \\ y_2' = -y_1 \end{cases}.$$

Find the fundamental matrix of the system and then find the solution to the following problem

$$\begin{cases} y_1' = y_2 \\ y_2' = -y_1 \\ y_1(0) = 1, y_2(0) = -1 \end{cases}.$$

Problem 6.8. Verify that vector functions $\vec{\phi}_1(x) = e^x \begin{pmatrix} \cos(x) \\ \sin(x) \end{pmatrix}$ and $\vec{\phi}_2(x) = e^x \begin{pmatrix} -\sin(x) \\ \cos(x) \end{pmatrix}$ are two core solutions to the system

$$\begin{cases} y_1' = y_1 - y_2 \\ y_2' = y_1 + y_2 \end{cases}.$$

Find the fundamental matrix of the system and then find the solution to the following problem

$$\begin{cases} y_1' = y_1 - y_2 \\ y_2' = y_1 + y_2 \\ y_1(0) = 1, y_2(0) = 0 \end{cases}.$$

Problem 6.9. Write down the following higher order equations in the form of a system of first order equations. If an equation is linear, write it in the matrix form.

i. $y'' + yy' + y = e^{-x}$
ii. $y'' + \omega^2 y = \sin(\omega x)$
iii. $y''' + xy'' + 2y = 0$
iv. $y''' + \sin(y) = y' - 1$

Problem 6.10. Consider the following system

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Find an equivalent second order equation for the above system and show that its characteristic is equal to $\det(A - \lambda \mathbb{I}) = 0$ where A is the coefficient matrix of the system.

Problem 6.11. Systems which are not fully coupled are sometimes easy to solve. For the following semi-coupled system, try to find the solution.

i.

$$\begin{cases} y_1' = 2y_1 + y_2 \\ y_2' = -y_2 \end{cases}$$

ii.

$$\begin{cases} y_1' = 3y_1 \\ y_2' = y_1(1 + y_2^2) \end{cases}$$

iii.

$$\begin{cases} y_1' = -y_1 + x + 1 \\ y_2' = y_2 / y_1 + y_2^2 / y_1^2 \\ y_1(1) = 1, y_2(1) = -1 \end{cases},$$

Problem 6.12. For each of the following vector fields, use a computer software to draw the trajectory passing through the given point. Use the method described in this section to construct approximate trajectory passing through the point (you can take the step size $h = 0.1$).

i. $V = (-2y, x), \quad p_0 = (1, 0)$
ii. $V = (-y + x, x), \quad p_0 = (1, 0)$
iii. $V = (y + x, -3y), \quad p_0 = (1, 0)$

Problem 6.13. Verify that for each the following vector fields, the given parametric curve is a trajectory

i. $V = (-y, x), \quad \gamma(t) = (\cos(t), \sin(t))$

ii. $V=(-y,2x-2y)$, $\gamma(t)=e^{-t}(\cos(t),\cos(t)+\sin(t))$

iii. $V=(y,2x+y)$, $\gamma(t)=(e^{2t}-e^{-t},2e^{2t}+e^{-t})$

iv. $V=(-2y,13x-2y)$, $\gamma(t)=e^{-t}(5\cos(5t)+\sin(5t),13\sin(5t))$,

Problem 6.14. Assume that $\gamma_1(t)$ and $\gamma_2(t)$ are trajectories of the following direction fields respectively

$$W_1=(\cos(y),\sin(x)-\sin(y)),$$
$$W_2=(\sin(y)-\sin(x),\cos(y)).$$

Prove that if $\gamma_1(t)$ and $\gamma_2(t)$ intersect at a point, then they intersect orthogonal.

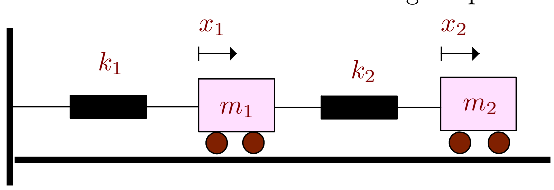
Problem 6.15. For the matrix $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, assume $|A|\neq 0$. Show that for arbitrary $\lambda\neq 0$, the integral curves of two systems $\vec{y}_1=A\,\vec{y}_1$ and $\vec{y}_2=\lambda A\,\vec{y}_2$ are parallel.

Problem 6.16. Let R_θ be the rotation matrix

$$R_\theta=\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Prove that the integral curves of systems $\vec{y}_1=R_\theta\,\vec{y}_1$ and $\vec{y}_2=R_{(\theta+\pi/2)}\,\vec{y}_1$ are orthogonal at their intersections.

Problem 6.17. Consider the following coupled mass spring system



The goal is to write equations describing $x_1(t)$, $x_2(t)$, the positions of the mass m_1 and m_2 respectively.

a) Use the NEWTON's second law for the mass m_1 and conclude

$$m_1x_1''=-(k_1+k_2)x_1+k_2x_2. \tag{6.10}$$

b) Use the NEWTON's second law for the mass m_2 and conclude

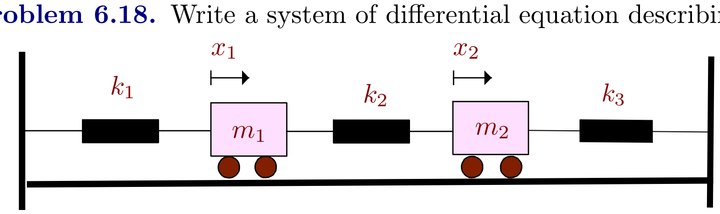
$$m_2\,x_2''=k_2x_1-k_2\,x_2. \tag{6.11}$$

c) By taking $v_1=x_1'$, $v_2=x_2'$, write down a first order system for the given mass-spring system. The answer should has the form

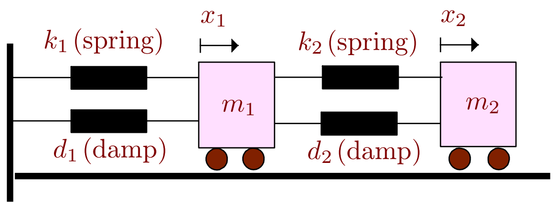
$$\vec{s}\,'=\begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{(k_1+k_2)}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2}{m_2} & 0 \end{pmatrix}\vec{s}, \tag{6.12}$$

for the state vector $\vec{s}=(x_1,v_1,x_2,v_2)$.

Problem 6.18. Write a system of differential equation describing the system shown below.



Problem 6.19. For the coupled damped mass-spring system shown below, write down the equation of motion as a linear system of first order equations.



Problem 6.20. Write down the following higher order equations in the form of a system of first order equations. If an equation is linear, write it in the matrix form.

i. $y''' + x y'' + 2y = 0$

ii. $y''' + \sin(y) = y' - 1$

Problem 6.21. Assume that $F(\boldsymbol{y})$ is a smooth vector field for $\boldsymbol{y} \in \mathbb{R}^n$. Prove that it is impossible that two trajectories of the system $\boldsymbol{y}' = F(\boldsymbol{y})$ intersect each other.

Problem 6.22. We observed that vector fields are tangent to the associated trajectories. From the physics point of view, the vector field of a first-order system is the *velocity field* of particle moving in the phase plane. This property can be used to construct the trajectories numerically as well. Assume a particle is located at (x_0, y_0) in the phase plane at time $t = 0$ following the equation

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = F(x, y).$$

The velocity vector at (x_0, y_0) is $\vec{v} = F(x_0, y_0)$. Therefore, one can use the *linear approximation* formula at $t = h \ll 1$ and write

$$\begin{pmatrix} x(h) \\ y(h) \end{pmatrix} \approx \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + h F(x_0, y_0).$$

Accordingly, we obtain the following recursive formula

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} \approx \begin{pmatrix} x_n \\ y_n \end{pmatrix} + h F(x_n, y_n),$$

where $x_n := x(n h)$, $y_n := y(n h)$ for small step size h.

a) Follow the above algorithm and draw the solution $x(t)$, $y(t)$ of the system (6.1) for $t \in [0, 2]$ using the step size $h = 0.1$, and the initial condition $x_0 = 8$, $y_0 = 5$. Take parameters as $r_1 = 4$, $k_1 = 0.4, r_2 = 2, k_2 = 0.2$.

b) Draw the obtained solution in the phase plane (x, y).

6.3 Linear homogeneous systems

We present a method to solve 2D systems with *constant coefficient matrices*, $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{Y} \in \mathbb{R}^2$. There is no general method to solve linear systems with variable coefficients.

6.3.1 Outline of the method

Assume that a particle is moving in the phase plane (y_1, y_2) subject to the law $\boldsymbol{y}' = A\boldsymbol{y}$, and assume that $\vec{v}$ is an eigenvector of A with eigenvalue λ. Consider the following initial value problem

$$\begin{cases} \frac{d}{dt}\boldsymbol{y} = A\boldsymbol{y} \\ \boldsymbol{y}(0) = \vec{v} \end{cases}.$$

Note that

$$\frac{d\boldsymbol{y}}{dt}(0) = A\boldsymbol{y}(0) = A\vec{v} = \lambda\vec{v},$$

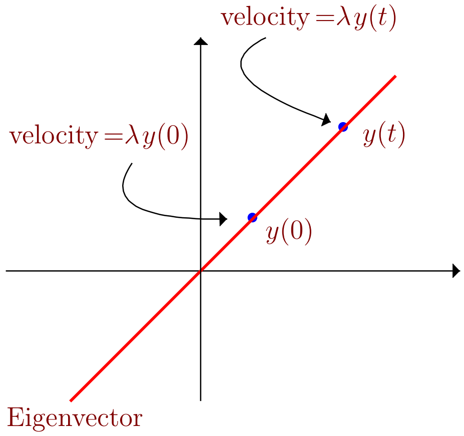
and thus the velocity vector of the particle at $t=0$ lies along the eigenvector $\vec{v}$. It implies that the particle remains along $\vec{v}$ for all t, and therefore, we can write the system as the scalar one *along* $\vec{v}$, that is,

$$\frac{dz}{dt}=\lambda z,$$

where z is a variable *along* $\vec{v}$. The solution of the above scalar equation is $z=e^{\lambda t}$, or equivalently

$$\boldsymbol{y}(t)=e^{\lambda t}\,\vec{v}.$$

The following figure shows the above argument schematically.



Proposition 6.1. *Consider the following initial value problem*

$$\begin{cases} \frac{d}{dt}\boldsymbol{y}=A\boldsymbol{y} \\ \boldsymbol{y}(0)=c\,\vec{v} \end{cases},$$

where $\vec{v}$ is an eigenvector of A with eigenvalue λ. Then the unique solution of the problem is

$$\phi(t)=c e^{\lambda t}\,\vec{v}.$$

Proof. The existence and uniqueness is verified by the existence and uniqueness theorem for the given initial value problem. We need only to verify that the given vector function solves the given problem. It satisfies the given initial condition, and furthermore, we have

$$\frac{d}{dt}\phi(t)=c\lambda e^{\lambda t}\vec{v},$$

and

$$A\phi(t)=c e^{\lambda t}A\,\vec{v}=c\lambda e^{\lambda t}\,\vec{v},$$

and thus $\boldsymbol{y}=\phi(t)$ satisfies the given problem. □

6.3.2 Two real distinct eigenvalues

Consider the following initial value problem

$$\begin{cases} \frac{d}{dt}\boldsymbol{y}=A\boldsymbol{y} \\ \boldsymbol{y}(0)=\boldsymbol{y}_0 \end{cases},$$

and assume A has two real distinct eigenvalues λ_1, λ_2. It is simply seen that their associated eigenvectors $\vec{v}_1, \vec{v}_2$ are linearly independent. Therefore, we can write the initial condition $\boldsymbol{y}_0$ as the linear combination

$$\boldsymbol{y}_0 = c_1 \vec{v}_1 + c_2 \vec{v}_2,$$

and thus the given initial value problem reads

$$\begin{cases} \frac{d}{dt}\boldsymbol{y} = A\boldsymbol{y} \\ \boldsymbol{y}(0) = c_1 \vec{v}_1 + c_2 \vec{v}_2 \end{cases}.$$

The superposition principle allows us to write the above problem as the summation of the following two sub-problems

$$(1)\begin{cases} \frac{d}{dt}\boldsymbol{y} = A\boldsymbol{y} \\ \boldsymbol{y}(0) = c_1 \vec{v}_1 \end{cases}, \quad (2)\begin{cases} \frac{d}{dt}\boldsymbol{y} = A\boldsymbol{y} \\ \boldsymbol{y}(0) = c_2 \vec{v}_2 \end{cases}.$$

Obviously, the first problem is solved for $\phi_1 = c_1\, e^{\lambda_1 t}\, \vec{v}_1$, and the second one for $\phi_2 = c_2\, e^{\lambda_2 t}\, \vec{v}_2$.

Example 6.3. Consider the following initial value problem

$$\begin{cases} \frac{d}{dt}\boldsymbol{y} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \boldsymbol{y} \\ \boldsymbol{y}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{cases}.$$

The coefficient matrix has eigenvalues $\lambda_1 = 1, \lambda_2 = 2$ with associated eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Note that we can write $\boldsymbol{y}_0$ as

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and thus

$$\phi(t) = 2e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} - e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2e^t - e^{2t} \\ -e^{-2t} \end{pmatrix}.$$

Example 6.4. Consider the following system

$$\begin{cases} y_1' = 3y_1 - 4y_2 \\ y_2' = y_1 - 2y_2 \\ y_1(0) = 1, y_2(0) = -1. \end{cases}, \tag{6.13}$$

The coefficient matrix A is

$$A = \begin{pmatrix} 3 & -4 \\ 1 & -2 \end{pmatrix}, \tag{6.14}$$

with the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 2$. The associated eigenvectors are $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$. Since

$$\boldsymbol{y}(0) = -\frac{5}{3}\vec{v}_1 + \frac{2}{3}\vec{v}_2, \tag{6.15}$$

we obtain

$$\boldsymbol{y}(t) = -\frac{5}{3}e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{2}{3}e^{2t}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 8e^{2t} - 5e^{-t} \\ 2e^{2t} - 5e^{-t} \end{pmatrix}. \tag{6.16}$$

6.3.3 Geometry of the trajectories

The following figure shows schematically the phase plane of a matrix A with two eigenvectors $\vec{v}_1$, $\vec{v}_2$ associated to eigenvalues $\lambda_1 < 0$ and $\lambda_2 > 0$. Consider a particle p moving in accordance with the system

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

If particle p lies initially at the direction of $\vec{v}_1$, it gradually approaches the origin as shown in the figure. On the other hand, it is it located at $\vec{v}_2$, it moves along the same direction away from the origin. Now, assume that it is located at $\boldsymbol{y}_0$. The vector $\boldsymbol{y}_0$ can be uniquely *decomposed* on the direction of $\vec{v}_1$ as p_1, and on the direction of $\vec{v}_2$ as p_2. After δt, the point p_1 reaches $p_1(\delta t)$, and p_2 to $p_2(\delta t)$. As it is shown in the figure, the initial point $\boldsymbol{y}_0$ moves to $y(\delta t)$ at $t = \delta t$.

