Chapter 5
Laplace Transform Method

In this chapter, we introduce one of the crucial tools in engineering mathematics called Laplace transform. This tool enables us to solve differential equations with discontinuous terms in an efficient and straightforward way. Besides solving ordinary differential equations, with the aid of the Laplace transform, we are able to define the concept of the transfer function of complex systems in engineering and the idea of the *fundamental solution*.

5.1 Definition of the transformation

The *unilateral* Laplace or L-transform of a function $f(t)$ is defined by the following integral

$$\mathsf{L}(f)(s) = \int_0^\infty e^{-st} f(t)\, dt, \tag{5.1}$$

as long as the integral exists for some values of s. Note that (5.1) transforms a function of t (that we can interpret as time) to a function of s (can be considered just as a parameter). In this chapter, we use notations $\hat{f}(s)$, $\mathsf{L}(f)$ for the transform. It is called unilateral because $f(t)$ for $t < 0$ (the history of f) does not affect the transformation.

Definition 5.1. *A function $f(t)$ is called* L-admissible *if there is an interval of s for which the integral (5.1) converges.*

Remark 5.1. The class of L-admissible functions is wide, however, the sub-exponential functions are well-known to be L-admissible (see the problem set). A function $f(t)$ is called sub-exponential if there are $a > 0, b > 0$ such that $|f(t)| < a e^{bt}$ for all $t \in [0, \infty)$.

At the first glance, formula (5.1) looks peculiar, nevertheless, the transform possesses nice properties that makes it desirable in engineering mathematics. For example, it transforms the derivative operation to an algebraic multiplication, that is, if $f'(t)$ is the derivative of a function $f(t)$, then

$$\mathsf{L}(f') = s\,\mathsf{L}(f) - f(0).$$

The above property finds its importance to solve ordinary differential equations.

5.1.1 L-Transform of basic functions

1. The HEAVISIDE or unit step function $u(t)$ is defined by the following two valued function

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}. \tag{5.2}$$

In some texts, the value $u(0)$ is considered $\frac{1}{2}$, or even 1, but $u(0)$ does not affect its transformation. We have

$$\hat{u}(s) = \int_0^\infty e^{-st} u(t)\, dt = \int_0^\infty e^{-st}\, dt = \frac{1}{s}\Big[\lim_{t\to\infty} e^{-st} - 1\Big]. \tag{5.3}$$

Clearly, the above integral converges to $\frac{1}{s}$ if $s > 0$. Note that that $\mathsf{L}(1) = \frac{1}{s}$ since two function $u(t)$ and 1 are equal for $t \geq 0$.

2. Consider the unit pulse

$$p(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}. \tag{5.4}$$

Its L-transform is

$$\hat{p}(s) = \int_0^1 e^{-st}\, dt = \frac{1 - e^{-s}}{s}. \tag{5.5}$$

Note that the value of $p(t)$ at the end points does not affect the transformation.

3. The transform of the ramp function $r(t) = t$ is

$$\mathsf{L}(t) = \int_0^\infty t e^{-st}\, dt = -\lim_{t\to\infty} \frac{t}{s} e^{-st} + \frac{1}{s}\int_0^\infty e^{-st}\, dt. \tag{5.6}$$

If $s > 0$, we have

$$\lim_{t\to\infty} \frac{t}{s} e^{-st} = 0,$$

and thus $\mathsf{L}(t) = \frac{1}{s^2}$.

4. The transform of the exponential function $f(t) = e^{at}$ is

$$\mathsf{L}(e^{at}) = \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt. \tag{5.7}$$

Clearly the integral converges for $s > a$ and then $\mathsf{L}(e^{-at}) = \frac{1}{s-a}$.

5. The L-transform of the function $f(t) = \sin(\omega t)$ is simply derived by the aid of the EULER formula

$$\sin(\omega t) = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}.$$

A simple algebraic simplification gives

$$\mathsf{L}(\sin(\omega t)) = \frac{1}{2i}\{\mathsf{L}(e^{i\omega t}) - \mathsf{L}(e^{-i\omega t})\} = \frac{\omega}{s^2 + \omega^2}. \tag{5.8}$$

A similar argument applies for the transform of $f(t) = \cos(\omega t)$

$$\mathsf{L}(\cos(\omega t)) = \frac{s}{s^2 + \omega^2}.$$

6. Remember the formula $\sinh(\omega t) = \frac{e^{\omega t} - e^{-\omega t}}{2}$. For $s > |\omega|$, we have

$$\mathsf{L}(\sinh(\omega t)) = \frac{1}{2}\{\mathsf{L}(e^{\omega t}) - \mathsf{L}(e^{-\omega t})\} = \frac{\omega}{s^2 - \omega^2}. \tag{5.9}$$

A similar argument applies for the transform of $\cosh(\omega t)$

$$\mathsf{L}(\cosh(\omega t)) = \frac{s}{s^2 - \omega^2}. \tag{5.10}$$

5.1.2 The inverse of LAPLACE transform

We saw that $\mathsf{L}(u(t)) = \frac{1}{s}$ for $s > 0$, and it implies that functions $u(t)$ in t-domain has the *representation* $\frac{1}{s}$ in s-domain through the LAPLACE transform; see the figure (5.1)

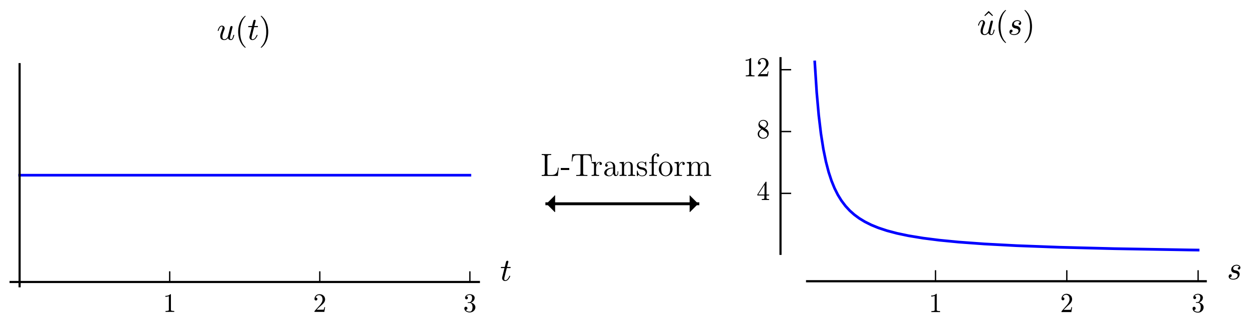


Figure 5.1.

An inverse transformation from s-domain to t-domain is also can be defined through the following relation

$$\mathsf{L}(f(t)) = \hat{f}(s) \Leftrightarrow f(t) = \mathsf{L}^{-1}(\hat{f}(s)).$$

However, since the transform is unilateral, we just write $\mathsf{L}^{-1}(\hat{f}(s)) = f(t)$ for $t > 0$. Accordingly, since $\mathsf{L}(t) = \frac{1}{s^2}$, we write $\mathsf{L}^{-1}\left(\frac{1}{s^2}\right) = t$, $t > 0$. There is still a minor problem here. Let $\tilde{f}(t)$ be the function

$$\tilde{f}(t) = \begin{cases} t & t \neq 1 \\ 0 & t = 1 \end{cases}.$$

It is simply seen that $\mathsf{L}(t) = \mathsf{L}(\tilde{f}) = \frac{1}{s^2}$, and therefore it is not clear that $\mathsf{L}^{-1}\left(\frac{1}{s^2}\right)$ how should be defined. In fact the transformation L is not one-to-one. Here we make the following convention.

Convention. The function $\mathsf{L}^{-1}(\hat{f}(s))$ refers to the *most possibly continuous* function $f(t)$ for $t > 0$ such that $\mathsf{L}(f(t)) = \hat{f}(s)$.

According to the above convention, we define $\mathsf{L}^{-1}\left(\frac{1}{s}\right) = 1$, $\mathsf{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at}$, $\mathsf{L}^{-1}\left(\frac{s}{s^2+\omega^2}\right) = \cos(t)$, and $\mathsf{L}^{-1}\left(\frac{\omega}{s^2+\omega^2}\right) = \sin(t)$ all for $t > 0$.

Proposition 5.1. *Assume that $f(t)$ is sub-exponential, then*

$$\lim_{s\to\infty} \hat{f}(s) = 0. \tag{5.11}$$

Proof. We have

$$\lim_{s\to\infty} \hat{f}(s) = \lim_{s\to\infty} \int_0^\infty e^{-st} f(t)\, dt. \tag{5.12}$$

Since the function $f(t)$ is sub-exponential, the function $|e^{-st}\, f(t)|$ is bounded. By the dominant convergence theorem, we can pass the limit inside the integral and write it as

$$\lim_{s\to\infty} \hat{f}(s) = \int_0^\infty \lim_{s\to\infty} e^{-st} f(t)\, dt = 0, \tag{5.13}$$

and this completes the proof. □

We have also the following fact.

Proposition 5.2. *Assume that $f(t)$ is sub-exponential, then $\hat{f}(s)$ is continuous in its domain of definition.*

Proof. Since the domain of $\hat{f}(s)$ is open, let us consider the fraction

$$\hat{f}(s+h) - \hat{f}(s) = \int_0^\infty e^{-st} f(t)(e^{-ht} - 1)\, dt.$$

Now, for $h \to 0$, we have

$$\lim_{h\to 0} (\hat{f}(s+h) - \hat{f}(s)) = \lim_{h\to 0} \int_0^\infty e^{-st} f(t)(e^{-ht} - 1)\, dt.$$

Now, if $|f(t)| < a e^{bt}$ for some $a, b > 0$ holds, then the condition of dominant convergence theorem (see the appendix to this book) holds for the above integral since

$$|e^{-st} f(t)(e^{-ht} - 1)| \le a e^{-(s-b)t}\, |e^{-ht} - 1| \le a e^{-(s-b+h)t} < a$$

if s is sufficiently large. Therefore

$$\lim_{h\to 0} \int_0^\infty e^{-st} f(t)(e^{-ht} - 1)\, dt = \int_0^\infty e^{-st} f(t) \lim_{h\to 0} (e^{-ht} - 1)\, dt = 0,$$

and this completes the proof. □

Problems

Problem 5.1. Assume that the function $f(t)$ is sub-exponential, that is, there are constants a, b such that $|f(t)| < a e^{bt}$ for all $t \in [0, \infty)$. Show that $f(t)$ is L-admissible.

Problem 5.2. Assume that $f(t)$ is a sub-exponential function and $f'(t)$ is L-admissible.

a) Use the definition and show

$$\mathsf{L}(f') = \int_0^\infty e^{-st} f'(t)\, dt + s \int_0^\infty e^{-st} f(t)\, dt.$$

b) Show the relation

$$\lim_{t\to\infty} e^{-st} f(t) = 0,$$

and conclude

$$\mathsf{L}(f') = s\,\mathsf{L}(f) - f(0).$$

Problem 5.3. Use the definition and find the L-transform of the following functions

i.

$$f(t) = \begin{cases} 1 & 0 < t < 1 \\ -1 & 1 < t < 2 \end{cases}.$$

ii.

$$f(t) = \begin{cases} e^t - 1 & 0 < t < 1 \\ 0 & t > 1 \end{cases}.$$

iii. $f(t) = e^t - e^{-2t}$.

