Chapter 4
Series Solution of Linear Equations

The solution of a linear equation with *analytic* coefficients is analytic. That is a deep result in the theory of linear differential equations. Although there is non guarantee that the solution of a *singular* differential equation is expanded as a *power series*, the FROBENIUS method provided us with conditions under which such equations possess solutions in the series form. In the last part of this chapter, we introduce some important equations of mathematical physics.

4.1 Introduction

4.1.1 Power series

The reader is referred to the appendix of this book for a detailed discussion on the topic. Here, we assume that the reader is familiar with the numeric series

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots,$$

and the notion of convergence and divergence of numeric series. Remember also the *ratio test* for the convergence of a numeric series. If

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1,$$

then the series convergence *absolutely*, that is, $\sum_{n=0}^{\infty} |a_n| = \bar{a} < \infty$. If

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1,$$

the series diverges. The general form of a *power series* centered at a point x_0 is

$$\sum_{n=0}^{\infty} c_n (x-x_0)^n = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \cdots. \tag{4.1}$$

Definition 4.1. *The series* $\sum_{n=1}^{\infty} c_n(x-x_0)^n$ *is convergent at a point* $x=a$ *if numeric series* $\sum_{n=1}^{\infty} c_n(a-x_0)^n$ *is convergent.*

Example 4.1. Consider the following series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots.$$

Here $c_n = 1$, and $x_0 = 0$. Note that the series is a *geometric series* for any $|x| < 1$, and thus the series converges to $\frac{1}{1-x}$.

Let us use the ratio test to determine the values of x for which power series (4.1) converges. According to the test, the series is absolutely convergent if

$$\lim_{k\to\infty}\left|\frac{c_{k+1}(x-x_0)^{k+1}}{c_k(x-x_0)^k}\right| < 1,$$

that implies

$$|x-x_0| \lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right| < 1,$$

or equivalently

$$|x-x_0| < \lim_{k\to\infty}\left|\frac{c_k}{c_{k+1}}\right|.$$

The above inequality gives a range for x for which the series converges *absolutely*, that is,

$$\sum_{n=0}^{\infty} |c_n(x-x_0)|^n,$$

convergent. The value $L := \lim_{k\to\infty}\left|\frac{c_k}{c_{k+1}}\right|$ is called the *radius of convergence* of the series, and thus the series converges for all x in $(x_0 - L,\ x_0 + L)$. This interval is called the *domain of convergence* of the series.

Example 4.2. Consider series $1 + x + x^2 + \cdots$. The test implies

$$L := \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = 1$$

and thus the series converges for all x in interval $(-1, 1)$. The series represents the function $f(x) = \frac{1}{1-x}$ expanded at $x_0 = 0$. The radius of convergence of series $\sum_{n=0}^{\infty} \frac{1}{n!}x^n$ is

$$L = \lim_{n\to\infty}\left|\frac{\frac{1}{n!}}{\frac{1}{(n+1)!}}\right| = \lim_{n\to\infty}(n+1) = \infty,$$

and thus the series converges for all $x \in (-\infty,\ \infty)$. The series represents function e^x expanded at $x_0 = 0$.

4.1.2 Analytic functions

Definition 4.2. *A real valued function f is called* analytic *at x_0 if there is an open interval $I=(x_0-L,x_0+L)$, for some $L>0$, and constants c_k such that*

$$f(x)=c_0+c_1(x-x_0)+c_2(x-x_0)^2+\cdots, \tag{4.2}$$

for all $x\in I$.

The definition states that for any $a\in I$, the following convergence holds

$$\lim_{n\to\infty}[c_0+c_1(a-x_0)+\cdots+c_n(a-x_0)^n]=f(a).$$

Problem 4.1. If a function f is analytic at a point x_0 with the domain of convergence I, show that f is analytic at all points $x\in I$.

Hint: Without loss of generality assume $x_0=0$ and for $a\in I$ write

$$f(x)=\sum_{n=0}^{\infty}c_n(x-a+a)^n.$$

Use binomial formula and write

$$f(x)=\sum_{k=0}^{\infty}d_k(x-a)^k,$$

where

$$d_k=\sum_{n=k}^{\infty}c_n\binom{n}{k}a^{n-k}.$$

Show $|d_k|<\infty$ for all k.