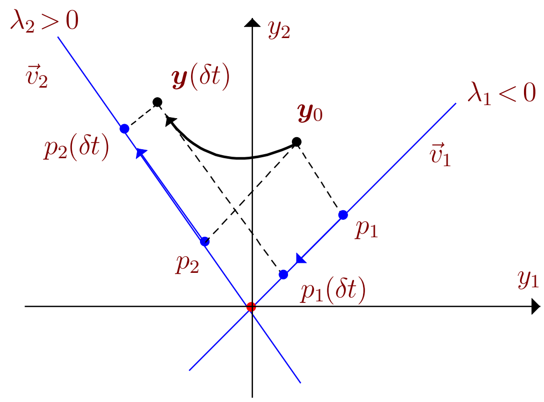


Figure 6.2.

There are three possible cases for a matrix of two distinct eigenvalues λ_1, λ_2.

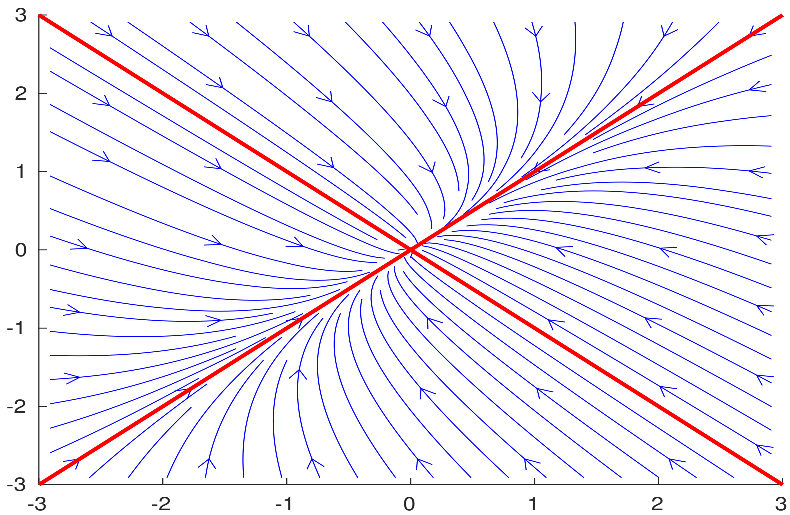
$\boldsymbol{\lambda_2 < \lambda_1 < 0}$**.** In this case, both terms $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ goes zero when $t \to \infty$ and then

$$\lim_{t\to\infty} \boldsymbol{y}(t) = 0.$$

The equilibrium point in this case is stable and is called a *nodal sink*. For example, consider the following system

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

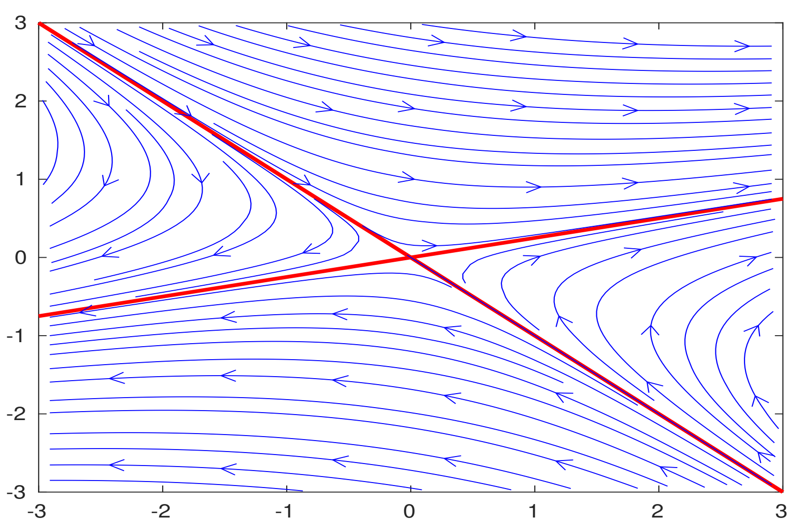
The eigenvalues are $\lambda_1 = -1$, $\lambda_2 = -3$ and eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. The following figure shows a few of trajectories in the phase plane. Note that $\lambda_2 < \lambda_1$ and thus $e^{\lambda_2 t}$ on the direction of $\vec{v}_2$ vanishes sooner than $e^{\lambda_1 t}$ on the direction of $\vec{v}_1$. Therefore, trajectories approaches the origin tangent to $\vec{v}_1$.



$\lambda_2 < 0 < \lambda_1$. In this case, the direction of $\vec{v}_1$ is *unstable* and the direction of $\vec{v}_2$ is stable. For example, consider the following system

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

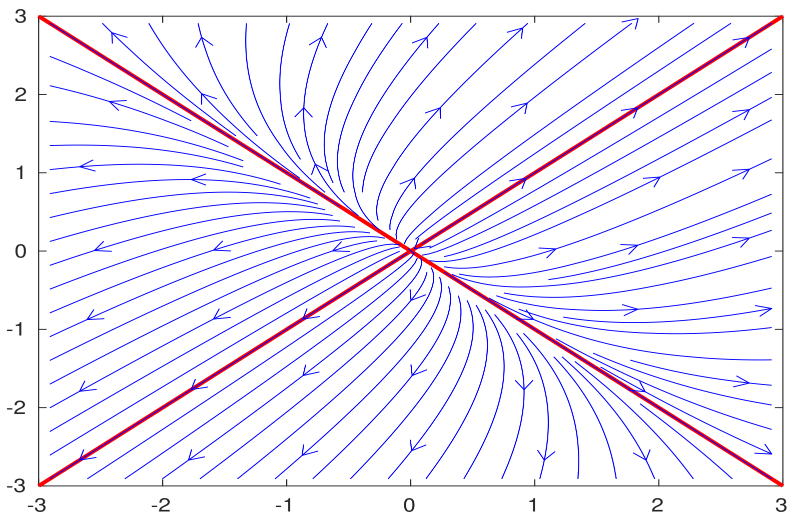
The coefficient matrix has eigenvalues $\lambda_1 = 3$, $\lambda_2 = -2$ and eigenvectors $\vec{v}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. The following figures show some trajectories in the phase plane. The origin in this case is called a *saddle point*.



$0 < \lambda_1 < \lambda_2$. For example, consider the following system

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

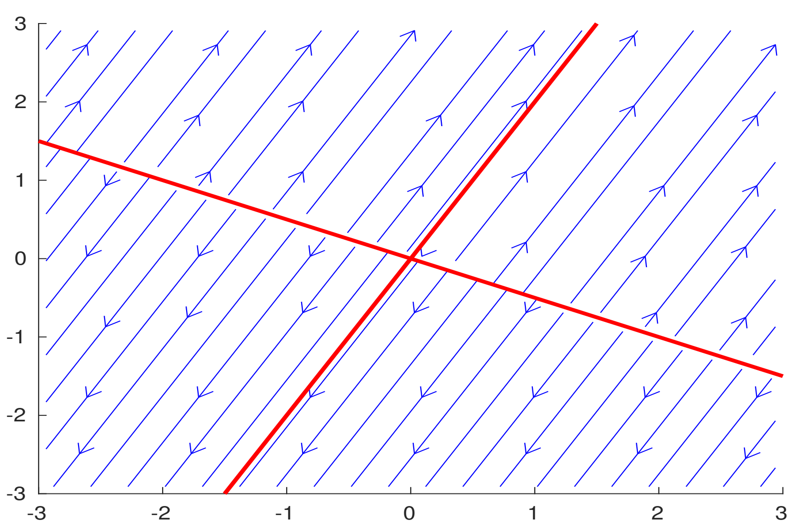
The eigenvalues ate $\lambda_1 = 1$, $\lambda_2 = 3$ with eigenvectors $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The following figure shows a few trajectories in the phase plane. In this case, the origin is called a *nodal source*. Note that how trajectories ten toward the dominant direction $\vec{v}_2$ with bigger eigenvalues.



Example 6.5. Let us draw the phase portrait of a system with a zero eigenvalue. Consider the following system

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

The coefficient matrix has eigenvalues-eigenvectors $\lambda_1 = 0$, $\vec{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $\lambda_2 = 5$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Since point son $\vec{v}_1$ do not move with respect to time, trajectories are straight lines perpendicular to the direction of $\vec{v}_1$ as shown in the following figure.



6.3.4 Repeated eigenvalues

If matrix $A_{2\times2}$ has a repeated eigenvalue λ then there are two possibilities for the set of eigenvectors:

1. that any vector of $\mathbb{R}^2$ is an eigenvector, and

2. there is only one eigenvector.

In fact, if A has two linearly independent eigenvectors $\vec{v}_1$, $\vec{v}_2$ with the same eigenvalues λ, then for arbitrary vector $\vec{w}$, there are constants c_1, c_2 such that

$$\vec{w} = c_1\vec{v}_1 + c_2\,\vec{v}_2,$$

and thus

$$A\vec{w} = c_1\,A\,\vec{v}_1 + c_2\,A\,\vec{v}_2 = \lambda\,(c_1\vec{v}_1 + c_2\,\vec{v}_2) = \lambda\,\vec{w}.$$

If so, two linearly independent solutions are

$$\phi_1(t) = e^{\lambda t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \phi_2(t) = e^{\lambda t}\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and thus the initial value problem

$$\begin{cases} \frac{d}{dt}\boldsymbol{y} = A\boldsymbol{y} \\ \boldsymbol{y}(0) = \boldsymbol{y}_0 \end{cases},$$

has the solution

$$\phi(t) = e^{\lambda t}\boldsymbol{y}_0.$$

The difficult part is the case when A has only one eigenvector. Assume that λ, $\vec{v}$ is the eigenvalue-eigenvector of a matrix A. Then the system

$$\frac{d}{dt}\boldsymbol{y} = A\boldsymbol{y},$$

has one solution

$$\phi_1(t) = e^{\lambda t}\,\vec{v}.$$

For the second solution we use a fact from linear algebra. Remember that if a matrix $A_{2\times2}$ has only one eigenvector $\vec{v}$, then there is a *generalized eigenvector* $\vec{w}$ such that

$$(A - \lambda\mathbb{I})\vec{w} = \vec{v}. \tag{6.17}$$

We claim that the second solution is as follows

$$\phi_2(t) = e^{\lambda t}(\vec{w} + t\,\vec{v}).$$

The verification is straightforward as given below. We have

$$\frac{d}{dt}\phi_2(t) = \lambda e^{\lambda t}(\vec{w} + t\,\vec{v}) + e^{\lambda t}\,\vec{v}.$$

On the other hand, we have

$$A\phi_2(t) = e^{\lambda t}\,A(\vec{w} + t\,\vec{v}),$$

and by the relation $A\vec{w} = \lambda\vec{w} + \vec{v}$, we obtain

$$A\phi_2(t) = e^{\lambda t}\,(\lambda\vec{w} + \vec{v} + t\lambda\,\vec{v}),$$

and hence

$$\frac{d}{dt}\phi_2(t) = A\phi_2(t).$$

Example 6.6. Consider the system

$$\begin{cases} \boldsymbol{y}' = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \boldsymbol{y} \\ \boldsymbol{y}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{cases}. \tag{6.18}$$

The coefficient matrix has eigenvalue-eigenvector $\lambda = 2$, $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and the generalized eigenvector is $\vec{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Therefore, two linearly independent solutions are

$$\phi_1(t) = e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \phi_2(t) = e^{2t}\begin{pmatrix} t+1 \\ t \end{pmatrix},$$

and thus the general solution is

$$\boldsymbol{y}(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\,e^{2t}\begin{pmatrix} t+1 \\ t \end{pmatrix}.$$

Applying the initial condition determined $c_1 = -1$, $c_2 = 2$ and hence the solution to the given initial value problem is

$$\phi(t) = e^{2t}\begin{pmatrix} 2t+1 \\ 2t-1 \end{pmatrix}.$$

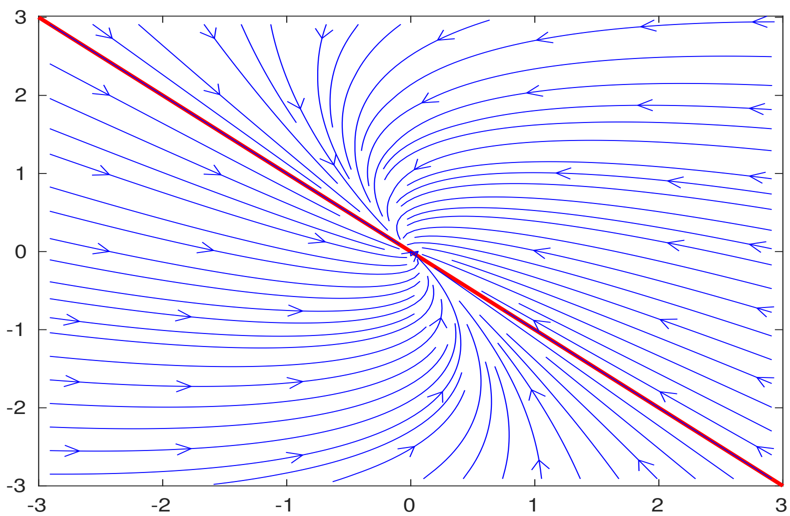
6.3.5 Trajectories in the phase plane

If A has only one eigenvector, the origin is called *improper* or *defective* point.

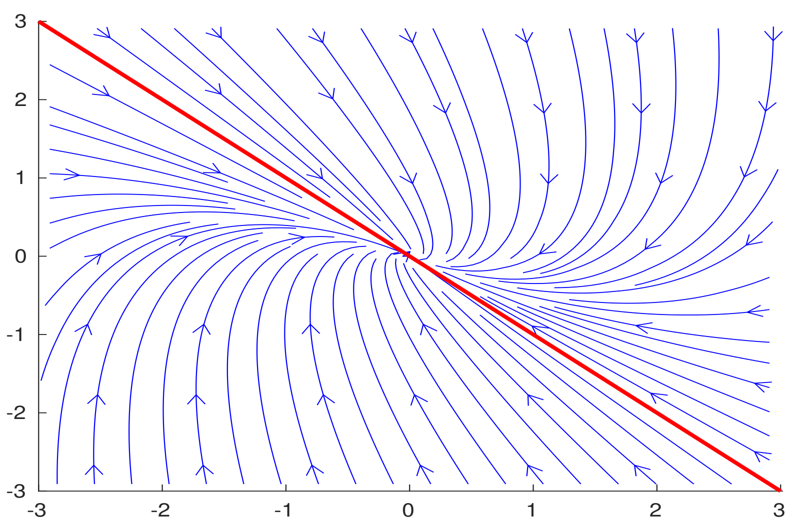
$\boldsymbol{\lambda < 0}$. In this case, all trajectories approaches the origin in long term

$$\lim_{t\to\infty} \boldsymbol{y}(t) = 0.$$

Furthermore, trajectories are tangent to the unique eigenvector as shown in the following figure for the matrix $A = \begin{pmatrix} -3 & -1 \\ 1 & -1 \end{pmatrix}$. In this case, the origin is called a *defective sink*.