Problem 5.4. Use the definition and calculate the transform of the following functions. For each function determine the domain of s for which the transform exists

a)

$$f(t) = \begin{cases} e^{2t} & 0 \le t < 1 \\ t & t > 1 \end{cases}$$

b) $f(t) = \cos^2(t)$

c) $f(t) = \sinh^2(t)$

d)

$$f(t) = \begin{cases} 1 & 1 < t < 2 \\ -1 & 2 < t < 3 \end{cases}$$

Problem 5.5. If $\hat{f}(s)$ is given we can recover the function $f(t)$ such that $\mathsf{L}(f(t)) = \hat{f}(s)$. For the following functions, find $f(t)$ such that $\mathsf{L}(f) = \hat{f}$.

i. $\hat{f}(s) = \frac{3}{s^2+1}$

ii. $\hat{f}(s) = \frac{2-3s}{s^2+4}$

iii. $\hat{f}(s) = \frac{3}{s-2}$

iv. $\hat{f}(s) = \frac{1}{(s+1)(s+2)}$

Problem 5.6. Is there any L-transform of the function $f(t) = e^{t^2}$? Is it possible the function $\hat{f}(s) = 1$ to be the transform of a sub-exponential function?

Problem 5.7. Determine which one of the following functions are sub-exponential in $[0, \infty)$

i. $f(x) = t\sin(t)$

ii. $f(x) = t\sin(1/t)$

iii. $f(x) = 1 + e^t + e^{2t} + \cdots + e^{nt}$.

iv. $f(x) = 1 + e^t + e^{2t} + \cdots$

v. $f(x) = u(t) + u(t-1) + u(t-2) + \cdots$

Problem 5.8. Show $\mathsf{L}(\sqrt{t}) = \frac{\sqrt{\pi}}{2} s^{-3/2}$. Hint: Remember the integral

$$\int_0^\infty e^{-u^2} du = \frac{\sqrt{\pi}}{2}.$$

5.2 Properties of LAPLACE transform

Here we study only important properties of L-transform that we need for solving initial value problems.

Linearity.

L-transform is linear in the sense that for any pair of admissible functions $f(t)$, $g(t)$ and any arbitrary constants c_1, c_2, it satisfies the relation

$$\mathsf{L}(c_1\, f + c_2\, g) = c_1\,\mathsf{L}(f) + c_2\,\mathsf{L}(g) \tag{5.14}$$

The property is directly proved by the definition (5.1):

$$\mathsf{L}(c_1\, f + c_2\, g) = \int_0^\infty e^{-st}(c_1\, f(t) + c_2\, g(t))dt = c_1 \int_0^\infty e^{-st} f(t)\,dt + \\ + c_2 \int_0^\infty e^{-st} g(t)\,dt = c_1\,\mathsf{L}(f) + c_2\,\mathsf{L}(g).$$

The linearity of L-transform implies the linearity of L^{-1} as well. In fact, by the relation

$$c_1\, f + c_2\, g = \mathsf{L}^{-1}\mathsf{L}(c_1\, f + c_2\, g) = \mathsf{L}^{-1}(c_1\,\hat{f} + c_2\,\hat{g}),$$

we obtain

$$\mathsf{L}^{-1}(c_1\,\hat{f}(s) + c_2\,\hat{g}(s)) = c_1\,\mathsf{L}^{-1}(\hat{f}(s)) + c_2\,\mathsf{L}^{-1}(\hat{g}(s)).$$

Example 5.1. Let us find $\mathsf{L}^{-1}\left(\frac{1}{s(s+1)}\right)$. According to the relation

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1},$$

we can write

$$\mathsf{L}^{-1}\left(\frac{1}{s(s+1)}\right) = \mathsf{L}^{-1}\left(\frac{1}{s}\right) - \mathsf{L}^{-1}\left(\frac{1}{s+1}\right) = (1 - e^{-t}), t > 0.$$

Multiplication by t.

If $f(t)$ is an sub-exponential functions then

$$\frac{d}{ds}\,\hat{f}(s) = \int_0^\infty \frac{d}{ds}\, e^{-st}\, f(t)\,dt = -\int_0^\infty e^{-st}\, t\, f(t)\,dt \tag{5.15}$$

In other word we can write

$$\frac{d}{ds}\hat{f}(s) = -\mathsf{L}(t\, f(t)), \tag{5.16}$$

or equivalently

$$\mathsf{L}(t\, f(t)) = -\frac{d}{ds}\hat{f}(s). \tag{5.17}$$

It is left as an exercise to the reader to verify the following formula by the mathematical induction

$$\mathsf{L}(t^n\, f(t)) = (-1)^n \frac{d^n}{ds^n}\hat{f}(s). \tag{5.18}$$

Example 5.2. We calculate the inverse transform of the following function

$$\hat{f}(s) = \ln\frac{s-1}{s}. \tag{5.19}$$

Taking derivative of $\hat{f}(s)$ gives

$$\frac{d\hat{f}}{ds}(s) = \frac{1}{s(s-1)} = \frac{1}{s-1} - \frac{1}{s}. \tag{5.20}$$

According to the linearity of L^{-1}, we have $\mathsf{L}^{-1}(\frac{d\hat{f}}{ds}) = e^t - 1$. Now use the property $\mathsf{L}^{-1}(\frac{d\hat{f}}{ds}) = -t f(t)$ and write

$$f(t) = \mathsf{L}^{-1}(\hat{f}(s)) = \frac{1-e^t}{t}, t > 0. \tag{5.21}$$

Note that $f(t)$ is a sub-exponential function.

Division by t.

Let $f(t)$ be a sub-exponential function that satisfies the relation

$$\lim_{t \downarrow 0} \left| \frac{f(t)}{t} \right| < \infty. \tag{5.22}$$

Consider the function $g(t) = \frac{f(t)}{t}$. By (5.16), we have

$$\frac{d}{ds}\hat{g}(s) = -\mathsf{L}\,(t g(t)) = -\mathsf{L}(f(t)) = -\hat{f}(s). \tag{5.23}$$

Integrating both sides of the above formula in $[s, \infty)$ gives

$$-\hat{g}(s) + \lim_{s\to\infty} \hat{g}(s) = -\int_s^\infty \hat{f}(v)\, dv. \tag{5.24}$$

By (5.11) we have

$$\lim_{s\to\infty} \hat{g}(s) = 0,$$

and thus

$$\mathsf{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty \hat{f}(v)\, dv. \tag{5.25}$$

Example 5.3. Some definite integrals can be calculated by the aid of the above property. For example, let us find the following integral

$$I = \int_0^\infty \frac{\sin(t)}{t}\, dt. \tag{5.26}$$

We can write

$$\int_0^\infty \frac{\sin(t)}{t}\, dt = \int_0^\infty \lim_{s\to 0} e^{-st}\frac{\sin(t)}{t}\, dt. \tag{5.27}$$

Since $\frac{\sin(t)}{t}$ is sub-exponential, we can take the limit out of the integral and write

$$\int_0^\infty \frac{\sin(t)}{t}\,dt = \lim_{s\to 0}\int_0^\infty e^{-st}\,\frac{\sin(t)}{t}\,dt = \lim_{s\to 0}\mathsf{L}\left(\frac{\sin(t)}{t}\right).$$

But, we have

$$\mathsf{L}\left(\frac{\sin(t)}{t}\right) = \int_s^\infty \frac{dv}{v^2+1} = \frac{\pi}{2} - \tan^{-1}(s). \tag{5.28}$$

Finally for $s\to 0$, we obtain

$$\int_0^\infty \frac{\sin(t)}{t}\,dt = \frac{\pi}{2} - \lim_{s\to 0}\tan^{-1}(s) = \frac{\pi}{2}. \tag{5.29}$$

Shift in t-domain.

Suppose that $f(t)$ is an admissible function and $g(t) = f(t-t_0)u(t-t_0)$ is the time shift of f by $t_0 \geq 0$; see the figure (5.2).

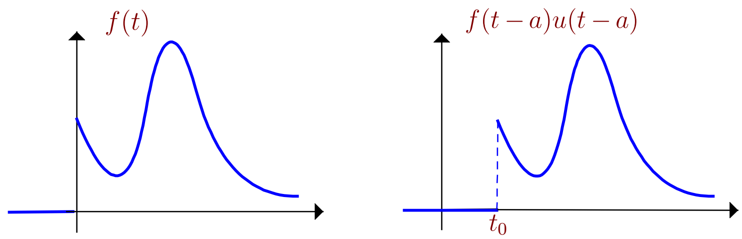


Figure 5.2. $f(t)$ and its shifted graph $f(t-t_0)$.

We have

$$\mathsf{L}(g) = \int_0^\infty e^{-st} f(t-t_0)u(t-t_0)\,dt = \int_{t_0}^\infty e^{-st} f(t-t_0)\,dt. \tag{5.30}$$

For $\tau = t - t_0$, we obtain

$$\int_{t_0}^\infty e^{-st} f(t-t_0)\,dt = \int_0^\infty e^{-s(\tau+t_0)} f(\tau)\,d\tau = e^{-t_0 s}\int_0^\infty e^{-s\tau} f(\tau)\,d\tau, \tag{5.31}$$

and then

$$\mathsf{L}(f(t-t_0)\,u(t-t_0)) = e^{-t_0 s}\,\hat{f}(s). \tag{5.32}$$

Example 5.4. The above property is used to calculate the inverse transform of some s-functions of the form $\hat{g}(s) = e^{-t_0 s}\,\hat{f}(s)$. For example,

$$\mathsf{L}^{-1}\left(e^{-2s}\,\frac{s}{s^2-1}\right) = \mathsf{L}^{-1}\left(\frac{s}{s^2-1}\right)|_{t\to t-2}\,u(t-2). \tag{5.33}$$

But,

$$\mathsf{L}^{-1}\left(\frac{s}{s^2-1}\right) = \cosh(t). \tag{5.34}$$

and hence

$$\mathsf{L}^{-1}\left(e^{-2s}\,\frac{s}{s^2-1}\right) = \cosh(t-2)\,u(t-2). \tag{5.35}$$

Example 5.5. Let $g(t)$ be the following function:

$$g(t)=\sum_{k=0}^{\infty} f(t-kt_0)u(t-kt_0), \tag{5.36}$$

and assume that $f(t)$ is sub-exponential. We have

$$|g(t)|\leq\sum_{k=0}^{\infty} |f(t-kt_0)|\,u(t-kt_0)\leq a\sum_{k=0}^{\infty} e^{b(t-kt_0)}\,u(t-kt_0)\leq$$

$$\leq a\,e^{bt}\sum_{k=0}^{\infty} e^{-bkt_0}\,u(t-kt_0).$$

Since

$$\sum_{k=0}^{\infty} e^{-bkt_0}\,u(t-kt_0)\leq\frac{1}{1-e^{-bt_0}}, \tag{5.37}$$

we conclude

$$|g(t)|\leq\frac{a}{1-e^{-bt_0}}\,e^{bt}. \tag{5.38}$$

The above argument guarantees that $g(t)$ is a sub-exponential function. Let us define

$$g_n(t)=\sum_{k=0}^{n} f(t-kt_0)u(t-kt_0), \tag{5.39}$$

and then

$$\hat{g}(s)=\lim_{n\to\infty}\hat{g}_n(s)=\lim_{n\to\infty}\int_0^{\infty} e^{-st}\,g_n(t)\,dt. \tag{5.40}$$

But, we have

$$\int_0^{\infty} e^{-st}\,g_n(t)\,dt=\sum_{k=0}^{n} \mathsf{L}(f(t-kt_0)\,u(t-kt_0))=\sum_{k=0}^{n} e^{-kt_0s}\,\hat{f}(s), \tag{5.41}$$

and finally

$$\hat{g}(s)=\hat{f}(s)\lim_{n\to\infty}\sum_{k=0}^{n} e^{-kt_0s}=\frac{1}{1-e^{-t_0s}}\,\hat{f}(s), \tag{5.42}$$

or equivalently

$$\mathsf{L}^{-1}\left(\frac{1}{1-e^{-t_0s}}\,\hat{f}(s)\right)=\sum_{k=0}^{\infty} f(t-kt_0)\,u(t-kt_0). \tag{5.43}$$

In particular, if $f(t)$ is a T-periodic function, that is, $f(t+T)=f(t)$, then

$$\mathsf{L}(f(t))=\frac{1}{1-e^{-sT}}\int_0^{T} e^{-st}\,f(t)\,dt.$$

Shift in s-domain.