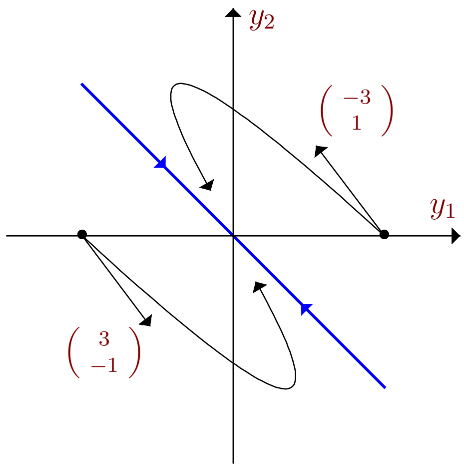
If we replace the matrix A with the matrix $B = \begin{pmatrix} -1 & 1 \\ -1 & -3 \end{pmatrix}$, the eigenvalue and eigenvector are the same, however, the form of trajectories are as follows



In order to determine the rotation of trajectories, we can do as follows. For the first matrix, let $\boldsymbol{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and thus

$$\boldsymbol{y}'(0) = \begin{pmatrix} -3 & -1 \\ 1 & -1 \end{pmatrix} \boldsymbol{y}_0 = \begin{pmatrix} -3 \\ 1 \end{pmatrix},$$

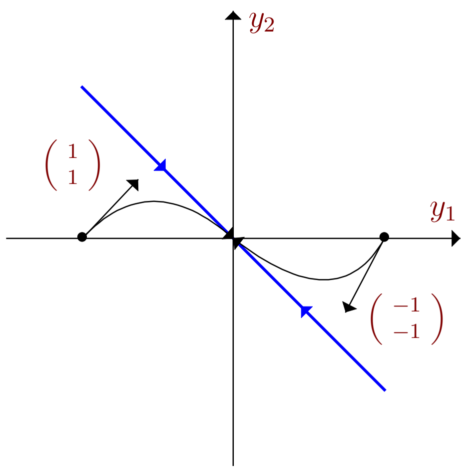
as shown below



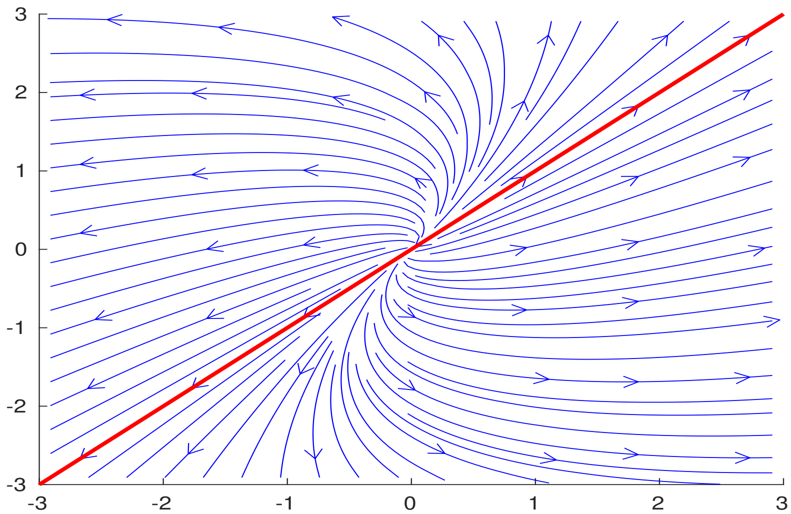
For the matrix B we have

$$\boldsymbol{y}(0) = \begin{pmatrix} -1 & 1 \\ -1 & -3 \end{pmatrix} \boldsymbol{y}(0) = \begin{pmatrix} -1 \\ -1 \end{pmatrix},$$

that is shown below



$\boldsymbol{\lambda > 0}$. In this case all trajectories goes unbounded when $t \to \infty$ as shown in the following figure The origin is called a *defective source*.



6.3.6 Complex eigenvectors

Complex eigenvalues are characteristic of rotational matrices. If a matrix $A_{2\times 2}$ has a complex eigenpair $(\lambda, \vec{v})$, then it posses the complex conjugate pair $(\bar{\lambda}, \bar{\vec{v}})$. Consider the following system

$$\frac{d}{dt}\boldsymbol{y} = A\boldsymbol{y},$$

where A has complex eigenpair $(\lambda, \vec{v})$. Then, the system has two solutions

$$\phi_1(t) = \operatorname{Re}\{e^{\lambda t}\vec{v}\}, \phi_2(t) = \operatorname{Im}(e^{\lambda t}\vec{v}).$$

Example 6.7. Consider the system

$$\begin{cases} \frac{d}{dt}\boldsymbol{y} = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}\boldsymbol{y} \\ \boldsymbol{y}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}. \tag{6.19}$$

The eigenvalue of the coefficient matrix is $\lambda = 1 + 2i$ with the complex eigenvector $\vec{v} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$. Therefore, two independent solutions are

$$\phi_1(t) = e^t \operatorname{Re}\left(e^{i2t}\begin{pmatrix} 1 \\ 1-i \end{pmatrix}\right) = e^t\begin{pmatrix} \cos(2t) \\ \cos(2t) + \sin(2t) \end{pmatrix},$$

$$\phi_2(t) = e^t \operatorname{Im}\left(e^{i2t}\begin{pmatrix} 1 \\ 1-i \end{pmatrix}\right) = e^t\begin{pmatrix} \sin(2t) \\ \sin(2t) - \cos(2t) \end{pmatrix}.$$

Therefore, the general solution is

$$\boldsymbol{y}(t) = c_1 e^t \begin{pmatrix} \cos(2t) \\ \cos(2t) + \sin(2t) \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin(2t) \\ \sin(2t) - \cos(2t) \end{pmatrix},$$

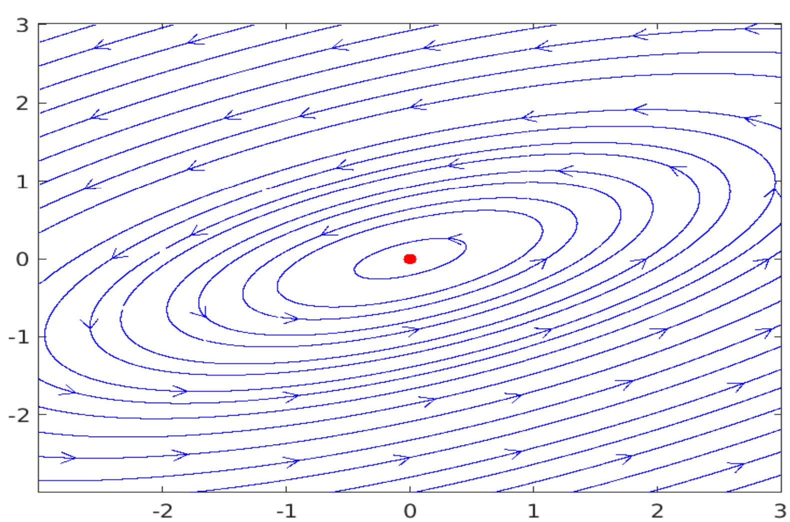
and applying the given initial condition yields $c_1 = 0$, $c_2 = -1$ and finally

$$\phi(t) = e^t \begin{pmatrix} -\sin(2t) \\ \cos(2t) - \sin(2t) \end{pmatrix}.$$

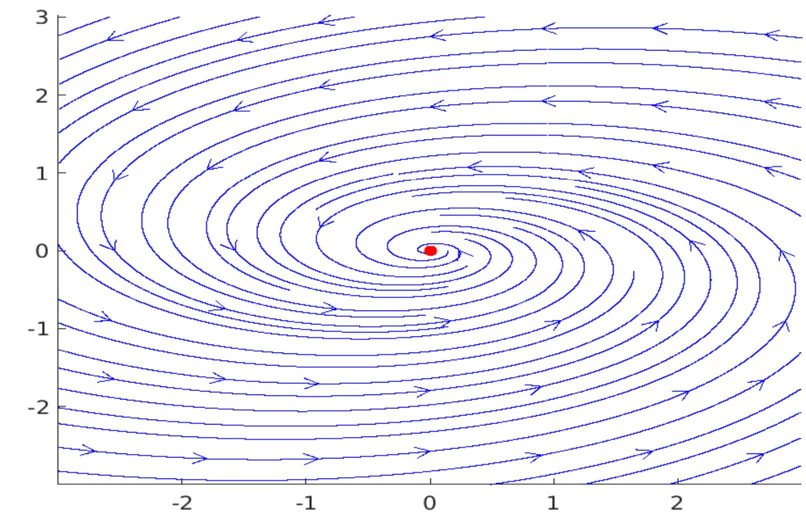
6.3.7 Trajectories in the phase plane

In this case, the trajectories form closed curves or spiral around the origin depending on the sign of σ.

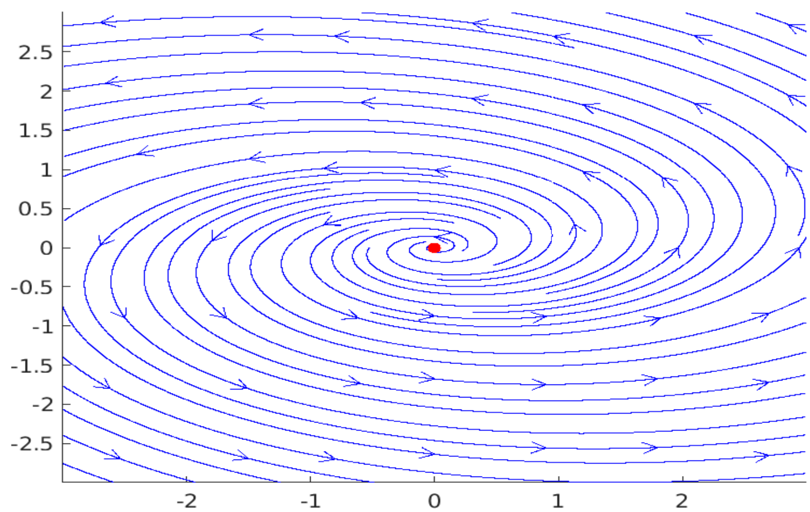
$\sigma = 0$. When the real part of the eigenvalue is zero, the exponential function $e^{\lambda x} = e^{i\omega x}$ is just a trigonometric functions and the trajectories form ellipses around the origin. The origin in this case is called a *center*. The following figure shows a few trajectories of a system with coefficient matrix $A = \begin{bmatrix} 1 & -3 \\ 1 & -1 \end{bmatrix}$



$\sigma < 0$. In this case, the rotation is multiplied by the factor $e^{\sigma x}$ and thus the trajectories goes to the origin when $t \to \infty$. The origin is called a *spiral sink* in this case. The following figures trajectories of a system with the coefficient matrix $A = \begin{bmatrix} -0.5 & -3 \\ 1 & -0.5 \end{bmatrix}$. Note that in this case, the value of σ is $\frac{1}{2}$.



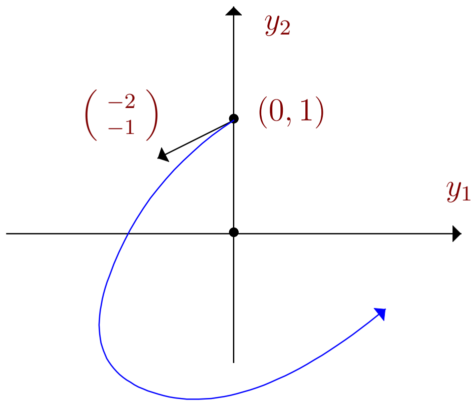
$\sigma > 0$. Similar behavior to $\sigma < 0$ but only the spirals are outward. The origin is called a *spiral source* in this case.



Remark 6.2. The direction of the rotation is simply determined by the aid of the coefficient matrix. For example, for a system with the coefficient matrix $\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$, we have

$$\boldsymbol{y}'(0) = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

and thus trajectories form a spiral source rotating counter-clockwise direction. This argument implies that if $c > 0$ in the coefficient matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the rotation is counter-clockwise and if $c < 0$, the rotation is clockwise.



Problems

Problem 6.23. Consider the following system

$$\frac{d}{dt}\boldsymbol{y} = A(t)\boldsymbol{y},$$

where $A(t) = [a_{ij}(t)]_{n\times n}$, and all $a_{ij}(t)$ are continuous. Prove that the system has exactly n-linearly independent solution vectors.

Problem 6.24. Rewrite each of the following initial value problem as a scalar equation along the appropriate eigenvector and then write the solution

i.

$$\begin{cases} y_1' = 2y_1 + y_2 \\ y_2' = 3y_1 + 4y_2 \\ y_1(0) = 1, y_2(0) = 3 \end{cases}$$

ii.

$$\begin{cases} y_1' = y_1 - y_2 \\ y_2' = -4y_1 + y_2 \\ 315\text{pt}y_1(0) = 2, y_2(0) = -4 \end{cases}$$

iii.

$$\begin{cases} y_1' = y_1 + 2y_2 \\ y_2' = 2y_1 + 4y_2 \\ y_1(0) = -2, y_2(0) = 1 \end{cases}$$

iv.

$$\begin{cases} y_1' = -y_1 + 8y_2 \\ y_2' = y_1 + y_2 \\ y_1(0) = 1, y_2(0) = -1 \end{cases}$$

Problem 6.25. Find two linearly independent solution vectors for each of the following systems

i.

$$\begin{cases} y_1' = 2y_1 + y_2 \\ y_2' = 3y_1 + 4y_2 \end{cases}$$

ii.

$$\begin{cases} y_1' = -3y_1 + 2y_2 \\ y_2' = -2y_1 + y_2 \end{cases}$$

iii.

$$\begin{cases} y_1' = 2y_1 - 5y_2 \\ y_2' = 4y_1 - 2y_2 \end{cases}$$

iv.

$$\begin{cases} y_1' = y_1 + y_2 \\ y_2' = -2y_1 - 2y_2 \end{cases}$$

Problem 6.26. For the following 2-dimensional systems, find the equilibrium point(s), determine the type of points, and draw the phase portrait of the system

a)

$$\begin{cases} y_1' = 2y_1 + y_2 \\ y_2' = 3y_1 + 4y_2 \end{cases},$$

b)

$$\begin{cases} y_1' = y_1 - y_2 \\ y_2' = -4y_1 + y_2 \end{cases},$$

c)

$$\begin{cases} y_1' = -y_1 + 8y_2 \\ y_2' = y_1 + y_2 \end{cases},$$

d)

$$\begin{cases} y_1' = -y_1 - 5y_2 \\ y_2' = y_1 + y_2 \end{cases},$$

e)

$$\begin{cases} y_1' = -3y_1 + 2y_2 \\ y_2' = -2y_1 + y_2 \end{cases},$$

f)

$$\begin{cases} y_1' = 5y_1 + 3y_2 \\ y_2' = -3y_1 - y_2 \end{cases},$$

g)

$$\begin{cases} y_1' = -6y_1 \\ y_2' = 2y_1 + y_2 \end{cases}$$

h)

$$\begin{cases} y_1' = -y_1 - 2y_2 \\ y_2' = 2y_1 + 3y_2 \end{cases}$$

i)

$$\begin{cases} y_1' = 2y_1 + 2y_2 \\ y_2' = -5y_1 - 4y_2 \end{cases}$$

j)

$$\begin{cases} y_1' = 2y_1 - 5y_2 \\ y_2' = 4y_1 - 2y_2 \end{cases}$$

k)

$$\begin{cases} y_1' = 4y_1 - y_2 \\ y_2' = 6y_1 - 2y_2 \end{cases}$$

Problem 6.27. Consider the system

$$\frac{d}{dt}\boldsymbol{y} = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}\boldsymbol{y}.$$

a) Show that the system has two linearly independent solution vectors $\vec{\phi}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\vec{\phi}_2 = \begin{pmatrix} t+\frac{1}{2} \\ 2t \end{pmatrix}$.

b) Deduce that the trajectories in the phase plane satisfies the equation $y_2 = 2y_1 + c$ for some constant c.