There is a beautiful duality between the shift in t domain and the shift in s domain sometimes called *phase shift*. For $g(t)=e^{at}\,f(t)$, we have

$$\mathsf{L}(g)=\int_0^{\infty} e^{-st}e^{at}f(t)\,dt=\int_0^{\infty} e^{-(s-a)t}f(t)\,dt=\hat{f}(s-a), \tag{5.44}$$

or equivalently

$$\mathsf{L}^{-1}(\hat{f}(s-a)) = e^{at}\,\mathsf{L}^{-1}(\hat{f}(s)). \tag{5.45}$$

Example 5.6. Let us use the above property to find

$$\mathsf{L}^{-1}\left(e^{-2s}\frac{s-1}{(s-1)^2-4}\right). \tag{5.46}$$

We have

$$\mathsf{L}^{-1}\left(e^{-2s}\frac{s-1}{(s-1)^2-4}\right) = \mathsf{L}^{-1}\left(\frac{s-1}{(s-1)^2-4}\right)|_{t\to t-2}\,u(t-2). \tag{5.47}$$

Since

$$\mathsf{L}^{-1}\left(\frac{s-1}{(s-1)^2-4}\right) = e^t\cosh(2t), \tag{5.48}$$

we obtain

$$\mathsf{L}^{-1}\left(e^{-2s}\frac{s-1}{(s-1)^2-4}\right) = e^{(t-2)}\cosh 2(t-2)u(t-2). \tag{5.49}$$

Derivative.

If $f(t)$ is a sub-exponential differentiable function, then

$$\int_0^\infty e^{-st} f'(t)\,dt = \lim_{t\to\infty} e^{-st} f(t) - \lim_{t\to 0^+} f(t) + s\int_0^\infty e^{-st} f(t)\,dt. \tag{5.50}$$

Since $f(t)$ is sub-exponential, we have $\lim_{t\to\infty} e^{-st} f(t) = 0$. If $f(t)$ is continuous at $t=0$, then $\lim_{t\to 0^+} f(t) = f(0)$, and then

$$\mathsf{L}(f') = s\,\hat{f}(s) - f(0). \tag{5.51}$$

Example 5.7. Consider the following problem

$$\begin{cases} y' + y = u(t-1) \\ y(0) = y_0 \end{cases}.$$

Taking the L-transform to the both sides of the equation gives

$$s\,\hat{y}(s) - y_0 + \hat{y}(s) = \frac{e^{-s}}{s},$$

or equivalently

$$(s+1)\hat{y}(s) = \frac{e^{-s}}{s} + y_0.$$

This yields $\hat{y}(s)$ as

$$\hat{y}(s) = \frac{1}{s+1}y_0 + \frac{e^{-s}}{s(s+1)}.$$

To obtain $y(t)$, we should take inverse transform of $\hat{y}(s)$. Note that

$$\mathsf{L}^{-1}\left(\frac{e^{-s}}{s(s+1)}\right) = (1-e^{-(t-1)})\,u(t-1),$$

and

$$\mathsf{L}^{-1}\left(\frac{1}{s+1}y_0\right) = y_0 e^{-t}, t>0.$$

Here we should take a notice. The term $\mathsf{L}^{-1}\big(\frac{1}{s+1}y_0\big)$ is associated to the initial condition of the equation at time $t=0$ and $\mathsf{L}^{-1}\Big(\frac{e^{-s}}{s(s+1)}\Big)$ is associated to the external term that is exercised for $t>1$. For this the homogeneous solution $y_0\,e^{-t}$ is not only defined for $t>0$ but also for all $t\in(-\infty,\infty)$. Therefore, we can write the solution to the equation as

$$y(t)=y_0\,e^{-t}+(1-e^{-(t-1)})\,u(t-1), \tag{5.52}$$

for all $t\in(-\infty,\,\infty)$. Note also that the solution is continuous everywhere and differentiable, but it is not differentiable at $t=1$ due t the term $u(t-1)$.

The following formula is simply proved by the mathematical induction:

$$\mathsf{L}(f^{(n)}(t))=s^n\,\hat{f}(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0). \tag{5.53}$$

Integration.

Let us derive the L-transform of the integral function

$$g(t)=\int_0^t f(\tau)\,d\tau\,,t>0. \tag{5.54}$$

Since $g'(t)=f(t)$ and $\lim_{t\to 0^+}g(t)=0$, we obtain

$$\hat{f}(s)=\mathsf{L}(g')=s\,\hat{g}(s) \tag{5.55}$$

and thus

$$\mathsf{L}\bigg(\int_0^t f(\tau)\,d\tau\bigg)=\frac{1}{s}\,\hat{f}(s), \tag{5.56}$$

or equivalently

$$\int_0^t f(\tau)\,d\tau=\mathsf{L}^{-1}\bigg(\frac{1}{s}\,\hat{f}(s)\bigg), \tag{5.57}$$

for $t>0$.

Example 5.8. By the aid of the integral property, we are able to calculate the inverse of some s-functions of the form $\hat{g}(s)=\frac{1}{s}\hat{f}(s)$. Consider the function

$$\hat{f}(s)=\frac{1}{s(s^2+1)}. \tag{5.58}$$

We have

$$\mathsf{L}^{-1}(\hat{f}(s))=\mathsf{L}^{-1}\bigg(\frac{1}{s}\,\frac{1}{(s^2+1)}\bigg)=\int_0^t \mathsf{L}^{-1}\bigg(\frac{1}{s^2+1}\bigg)=\int_0^t \sin{(v)}\,dv=(1-\cos(t))u(t). \tag{5.59}$$

Example 5.9. Let us calculate $\mathsf{L}^{-1}\Big(\frac{\hat{f}(s)}{s+a}\Big)$. According to the phase shift property, we have

$$\mathsf{L}\bigg(e^{-at}\int_0^t f(\tau)\bigg)=\mathsf{L}\bigg(\int_0^t f(\tau)\bigg)|_{s\to s+a}=\frac{\hat{f}(s)}{s}|_{s\to s+a}=\frac{\hat{f}(s+a)}{s+a}.$$

Since we need $\hat{f}(s)$ in the top, we write

$$\mathsf{L}\left(e^{-at}\int_0^t e^{a\tau}f(\tau)\right)=\mathsf{L}\left(\int_0^t e^{a\tau}f(\tau)\right)|_{s\to s+a}.$$

But

$$\mathsf{L}\left(\int_0^t e^{a\tau}f(\tau)\right)=\frac{1}{s}\mathsf{L}(e^{at}f(t))=\frac{\hat{f}(s-a)}{s},$$

and therefore

$$\mathsf{L}\left(e^{-at}\int_0^t e^{a\tau}f(\tau)\right)=\frac{\hat{f}(s-a)}{s}|_{s\to s+a}=\frac{\hat{f}(s)}{s+a},$$

or equivalently

$$\mathsf{L}^{-1}\left(\frac{\hat{f}(s)}{s+a}\right)=\left(e^{-at}\int_0^t e^{a\tau}f(\tau)\,d\tau\right)u(t)$$

5.2.1 DIRAC delta function and its LAPLACE transform

The DIRAC delta function δ, introduced by the English physicist PAUL DIRAC (1902-1984) is a specific type of singularity used in advanced physics. There are different ways to define the DIRAC delta function. One way that we adopt here is to define it as the limit of a sequence of functions called the delta-sequence functions. Consider the following sequence

$$\delta_n(t)=\begin{cases}\frac{n}{2} & -\frac{1}{n}<t<\frac{1}{n}\\ 0 & \text{otherwise}\end{cases}, \tag{5.60}$$

for $n=1,2,3,\cdots$. The graph is shown below.

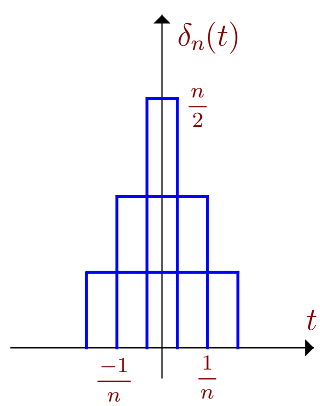


Figure 5.3. An example of δ_n-sequence functions.

Assume that $f(t)$ is a continuous function. We can write

$$\min_{t\in\left[\frac{-1}{n},\frac{1}{n}\right]} f(t)\le\int_{-\infty}^{\infty}f(t)\delta_n(t)\,dt\le\max_{t\in\left[\frac{-1}{n},\frac{1}{n}\right]} f(t). \tag{5.61}$$

Since

$$\lim_{n\to\infty} \min_{t\in\left[\frac{-1}{n},\frac{1}{n}\right]} f(t) = \lim_{n\to\infty} \max_{t\in\left[\frac{-1}{n},\frac{1}{n}\right]} f(t) = f(0),$$

then by the squeeze theorem, we can write

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(t)\delta_n(t)\,dt = f(0).$$

As it is seen, $\delta_n(t)$ converges to a spike at $t=0$ with infinite height and zero width with the property

$$\int_{-\infty}^{\infty} \delta_n(t)\,dt = 0, \forall n \geq 1.$$

This limiting *function* which is a singularity at $t=0$ is called the Dirac delta function. The above defined sequence is not unique and there are other sequences with the same property. For example, consider the sequence of function $G_n(t) = \frac{n}{\sqrt{\pi}}\, e^{-n^2t^2}$ for $n=1, 2, \cdots$ shown below. Note that all $G_n(x)$ are derived from the Gaussian function $G(t) = \frac{1}{\sqrt{\pi}} e^{-t^2}$ by the relation $G_n(t) = nG(nt)$.

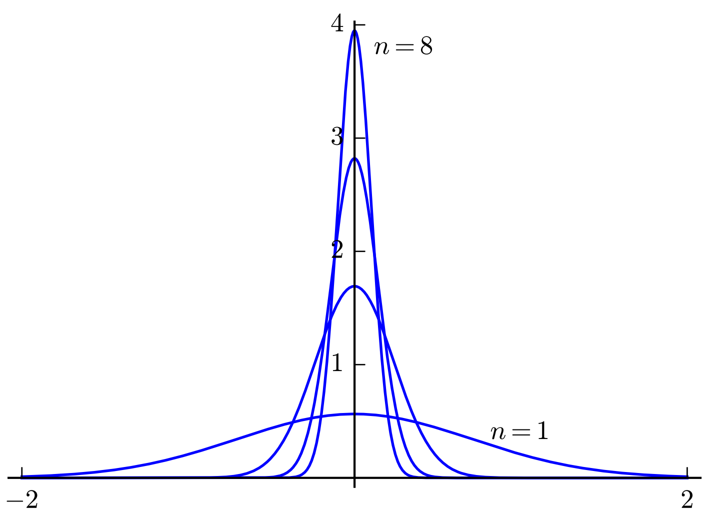


Figure 5.4.