Problem 6.28. Let $A_{2\times 2}$ has a complex eigenvalue $\lambda = \sigma + i\omega$.

i. For the matrix $Q = \frac{1}{2}[\mathrm{Re}(\vec{v})|\mathrm{Im}(\vec{v})\,]$ show that $Q^{-1}AQ = \begin{pmatrix} \sigma & -\omega \\ \omega & \sigma \end{pmatrix}$.

ii. Conclude that $A = QBQ^{-1}$ for the matrix $B = |\lambda|\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ for some suitable angle θ.

6.4 Non-homogeneous systems

Consider the following system

$$\frac{d}{dt}\boldsymbol{y} = A\boldsymbol{y} + \boldsymbol{r}(t). \tag{6.20}$$

As we expect, the general solution should has the following form

$$\boldsymbol{y}(t) = \boldsymbol{y}_h(t) + \boldsymbol{y}_p(t), \tag{6.21}$$

where $\boldsymbol{y}_h$ is the homogeneous solution when $\boldsymbol{r}$ is identically zero, and $\boldsymbol{y}_p$ is a particular solution of the system. We present three different methods to find $\boldsymbol{y}_p$.

6.4.1 Eigenvector decomposition method

Consider the following system

$$\begin{cases} \frac{d}{dt}\boldsymbol{y} = A_{2\times 2}\boldsymbol{y} + \boldsymbol{r}(t) \\ \boldsymbol{y}(0) = \boldsymbol{y}_0 \end{cases}.$$

If A has two linearly independent eigenvectors $\vec{v}_1$, $\vec{v}_2$, then we can *decompose* $\boldsymbol{r}(t)$ in a unique way as

$$\boldsymbol{r}(t) = r_1(t)\,\vec{v}_1 + r_2(t)\,\vec{v}_2,$$

and thus the given non-homogeneous system can be solved separately as

$$\begin{cases} \frac{d}{dt}\boldsymbol{y} = A_{2\times 2}\boldsymbol{y} + r_1(t)\,\vec{v}_1 \\ \boldsymbol{y}(0) = c_1\,\vec{v}_1 \end{cases}, \begin{cases} \frac{d}{dt}\boldsymbol{y} = A_{2\times 2}\boldsymbol{y} + r_2(t)\,\vec{v}_2 \\ \boldsymbol{y}(0) = c_2\,\vec{v}_2 \end{cases},$$

where $\boldsymbol{y}_0 = c_1\,\vec{v}_1 + c_2\,\vec{v}_2$ for some c_1, c_2. The first system reduces to a scalar equation *along* $\vec{v}_1$, that is,

$$\begin{cases} \frac{dz}{dt} = \lambda_1\,z + r_1(t) \\ z(0) = c_1 \end{cases},$$

and the second system reads

$$\begin{cases} \frac{dz}{dt} = \lambda_1 z + r_2(t) \\ z(0) = c_2 \end{cases}.$$

Note that, the solution of the first equation is a vector solution along $\vec{v}_1$, and of the second one is a solution along $\vec{v}_2$, that is,

$$\phi_1(t) = \left[c_1 e^{-\lambda_1 t} + e^{-\lambda_1 t} \int_0^t e^{\lambda_1 \tau} r_1(\tau) d\tau \right] \vec{v}_1,$$

$$\phi_2(t) = \left[c_2 e^{-\lambda_2 t} + e^{-\lambda_2 t} \int_0^t e^{\lambda_2 \tau} r_2(\tau) d\tau \right] \vec{v}_2.$$

Example 6.8. Let $A = \left(\begin{smallmatrix} 3 & -2 \\ 2 & -2 \end{smallmatrix} \right)$, $\boldsymbol{y}(0) = 0$, and $\boldsymbol{r} = \left(\begin{smallmatrix} e^t \\ t \end{smallmatrix} \right)$. Since $\vec{v}_1 = \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \right)$, $\vec{v}_2 = \left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \right)$, we have

$$\boldsymbol{r} = \left(\frac{-1}{3} e^t + \frac{2}{3} t \right) \vec{v}_1 + \left(\frac{2}{3} e^t - \frac{1}{3} t \right) \vec{v}_2. \tag{6.22}$$

Since $\lambda_1 = -1$ and $\lambda_2 = 2$, the associated sub-systems along their associated eigenvectors are

$$\begin{cases} z' = -z - \frac{1}{3} e^t + \frac{2}{3} t \\ z(0) = 0 \end{cases}, \begin{cases} z' = 2z + \frac{2}{3} e^t - \frac{1}{3} t \\ z(0) = 0 \end{cases}.$$

These two scalar equations are simply solved for

$$\phi_1(t) = \left[\frac{5}{6} e^{-t} - \frac{1}{6} e^{-t} + \frac{2}{3} t - \frac{2}{3} \right] \vec{v}_1, \phi_2(t) = \left[-\frac{11}{36} e^{-2t} + \frac{2}{9} e^t - \frac{1}{6} t + \frac{1}{12} \right] \vec{v}_2,$$

and thus the final solution is $\phi(t) = \phi_1(t) + \phi_2(t)$.

As it is seen, this method works very well if $A_{2\times 2}$ has two real distinct eigenvalues. However, the application is limited if A has a repeated or complex eigenvalues.

6.4.2 Variation of parameters

Let $\phi_1(t), \phi_2(t)$ be two linearly independent of the homogeneous system

$$\frac{d}{dt} \boldsymbol{y} = A \boldsymbol{y}.$$

Then we write the particular solution of the non-homogeneous system

$$\frac{d}{dt} \boldsymbol{y} = A \boldsymbol{y} + \boldsymbol{r}(t)$$

as follows for some undetermined functions $c_1(t), c_2(t)$

$$\boldsymbol{y}_p(t) = c_1(t)\phi_1 + c_2(t)\phi_2,$$

Substituting $\boldsymbol{y}_p(t)$ into the system, gives

$$c_1'(t)\phi_1 + c_1(t)\phi_1' + c_2'(t)\phi_2 + c_2(t)\phi_2' = c_1(t)A\phi_1 + c_2(t)\,A\phi_2 + \boldsymbol{r}(t).$$

Note that $\phi_1' = A\phi_1$ and $\phi_2' = A\phi_2$, and thus

$$c_1'(t)\phi_1 + c_2'(t)\phi_2 = \boldsymbol{r}(t). \tag{6.23}$$

In the matrix form, we can write the above system as follows

$$\Phi(t)\begin{pmatrix} c_1'(t) \\ c_2'(t) \end{pmatrix} = \boldsymbol{r}(t),$$

where $\Phi(t) = [\phi_1|\phi_2]$. Therefore,

$$\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \int \Phi^{-1}(t)\,\boldsymbol{r}(t)\,dt.$$

Example 6.9. Let us find a particular solution to the system $\boldsymbol{y}' = A\boldsymbol{y} + \begin{pmatrix} 0 \\ t \end{pmatrix}$ where $A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$. It is simply seen that the homogeneous system has two vector solution $\phi_1(t) = e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, and $\phi_2(t) = e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. We write the particular solution as

$$\boldsymbol{y}_p(t) = c_1(t)e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2(t)e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

where c_1, c_2 satisfy the following equation

$$c_1'(t)\,e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2'(t)\,e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ t \end{pmatrix},$$

and thus

$$\begin{cases} e^{-t}c_1'(t) + 2e^{2t}c_2'(t) = 0 \\ 2e^{-t}c_1'(t) + e^{2t}c_2'(t) = t \end{cases}.$$

By eliminating c_2', we obtain $3e^{-t}c_1' = 2t$ and hence

$$c_1(t) = \frac{2}{3}\int t e^t = \frac{2}{3}(t-1)e^t.$$

Similarly, we obtain

$$c_2(t) = -\frac{1}{3}\int t e^{-2t} = -\frac{1}{3}\left(\frac{-1}{2}t e^{-2t} - \frac{1}{4}e^{-2t}\right) = e^{-2t}\left(\frac{1}{6}t + \frac{1}{12}\right),$$

and finally

$$\boldsymbol{y}_p(t) = \begin{pmatrix} t - \frac{1}{2} \\ \frac{3}{2}t - \frac{5}{4} \end{pmatrix}.$$

Theorem 6.2. *Let $\Phi=[\phi_1(t)|\phi_2(t)]$ be a solution matrix of the homogeneous system*

$$\frac{d}{dt}\boldsymbol{y}=A\boldsymbol{y}.$$

Then, the solution of the non-homogeneous system

$$\begin{cases} \frac{d}{dt}\boldsymbol{y}=A\boldsymbol{y}+\boldsymbol{r}(t) \\ \boldsymbol{y}(0)=\boldsymbol{y}_0 \end{cases}, \tag{6.24}$$

is

$$\boldsymbol{y}(t)=\Phi(t)\Phi^{-1}(0)\boldsymbol{y}_0+\Phi(t)\int_0^t \Phi^{-1}(\tau)\,\boldsymbol{r}(\tau)\,d\tau. \tag{6.25}$$

Proof. The verification process is straightforward as follows. Fist, the given solution satisfies the initial condition at $t=0$. Furthermore, we have

$$\frac{d}{dt}\boldsymbol{y}(t)=\Phi'(t)\Phi^{-1}(0)\boldsymbol{y}_0+\Phi'(t)\int_0^t \Phi^{-1}(\tau)\,\boldsymbol{r}(\tau)\,d\tau+\boldsymbol{r}(t).$$

For the last term, we used the fundamental theorem of calculus. On the other hand, $\Phi(t)$ is the homogeneous *matrix* solution of the homogeneous system, that is,

$$\frac{d}{dt}\Phi(t)=A\Phi(t),$$

and thus

$$\frac{d}{dt}\boldsymbol{y}(t)=A\left[\Phi(t)\Phi^{-1}(0)\boldsymbol{y}_0+\Phi(t)\int_0^t \Phi^{-1}(\tau)\,\boldsymbol{r}(\tau)\,d\right]\tau+\boldsymbol{r}(t)=A\boldsymbol{y}(t)+\boldsymbol{r}(t),$$

and this completes the proof. □

6.4.3 Undetermined coefficient method

This method is applied if the forcing terms is of the following forms

- polynomials,
- exponential,
- trigonometric sine and cosine functions.

polynomials. If $\boldsymbol{r}(t)$ has the form $\vec{a}_n t^n+\cdots+\vec{a}_0$ and $\lambda=0$ is not an eigenvalue of the coefficient matrix A, then the particular solution is $\boldsymbol{y}_p=\vec{c}_n t^n+\cdots+\vec{c}_0$ for some undetermined vectors $\vec{c}_n$. If $\lambda=0$ is a simple eigenvalue of A then $\boldsymbol{y}_p=\vec{c}_{n+1}t^{n+1}+\vec{c}_n t^n+\cdots+\vec{c}_0$.

Exponential. If $\boldsymbol{r}(t)$ is an exponential function $\boldsymbol{r}(t)=\vec{a}\,e^{bt}$ and $\lambda=b$ is not an eigenvalue of A, then $\boldsymbol{y}_p=\vec{c}e^{bt}$ for an undetermined vector $\vec{c}$. If $\lambda=b$ is a simple eigenvalue of A, then $\boldsymbol{y}_p=\vec{c}_1 t\,e^{bt}+\vec{c}_0\,e^{bt}$. If $\lambda=b$ is a repeated eigenvalue then

$$\boldsymbol{y}_p=\vec{c}_2\,t^2e^{bt}+\vec{c}_1 t\,e^{bt}+\vec{c}_0\,e^t.$$

Trigonometric. If $\boldsymbol{r}(t)$ has the form $\vec{a}\sin(\omega t)$ or $\vec{a}\cos(\omega t)$ and $\lambda = i\omega$ is not an eigenvalue of A, then $\boldsymbol{y}_p = \vec{c}_1 \sin(\omega t) + \vec{c}_2 \cos(\omega t)$. If $\lambda = i\omega$ is an eigenvalue of A, then

$$\boldsymbol{y}_p = (\vec{c}_1\, t + \vec{d}_1) \sin(\omega t) + (\vec{c}_2\, t + \vec{d}_2) \cos(\omega t).$$

Example 6.10. Let us solve the system given in the previous example, that is, $A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} e^t \\ t \end{pmatrix}$. It is better to rewrite $\boldsymbol{r}$ as $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + t \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The particular solution associated to the first term is $\boldsymbol{y}_p = e^t\, \vec{c}_1$ where $\vec{c}_1$ satisfies the following relation

$$e^t\, \vec{c}_1 = e^t A\, \vec{c}_1 + e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and thus $(A - \mathrm{Id})\vec{c}_1 = -\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. This gives

$$\vec{c}_1 = -(A - \mathrm{Id})^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{-3}{2} \\ -1 \end{pmatrix}.$$

A particular solution associated to the second term is $\boldsymbol{y}_p = \vec{c}_2 t + \vec{c}_3$ where $\vec{c}_2, \vec{c}_3$ satisfy the relation $\vec{c}_2 = t A \vec{c}_2 + A \vec{c}_3 + t \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Thus, we obtain $A\vec{c}_2 = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $A\vec{c}_3 = \vec{c}_2$. These relations determine $\vec{c}_2 = \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}$, and $\vec{c}_3 = \begin{pmatrix} \frac{-1}{2} \\ \frac{-5}{4} \end{pmatrix}$. Hence, the particular solution is

$$\boldsymbol{y}_p = e^t \begin{pmatrix} \frac{-3}{2} \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{5}{4} \end{pmatrix}.$$

Problems

Problem 6.29. Find the general solution of the following systems by the eigenvector decomposition method

a)

$$\boldsymbol{y}' = \begin{pmatrix} 3 & -4 \\ 1 & -2 \end{pmatrix} \boldsymbol{y} + \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

b)

$$\boldsymbol{y}' = \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix} \boldsymbol{y} + \begin{pmatrix} t+1 \\ e^{-t} \end{pmatrix}.$$

c)

$$\boldsymbol{y}' = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix} \boldsymbol{y} + \begin{pmatrix} e^{-t} \\ e^{t} \end{pmatrix}.$$

d)

$$\boldsymbol{y}' = \begin{pmatrix} 2 & 5 \\ 1 & -2 \end{pmatrix} \boldsymbol{y} + \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

e)

$$y' = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} y + \begin{pmatrix} \frac{e^t}{1+e^{2t}} \\ \frac{e^t}{1+e^{2t}} \end{pmatrix}$$

Problem 6.30. Find the general solution of the following systems by the variation of parameters method

a)

$$y' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} y + \begin{pmatrix} e^t \\ e^{-t} \end{pmatrix}.$$

b)

$$y' = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} y + \begin{pmatrix} \frac{e^{2t}}{1+e^t} \\ \frac{e^{2t}}{1+e} \end{pmatrix}.$$

c)

$$y' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} y + \begin{pmatrix} \frac{e^t}{\cos t} \\ \frac{e^t}{\sin t} \end{pmatrix}.$$

d)

$$y' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} y + \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

e)

$$y' = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} y + \begin{pmatrix} \frac{e^t}{1+e^{2t}} \\ \frac{e^t}{1+e^{2t}} \end{pmatrix}$$