When n approaches infinity, the function approaches to a spike with the width 0 and the height infinity at $t=0$. Let $f(t)$ be a continuous and *bounded* function. We have

$$\int_{-\infty}^{\infty} f(t)\, G_n(t)\,dt = \int_{-\infty}^{\infty} f(t)\, \frac{n}{\sqrt{\pi}}\, e^{-n^2t^2}\,dt,$$

and by taking $\tau = nt$, we obtain

$$\int_{-\infty}^{\infty} f(t)G_n(t)\,dt = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f\left(\frac{\tau}{n}\right) e^{-\tau^2}\,d\tau.$$

For $n \to \infty$, we can write

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f(t)G_n(t)\,dt = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\lim_{n\to\infty} f\left(\frac{\tau}{n}\right)e^{-\tau^2}\,d\tau = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(0)e^{-\tau^2}\,d\tau = f(0).$$

Note that in the above calculations, we took the limit inside the integral which is justifiable by the assumption on f. Note also the relation

$$\int_{-\infty}^{\infty} G_n(t)\,dt = 1, \forall n \geq 1.$$

Definition 5.2. *A sequence of function $(\delta_n(x))$ is called a δ-sequence function if the following relation holds for any bounded and continuous function f*

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f(t)\,\delta_n\,(t)dt = f(0).$$

The limit function of a δ-sequence function is called the DIRAC *delta function and is denoted by $\delta(t)$.*

An alternative way for defining δ is by the following equality

$$\int_{-a}^{a} f(t)\,\delta(t)dt = f(0), \tag{5.62}$$

for any $a > 0$ and any continuous function f. The formula (5.62) gives immediately the following relation

$$\int_{t-a}^{t+a} f(\tau)\delta(t-\tau)d\tau = f(t), \tag{5.63}$$

or equivalently

$$\int_{-\infty}^{\infty} f(\tau)\,\delta(t-\tau)\,d\tau = f(t).$$

That $\delta(t)$ is not a function in the usual sense is seen from the following observation. If δ was a classical function then it would had to satisfy the following condition

$$\lim_{a\to 0}\int_{-a}^{a} f(t)\,\delta(t)\,dt = 0,$$

which contradicts the relation (5.62). In engineering literature, $\delta(t)$ is called an *impulse* and is usually denoted by a unit arrow; see Fig.5.5. For this reason, $\delta(t)$ is sometime defined (and it is not technically correct) by the following relation

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & \text{otherwise} \end{cases}. \tag{5.64}$$

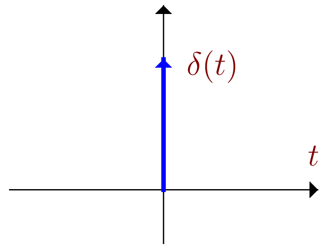


Figure 5.5. The graph of δ function.

For any $\tau > 0$, the relation (5.63) implies

$$\mathsf{L}(\delta(t-a)) = \int_0^\infty e^{-st}\,\delta(t-a)\,dt = e^{-as}.$$

We accept this definition for $a = 0$ as well and write

$$\mathsf{L}(\delta) = 1, \quad \text{and} \quad \mathsf{L}^{-1}(1) = \delta(t). \tag{5.65}$$

By this definition, the following relation is immediately followed

$$\mathsf{L}(f(t)\delta(t-a)) = f(a)e^{-as}. \tag{5.66}$$

Problems

Problem 5.9. Find the transform of the following functions

i. $f(t) = e^t\, u(t-1)$
ii. $f(t) = t\, e^{-t}\, u(t-1)$
iii. $f(t) = u(t) - e^t u(t{-}1)$
iv. $f(t) = e^{\sin(\pi t)}\,\delta(t-2)$
v. $f(t) = (1 - u(t-2))\, u(t-1)$.

Problem 5.10. Compare the transformation of the function

$$f(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & \text{otherwise} \end{cases},$$

and the function

$$g(t) = \begin{cases} t & 0 < t < 1 \\ 0 & \text{otherwise} \end{cases},$$

and conclude $\hat{g}(s) = -\frac{d}{ds}\hat{f}(s)$.

Problem 5.11. Use the definition and derive the Laplace transform of the function

$$f(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}, g(t) = \begin{cases} 1 & 1 < t < 2 \\ 0 & \text{otherwise} \end{cases},$$

and conclude $\hat{g}(s) = e^{-s}\hat{f}(s)$.

Problem 5.12. Use the Laplace transform properties to calculate

i. $\mathsf{L}(t e^{at})$
ii. $\mathsf{L}(t \sin(\omega t))$

iii. $\mathsf{L}(t\cos(\omega t))$

iv. $\mathsf{L}(t\sinh(\omega t))$

v. $\mathsf{L}(t\cosh(\omega t))$

Problem 5.13. Find the transformation of the following functions

i. $\mathsf{L}(t^{3/2})$.

ii. $\mathsf{L}(t^{-1/2})$

iii. $\mathsf{L}(e^{-t}t^{-1/2})$

Problem 5.14. Use mathematical induction to prove the identities

$$\mathsf{L}(t^n f(t)) = (-1)^n \frac{d^n}{ds^n}\hat{f}(s),$$

and conclude

i. $\mathsf{L}(t^n) = \frac{n!}{s^{n+1}}$,

ii. $\mathsf{L}(t^n e^{-at}) = \frac{n!}{(s+a)^{n+1}}$.

Problem 5.15. Use partial fraction to find the inverse transformation of the following functions:

i. $\hat{f}(s) = \frac{s}{s^2-3s+2}$

ii. $\hat{f}(s) = \frac{1}{s(s^2+4)}$

iii. $\hat{f}(s) = \frac{s+2}{s^3-s^2+s-1}$.

Problem 5.16. Use the properties of LAPLACE transform to obtain the following integrals:

i. $\int_0^\infty \frac{1-e^{-t}}{t} e^{-t}\, dt$,

ii. $\int_0^\infty \frac{1-\cos(t)}{t} e^{-t}\, dt$,

iii. $\int_0^\infty \frac{\sin(t)}{t} e^{-t}\, dt$

iv. $\int_{-\infty}^\infty e^{-|t|} \frac{1-\cos(t)}{\ln(2)\,|t|}\, dt$

Problem 5.17. Find the inverse transformation of the following functions:

i. $\hat{f}(s) = \frac{s}{s^2-4s+5}$

ii. $\hat{f}(s) = \frac{s+3}{s^2+2s+5}$

iii. $\hat{f}(s) = \frac{s}{(s-1)^2+3}$

iv. $\hat{f}(s) = e^{-s}\log\left(\frac{s+1}{s}\right)$

v. $\hat{f}(s) = e^{-2s}\frac{s}{(s^2+3)^2}$,

vi. $\hat{f}(s) = e^{-s}\frac{s}{s^2+3s+2}$.

Problem 5.18. Find the inverse transformation of following functions:

i. $\hat{f}(s) = \frac{1}{1-e^{-s}}\frac{1}{s^2-1}$

ii. $\hat{f}(s) = \frac{1+e^{-s}}{1-e^{-2s}}\frac{1}{s^2+1}$.

Problem 5.19. If we define

$$g(t) = \sum_{k=0}^{\infty} (-1)^k f(t-ka)u(t-ka)$$

show that

$$\hat{g}(s) = \frac{1}{1+e^{-as}}\hat{f}(s).$$

Using the above argument calculate the following inverse transformation

$$\mathsf{L}^{-1}\left(\frac{1}{1+e^{-2s}}\frac{3!}{(s-3)^4}\right).$$

Problem 5.20. Use mathematical induction to prove

$$\mathsf{L}(f^{(n)}(t)) = s^n \hat{f}(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0).$$

Problem 5.21. Use LAPLACE transform of $t\sin(\omega t)$ and $t\cos(\omega t)$ to find the inverse transformation of the following functions:

i. $\hat{f}(s) = \frac{2}{(s^2+1)^2}$,

ii. $\hat{f}(s) = \frac{2}{((s-2)^2+1)^2}$

Problem 5.22. Find the transformation of the following periodic functions:

i. $f(t) = t, 0 < t < 1, \quad f(t+1) = f(t)$,

ii. $f(t) = \left\{\begin{smallmatrix} 1 & 0<t<1 \\ -1 & 1<t<2 \end{smallmatrix}\right., f(t+2) = f(t)$.

Problem 5.23. Show the following relation

$$\mathsf{L}^{-1}\left(\frac{\hat{f}(s)}{(s+a)^2}\right) = e^{-at}\int_0^t \left(\int_0^\tau e^{as} f(s)\, ds\right) d\tau.$$

Use integration by part formula and rewrite the right hand side of the above formula as

$$e^{-at}\int_0^t \left(\int_0^\tau e^{as} f(s)\, ds\right) d\tau = \int_0^t (t-\tau) e^{-a(t-\tau)} f(\tau)\, d\tau.$$

Problem 5.24. Even though the derivative of the unit step function $u(t-a)$ is not defined at $t = a$ (it is not even continuous at this point), show that $\delta(t-a)$ and $u(t-a)$ are related in the following way

$$\mathsf{L}(\delta(t-a)) = s\,\mathsf{L}(u(t-a)),$$

that is $\delta(t-a)$ can be considered as the *generalized derivative* of $u(t-a)$.

Problem 5.25. Consider the sequence of functions

$$f_n(t) = \begin{cases} nt & 0 \le t \le \frac{1}{n} \\ 1 & t \ge \frac{1}{n} \end{cases}.$$

i. Find $\hat{f}_n(s)$ and then obtain $\lim_{n\to\infty} \hat{f}_n(s)$.

ii. What is the limit function $\lim_{n\to\infty} f_n(t)$?

iii. Compare $\lim_{n\to\infty} \hat{f}_n(s)$ with the the L-transform of the limiting function you obtained in part (ii).

Problem 5.26. In the proposition (5.2) we proved that $\hat{f}(s)$ is continuous in its domain of definition.

i. For $f(t) = e^{-t}\sin(t)$ find $\int_0^\infty f(t) dt$.

ii. Find $\hat{f}(s)$ and let $\lim_{s\to 0}\hat{f}(s)$ and explain why it gives the same result in part (i).

iii. Now for $f(t)=\sin(t)$ we know $\hat{f}(s)=\frac{1}{s^2+1}$ and $\lim_{s\to 0}\hat{f}(s)=1$. Does it imply that $\int_0^\infty \sin(t)\,dt = 1$? why?

Problem 5.27. It is not always allowed to pass the limit inside the integral.

i. Let $f_n(t)=\frac{n}{1+n^2t^2}$. Find $\int_0^\infty f_n(t)$.

ii. Compare $\lim_{n\to\infty}\int_0^\infty f_n(t)dt$ and $\int_0^\infty \lim_{n\to\infty} f_n(t)dt$ and explain why they are not equal.

Problem 5.28. Let $f_n(t)=\sin(nt)$.

i. Find $\lim_{n\to\infty}\hat{f}_n(s)$.

ii. Can we pass the limit inside the integral of L-transform?

5.3 Initial value problems and L-transform

In this section, we solve initial value problems by the aid of the LAPLACE transform. As we will see, this method offers considerable advantage over the classical methods for solving initial value problem, specially when the forcing terms are discontinuous. Consider the following initial value problem

$$\begin{cases} y''+a\,y'+by=f(t) \\ y(0)=y_0,\, y'(0)=y_1 \end{cases}, \tag{5.67}$$

where a,b are constants. The L-transform of the equation transforms the equation into the following algebraic one

$$\mathsf{L}(y'')+a\,\mathsf{L}(y')+b\,\mathsf{L}(y)=\hat{f}(s). \tag{5.68}$$

Using the relations

$$\mathsf{L}(y'')=s^2\hat{y}(s)-s\,y_0-y_1, \text{ and } \mathsf{L}(y')=s\hat{y}(s)-y_0,$$

we obtain

$$s^2\,\hat{y}(s)-s\,y_0-y_1+a\,s\,\hat{y}(s)-a\,y_0+b\,\hat{y}(s)=\hat{f}(s), \tag{5.69}$$

and thus we reach

$$(s^2+a\,s+b)\,\hat{y}(s)=(s+a)y_0+y_1+\hat{f}(s). \tag{5.70}$$

Note that the coefficient of $\hat{y}(s)$ is the characteristic polynomial of the differential equation. Therefore,

$$\hat{y}(s)=\frac{\hat{f}(s)}{s^2+a\,s+b}+\frac{s+a}{s^2+a\,s+b}y_0+\frac{1}{s^2+a\,s+b}y_1, \tag{5.71}$$

and finally

$$y(t)=y_0\,\mathsf{L}^{-1}\left(\frac{s+a}{s^2+a\,s+b}\right)+y_1\,\mathsf{L}^{-1}\left(\frac{1}{s^2+a\,s+b}\right)+\mathsf{L}^{-1}\left(\frac{\hat{f}(s)}{s^2+a\,s+b}\right) \tag{5.72}$$

Example 5.10. Consider the following problem

$$\begin{cases} y''+y=u(t-2)-u(t-8) \\ y(0)=y'(0)=0 \end{cases}. \tag{5.73}$$

Here a pulse in the period $(2, 8)$ is applied to a harmonic oscillator. By the method of LAPLACE transform, the problem reduces to the following algebraic one

$$(s^2+1)\hat{y}(s)=\frac{e^{-2s}-e^{-8s}}{s},$$

and thus

$$y(t)=\mathsf{L}^{-1}\left(\frac{e^{-2s}-e^{-8s}}{s(s^2+1)}\right)=\mathsf{L}^{-1}\left(\frac{e^{-2s}}{s(s^2+1)}\right)-\mathsf{L}^{-1}\left(\frac{e^{-8s}}{s(s^2+1)}\right).$$

By the property

$$\mathsf{L}^{-1}(e^{-as}\hat{f}(s))=f(t-a)\,u(t-a),$$

we have

$$\mathsf{L}^{-1}\left(\frac{e^{-2s}}{s(s^2+1)}\right)=\mathsf{L}^{-1}\left(\frac{1}{s(s^2+1)}\right)|_{t=t-2}\,u(t-2).$$

Since

$$\mathsf{L}^{-1}\left(\frac{1}{s(s^2+1)}\right)=\int_0^t \mathsf{L}^{-1}\left(\frac{1}{s^2+1}\right)=\int_0^t \sin v\,dv=1-\cos t,$$

we conclude

$$y(t)=[1-\cos(t-2)]\,u(t-2)-[1-\cos(t-8)]\,u(t-8). \tag{5.74}$$

The figure (5.6) shows the graph of this solution.