Problem 6.31. Find the general solution of the following systems by the eigenvector decomposition method

a)

$$y' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} y + \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

b)

$$y' = \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix} y + \begin{pmatrix} t+1 \\ e^{-t} \end{pmatrix}.$$

c)

$$y' = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix} y + \begin{pmatrix} e^{-t} \\ e^t \end{pmatrix}.$$

d)

$$y' = \begin{pmatrix} 2 & 5 \\ 1 & -2 \end{pmatrix} y + \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

e)

$$y' = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} y + \begin{pmatrix} t \\ t+e \end{pmatrix}.$$

f)

$$y' = \begin{pmatrix} 3 & -4 \\ 1 & -2 \end{pmatrix} y + \begin{pmatrix} -1 \\ t \end{pmatrix}.$$

g)

$$y' = \begin{pmatrix} 3 & -6 \\ 3 & -3 \end{pmatrix} y + \begin{pmatrix} \sin t \\ 0 \end{pmatrix}.$$

Problem 6.32. Solve the following initial value problems

i.
$$\begin{cases} y_1' = 2y_1 + 3y_2 \\ y_2' = y_1 + 2y_2 + 1 \\ y_1(0) = y_2(0) = 0 \end{cases}$$

ii.
$$\begin{cases} y_1' = -2y_1 + 3y_2 + 1 \\ y_2' = -y_1 + 2y_2 \\ y_1(0) = y_2(0) = 0 \end{cases}$$

iii.
$$\begin{cases} y_1' = 2y_2 + t \\ y_2' = -y_1 - 2y_2 \\ y_1(0) = 0, y_2(0) = 1 \end{cases}$$

iv.
$$\begin{cases} y_1' = y_2 + 1 \\ y_2' = -y_1 \\ y_1(0) = y_2(0) = 0 \end{cases}$$

v.
$$\begin{cases} y_1' = 5y_1 + 2y_2 + 1 \\ y_2' = -2y_1 + y_2 + 1 \\ y_1(0) = y_2(0) = 0 \end{cases}$$

vi.
$$\begin{cases} y_1' = 6y_1 + 4y_2 + 3 \\ y_2' = -y_1 + 2y_2 - 2 \\ y_1(0) = 1, y_2(0) = 0 \end{cases}$$

vii.
$$\begin{cases} y_1' = -3y_1 + 2y_2 + t \\ y_2' = -y_1 - y_2 \\ y_1(0) = y_2(0) = 0 \end{cases}$$

Problem 6.33. Consider the system $\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{r}(t)$, where $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$. Note that $\lambda = 0$ is an eigenvalue of A.

i. Find two linearly independent solution vectors to the homogeneous system.
ii. Use undetermined coefficient method to find a particular solution to the given system if $\boldsymbol{r}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
iii. Repeat the problem when $\boldsymbol{r}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
iv. Now an arbitrary constant matrix $\boldsymbol{r} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$ can be uniquely decomposed in directions $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and thus we expect the particular solution to be of the form $\vec{c}_1 x + \vec{c}_0$. Find a particular solution when $\boldsymbol{r} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

Problem 6.34. Consider the system $\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{r}(t)$, where $A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$.

i. Find two linearly independent solutions to the homogeneous system.
ii. Use undetermined coefficient method to find a particular solution to the given system if $\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$.
iii. Repeat the problem if $\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t}$.
iv. Now find a particular solution if $\boldsymbol{r} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$.

6.5 Nonlinear systems

Despite linear systems, there is no general method to solve nonlinear ones. In this section, we discuss the properties of nonlinear systems without attempting to solve them.

6.5.1 Linearization

In the first section of this chapter, we studied the method of linearization for nonlinear systems around an equilibrium point. By linearization, we gain some information about the behavior of the nonlinear system. Consider the following system

$$\begin{cases} x' = y \\ y' = -2y - x^2 + 4 \end{cases}.$$

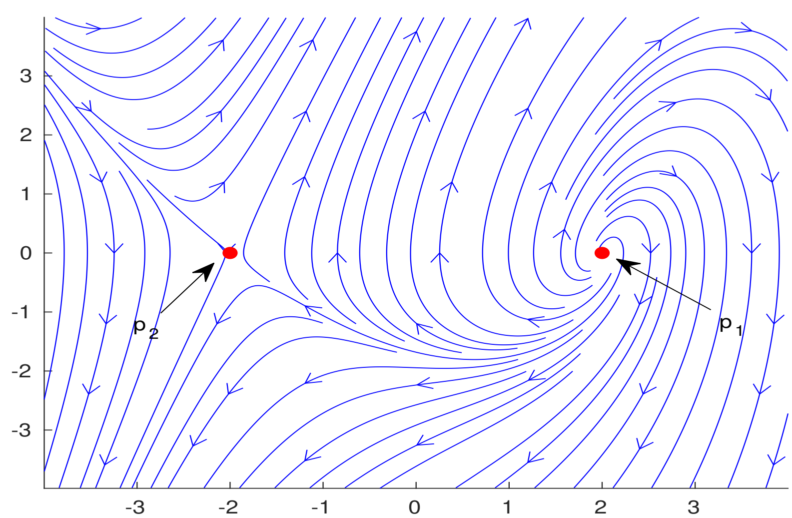
The system has two equilibrium points $p_1 = (2, 0)$, $p_2 = (-2, 0)$. The linearzed version of the system around p_1 is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix}\begin{pmatrix} x-2 \\ y \end{pmatrix}.$$

The eigenvalues of the coefficient matrix are $\lambda = -1 \pm i\sqrt{3}$ and thus p_1 is a spiral sink for the *linearized system*. The linear system around p_2 is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & -2 \end{bmatrix}\begin{pmatrix} x+2 \\ y \end{pmatrix},$$

with eigenvalues $\lambda_{1,2} = -1 \pm \sqrt{5}$ and thus p_2 is a saddle point for the linear system. Therefore, the behavior of the original system around the equilibrium points is very similar to one of the linearized versions. Note that the behavior may be different at points far from the equilibrium. The following figure shows the trajectories of the nonlinear system in the phase plane.



6.5.2 Integrable systems

Consider the following autonomous system

$$\begin{cases} \frac{dx}{dt} = f(x, y) \\ \frac{dy}{dt} = g(x, y) \end{cases}. \tag{6.26}$$

If there is a scalar function $H(x,\ y_2)$ such that $f = -\frac{\partial H}{\partial y}$, and $g = \frac{\partial H}{\partial x}$, then the above system is called *integrable*. Consider the following system

$$\begin{cases} \frac{dx}{dt} = -\frac{\partial H}{\partial y} \\ \frac{dy}{dt} = \frac{\partial H}{\partial x} \end{cases}.$$

The equation in the phase plane (x, y) is

$$\frac{dy}{dx} = -\frac{\frac{\partial H}{\partial x}}{\frac{\partial H}{\partial y}},$$

or equivalently

$$\frac{\partial H}{\partial x}dx + \frac{\partial H}{\partial y}dy = 0,$$

which is solved as $H(x,\ y) = \mathrm{const}$. The condition that the system (6.26) is integrable in an open domain $D \subset \mathbb{R}^2$, the following condition must be satisfied

$$\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y},$$

in addition to the continuity of f, g, and $\frac{\partial f}{\partial x}, \frac{\partial g}{\partial y}$ on D.

Example 6.11. Consider the equation of a pendulum

$$\theta'' + \frac{g}{l}\sin(\theta) = 0, \tag{6.27}$$

and let us rewrite the equation in the following form

$$\begin{cases} \theta_1' = \theta_2 \\ \theta_2' = -\frac{g}{l}\sin(\theta_1) \end{cases}. \tag{6.28}$$

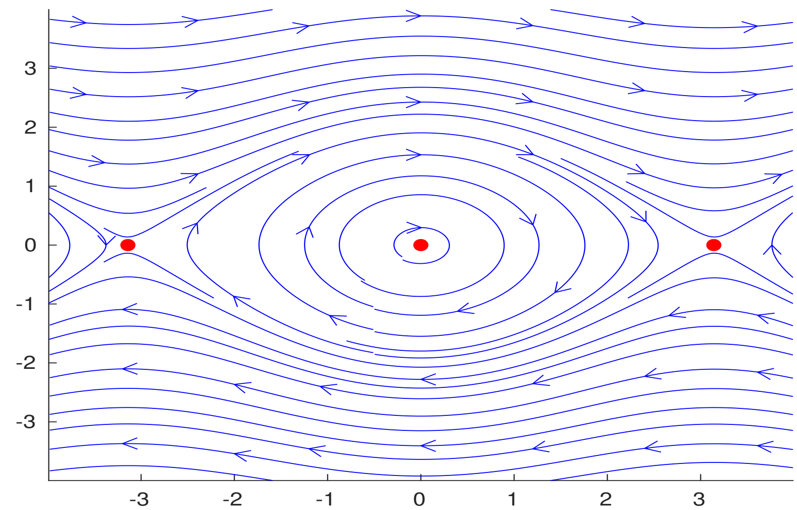
The above system is integrable as it satisfies the continuity condition and the relation

$$\frac{\partial}{\partial \theta_1}\theta_2 = -\frac{\partial}{\partial \theta_2}\Big(-\frac{g}{l}\sin(\theta_1)\Big).$$

It is simply verified that the scalar function H is as follows

$$H(\theta_1, \theta_2) = \frac{1}{2}(\theta_2)^2 - \frac{g}{l}\cos(\theta_1),$$

and thus $\frac{1}{2}(\theta_2)^2 - \frac{g}{l}\cos(\theta_1) = C$ is the solution of the system in the phase plane (θ_1, θ_2).



Observe that some trajectories are closed around the point $(0,\ 0)$. For this trajectories, the solutions $\theta_1 = \theta_1(t)$ and $\theta_2 = \theta_2(t)$ in the (t, θ)-plane are periodic.

6.5.3 Stability

Definition 6.2. *A point $\bar{p} = (\bar{x},\ \bar{y})$ is called an equilibrium point of the system (6.26) if $f(\bar{p}) = g(\bar{p}) = 0$. An equilibrium point $\bar{p}$ is called an* isolated equilibrium *if there is an open neighborhood of $\bar{p}$ containing no other equilibrium other than $\bar{p}$.*

Usually it is desired to gain some information about the behavior of a system about one of its equilibrium points. For example, the system (6.28) has equilibrium points of the form $(n\pi, 0)$ for (θ_1, θ_2). The trajectories near the equilibrium $(0, 0)$ are periodic, while at $(\pi, 0)$ or $(-\pi, 0)$ are unstable.

Definition 6.3. *An isolated equilibrium $\bar{p}$ is called* stable *if there is an open disk D centered at $\bar{p}$ such that all trajectories starting at D remain inside D or in a bigger disk D_1. An equilibrium $\bar{p}$ is called* asymptotically stable *id there is an open disk D centered at $\bar{p}$ such that all trajectories starting inside D approach $\bar{p}$ when $t \to \infty$.*

An important result about the asymptotically stable equilibria is given in the following theorem.

Theorem 6.3. *Let $\bar{p} = (\bar{x},\ \bar{y})$ be an isolated equilibrium of the system (6.26) and assume that the eigenvalues of the Jacobi matrix $J_{(f,g)}(\bar{p})$ has negative real part. Then $\bar{p}$ is an asymptotically stable equilibrium of the system.*

Example 6.12. Consider the following system

$$\begin{cases} \frac{dx}{dt} = -y - \frac{1}{2}x(1-y^2) \\ \frac{dy}{dt} = x - \frac{1}{2}y(1-y^2) \end{cases}.$$

The Jacobi matrix at $\bar{p}=(0,0)$ is

$$J=\begin{bmatrix} -\frac{1}{2} & -1 \\ 1 & -\frac{1}{2} \end{bmatrix},$$

with eigenvalues

$$\lambda=-\frac{1}{2}\pm i\frac{\sqrt{2}}{2}.$$

Therefore, $\bar{p}$ is asymptotically stable.

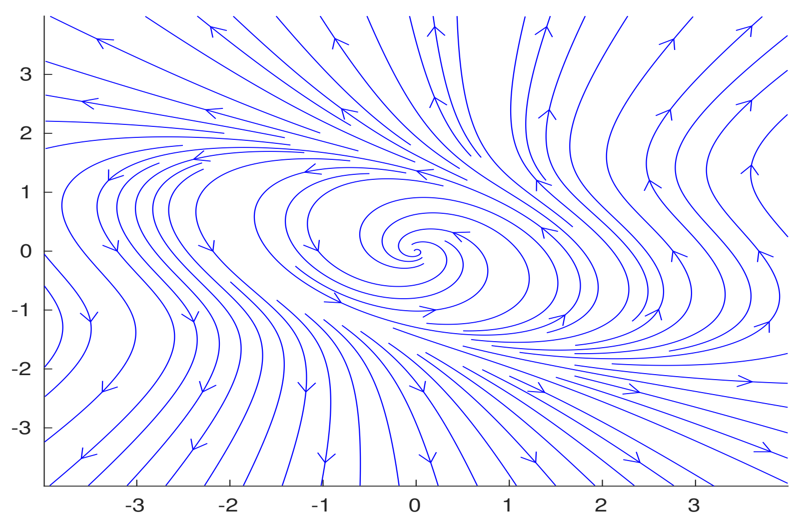
Although, the above theorem provides us with a powerful tool to decide whether if an equilibrium is asymptotically stable or not, it does not provide us with any estimation of the domain of the stability. For the above example, consider the following scalar function

$$V(x(t),y(t))=|x(t)|^2+|y(t)|^2,$$

which is the magnitude of the solution of the system. We have

$$\frac{dV}{dt}=2x(t)\,x'(t)+2\,y(t)\,y'(t)=-(x^2+y^2)\,(1-y^2).$$

Obviously, $\frac{dV}{dt}<0$ for $-1<y<1$ and thus we expect that trajectories starting or entering in the disk $x^2+y^2<1$ approach the origin when $t\to\infty$.



Example 6.13. Consider the following damped pendulum equation

$$\theta''+\varepsilon\theta'+\frac{g}{l}\sin(\theta)=0. \tag{6.29}$$

As we know, the system dissipate its energy and approaches the equilibrium $(0,0)$. In fact, if we multiply the equation by θ', we obtain

$$\frac{d}{dt}\left\{\frac{1}{2}|\theta'|^2+\frac{g}{l}(1-\cos(\theta))\right\}=-\varepsilon\,|\theta'|^2.$$

If we take

$$V(\theta,\theta')=\frac{1}{2}|\theta'|^2+\frac{g}{l}(1-\cos(\theta)),$$

then

$$V(\theta,\theta')=V_0-\varepsilon\int_0^t|\theta'(s)|^2ds,$$

where $V_0=V(\theta(0),\,\theta'(0))$ is the initial energy of the pendulum. Obviously, $|\theta'(t)|\to 0$ for $t\to\infty$, and thus $V(\theta(t),\,\theta'(0))\to 0$ for $t\to\infty$ (why?). This implies that the pendulum approaches the equilibrium in long term. Let us write the equation in the following form

$$\begin{cases}\theta_1'=\theta_2\\ \theta_2'=-\varepsilon\ \theta_2-\frac{g}{l}\sin(\theta_1)\end{cases}.$$

The linearized version of the system is

$$\begin{pmatrix}\theta_1'\\ \theta_2'\end{pmatrix}=\begin{pmatrix}0 & 1\\ \frac{-g}{l} & -\varepsilon\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}.$$

The eigenvalues of the coefficient matrix are

$$\lambda_{1,2}=\frac{-\varepsilon\pm\sqrt{\varepsilon^2-4\frac{g}{l}}}{2}. \tag{6.30}$$

If ε is sufficiently small, then $\lambda_{1,2}$ are complex with the real part $\frac{-\varepsilon}{2}$. This implies that the point $(0,0)$ is a spiral sink as we expected from the given non-linear equation.

If the real part of the Jacobi matrix is zero, we can not say that the equilibrium point is a center, asymptotically stable or unstable. The following example explains this case.

Example 6.14. Consider the following damped pendulum equation

$$\theta''+\varepsilon\theta'^3+\frac{g}{l}\sin\theta=0. \tag{6.31}$$

Here the drag force is assumed to be of order 3 of the velocity. It is simply seen that this nonlinear system dissipate energy in time and approaches the origin for small $\varepsilon>0$. Let us write the equation in the system form as follows

$$\begin{cases}\theta_1'=\theta_2\\ \theta_2'=-\varepsilon\ \theta_2^3-\frac{g}{l}\sin(\theta_1)\end{cases}.$$

The linearize version of the system at $(0,0)$ is

$$\begin{cases}\theta_1'=\theta_2\\ \theta_2'=-\frac{g}{l}\theta_1\end{cases},$$

and thus $(0,0)$ is a center point for the linearized system while it is a sink for the original nonlinear system.

Problems

Problem 6.35. For the following linear system, find the general solution in the phase plane and compare the results with the solutions obtained by the direct method of solving linear systems.

a)

$$\begin{cases} \frac{dy_1}{dt} = 3y_1 - 5y_2 \\ \frac{dy_2}{dt} = y_1 - 3y_2 \end{cases}.$$

b)

$$\begin{cases} \frac{dy_1}{dt} = 3y_1 - y_2 \\ \frac{dy_2}{dt} = y_1 + y_2 \end{cases}.$$

c)

$$\begin{cases} \frac{dy_1}{dt} = y_1 - y_2 \\ \frac{dy_2}{dt} = y_1 + y_2 \end{cases}.$$

Problem 6.36. Verify that the following systems are integrable and then derive the potential H and write the solution given the initial condition.

a)

$$\begin{cases} \frac{dx}{dt} = -y \\ \frac{dy}{dt} = x \end{cases}, (x(0), y(0)) = (1, -1)$$

b)

$$\begin{cases} \frac{dx}{dt} = x - y^2 + 1 \\ \frac{dy}{dt} = x^2 - y - 1 \end{cases}, (x(0), y(0)) = (0, 1)$$

c)

$$\begin{cases} \frac{dx}{dt} = -\frac{y}{x^2 + y^2} \\ \frac{dy}{dt} = \frac{x}{x^2 + y^2} \end{cases}, (x(0), y(0)) = (1, 1)$$

Problem 6.37. For each of the following system, find equilibrium points and determine the type of each equilibrium of the correspond linearized system

i.

$$\begin{cases} y_1' = y_1\, y_2 \\ y_2' = y_1^2 + y_2^2 - 1 \end{cases}$$

ii.

$$\begin{cases} y_1' = y_1 y_2 \\ y_2' = y_1^2 - 3y_1 + 2 \end{cases}$$

iii.

$$\begin{cases} y_1' = y_1 \sin(y_2) \\ y_2' = -y_2 \sin(y_1) \end{cases}$$

iv.

$$\begin{cases} y_1' = y_1(1 - y_2^2) \\ y_2' = y_2(1 + y_1) \end{cases}$$

v.

$$\begin{cases} y_1' = y_1^2 + 4y_2 \\ y_2' = y_1^2 - y_2^2 \end{cases}$$

Problem 6.38. Prove that $(0, 0)$ is an asymptotically stable point of the following system

$$\begin{cases} \frac{dx}{dt} = -x e^y + y \\ \frac{dy}{dt} = -x - y \end{cases}$$

Problem 6.39. Linearize the following system at its equilibrium point(s) and determine the type of the linearized system.

$$\begin{cases} x' = y(x - 1) \\ y' = x(y - 1) \end{cases}.$$

6.6 Higher dimensional systems: fundamental matrix

A matrix $\Phi(x)$ is called the fundamental matrix of another matrix $A_{n\times n}$ if

i. $\Phi_{n\times n}(0) = \mathbb{I}$,

ii. $\Phi'(t) = A_{n\times n}\Phi(t)$ for all $t \in (-\infty, \infty)$

Consider the following initial value problem

$$\begin{cases} \frac{d}{dt}\boldsymbol{y} = A\boldsymbol{y} + \boldsymbol{r}(t) \\ \boldsymbol{y}(0) = \boldsymbol{y}_0 \end{cases}.$$

If $\Phi(t)$ is the fundamental matrix of A, then

$$\boldsymbol{y}(t) = \Phi(t)\boldsymbol{y}_0 + \Phi(t)\int_0^t \Phi(-\tau)\boldsymbol{r}(\tau)\,d\tau.$$

For a 2×2 matrix A, the fundamental matrix Φ is just

$$\Phi = [\phi_1(t)|\phi_2(t)]\,[\phi_1(0)|\phi_2(0)]^{-1},$$

where ϕ_1, ϕ_2 are two linearly solutions of the system $\boldsymbol{y}' = A\boldsymbol{y}$. Here we introduce a method to derive the fundamental matrix for general matrices.

6.6.1 Exponential formula

If A is a matrix, we define the exponential matrix e^A through the following series

$$e^A = \mathbb{I} + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdots. \tag{6.32}$$

Apparently, we need to justify that the infinite series in the right hand side of the above equality converges. In the appendix, we prove that the series is convergent.

Theorem 6.4. *Let A be a $n\times n$ constant matrix. The unique fundamental matrix $\Phi(t)$ of the A is $\Phi(t)=e^{At}$.*

Proof. According to the definition of a fundamental matrix and of the matrix e^{At}, it is immediately follows that $\Phi(0)=\mathbb{I}$. To justify the second property of a fundamental matrix, we need the following fact from the linear algebra: if A, B are two $n\times n$ matrices that satisfies the condition $AB=BA$ then $e^A e^B=e^{A+B}$. Using that fact, we justify the second property as follows

$$\frac{d}{dt}e^{At}=\lim_{h\to 0}\frac{e^{A(t+h)}-e^{At}}{h}=\lim_{h\to 0}\frac{e^{Ah}e^{At}-e^{At}}{h}=\lim_{h\to 0}\frac{e^{Ah}-I}{h}e^{At}.$$

According to the definition of e^{Ah}, we can write

$$\frac{e^{Ah}-I}{h}=A+h\left(\frac{1}{2}A^2+\cdots+\frac{1}{n!}A^n h^{n-2}+\cdots\right).$$

The expression in the bracket converges to a $n\times n$ matrix, say B, and then

$$\lim_{h\to 0}\frac{e^{Ah}-I}{h}=A+\lim_{h\to 0}hB=A.$$

This implies that $\frac{d}{dt}e^{At}=Ae^{At}$ and thus e^{At} is a fundamental matrix of the matrix A. We now show that this matrix is unique. If Φ_1, Φ_2 are two fundamental matrices, then for any arbitrary $\vec{c}\in\mathbb{R}^n$, the vectors $\boldsymbol{y}_1=\Phi_1(t)\vec{c}$ and $\boldsymbol{y}_2=\Phi_2(t)\vec{c}$ are solutions to the initial value problem

$$\begin{cases}\frac{d}{dt}\boldsymbol{y}=A\boldsymbol{y}\\ \boldsymbol{y}(0)=\vec{c}\end{cases}.$$

According to the uniqueness theorem, we have $\Phi_1(t)\vec{c}=\Phi_2(t)\vec{c}$ for arbitrary $\vec{c}$, and thus $\Phi_1=\Phi_2$. □

Below, we present an algorithm to calculate the summation of the infinite sums.

6.6.2 Calculation of the fundamental matrix

In order to derive the fundamental matrix in a closed form, we use the Jordan *normal form* of a matrix; see the appendix of this book. Firs let us fix our notation. If $\vec{v}_1, \ldots, \vec{v}_n$ are the columns of a matrix Q, we represent Q by $Q=[\vec{v}_1|\vec{v}_2|\cdots|\vec{v}_n]$ and by this notation, the matrix multiplication AQ (in the case of the proper dimensionality) is defined by the relation

$$AQ=[A\vec{v}_1|A\vec{v}_2|\cdots|A\vec{v}_n]. \tag{6.33}$$

We need the following fact in our subsequent discussion.

Proposition 6.2. *If C is an invertible matrix then for an arbitrary matrix A we have*

$$e^{CAC^{-1}}=Ce^A C^{-1}. \tag{6.34}$$

Proof. Direct calculation gives the result. In fact, we have

$$e^{CAC^{-1}}=\mathbb{I}+CAC^{-1}+\cdots+\frac{1}{n!}(CAC^{-1})^n+\cdots. \tag{6.35}$$

Since

$$(CAC^{-1})^k = CAC^{-1}CAC^{-1}\cdots CAC^{-1} = CA^kC^{-1}, \tag{6.36}$$

we obtain

$$e^{CAC^{-1}} = \mathbb{I} + \cdots + \frac{1}{n!}CA^kC^{-1} + \cdots = C\left(\mathbb{I} + \cdots + \frac{1}{n!}A^k + \cdots\right)C^{-1} = Ce^AC^{-1}, \tag{6.37}$$

and this completes the proof. □

Matrix *A* with real distinct eigenvalues.

Assume that $A_{n\times n}$ has n real distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. In this case A has n linearly independent eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$. Moreover, for the eigenvector matrix $Q = [\vec{v}_1|\vec{v}_2|\cdots|\vec{v}_n]$, we have

$$Q^{-1}AQ = \operatorname{diag}(\lambda_1, \ldots, \lambda_n).$$

The proof is straightforward and we leave it as an exercise to the reader.

Consider the following system

$$\frac{d}{dt}\boldsymbol{y} = A\boldsymbol{y},$$

and let Q be the eigenvector matrix $Q = [\vec{v}_1|\cdots|\vec{v}_n]$. By the axis transformation $\boldsymbol{Y} = Q^{-1}\boldsymbol{y}$, the above system is transformed to the following one

$$Q\frac{d}{dt}\boldsymbol{Y} = AQ\boldsymbol{Y},$$

and therefore,

$$\frac{d}{dt}\boldsymbol{Y} = Q^{-1}AQ\boldsymbol{Y} = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)\boldsymbol{Y}.$$

The fundamental matrix of the later system is

$$\Psi(x) = e^{\operatorname{diag}(\lambda_1 t, \ldots, \lambda_n t)} = \operatorname{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t}).$$

The above equality is directly follows from the definition of the exponential matrix e^A. On the other hand, since $A = Q\Lambda Q^{-1}$, we obtain

$$\Phi(t) = e^{Q\operatorname{diag}(\lambda_1 t, \ldots, \lambda_n t)Q^{-1}} = Q\operatorname{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t})\,Q^{-1}.$$

The last equality obtained by the proposition (6.2).

Example 6.15. Consider the following system

$$\frac{d}{dt}\boldsymbol{y} = \begin{pmatrix} 1 & 0 & 4 \\ 3 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\boldsymbol{y}. \tag{6.38}$$

The eigenvalues of the coefficient matrix are $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$. The associated eigenvectors are

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 3 \\ \frac{1}{2} \end{pmatrix}. \tag{6.39}$$

Therefore, Φ is obtained as follows

$$\Phi = Q\begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}Q^{-1} = \begin{pmatrix} e^t & 0 & 2e^{3t}-2e^t \\ 3e^{2t}-3e^t & e^{2t} & 6e^{3t}-12e^{2t}+6e^t \\ 0 & 0 & e^{3t} \end{pmatrix}.$$

Matrix A with repeated eigenvalues.

One eigenvector.

Assume that A has only one eigenvector $\vec{v}_1$. This implies that the characteristic polynomial $p(\lambda)$ of A has only one root, say λ_1, that is, $p(\lambda) = (\lambda - \lambda_1)^n$. It is known (see the appendix of the book) that there are exactly $n-1$ *generalized eigenvectors* $\vec{v}_2, \ldots, \vec{v}_n$ for A satisfying the following relation

$$(A - \lambda \mathbb{I})\vec{v}_k = \vec{v}_{k-1}, k = 2, \ldots, n.$$

Moreover, we have the following fact.

Proposition 6.3. *Assume that $A_{n\times n}$ has only one eigenvector $\vec{v}_1$ and one eigenvalue λ. Let $\vec{v}_2, \ldots, \vec{v}_n$ be the generalized eigenvectors of $A_{n\times n}$. Then $Q^{-1}AQ = \lambda\mathbb{I} + [\delta_{i,j-1}]$, where $\delta_{i,j} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$ and $Q = [\vec{v}_1|\cdots|\vec{v}_n]$.*

Proof. We have

$$AQ = [A\vec{v}_1|A\vec{v}_2|\cdots|A\,\vec{v}_n] = [\lambda\,\vec{v}_1|\vec{v}_1 + \lambda\vec{v}_2|\cdots|\vec{v}_{n-1} + \lambda\vec{v}_n].$$

On the other hand, we have

$$[\lambda\,\vec{v}_1|\vec{v}_1 + \lambda\vec{v}_2|\cdots|\vec{v}_{n-1} + \lambda\vec{v}_n] = [\lambda\,\vec{v}_1|\lambda\vec{v}_2|\cdots|\lambda\vec{v}_n] + [0|\vec{v}_1|\cdots|\vec{v}_{n-1}] = \\ = \lambda Q + [0|\vec{v}_1|\cdots|\vec{v}_{n-1}].$$

It is simply seen that

$$[0|\vec{v}_1|\cdots|\vec{v}_{n-1}] = Q\,[0|\vec{e}_1|\vec{e}_2|\cdots|\vec{e}_{n-1}].$$

Thus $Q^{-1}AQ = \lambda\mathbb{I} + [0|\vec{e}_1|\vec{e}_2|\cdots|\vec{e}_{n-1}]$ and this is the JORDAN form of a matrix with only one eigenvector. □

We use the notation $S = [0|\vec{e}_1|\vec{e}_2|\cdots|\vec{e}_{n-1}]$ in our subsequent calculation. It is simply seen that $S^2 = [0|0|\vec{e}_1|\cdots|\vec{e}_{n-2}]$, and $S^3 = [0|0|0|\vec{e}_1|\cdots|\vec{e}_{n-3}]$. This implies that $S^n = \mathbf{0}_{n\times n}$. Now, we have

$$Q^{-1}e^{At}\,Q = e^{Q^{-1}AtQ} = e^{\lambda t}\,e^{St} = e^{\lambda t}\left(\mathbb{I} + t\,S + \frac{1}{2!}t^2S^2 + \cdots + \frac{1}{(n-1)!}t^{n-1}\,S^{n-1}\right). \tag{6.40}$$

This implies

$$\Phi(t) = e^{\lambda t}\,Q\left(\mathbb{I} + t\,S + \frac{1}{2!}t^2S^2 + \cdots + \frac{1}{(n-1)!}t^{n-1}\,S^{n-1}\right)Q^{-1}. \tag{6.41}$$

Example 6.16. Let us find the fundamental matrix of $A = \begin{pmatrix} 2 & 3 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$. The matrix has repeated eigenvalue $\lambda = 2$ with the eigenvector $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. The generalized eigenvector $\vec{v}_2$ is determined by solving the equation $(A - 2\mathbb{I})\vec{v}_2 = \vec{v}_1$. It is simply verified that $\vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$. Similarly, for the generalized eigenvector $\vec{v}_3$, we solve the equation $(A - 2\mathbb{I})\vec{v}_3 = \vec{v}_2$ and obtain $\vec{v}_3 = \begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix}$. For $Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & -3 \end{pmatrix}$, the fundamental matrix is

$$\Phi = e^{2t} Q \begin{pmatrix} 1 & t & \frac{1}{2!}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} Q^{-1} = e^{2t} \begin{pmatrix} 1 & 3t - \frac{1}{2}t^2 & -t \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix}. \tag{6.42}$$

Multiple eigenvectors.