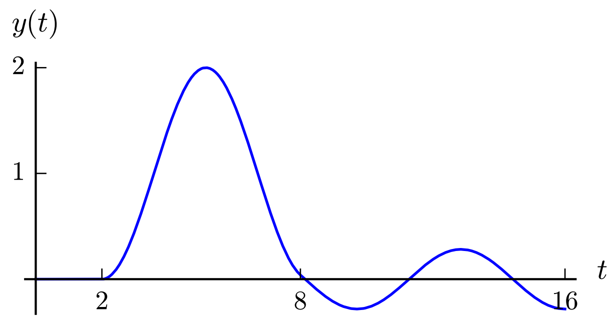


Figure 5.6. The graph of the solution $y(t)$.

Observe from the figure that $y(t)\equiv 0$ for $t<2$ that confirms our expectation because the system is at rest for $t<2$. The solution $y(t)$ is smooth of order 1 at $t=2$ since $y''(t)$ has a finite jump at that time. A similar argument holds at $t=8$ where $f(t)$ jumps from 1 to 0. By the classical method of previous chapters, to solve the problem, we had to split the problem into the three dub-domains $t<2$, $2<t<8$ and $t>8$ as follows. For $t<0$ the problem is

$$\begin{cases} y''+y=0, \\ y(0)=y'(0)=0 \end{cases}, \tag{5.75}$$

and thus the unique solution is $y(t)=0$. In the interval $(2,8)$, the problem reads

$$\begin{cases} y''+y=1, \\ y(2)=y'(2)=0 \end{cases}, \tag{5.76}$$

with the solution $y(t)=1-\cos(t-2)$. In the interval $(8,\infty)$, we have

$$\begin{cases} y''+y=0, \ \text{for } t>8 \\ y(8)=1-\cos(6), y'(8)=\sin(6) \end{cases}, \tag{5.77}$$

with the solution $y=\cos(t-2)-\cos(t-8)$. Note the initial conditions in the above problem.

Example 5.11. Let us solve the following problem

$$\begin{cases} y''+y=u(t-1) \\ y(1)=1, y'(1)=0 \end{cases}.$$

Here the initial data is given at $t=1$ instead of $t=0$. To solve the problem, we process as if conditions $y(0)=y_0$ and $y'(0)=y_1$ are known. Taking L-transform we obtain

$$\hat{y}(s)=\frac{e^{-s}}{s(s^2+1)}+y_0\frac{s}{s^2+1}+y_1\frac{1}{s^2+1}.$$

This implies

$$y(t)=(1-\cos(t-1))u(t-1)+y_0\cos(t)+y_1\sin(t).$$

Now, we apply the data $y(1)$ and $y'(1)$ and get $y_0=\cos(1)$ and $y_1=\sin(1)$ and thus

$$y(t)=(1-\cos(t-1))u(t-1)+\cos(t-1)=\begin{cases} \cos(t-1) & t\le 1 \\ 1 & t\ge 1 \end{cases}.$$

The graph of the solution for $t>1$ is shown in the figure (5.7).

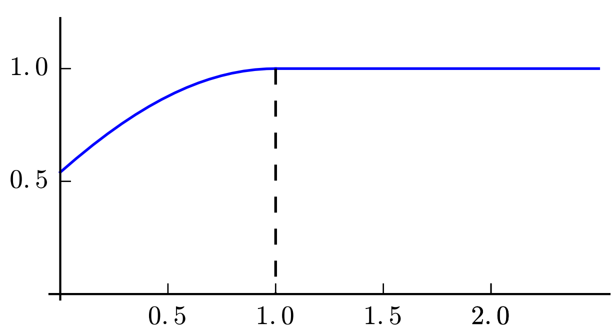


Figure 5.7.

Note that two branches of the solution for $t\le 1$ and for $t\ge 1$ connect together smoothly of first order. This is because y'' has a finite jump at $t=1$ due to the term $u(t-1)$ and therefore y' is continuous at this point. Note that the initial condition at $t=1$ determines $y(t)$ for $t<1$.

Example 5.12. Let $f(t)$ be the function

$$f(t) = u(t) - 2u\left(t - \frac{\pi}{2}\right) + u(t - \pi),$$

and let $g(t)$ be the π-periodic extension of f in $(0, \infty)$; see the figure (5.8).

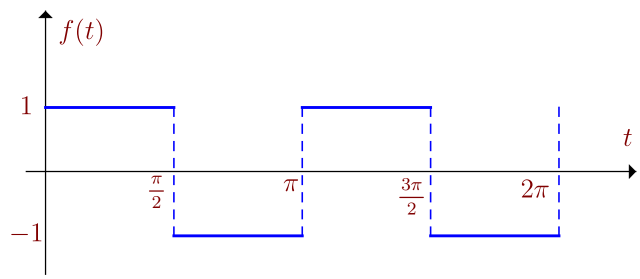


Figure 5.8.

Note that g has the representation

$$g(t) = \sum_{k=0}^{\infty} f(t - k\pi)\, u(t - k\pi).$$

Now consider the problem

$$\begin{cases} y'' + y = f(t) \\ y(0) = y'(0) = 0 \end{cases}. \tag{5.78}$$

We have

$$\mathsf{L}(f) = \sum_{k=0}^{\infty} \mathsf{L}(g(t - k\pi)u(t - k\pi)) = \hat{g}(s) \sum_{k=0}^{\infty} e^{-k\pi s} = \frac{1}{1 - e^{-\pi s}} \hat{g}(s).$$

By the L-transform method, we have

$$\hat{y}(s) = \frac{1}{1 - e^{-\pi s}} \frac{\hat{g}(s)}{s^2 + 1},$$

and thus

$$y(t) = \mathsf{L}^{-1}\left(\frac{1}{1 - e^{-\pi s}} \frac{\hat{g}(s)}{s^2 + 1}\right). \tag{5.79}$$

We use the formula (5.42) to write $y(t)$ as

$$y(t) = \sum_{k=0}^{\infty} \mathsf{L}^{-1}\left(\frac{\hat{g}(s)}{s^2 + 1}\right)\Big|_{t \to t - k\pi} u(t - k\pi). \tag{5.80}$$

But

$$\hat{g}(s) = \frac{1 - 2e^{-\pi s/2} + e^{-\pi s}}{s},$$

and therefore

$$\mathsf{L}^{-1}\left(\frac{\hat{g}(s)}{s^2 + 1}\right) = (1 - \cos(t))\, u(t) - 2\,(1 - \sin(t))\, u\left(t - \frac{\pi}{2}\right) + (1 + \cos(t))\, u(t - \pi).$$

The figure (5.9) shows the solution to the problem.

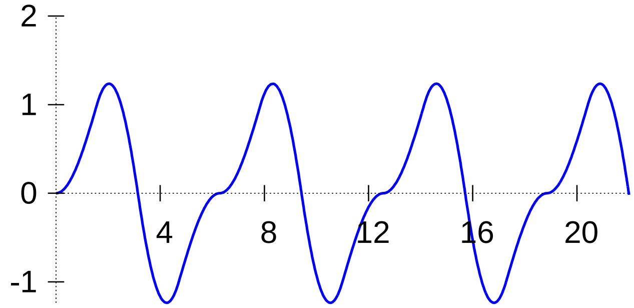


Figure 5.9. The graph of the solution $y(t)$.

Example 5.13. Consider the initial value problem

$$\begin{cases} y'' + y = e^{\sin(\log(t/\pi))}\,\delta(t-\pi) \\ y(0) = y'(0) = 0 \end{cases}. \tag{5.81}$$

The transform reduces the equation to the following algebraic one

$$(s^2+1)\hat{y}(s) = \mathsf{L}(e^{\sin(\log(t/\pi))}\,\delta(t-\pi)). \tag{5.82}$$

It is simply seen that

$$\mathsf{L}(e^{\sin(\log(t/\pi))}\,\delta(t-\pi)) = e^{\sin(\log(\pi/\pi))}e^{-\pi s} = e^{-\pi s}, \tag{5.83}$$

and thus $\hat{y}(s) = \frac{e^{-\pi s}}{s^2+1}$ which gives

$$y = \sin(t-\pi)\,u(t-\pi). \tag{5.84}$$

Observe that the solution starts at $t=\pi$, the time when the external source is applied.

Problems

Problem 5.29. Use the LAPLACE transform method to solve the following initial value problems

i.

$$\begin{cases} y' + y = \sin(t-1)\,u(t-1) \\ y(0) = 0. \end{cases}$$

ii.

$$\begin{cases} y'' + 4y' + 3y = e^t \\ y(1) = 0,\, y'(1) = 1 \end{cases}$$

iii.

$$\begin{cases} y'' + y = \sin(t) \\ y(0) = 0,\, y'(0) = 1 \end{cases}$$

iv.

$$\begin{cases} y'' + y = u(t-1) \\ y(0) = 0,\, y'(1) = 1 \end{cases}$$

v.

$$\begin{cases} y'' + 3y' + 2y = 2u(t-1) + e^t\delta(t-2) \\ y(0) = y'(0) = 0 \end{cases}$$

vi.

$$\begin{cases} y'' + y = u(t-1) - u(t-2), \\ y(0) = y(\frac{\pi}{2}), y'(0) = 1. \end{cases}$$

vii.

$$\begin{cases} y'' + y = \sin(t)\, u(t-\pi) \\ y(0) = y'(0) = 0 \end{cases},$$

viii.

$$\begin{cases} y''' - 3y'' + 3y' + y = e^t(1+t) \\ y(0) = y'(0) = y''(0) = 0 \end{cases}.$$

Problem 5.30. Let us solve the problem (5.11) as follows. Let $\tau = t - 1$ and defined $z(\tau) = y(t)$. Then we have

$$\begin{cases} z''(\tau) + z(\tau) = u(\tau) \\ z(0) = 1, z'(0) = 0 \end{cases}.$$

The L-transform of the problem gives

$$\hat{z}(s) = \frac{1}{s(s^2+1)} + \frac{s}{s^2+1} = \frac{1}{s}.$$

Therefore $z(\tau) = u(\tau)$ and $y(t) = u(t-1)$ (!) What is wrong here?