With only one eigenvalue, the matrix $A_{n\times n}$ can have one, two or even n independent eigenvectors based on the *algebraic multiplicity* of the eigenvalue.

Definition 6.4. *Let $A_{n\times n}$ be a matrix. The algebraic multiplicity of an eigenvalue λ_* of A is the value m such that $p(\lambda)$ is of the form*

$$p(\lambda) = (\lambda - \lambda_*)^m q(\lambda), \tag{6.43}$$

where $q(\lambda)$ does not have any factor of $\lambda - \lambda_*$. The geometric multiplicity r of λ_* is the dimension of the null space of the map $(A - \lambda_*\mathbb{I})$, i.e., $r = \ker(A - \lambda_*\mathbb{I})$.

Remark 6.3. Note that r is always less than or equal m. If $r < m$ then there exist $m - r$ generalized eigenvectors $w_1, \ldots, w_{m-r}$ such that $(A - \lambda_*\mathbb{I})w_1$ is an eigenvector and

$$(A - \lambda_*\mathbb{I})w_k = w_{k-1} \text{ for } k = 2, \ldots, m - r. \tag{6.44}$$

Example 6.17. Consider the following matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.45}$$

This matrix has repeated eigenvalue $\lambda = 1$ with the algebraic multiplicity $m = 3$. It is simply verified that vectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ are eigenvectors associated to $\lambda = 1$ and thus $\lambda = 1$ has geometric multiplicity $r = 2$. Note that for any vector $\vec{v}$ in $\text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right\}$, we have $(\lambda - \mathbb{I})\vec{v} = 0$. Moreover, we have $(A - \mathbb{I})\vec{w} = \vec{v}_1$ for $\vec{w} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and thus $\vec{w}$ is a generalized eigenvector of A.

Example 6.18. Consider the system

$$\frac{d}{dt}\boldsymbol{y} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\boldsymbol{y}. \tag{6.46}$$

As we saw above, the matrix has two eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ with eigenvalue $\lambda = 1$. Since $(A - \mathbb{I})w = \vec{v}_1$ for $w = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, the matrix Q is $Q = [\vec{v}_1|\vec{w}|\vec{v}_2]$ and then

$$Q^{-1}AQ = \begin{pmatrix} \boxed{1} & \boxed{1} & 0 \\ \boxed{0} & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{pmatrix}. \tag{6.47}$$

Observe how a JORDAN block is formed in the upper sub-matrix of $Q^{-1}AQ$. Accordingly, we have

$$e^{Ax} = Q\begin{pmatrix} \boxed{e^t} & \boxed{te^t} & 0 \\ \boxed{0} & \boxed{e^t} & 0 \\ 0 & 0 & \boxed{e^t} \end{pmatrix}Q^{-1} = e^t\begin{pmatrix} 1 & t & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.48}$$

Matrix A with complex eigenvalues.

We describe first the method for a 2×2 matrix and then generalize the result for higher dimensional systems. First, note that if $A_{2\times 2}$ has complex eigenvalues $\lambda_{1,2} = \sigma \pm i\omega$ then its associated eigenvectors $\vec{v}_1$, $\vec{v}_2$ are in conjugate form, i.e., $\vec{v}_2 = \bar{\vec{v}}_1$. For simplicity, we denote the eigenvalue by $\lambda = \sigma + i\omega$ and the complex eigenvector by $\vec{v}$. In this case, the matrix $Q = [\vec{v}|\ \bar{\vec{v}}]$ is complex. We transform this matrix to a real one by a simple trick. Define Q as

$$Q = \frac{1}{2}[\vec{v}|\,\bar{\vec{v}}]\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} = \frac{1}{2}[-i(\vec{v} - \bar{\vec{v}})|\vec{v} + \bar{\vec{v}}] = [\mathrm{Im}(\vec{v})|\,\mathrm{Re}(\vec{v})]. \tag{6.49}$$

Proposition 6.4. *For Q defined in (6.49), we have*

$$Q^{-1}AQ = \begin{pmatrix} \sigma & -\omega \\ \omega & \sigma \end{pmatrix}. \tag{6.50}$$

Proof. Direct calculation proves the claim. In fact, we have

$$AQ = A[\vec{v}|\,\bar{\vec{v}}]\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} = [\lambda\vec{v}|\bar{\lambda}\bar{\vec{v}}]\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} = [\vec{v}|\bar{\vec{v}}]\begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} =$$

$$= [\vec{v}|\bar{\vec{v}}]\begin{pmatrix} \sigma + i\omega & 0 \\ 0 & \sigma - i\omega \end{pmatrix}\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} = [\vec{v}|\bar{\vec{v}}]\begin{pmatrix} \omega - i\sigma & \sigma + i\omega \\ \omega + i\sigma & \sigma - i\omega \end{pmatrix}.$$

On the other hand, we have

$$Q\begin{pmatrix} \sigma & -\omega \\ \omega & \sigma \end{pmatrix} = [\vec{v}|\bar{\vec{v}}]\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}\begin{pmatrix} \sigma & -\omega \\ \omega & \sigma \end{pmatrix} = [\vec{v}|\bar{\vec{v}}]\begin{pmatrix} \omega - i\sigma & \sigma + i\omega \\ \omega + i\sigma & \sigma - i\omega \end{pmatrix}.$$

Therefore $AQ = Q\begin{pmatrix} \sigma & -\omega \\ \omega & \sigma \end{pmatrix}$ and this completes the proof. □

Accordingly, we have the equality

$$A = Q\begin{pmatrix} \sigma & -\omega \\ \omega & \sigma \end{pmatrix} Q^{-1}, \tag{6.51}$$

and then

$$\Phi = Q e^{\begin{pmatrix} \sigma & -\omega \\ \omega & \sigma \end{pmatrix} t} Q^{-1}. \tag{6.52}$$

Now, we have the following proposition.

Proposition 6.5. *The following relation holds*

$$e^{\begin{pmatrix} \sigma & -\omega \\ \omega & \sigma \end{pmatrix} t} = e^{\sigma t} \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix}. \tag{6.53}$$

Proof. Consider the following decomposition

$$\begin{pmatrix} \sigma & -\omega \\ \omega & \sigma \end{pmatrix} = \sigma \mathbb{I}_{2\times 2} + \omega D, \tag{6.54}$$

where $D = [\vec{e}_2 | -\vec{e}_1]$. Then we can write

$$e^{\begin{pmatrix} \sigma & -\omega \\ \omega & \sigma \end{pmatrix} t} = e^{\sigma t} e^{\omega t D}. \tag{6.55}$$

But we have

$$e^{\omega t D} = \mathbb{I}_{2\times 2} + \omega t\, D + \frac{1}{2!}\omega^2 t^2 D^2 + \cdots. \tag{6.56}$$

Simple calculation shows $D^2 = -\mathbb{I}$ and then

$$D^3 = -D, D^4 = \mathbb{I}_{2\times 2}, D^5 = D, \cdots. \tag{6.57}$$

Let us denote S the matrix $e^{\omega t D}$. We have then

$$S_{1,1} = 1 - \frac{1}{2!}\omega^2 t^2 + \frac{1}{4!}\omega^4 t^4 - \cdots = \cos(\omega t), \tag{6.58}$$

and

$$S_{12} = -\omega t + \frac{1}{3!}\omega^3 t^3 - \frac{1}{5!}\omega^5 t^5 + \cdots = -\sin(\omega t). \tag{6.59}$$

Similarly we obtain $S_{2,1} = \sin(\omega t)$ and $S_{22} = \cos(\omega t)$. □

By the above proposition, the fundamental matrix Φ is

$$\Phi = e^{\sigma t} Q \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix} Q^{-1}. \tag{6.60}$$

Example 6.19. Let us find the fundamental matrix of $A = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix}$. This matrix has eigenvalue $\lambda = -1 + i$ and thus $\sigma = -1$ and $\omega = 1$. The associated eigenvector is $\vec{v} = \begin{pmatrix} 5 \\ 2-i \end{pmatrix}$. For the matrix Q we have

$$Q = [\mathrm{Im}(\vec{v}) | \mathrm{Re}(\vec{v})] = \begin{pmatrix} 0 & 5 \\ -1 & 2 \end{pmatrix}. \tag{6.61}$$

Therefore Φ is

$$\Phi = e^{-t}\begin{pmatrix} 0 & 5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}\begin{pmatrix} 0 & 5 \\ -1 & 2 \end{pmatrix}^{-1} = \\ = e^{-t}\begin{pmatrix} \cos(t)+2\sin(t) & -5\sin(t) \\ \sin(t) & \cos(t)-2\sin(t) \end{pmatrix}.$$

Now, we follow the same method to find the fundamental matrix of a 4×4 matrix.

Example 6.20. We find the fundamental matrix associated to the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \tag{6.62}$$

The eigenvalue of the matrix is $\lambda = 1 + i$ with the algebraic multiplicity $m = 2$. The eigenvector of the matrix is $\vec{v} = (1,\, i,\, 0,\, 0)$. So we have two eigenvectors $\vec{v}$ and $\bar{\vec{v}}$. We find the generalized eigenvector $\vec{w}$ such that

$$(A - \lambda \mathbb{I}_{4\times 4})\vec{w} = \vec{v}. \tag{6.63}$$

A simple calculation gives $\vec{w} = (1, i, i, 1)$. The matrix Q is

$$Q = [\mathrm{Im}(\vec{v})|\mathrm{Re}(\vec{v})|\mathrm{Im}(\vec{w})|\mathrm{Re}(\vec{w})] = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is seen that

$$\Lambda = Q^{-1}AQ = \begin{pmatrix} 1 & -1 & \boxed{1} & \boxed{0} \\ 1 & 1 & \boxed{0} & \boxed{1} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \tag{6.64}$$

Note the identity block in the upper diagonal of the matrix Λ. We have

$$e^{\Lambda t} = e^{t}\begin{pmatrix} \cos(t) & -\sin(t) & \boxed{t\cos(t)} & \boxed{-t\sin(t)} \\ \sin(t) & \cos(t) & \boxed{t\sin(t)} & \boxed{t\cos(t)} \\ 0 & 0 & \cos(t) & -\sin(t) \\ 0 & 0 & \sin(t) & \cos(t) \end{pmatrix},$$

and finally the fundamental matrix is

$$\Phi = Qe^{\Lambda t}Q^{-1} = e^{t}Q\begin{pmatrix} \cos(t) & \sin(t) & t\sin(t) & t\cos(t) \\ -\sin(t) & \cos(t) & t\cos(t) & -t\sin(t) \\ 0 & 0 & \cos(t) & -\sin(t) \\ 0 & 0 & \sin(t) & \cos(t) \end{pmatrix}Q^{-1}. \tag{6.65}$$

6.6.3 General matrices

Let us show the method for a general matrix by solving an example. Consider the system $\boldsymbol{y} = A\boldsymbol{y}$, where A is

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & -1 & 3 & 4 \\ 0 & 1 & 3 & 2 & 4 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 1 & -3 \end{pmatrix}. \tag{6.66}$$

The eigenvalues of the matrix are

$$\lambda_1 = 1, \lambda_{2,3} = 2, \lambda_{4,5} = -1 \pm i, \tag{6.67}$$

with the associated eigenvectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \vec{v}_{4,5} = \begin{pmatrix} -1.36 \pm i0.58 \\ -3.84 \mp i3.38i \\ -1.06 \pm i0.08 \\ 2 \pm i \\ 1 \end{pmatrix} \tag{6.68}$$

The generalized eigenvector $\vec{v}_3$ is obtained by solving the equation

$$(A - 2\mathbb{I})\,\vec{v}_3 = \vec{v}_2, \tag{6.69}$$

that gives $\vec{v}_3 = (1, 1, 0, 0, 0)$. Thus, the matrix Q is

$$Q = [\vec{v}_1|\vec{v}_2|\vec{v}_3|\mathrm{Im}(\vec{v}_4)|\mathrm{Re}(\vec{v}_4)] = \begin{pmatrix} 1 & 1 & 1 & 0.58 & -1.36 \\ 0 & -1 & 1 & -3.38 & -3.84 \\ 0 & 1 & 0 & 0.0.8 & -1.06 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

It is simply verified that

$$Q^{-1}AQ = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}. \tag{6.70}$$

So $\Lambda = Q^{-1}AQ$ is a JORDAN 3-block matrix. Therefore we have

$$e^{\Lambda t} = \begin{pmatrix} e^t & 0 & 0 & 0 & 0 \\ 0 & e^{2t} & t\,e^{2t} & 0 & 0 \\ 0 & 0 & e^{2t} & 0 & 0 \\ 0 & 0 & 0 & e^{-t}\cos(t) & -e^{-t}\sin(t) \\ 0 & 0 & 0 & e^{-t}\sin(t) & e^{-t}\cos(t) \end{pmatrix}. \tag{6.71}$$

The matrix Φ is obtained by the formula $\Phi = Q\,e^{\Lambda t}\,Q^{-1}$.

Problems

Problem 6.40. Show that the fundamental matrix of a system is unique. That is, if Φ_1, Φ_2 are two fundamental matrices of a matrix A, then $\Phi_1 = \Phi_2$.

Problem 6.41. If Φ is the fundamental matrix of matrix A, show that the solution to the problem

$$\begin{cases} \boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{r}(t) \\ \boldsymbol{y}(0) = \boldsymbol{y}_0 \end{cases},$$

is obtained by the relation

$$\boldsymbol{y}(t) = \Phi(t)\,\boldsymbol{y}_0 + \Phi(t)\int_0^t \Phi(-\tau)\boldsymbol{r}(\tau)d\tau. \tag{6.72}$$

Problem 6.42. Show that if $\phi = [\phi_1(t)|\phi_2(t)]$ is a solution matrix of the system $\boldsymbol{y}' = A\boldsymbol{y}$, then $\Phi = \phi(t)\phi^{-1}(0)$ is the fundamental matrix of A.