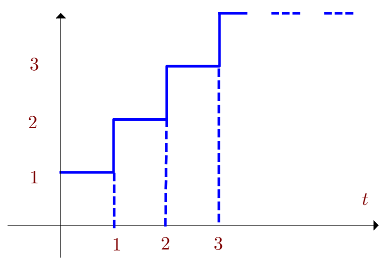
Problem 5.31. Solve the following equations and use a computer software to draw the solutions

i. $y'' + 2y' + y = f(t), y(0) = y'(0) = 0$ and $f(t)$ is a periodic function as follows

$$f(t) = u(t) - u(t-1),\ f(t+2) = f(t)$$

ii. $y'' + y = f(t),\ f(t) = t$ for $0 < t < 1$ and $f(t+1) = f(t)$

iii. $y'' + y = f(t)$ where $f(t)$ is given in the following figure



Problem 5.32. Assume that $s(t)$ is a continuous function such that

$$\int_{-\infty}^{\infty} s(t)\, dt = 1. \tag{5.85}$$

Prove that the sequence $s_n(t) = n\, s(n\, t)$ is a δ-sequence functions for the class of continuous and bounded functions $f(t)$. Use this property to conclude that

$$\int_{-\infty}^{\infty} \frac{n\cos(t)}{\pi(1+n^2 t^2)}\, dt \xrightarrow{n\to\infty} 1.$$

Problem 5.33. If $\mathsf{L}^{-1}(F(s)) = f(t)$, show that

$$\mathsf{L}^{-1}(e^{-2s} s F(s)) = f'(t-2) u(t-2) + f(0)\, \delta(t-2).$$

Problem 5.34. Solve the following problem and draw the solution $y(t)$ in the interval $[0, 10]$

$$\begin{cases} y'' + y = \sum_{k=1}^{\infty} \delta(t-k) \\ y(0) = y'(0) = 0 \end{cases}.$$

Problem 5.35. Solve the following problem and draw the solution $y(t)$ in the interval $[0,10]$

$$\begin{cases} y^{(4)}+2y''+y=\sum_{k=1}^{\infty}2^{-k}\delta(t-k) \\ y^{k}(0)=0, k=0,1,2,3 \end{cases}.$$

5.4 Impulse response and convolution

5.4.1 Convolution

Definition 5.3. *Assume that $f(t)$ and $g(t)$ are two integrable functions defined on $(-\infty,\infty)$. The convolution $f*g$ is defined by the following relation integral relation*

$$(f*g)(t)=\int_{-\infty}^{\infty}f(\tau)\,g(t-\tau)\,d\tau. \tag{5.86}$$

Although it looks odd at first glance, we will see the importance of convolution in real applications. Before that, let us solve a few examples.

Example 5.14. If $f(t)=\delta(t)$, then for arbitrary integrable function $g(t)$, we have

$$(\delta*g)(t)=\int_{-\infty}^{\infty}\delta(\tau)\,g(t-\tau)\,d\tau=g(t). \tag{5.87}$$

If $f(t)=u(t)$ then for arbitrary integrable function $g(t)$ we have

$$(u*g)(t)=\int_{-\infty}^{\infty}u(\tau)\,g(t-\tau)\,d\tau=\int_{0}^{\infty}g(t-\tau)\,d\tau=\int_{-\infty}^{t}g(\tau)\,d\tau, \tag{5.88}$$

and in particular, if $g(t)=0$ for $t<0$, then

$$(u*g)(t)=\int_0^t g(\tau)d\tau.$$

Assume $f(t)=e^{-t}u(t)$, then

$$(f*f)(t)=\int_{-\infty}^{\infty}e^{-\tau}u(\tau)\,e^{-(t-\tau)}\,u(t-\tau)\,d\tau=\int_0^t e^{-t}\,d\tau=t\,e^{-t}.$$

Remark 5.2. If f,g are zero for $t<0$ (as we usually assume for the LAPLACE transform) the integral reads

$$(f*g)(t)=\int_0^t f(\tau)\,g(t-\tau)\,d\tau. \tag{5.89}$$

Remark 5.3. It is simply verified that the convolution relation is commutative, that is, $f*g=g*f$

$$(f*g)(t)=\int_{-\infty}^{\infty}f(\tau)g(t-\tau)\,d\tau=\int_{-\infty}^{\infty}f(t-\tau)\,g(\tau)\,d\tau=(g*f)(t).$$

The equality is verified simply by the change of variable $v = t - \tau$.

Theorem 5.1. *For any admissible function $f(t)$ and $g(t)$ we have*

$$\mathsf{L}(f * g) = \hat{f}(s)\hat{g}(s). \tag{5.90}$$

Proof. By definition, we can write

$$\mathsf{L}(f * g) = \int_0^\infty \int_0^t e^{-st} f(\tau)\, g(t-\tau)\, d\tau dt. \tag{5.91}$$

Remember the following change of order of the integration from calculus

$$\int_0^b \int_0^t h(t,\tau)\, d\tau dt = \int_0^b \int_\tau^b h(t,\tau)\, dt d\tau. \tag{5.92}$$

The following figure justifies the above relation

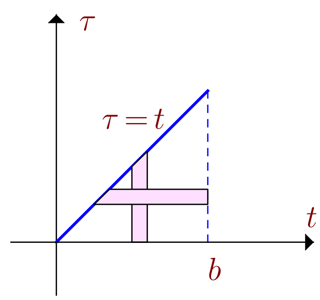


Figure 5.10.

Using the above formula for $b = \infty$, we can write

$$\int_0^\infty \int_0^t e^{-st} f(\tau)\, g(t-\tau)\, d\tau dt = \int_0^\infty \int_\tau^\infty e^{-st} f(\tau)\, g(t-\tau)\, dt d\tau =$$
$$= \int_0^\infty f(\tau) \left(\int_\tau^\infty e^{-st}\, g(t-\tau)\, dt \right) d\tau.$$

By the change of variable $t - \tau = v$, we reach

$$\int_0^\infty \int_0^t e^{-st} f(\tau)\, g(t-\tau)\, d\tau dt = \left(\int_0^{\infty b} e^{-s\tau} f(\tau)\, d\tau \right) \left(\int_0^\infty e^{-sv}\, g(v)\, dv \right) = \hat{f}(s)\, \hat{g}(s),$$

and this completes the proof. □

As it is seen, the Laplace transform, transforms the convolution relation to a simple algebraic multiplication. As we will see below, this relation offers significant a simplification in the calculations of *linear systems*. But before that, let us solve a few examples.

Example 5.15. Let us start with the following inverse transformation

$$\mathsf{L}^{-1}\left\{ \frac{1}{s}\, \hat{f}(s) \right\}.$$

As we know, it is equal to

$$\mathsf{L}^{-1}\left\{\frac{1}{s}\,\hat{f}(s)\right\}=\int_0^t f(t)\,d\tau.$$

By the convolution relation, we can write

$$\mathsf{L}^{-1}\left\{\frac{1}{s}\hat{f}(s)\right\}=\mathsf{L}^{-1}\left\{\frac{1}{s}\right\}*\mathsf{L}^{-1}\{\hat{f}(s)\}=u(t)*f(t)=\int_0^t f(\tau)u(t-\tau)\,d\tau=\int_0^t f(\tau)d\tau,$$

for $t>0$. Similarly, we can write

$$\mathsf{L}^{-1}\left\{\frac{1}{s+a}\,\hat{f}(s)\right\}=\mathsf{L}^{-1}\left\{\frac{1}{s+a}\right\}*\mathsf{L}^{-1}\{\hat{f}(s)\}=e^{-at}*f(t)=$$
$$\int_0^t f(\tau)\,e^{-a(t-\tau)}\,d\tau=e^{-at}\int_0^t f(\tau)\,e^{a\tau}\,d\tau.$$

The above relation is the solution of the familiar initial value problem

$$\begin{cases} y'+ay=f(t)\\ y(0)=0\end{cases},$$

which is by the LAPLACE transform is equivalent to

$$\hat{y}(s)=\frac{\hat{f}(s)}{s+a}.$$

Example 5.16. Let us solve the following integral equation

$$y-\int_0^t e^{t-\tau}y(\tau)\,d\tau=\delta(t-\pi). \tag{5.93}$$

Since the integral term in the equation is equal to e^t*y, the LAPLACE transform of the equation is

$$\hat{y}(s)-\mathsf{L}(e^t*y)=e^{-\pi s}, \tag{5.94}$$

and therefore

$$\hat{y}(s)-\frac{\hat{y}(s)}{s-1}=e^{-\pi s}. \tag{5.95}$$

The above equation is solved for $\hat{y}$ as

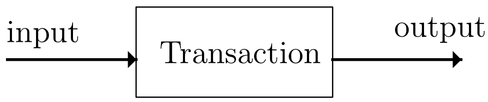
$$\hat{y}(s)=\frac{s-1}{s-2}e^{-\pi s}, \tag{5.96}$$

and finally

$$y(t)=\mathsf{L}^{-1}\left\{\frac{s-1}{s-2}e^{-\pi s}\right\}=\delta(t-\pi)+e^{2(t-\pi)}\,u(t-\pi). \tag{5.97}$$

5.4.2 System approach and transfer function

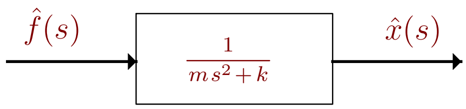
The word *system* is a prevalent term in the whole of applied sciences. Almost all physical systems, including man-machine and natural systems, are formulated in terms of three fundamental terms: *input*, *output*, and *transaction*.



For example, the movement of a mass connected to a spring of stiffness k under the influence of an external force $f(t)$, or the change of inflation rate in an economy, or the reaction of a body to some virus all of them can be interpreted in terms of the above simple block-diagram. The advantage of block-diagram representation is find out the existent *functional similarities* between seemingly different systems. For example, the dynamic of a mass-spring system is described by the following differential equation

$$m\frac{d^2x}{dt^2} + kx = f(t), \tag{5.98}$$

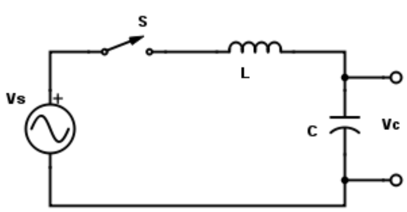
where k is the stiffness of the spring, m is the mass of the body, and $f(t)$ is the external force exerting to the mass. In the block-diagram representation, we can show the above system as follows



In the above block-diagram, the transaction is represented by the Laplace transform of the differential equation

$$\hat{x}(s) = \frac{1}{ms^2 + k}\hat{f}(s). \tag{5.99}$$

The expression $\hat{h}(s) = \frac{1}{ms^2+k}$ is also called the *transfer function* of the mass-spring system. Now, let us consider the following LC circuit and assume that the switch S connects at $t = 0$.



According to the voltage-current relationships of the capacitor C and the inductor L, we can write

$$LC\frac{d^2V_c}{dt^2} + V_c = v_s(t)\,u(t). \tag{5.100}$$

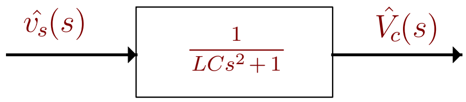
Assuming that the system is at rest, the Laplace transform of the equation is

$$(LCs^2 + 1)\hat{V_c}(s) = \hat{v}_s(s), \tag{5.101}$$

or equivalently

$$\hat{V_c}(s) = \frac{1}{LCs^2 + 1}\,\hat{v}_s(s), \tag{5.102}$$

which has the following block-diagram.



As it is seen, two systems have the completely similar transfer function and thus they behave completely similar from the functional point of view.

Note that if the input in the above examples is $\delta(t)$, then the LAPLACE transform of the output will be $\hat{h}(s)$. In fact, for the mass-spring functional equation (5.99), if $f(t) = \delta(t)$, then $\hat{f}(s) = 1$ and thus $\hat{x}(s) = \frac{1}{ms^2+k} := \hat{h}(s)$.