Problem 6.43. Consider the problem $\begin{cases} \boldsymbol{y}' = A\boldsymbol{y} \\ \boldsymbol{y}(0) = \boldsymbol{y}_0 \end{cases}$ and assume A has two real distinct eigenvalue λ_1, λ_2 with associated eigenvalue $\vec{v}_1, \vec{v}_2$.

i. Show that the solution can be written as follows

$$\boldsymbol{y}(t) = e^{\lambda_1 t}\,\boldsymbol{y}_0 + (e^{\lambda_2 t} - e^{\lambda_1 t})\,c\vec{v}_2,$$

for some suitable constant c.

ii. Show that

$$c\,\vec{v}_2 = \frac{1}{\lambda_2 - \lambda_1}(A - \lambda_1\mathbb{I})\,\boldsymbol{y}_0,$$

and conclude that

$$\boldsymbol{y}(t) = \left[e^{\lambda_1 t}\,\mathbb{I} + \frac{(e^{\lambda_2 t} - e^{\lambda_1 t})}{\lambda_2 - \lambda_1}(A - \lambda_1\mathbb{I})\right]\boldsymbol{y}_0. \tag{6.73}$$

iii. Show that

$$\boldsymbol{y}(t) = \left[\frac{e^{\lambda_1 t}}{\lambda_1 - \lambda_2}(A - \lambda_1\mathbb{I}) + \frac{e^{\lambda_2 t}}{\lambda_2 - \lambda_1}(A - \lambda_2\mathbb{I})\right]\boldsymbol{y}_0 \tag{6.74}$$

iv. Assume that $A_{2\times 2}$ has only one eigenvector $\vec{v}$ with eigenvalue λ. Use formula (6.73) and show that the fundamental matrix $\Phi(x)$ of A is

$$\Phi(t) = e^{\lambda t}\,[\mathbb{I} + t\,(A - \lambda\mathbb{I})]. \tag{6.75}$$

Hint: For $\lambda_1 = \lambda$ let $\lambda_2 \to \lambda$ and calculate the limit.

v. Assume that A has a complex eigenvalue $\lambda = \sigma + i\omega$ and a complex eigenvector $\vec{v}$. Use formula (6.74) and show that the fundamental matrix $\Phi(t)$ of A is

$$\Phi(t) = e^{\sigma t}\left[\cos(\omega t)\mathbb{I} + \frac{1}{\omega}\sin(\omega t)(A - \sigma\mathbb{I})\right].$$

Problem 6.44. Use the formula $\Phi = \phi(t)\phi^{-1}(0)$ to determine the fundamental matrix of the following system. Then use the exponential form of the fundamental matrix and verify it is equal to the obtained one.

a)

$$\boldsymbol{y}' = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}\boldsymbol{y}.$$

b)

$$\boldsymbol{y}' = \begin{pmatrix} -2 & 3 \\ -1 & 2 \end{pmatrix}\boldsymbol{y}.$$

c)

$$\boldsymbol{y}' = \begin{pmatrix} 0 & 2 \\ -1 & -2 \end{pmatrix} \boldsymbol{y}.$$

d)

$$\boldsymbol{y}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \boldsymbol{y}.$$

e)

$$\boldsymbol{y}' = \begin{pmatrix} 6 & 3 \\ -2 & 1 \end{pmatrix} \boldsymbol{y}.$$

f)

$$\boldsymbol{y}' = \begin{pmatrix} 5 & 2 \\ -2 & 1 \end{pmatrix} \boldsymbol{y}.$$

g)

$$\boldsymbol{y}' = \begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix} \boldsymbol{y}.$$

h)

$$\boldsymbol{y}' = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \boldsymbol{y}.$$

i)

$$\boldsymbol{y}' = \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix} \boldsymbol{y}.$$

Problem 6.45. Write down the following equations in a system form, find the fundamental matrix of the system and then find the solution of the system satisfying the initial conditions

i. $y'' + 3y' + 2y = \sin(x), \quad y(0) = 0,\ y'(0) = 0.$

ii. $y'' - y = e^x, \quad y(0) = 0,\ y'(0) = 1.$

iii. $y'' - 2y' - 3y = x, \quad y(0) = 1,\ y'(0) = -1.$

Problem 6.46. Consider the system

$$\begin{cases} \frac{d}{dt}\boldsymbol{y} = A\boldsymbol{y} \\ \boldsymbol{y}(0) = \boldsymbol{y}_0 \end{cases}.$$

Divide the segment $[0, t]$ into n sub-interval with the length $h = \frac{t}{n}$ and use the approximation

$$\boldsymbol{y}((k+1)h) = \boldsymbol{y}(kh) + \boldsymbol{y}'(kh)\, h,$$

to conclude

$$\boldsymbol{y}(t) = (\mathbb{I} + hA)^{\frac{t}{h}} \boldsymbol{y}_0.$$

Let $n \to \infty$ and conclude $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$.

Problem 6.47. Consider the system in the previous problem. Rewrite the system in the integral form

$$\boldsymbol{y}(t) = \boldsymbol{y}_0 + A\int_0^t \boldsymbol{y}(\tau)\, d\tau.$$

Use the PICARD approximation and conclude

$$\boldsymbol{y}_{n+1}(t) = \left(\mathbb{I} + A + \frac{1}{2}A^2 + \cdots + \frac{1}{n!}A^n\right)\boldsymbol{y}_0.$$

Let $n \to \infty$ and conclude $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$.

Problem 6.48. Assume that $A_{n\times n}$ has only one eigenvalue λ_*. This implies $p(\lambda) = (\lambda - \lambda_*)^n$ and by the CAYLEY-HAMILTON theorem we know $(A - \lambda_*\mathbb{I})^n = 0$. Use this identity to show

$$\Phi(t) = e^{\lambda_* t} \sum_{k=1}^{n-1} \frac{1}{k!}(A - \lambda_*\mathbb{I})^k. \tag{6.76}$$

Problem 6.49. Obtain the fundamental matrix of the following system

$$\boldsymbol{y}' = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \boldsymbol{y}.$$

Problem 6.50. Obtain the fundamental matrix of the following system and solve it:

$$\boldsymbol{y}' = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \boldsymbol{y} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \quad \boldsymbol{y}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Problem 6.51. Find the fundamental matrix of the following system

$$\begin{cases} y_1' = y_1 + 5\, y_3 + t \\ y_2' = 2y_2 + 6y_3 \\ y_3' = 3y_1 + 3y_3 + \sin(t) \end{cases}$$

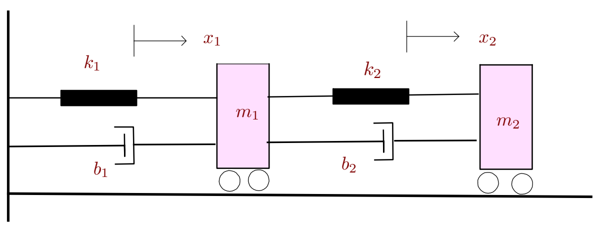
and find $y_1(t)$ if the initial condition is $(y_1(0), y_2(0), y_3(0)) = (0, 0, 0)$.

Problem 6.52. Find the fundamental matrix of the following system

$$\begin{cases} y_1' = 2y_1 + 3y_3 + t \\ y_2' = 2y_2 + y_3 - e^{-t} \\ y_3' = x - 4y_2 + 4y_3 \end{cases}$$

and find y_3 if the initial condition is $(y_1(0), y_2(0), y_3(0)) = (0, 0, 0)$.

Problem 6.53. For the following system, write down the equation of motion. It gives a system of second order equations. If $b_1 = b_2 = 2$, make a substitution to reduce the system to a decoupled system. Analyze the obtained system in terms of k_1, k_2.



Appendix A
Repeated eigenvalue: A justification

In the case of repeated roots for the characteristic polynomial $p(\lambda)$, the matrix $A_{2\times 2}$ may have only one eigenvector $\vec{v}$. In this case, $p(\lambda)$ has the form

$$p(\lambda) = (\lambda - \lambda_*)^2. \tag{A.1}$$

Proposition A.1. *Assume λ_* is a repeated eigenvalue of the matrix $A_{2\times 2}$ with only one eigenvector $\vec{v}$. Then there exist $\vec{w}$ such that*

$$(A - \lambda_* \mathbb{I})\,\vec{w} = \vec{v}. \tag{A.2}$$

Proof. Recall the CAYLEY-HAMILTON theorem which states that every matrix satisfies in its characteristic polynomial, i.e.,

$$(A - \lambda_* \mathbb{I})^2 = \mathbf{0}. \tag{A.3}$$

If $\mathbb{V}$ is the one dimensional space containing $\vec{v}$, choose arbitrary vector $\vec{w} \notin \mathbb{V}$. For the vector $\vec{u} = (A - \lambda_* \mathbb{I})\vec{w}$, we have $(A - \lambda_* \mathbb{I})\,\vec{u} = 0$. This implies that $\vec{u} = c\,\vec{v}$ for some constant $c \neq 0$ (note that if $c = 0$ then $\vec{u} = 0$ or equivalently $\vec{w} \in \mathbb{V}$). Now choose $\vec{w}$ such that $c = 1$ and therefore $(A - \lambda_* \mathbb{I})\vec{w} = \vec{v}_1$. □

Let $\mathbb{V}$ and $\mathbb{W}$ be one dimensional spaces of $\vec{v}$ and $\vec{w}$ respectively. If $\boldsymbol{y}(0) = \vec{v}_1$ then according to the relation $\boldsymbol{y}' = \lambda_*\,\boldsymbol{y}$ and we obtain $\boldsymbol{y}(t) = e^{\lambda_* t}\,\vec{v}_1$. If $\boldsymbol{y}(0) = \vec{v}_2$ then we have

$$\boldsymbol{y}'(0) = \lambda_*\,\vec{v}_2 + \vec{v}_1. \tag{A.4}$$

In order to obtain $\boldsymbol{y}(t)$ when $\boldsymbol{y}(0) = \vec{w} \in \mathbb{W}$, we do as follows. Divide the segment $[0, t]$ into n sub-interval $\delta x = \frac{t}{n}$. The approximation value of $\boldsymbol{y}(\delta t)$ is

$$\boldsymbol{y}(\delta t) \cong \boldsymbol{y}(0) + \boldsymbol{y}'(0)\,\delta t = (1 + \lambda_*\,\delta t)\vec{w} + \delta \mathrm{tx}\,\vec{v}. \tag{A.5}$$

At $2\delta t$, the approximation is

$$\boldsymbol{y}(2\delta t) = \boldsymbol{y}(\delta t) + \boldsymbol{y}'(\delta t)\,\delta t = (1 + \lambda_*\,\delta t)^2\,\vec{w} + 2(1 + \lambda_* \delta t)\,\vec{v}. \tag{A.6}$$

It is seen that for $t = n\,\delta t$

$$\boldsymbol{y}(t) = (1 + \lambda_*\,\delta t)^n\;\vec{w} + n(1 + \lambda_* \delta t)^{n-1}\,\vec{v}. \tag{A.7}$$

Now let $n \to \infty$, then according to the formula

$$\lim_{n\to\infty}\left(1+\lambda_*\frac{t}{n}\right)^n = e^{\lambda_* t}, \tag{A.8}$$

and

$$\lim_{n\to\infty} n(1+\lambda_*\delta t)^{n-1} = t e^{\lambda_* t}, \tag{A.9}$$

we obtain the solution

$$\boldsymbol{y}(t) = e^{\lambda_* t}\vec{w} + t e^{\lambda_* t}\vec{v}. \tag{A.10}$$

In general if $\boldsymbol{y}(0) = a_1\vec{v} + a_2\vec{v}$, we obtain $\boldsymbol{y}(t)$ as

$$\boldsymbol{y}(t) = e^{\lambda_* t} a_1\vec{v} + e^{\lambda_* t}(a_2 t\vec{v} + a_2\vec{w}). \tag{A.11}$$

We can write (A.11) as

$$\boldsymbol{y}(t) = e^{\lambda_* t}\boldsymbol{y}_0 + t e^{\lambda_* t} a_2\vec{v}. \tag{A.12}$$

On the other hand, according to the relation (A.2) we can write

$$a_2\vec{v} = (A - \lambda_*\mathbb{I})\boldsymbol{y}_0, \tag{A.13}$$

and this justifies formula we obtained for repeated eigenvalue.

A.1 Convergence of matrix series

Theorem A.1. *If $A_{n\times n}$ is a constant matrix, the matrix series*

$$e^A = \mathbb{I} + A + \frac{1}{2!}A^2 + \cdots + \frac{1}{n!}A^n + \cdots,$$

converges to a $n\times n$ matrix.

We define the norm of a matrix $A = [a_{ij}]_{n\times n}$ as

$$\|A\| = \sum_{i,j} |a_{ij}|. \tag{A.14}$$

Lemma A.1. *If $A = [a_{ij}]_{n\times n}$, $B = [b_{ij}]_{n\times n}$ are two matrices then*

i. $\|A + B\| \le \|A\| + \|B\|$

ii. $\|AB\| \le \|A\|\,\|B\|$.

Proof. The proof of the part (i) is straightforward and is left as an exercise to the reader. For the part (ii), first note that if

$$a = (a_1, \ldots, a_n), \text{ and } b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix},$$

then

$$ab \le (|a_1| + \cdots + |a_n|)\,(|b_1| + \cdots + |b_n|). \tag{A.15}$$

If $\tilde{A}_i$ denotes the i^{th} row of the matrix A and B_j denotes the j^{th} column of the matrix B then $AB = [\tilde{A}_i B_j]_{n\times n}$ and

$$\|AB\| = \sum_{ij} |\tilde{A}_i B_j| \leq \sum_{ij} |\tilde{A}_i|\,|B_j| = (|\tilde{A}_1| + \cdots + |\tilde{A}_n|)\,(|B_1| + \cdots + |B_n|), \tag{A.16}$$

where $|\tilde{A}_i| = \sum_j |a_{ij}|$ and $|B_j| = \sum_i |b_{ij}|$. Since

$$\|A\| = \sum_i (|\tilde{A}_i|), \text{ and } \|B\| = \sum_j (|B_j|), \tag{A.17}$$

we obtain

$$\|AB\| \leq \|A\|\,\|B\|, \tag{A.18}$$

and this completes the proof. □

Define the matrix S_N as

$$S_N = \mathbb{I} + A + \cdots + \frac{1}{N!}A^N. \tag{A.19}$$

Using the above lemma we have

$$\|S_N\| \leq \|I\| + \|A\| + \cdots + \frac{1}{N!}\|A^N\| \leq n + \|A\| + \cdots + \frac{1}{N!}\|A\|^N \tag{A.20}$$

But we have

$$e^{\|A\|} = 1 + \|A\| + \cdots + \frac{1}{N!}\|A\|^N + \cdots, \tag{A.21}$$

and then

$$\|S_N\| \leq (n-1) + e^{\|A\|}. \tag{A.22}$$

Let $N \to \infty$ and then we have $\|S\| \leq (n-1) + e^{\|A\|}$. In particular, for $S = [s_{ij}]_{n\times n}$ we have $|s_{ij}| < \infty$ and then S_N converges to the $n \times n$ matrix S.