Definition 5.4. *The response of a system to the input $\delta(t)$ is called the* impulse response *of the system and is usually denoted by $h(t)$. The* LAPLACE *transform of the impulse response is called the* transfer function *of a system and is denoted by $\hat{h}(s)$.*

For example, for the *impulse response* of the mass-spring system is

$$h(t) = \mathsf{L}^{-1}(\hat{h}(s)) = \sqrt{\frac{1}{km}}\sin\left(\sqrt{\frac{k}{m}}\,t\right)u(t), \tag{5.103}$$

and the impulse response of the above LC circuit is

$$h(t) = \frac{1}{\sqrt{LC}}\sin\left(\frac{t}{\sqrt{LC}}\right)u(t). \tag{5.104}$$

Example 5.17. Consider the following initial value problem

$$\begin{cases} y' + y = \delta(t-\tau), \tau > 0 \\ y(0) = 0 \end{cases}.$$

By LAPLACE transform method, the response of the system to the input $\delta(t-\tau)$ is

$$h(t-\tau) = e^{-(t-\tau)}u(t-\tau).$$

Note that since $y'(t)$ is a DIRAC singularity at $t = \tau$, the response $y(t)$ is of a finite jump at that time. For $\tau \to 0$, we take

$$h(t) = \begin{cases} e^{-t} & t > 0 \\ 0 & t \le 0 \end{cases}.$$

On the other hand, the response of the system to input $r(t)$ is

$$\hat{y}(s) = \frac{\hat{r}(s)}{s+1},$$

and hence,

$$y(t) = h(t) * r(t) = e^{-t}\int_0^t e^{\tau} r(\tau)\,d\tau.$$

5.4.3 LTI systems

General systems are usually classified into the following classes:

a) Causal system: The output y at time t depends only on the input $x(\tau)$ for $\tau \le t$ and is independent of $\tau > t$.

b) Memory-less system: The output y at time t depends only on the input x at time t, not on the previous or future values of t.

c) Time Invariant: The output y is not sensitive to the time of applying input x. In other word, if $y(t)$ is the response to the input $x(t)$, then $y(t-\tau)$ will be the response of $x(t-\tau)$.

d) Linear system: If y_1, y_2 are respectively the response to inputs x_1, x_2, then the response of the system to $\alpha x_1 + \beta x_2$ for arbitrary constants α, β is $\alpha y_1 + \beta y_2$.

Example 5.18. Suppose that the transaction T is a derivative operator, that is,

$$y(t) = T\{x(t)\} = x'(t).$$

The system is not causal and memory-less because of the definition of the derivative:

$$x'(t) = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}.$$

The system is linear due to the relation $T\{a x_1 + b x_2\} = a x_1' + b x_2' = a y_1 + b y_2$. In addition, this system is time invariant $y(t - t_0) = x'(t - t_0)$. Similarly, an integrator

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau,$$

is a linear, time invariant and causal but not memory-less.

Definition 5.5. *Consider a system (S) with input $x(t)$ and output $y(t)$. The system is called linear if it satisfies the following two conditions:*

1. *for arbitrary $\lambda \in \mathbb{R}$, the response to input $\lambda x(t)$ is $\lambda y(t)$, (*homogeneity property*)*

2. *if y_1, y_2 are responses to inputs x_1 and x_2 respectively, the response to the input $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$ (*additive property*)*

This concept is shown in the figure (5.11) schematically.

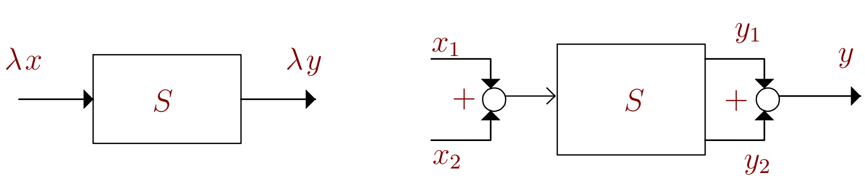


Figure 5.11. A linear system.

Definition 5.6. *A system (S) with input $x(t)$ and output $y(t)$ is called Time Invariant (TI) if response to $x(t-\tau)$ is $y(t-\tau)$ for arbitrary $\tau \geq 0$.*

Example 5.19. Consider the mass-spring system (5.99). It is almost straightforward to verify that the system is linear. In fact, if $f(t)$ is replaced by $\lambda f(t)$ (for arbitrary constant λ), the response changes to $\lambda x(t)$. Furthermore, if $f(t)$ is replaced by $f_1(t) + f_2(t)$, then by superposition principle, we can write

$$x(t) = x_1(t) + x_2(t). \tag{5.105}$$

In addition, if $f(t)u(t)$ is replaced by $f(t-\tau)u(t-\tau)$ for some $\tau \geq 0$, then the response of the system changes to

$$\mathsf{L}^{-1}\left(\frac{e^{-s\tau}\hat{f}}{ms^2+k}\right) = \mathsf{L}^{-1}\left(\frac{\hat{f}}{ms^2+k}\right)|_{t\to t-\tau}u(t-\tau) = x(t-\tau)\,u(t-\tau). \tag{5.106}$$

Therefore, the mass spring system is an LTI system. The reader can check that the LC circuit in the previous example is LTI as well.

Example 5.20. Consider the following initial value problem

$$\begin{cases} y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = x(t) \\ y^{(k)}(0) = 0, \text{for } k = 0, \ldots, n-1 \end{cases}, \tag{5.107}$$

where a_k are some constants. We write the equation as $L[y] = x(t)$, for L the operator

$$L := D^n + a_1 D^{n-1} + \cdots + a_n. \tag{5.108}$$

It is simply verified that the equations represents an LTI system with $x(t)$, $y(t)$ as its input and output respectively. In fact, for arbitrary $\lambda_1, \lambda_2 \in \mathbb{R}$, the solution to the input $x(t) = \lambda_1 x_1(t) + \lambda_2 x_2(t)$, is $y(t) = \lambda_1 y_2(t) + \lambda_2 y_2(t)$, where $L[y_1] = x_1(t)$ and $L[y_2] = x_2(t)$. Furthermore, according to the relation

$$y^{(n)}(t-\tau) + a_1\, y^{(n-1)}(t-\tau) + \cdots + a_n\, y(t-\tau) = x(t-\tau), \tag{5.109}$$

we have $L[y](t-\tau) = x(t-\tau)$. The transfer function of this system is:

$$\hat{h}(s) = \frac{1}{s^n + a_1 s^{n-1} + \cdots + a_n}. \tag{5.110}$$

In this section, we study *causal linear time invariant* (LTI) systems described by linear differential equations with constant coefficients

$$\begin{cases} y'' + ay + by = r(t) \\ y(t_0) = 0, y'(t_0) = 0 \end{cases}. \tag{5.111}$$

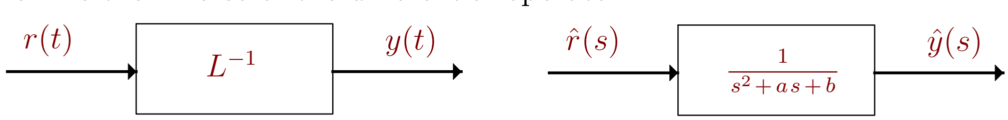
Or in the operator form as

$$L[y](t) = r(t),$$

where L stands for the differential operator $\frac{d^2}{dt^2} + a\frac{d}{dt} + b$. Accordingly, we can write

$$y(t) = L^{-1}[r(t)],$$

where L is the inverse of the differential operator L.



5.4.4 Convolution and LTI control systems

The importance of convolution lies in the following theorem.

Theorem 5.2. *Assume that $h(t)$ is the impulse response of an LTI system. Then the response of the system to an arbitrary input $x(t)$ is determined by the following convolution*

$$y(t) = \int_0^t h(t-\tau)\, x(\tau)\, d\tau. \tag{5.112}$$

Proof. Note that we can write $x(t)$ as the convolution as

$$x(t) = \int_0^t x(\tau)\, \delta(t-\tau)\, d\tau. \tag{5.113}$$

Let us write the above integral as the following Riemann sum

$$\int_0^t x(\tau)\, \delta(t-\tau)\, d\tau = \lim_{n\to\infty} \sum_{k=1}^{n} x(\tau_k)\, \delta(t-\tau_k)\, \Delta\tau_k, \tag{5.114}$$

and define $x_n(t)$ by the following partial sum

$$x_n(t) = \sum_{k=1}^{n} x(\tau_k)\delta(t-\tau_k)\, \Delta\tau_k. \tag{5.115}$$

Since the system is linear time-invariant, $y_n(t)$ is determined by the following summation

$$y_n(t) = \sum_{k=1}^{n} x(\tau_k)\, h(t-\tau_k)\, \Delta\tau_k, \tag{5.116}$$

where $h(t-\tau_k)$ is the response to the impulse $\delta(t-\tau_k)$. Now, we can write

$$y(t) = \lim_{n\to\infty} y_n(t) = \lim_{n\to\infty} \sum_{k=1}^{n} x(\tau_k)\, h(t-\tau_k)\, \Delta\tau_k = \int_0^t x(\tau)\, h(t-\tau)\, d\tau, \tag{5.117}$$

and this completes the proof. □

Example 5.21. For a LTI system, we know that the impulse response function $h(t)$ is

$$h(t) = e^{-t} \sin(t)\, u(t).$$

We would like to find the response of the system to the input function $r(t) = e^{-t}\, u(t)$. According to the above theorem, we can write

$$y(t) = r(t) * h(t) = \int_0^t e^{-(t-\tau)}\, e^{-\tau} \sin(\tau)\, d\tau = e^{-t}(1-\cos(t))\, u(t).$$

Problem 5.36. Following problems:

a) If we know that the impulse response of an LTI system is $\sin(t)$, find the response to the input $x(t) = \cos(t)$.

b) If we know that the response of an LTI system to input $u(t)$ is $\sin(t)$, find the response to the input $x(t) = \cos(t)$.

Working with convolution integrals is not always as simple as above, specially for *control systems*. For example consider a simple feedback control system shown below.

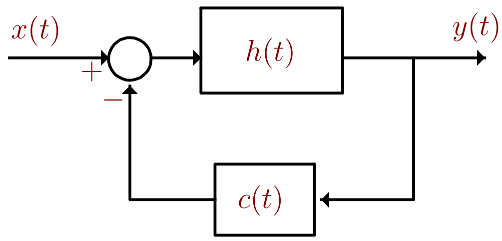


Figure 5.12.

By a a straightforward calculation, we can write $y(t)$ as the following convolution:

$$y(t) = (x(t) - c(t) * y(t)) * h(t), \tag{5.118}$$

or equivalently in the integral form as

$$y(t) = \int_0^t \left[x(\tau) - \int_0^\tau c(\tau - v) y(v)\, dv \right] h(t - \tau)\, d\tau.$$

As it is seen, the formula looks very complicated even for such a simple feedback system. Here we see how the LAPLACE transform simplifies the calculations. Remember the relation

$$\mathsf{L}\{f * g\} = \hat{f}(s)\, \hat{g}(s),$$

and thus for (5.118), we can write

$$\hat{y}(s) = \hat{h}(s)[\hat{x}(s) - \hat{c}(s)\, \hat{y}(s)],$$

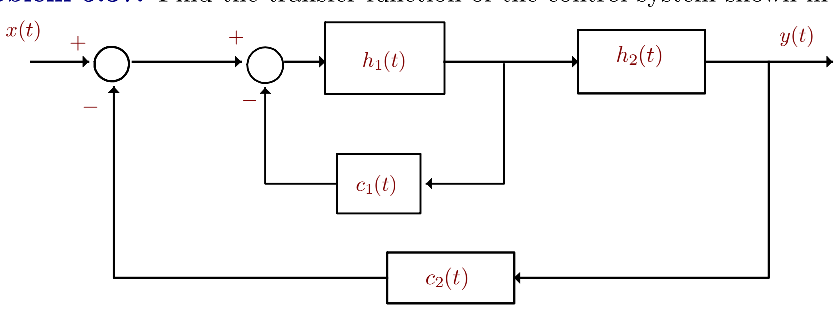
which is solved simply for $\hat{y}(s)$ as

$$\hat{y}(s) = \frac{\hat{h}(s)}{1 + \hat{c}(s)\, \hat{h}(s)} \hat{x}(s).$$

Therefore, the transfer function of the above control system is

$$\hat{h}_c(s) = \frac{\hat{h}(s)}{1 + \hat{c}(s)\, \hat{h}(s)}.$$

Problem 5.37. Find the transfer function of the control system shown in the following diagram



Problems

Problem 5.38. Show that an integrator

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau,$$

is time invariant system.

Problem 5.39. Consider the *discrete system* $y[n] = x[2n]$ for $n \in \mathbb{Z}$. Give and example that shows the system is not time invariant.

Problem 5.40. Consider the equation

$$y' + ay = r(t),$$

where $r(t) = 0$ for $t < 0$. Show that the transaction L^{-1} defined by

$$L^{-1}[r](t) = \int_{0}^{t} e^{-a(t-\tau)} r(\tau) d\tau,$$

is a system representation of the given equation. Show that the system L^{-1} is linear time invariant.

Problem 5.41. Consider a system that its response is described by the following differential equation

$$y' + a\,y = r(t).$$

i. Find the impulse response of the system.

ii. Write the solution in the convolution form.

The solution is the same as we obtained through solving a linear first order equation in previous chapters.

Problem 5.42. Consider a system that its response is described by the following differential equation

$$y'' + y = r(t).$$

i. Find the impulse response of the system.

ii. Write the solution in the convolution form.

iii. Verify that the obtained formula is a solution to the problem

$$\begin{cases} y'' + y = r(t) \\ y(0) = y'(0) = 0 \end{cases}.$$

Problem 5.43. Consider the following initial value problem

$$\begin{cases} y'' + a_1 y' + a_2 y = r(t) \\ y(0) = y'(0) = 0 \end{cases}.$$

i. If the characteristic polynomial of the equation has two distinct roots $\lambda_1 \neq \lambda_2$, use convolution theorem to show that the solution to the equation is

$$\phi(t) = \frac{1}{\lambda_1 - \lambda_2} (e^{\lambda_1 t} * r(t) - e^{\lambda_2 t} * r(t)).$$

ii. If the characteristic polynomial has a repeated root λ, use convolution theorem to show that

$$\phi(t) = t e^{\lambda t} * r(t).$$

Problem 5.44. For the RC circuit shown in the figure (5.13), find the impulse response $h(t)$ and the transfer function $\hat{h}(s)$ if $v_o(t)$ is considered as the response of the system. Find $v_o(t)$ if $v(t) = 1$ and the switch S connects at time $t = 1$.

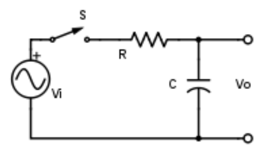


Figure 5.13.

Problem 5.45. Consider the circuit shown in the figure (5.14) where $v_i(t)$ and $v_o(t)$ are the input and output of the system respectively. Find the transfer function and the impulse response of the system. If $R=L=C=1$, find the response $v_o(t)$ provided that $v_i(t)=\sin(t)$.

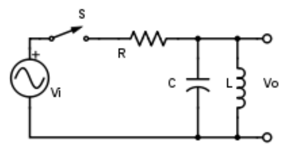


Figure 5.14.

Problem 5.46. Use the definition of convolution to calculate the convolution of following function. Use LAPLACE transform method for convolution and compare the results.

i. $f(t)=tu(t),\ g(t)=u(t)$

ii. $f(t)=e^{-t},\ g(t)=u(t)-u(t-1)$

iii. $f(t)=u(t),\ g(t)=t[u(t)-u(t-1)]$

iv. $f(t)=u(t)-u(t-1),\ g(t)=u(t-1)-u(t-2)$

Problem 5.47. Use convolution property to calculate the inverse transformation of the following function

i.
$$\mathsf{L}^{-1}\left(\frac{1}{s(s^2+1)}\right)$$

ii.
$$\mathsf{L}^{-1}\left(\frac{1}{(s^2+1)^2}\right)$$

iii.
$$\mathsf{L}^{-1}\left(\frac{e^{-s}}{(s-1)(s^2+2s+2)}\right)$$

Problem 5.48. Show that convolution is commutative, that is, $f*g=g*f$.

Problem 5.49. Us LAPLACE transform method to solve the following integro-differential equations. The initial condition(s) for all equations is assumed zero.

i. $y+3\int_0^t y(v)\sin(t-v)\,dv=u(t-1),$

ii. $y'+e^t\int_0^t e^{-\tau}y(\tau)\,d\tau=\cos(t)\,\delta(t-\pi)$

iii. $y+\int_0^t y'(\tau)\,e^{t-\tau}d\tau=u(t-1).$

iv. $y''+\int_0^t y(v)\,u(t-v)\,dv=\delta(t-1)+u(t-1)$

5.5 Systems of differential equations

In this section we see how the method is employed to solve the linear systems of differential equations with constant coefficients. Consider the following system

$$\begin{cases} y_1' = a_{11}\, y_1 + a_{12}\, y_2 + b_1(t) \\ y_2' = a_{21}\, y_1 + a_{22}\, y_2 + b_2(t) \\ y_1(0) = y_2(0) = 0 \end{cases}. \tag{5.119}$$

The Laplace transform of the system reduces it to the following algebraic system

$$\begin{cases} (s - a_{11})\, \hat{y}_1(s) - a_{12}\hat{y}_2(s) = \hat{b}_1(s) \\ (s - a_{22})\, \hat{y}_2(s) - a_{21}\hat{y}_1(s) = \hat{b}_2(s) \end{cases}, \tag{5.120}$$

that can be put in the following matrix form in turn

$$\begin{pmatrix} s - a_{11} & -a_{12} \\ -a_{21} & s - a_{22} \end{pmatrix} \begin{pmatrix} \hat{y}_1(s) \\ \hat{y}_2(s) \end{pmatrix} = \begin{pmatrix} \hat{b}_1(s) \\ \hat{b}_2(s) \end{pmatrix}. \tag{5.121}$$

The system (5.121) is solvable if the coefficient matrix is invertible. If so, we can write

$$\begin{pmatrix} \hat{y}_1(s) \\ \hat{y}_2(s) \end{pmatrix} = \frac{1}{p(s)} \begin{pmatrix} s - a_{22} & a_{12} \\ a_{21} & s - a_{11} \end{pmatrix} \begin{pmatrix} \hat{b}_1(s) \\ \hat{b}_2(s) \end{pmatrix}, \tag{5.122}$$

where $p(s)$ is the characteristic polynomial of the matrix

$$p(s) = s^2 - (a_{11} + a_{22})s + a_{11}a_{22} - a_{12}a_{21}. \tag{5.123}$$

Now, $y_1(t)$ and $y_2(t)$ can be restored by the inverse transform of $\hat{y}_1(s)$ and $\hat{y}_2(s)$ respectively.

Example 5.22. Consider the following system

$$\begin{cases} y_1' = y_1 - y_2 \\ y_2' = y_1 + \delta(t-1) \\ y_1(0) = y_2(0) = 0 \end{cases}. \tag{5.124}$$

By Laplace transform, we write the above system as the following algebraic one

$$\begin{cases} (s-1)\, \hat{y}_1(s) + \hat{y}_2(s) = 0 \\ s\hat{y}_2(s) - \hat{y}_1(s) = e^{-s} \end{cases},$$

or in the matrix form

$$\begin{pmatrix} s-1 & 1 \\ -1 & s \end{pmatrix} \begin{pmatrix} \hat{y}_1(s) \\ \hat{y}_2(s) \end{pmatrix} = \begin{pmatrix} 0 \\ e^{-s} \end{pmatrix}.$$

The above system is solved for the vector $(\hat{y}_1, \hat{y}_2)$ by the formula

$$\begin{pmatrix} \hat{y}_1(s) \\ \hat{y}_2(s) \end{pmatrix} = \frac{1}{s^2 - s + 1} \begin{pmatrix} s & -1 \\ 1 & s-1 \end{pmatrix} \begin{pmatrix} 0 \\ e^{-s} \end{pmatrix} = \frac{1}{s^2 - s + 1} \begin{pmatrix} -e^{-s} \\ (s-1)\, e^{-s} \end{pmatrix}.$$

Thus $\hat{y}_1(s) = \frac{-e^{-s}}{s^2 - s + 1}$ and therefore

$$y_1(t) = \frac{-2}{\sqrt{3}}\, e^{\frac{t-1}{2}} \sin\!\left(\frac{\sqrt{3}}{2}(t-1)\right) u(t-1).$$

Likewise, we have $\hat{y}_2(s) = \frac{(s-1)e^{-s}}{s^2-s+1}$ and therefore

$$y_2(t) = e^{(t-1)/2}\cos\left(\frac{\sqrt{3}}{2}(t-1)\right)u(t-1) - \frac{1}{\sqrt{3}}e^{\frac{t-1}{2}}\sin\left(\frac{\sqrt{3}}{2}(t-1)\right)u(t-1).$$

The method is applied to higher order systems as well. The following example presents the method for a second order system.

Example 5.23. Consider the following system

$$\begin{cases} y_1'' = 2y_2 + u(t) \\ y_2'' = 8y_1 \\ y_1(0) = y_2(0) = y_1'(0) = y_2'(0) = 0 \end{cases}.$$

The system reduces to the algebraic one

$$\begin{cases} s^2\hat{y}_1(s) = 2\hat{y}_2(s) + \frac{1}{s} \\ s^2\,\hat{y}_2(s) = 8\hat{y}_1(s) \end{cases}.$$

Solving the system for $\hat{y}_1(s)$ and $\hat{y}_2(s)$ gives

$$\hat{y}_1(s) = \frac{s}{s^4-16}, \quad \hat{y}_2(s) = \frac{8}{s(s^4-16)}.$$

The inverse transform yields $y_1(t)$ and $y_2(t)$ as follows

$$y_1(t) = \frac{1}{8}\cosh(2t) - \frac{1}{8}\cos(2t) \quad \text{and} \quad y_2(t) = \frac{1}{4}\cosh(2t) + \frac{1}{4}\cos(2t) - \frac{1}{2}.$$

Problems

In the following problems, use LAPLACE transform method to solve the following systems.

Problem 5.50. $\begin{cases} y_1' = y_2 + \delta(t-1) \\ y_2' = -y_1 + u(t) \end{cases}, \quad y_1(0) = y_2(0) = 0$

Problem 5.51. $\begin{cases} y_1' = -y_1 - y_2 + \delta(t-1) \\ y_2' = -2y_1 \end{cases}, y_1(0) = 0, y_2(0) = 1$

Problem 5.52. $\begin{cases} y_1'' = 2y_1 + y_2 \\ y_2'' = 12y_1 - 2y_2 \end{cases}, \quad y_1(0) = 1, y_2(0) = y_1'(0) = y_2'(0) = 0$

Problem 5.53. $\begin{cases} y_1'' = 2y_2 - y_1' + y_2' \\ y_1' - y_2' = -y_1 \end{cases}, \quad y_1(0) = y_2(0) = 1, y_1'(0) = y_2'(0) = 0$

Problem 5.54. $\begin{cases} y_1'' + 3y_2'' = y_1 + \delta(t-1) \\ y_1' + 3y_2' = 2y_2 \end{cases}, y_1(0) = y_2(0) = y_1'(0) = y_2'(0) = 0$

Problem 5.55. Use the LAPLACE transform method to prove that the following two systems have the same solution

1. $\vec{y}' = A_{2\times 2}\,\vec{y},\ \ \vec{y}(0) = \vec{y}_0$
2. $\vec{y}' = A_{2\times 2}\,\vec{y} + \delta(t)\vec{y}_0,\ \ \vec{y}(0) = 0$.

Here $\vec{y}$ stands for the vector $\left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right)$ and the matrix $A_{2\times 2}$ is a constant matrix

$$A_{2\times 2} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$