Theorem 4.1. *Assume that f is analytic at x_0 and*

$$f(x)=\sum_{n=0}^{\infty}c_n(x-x_0)^n.$$

Then f and f' are continuous at x_0 and furthermore

$$f'(x)=\sum_{n=1}^{\infty}nc_n(x-x_0)^{n-1}.$$

Proof. Without loss of generality, let us assume $x_0=0$. Consider the series

$$f(x)=\sum_{n=0}^{\infty}c_nx^n, \tag{4.3}$$

and assume it converges in I. Choose δx sufficiently small such that $x+\delta x\in I$. Therefore, the series

$$\sum_{n=0}^{\infty}c_n(x+\delta x)^n, \tag{4.4}$$

converges to $f(x+\delta x)$ and we have

$$f(x+\delta x)-f(x)=\sum_{n=1}^{\infty}c_n[(x+\delta x)^n-x^n]. \tag{4.5}$$

Use the intermediate value theorem to write

$$(x+\delta x)^n - x^n = n(x+\xi_n)^{n-1}\,\delta x, \tag{4.6}$$

for some ξ_n in the segment $<x, x+\delta x>$. Thus

$$|f(x+\delta x) - f(x)| \le \delta x \sum_{n=1}^{\infty} n\,|c_n|\,|x+\delta x|^{n-1}.$$

Note that the series in the right hand side is convergent according to the ratio test. Therefore, we have

$$\lim_{\delta x\to 0} f(x+\delta x) - f(x) = 0,$$

and this proves the continuity of f. Now, define function g as

$$g(x) = \sum_{n=1}^{\infty} n\,c_n\,x^{n-1}. \tag{4.7}$$

Clearly g is well defined because the series in the right hand side is convergent. We have

$$\left|\frac{f(x+\delta x) - f(x)}{\delta x} - g(x)\right| \le \sum_{n=2}^{\infty} n\,|c_n|\,|(x+\xi_n)^{n-1} - x^{n-1}|. \tag{4.8}$$

Use the intermediate value theorem and write

$$|(x+\xi_n)^{n-1} - x^{n-1}| \le (n-1)\,|x+\delta x|^{n-2}\,|\delta x|. \tag{4.9}$$

Therefore

$$\left|\frac{f(x+\delta x) - f(x)}{\delta x} - g(x)\right| \le |\delta x| \sum_{n=2}^{\infty} (n-1)\,n\,|c_n|\,|x+\delta x|^{n-2}.$$

Since the series in the right hand side converges, we have

$$\left|\frac{f(x+\delta x) - f(x)}{\delta x} - g(x)\right| \xrightarrow{\delta x\to 0} 0. \tag{4.10}$$

Therefore, $g(x)$ is the derivative of $f(x)$. The continuity of g is proved by a similar argument. □

Corollary 4.1. *If a function f is analytic, then it is differentiable of any order and all $f^{(n)}$ are analytic with the same domain of convergence.*

Example 4.3. Function $f(x) = x|x|$ is not analytic at $x_0 = 0$. In fact, $f''(0)$ does not exists and then f_2 can not be determined. Note that every analytic function is continuously differentiable of any order. These functions are generally called C^∞ functions. If f is analytic, it is C^∞, however, every C^∞ function is not analytic. For example, the function $f(x) = e^{-1/x^2}$ is C^∞ in any open interval around $x_0 = 0$ but it is not analytic at this point. In fact, if we write

$$e^{-1/x^2} = c_0 + c_1\,x + c_2\,x^2 + \cdots, \tag{4.11}$$

then $c_0 = e^{-1/x^2}|_{x=0} = 0$, and

$$c_1 = \frac{d}{dx} e^{-1/x^2}\bigg|_{x=0} = 0,$$

and similarly we obtain all $c_n = 0$. Thus, e^{-1/x^2} is not analytic at $x_0 = 0$ even though it is C^∞ at $x_0 = 0$.

We use the following proposition in our subsequent discussions. The proof is straightforward and the reader is asked for prove it.

Proposition 4.1. *If f and g are analytic functions in an open interval I, then functions $f \pm g$ and $f.g$ are analytic in I as well.*

Theorem 4.2. *Assume that $f(x)$ is analytic at x_0. Then c_n, the coefficients of the series of f at x_0 are as follows*

$$c_n = \frac{1}{n!} f^{(n)}(x_0), \tag{4.12}$$

where $f^{(n)}$ stands for the n^{th} derivative of f. Moreover, $f'(x)$ has the following series representation

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x - x_0)^{n-1}.$$

Therefore, an analytic function f at x_0 can be *represented* as the following series

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n, \tag{4.13}$$

for all x in an open interval I at x_0. The above representation is called the TAYLOR's series of f at x_0.

Example 4.4. Function $f(x) = e^x$ is analytic everywhere. At $x_0 = 0$, the function has the familiar expansion

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots.$$

Note that $f^{(n)}(0) = 1$ and thus $c_n = \frac{1}{n!}$. The domain of convergence for the expansion is

$$L = \lim_{n \to \infty} \left| \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} \right| = \infty,$$

and thus the series converges for $x \in (-\infty, \infty)$. Similarly, function $\sin(x)$ is analytic everywhere. It is simply seen that

$$\frac{d^n}{dx^n} \sin(x)|_{x=0} = \begin{cases} (-1)^{n+1/2} & n\text{: odd} \\ 0 & n\text{: even} \end{cases},$$

and thus

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots.$$

Function $\cos(x)$ has the following series expansion

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots.$$

The domain of convergence of sine and cosine functions are ∞.

4.1.3 Partial sums and convergence

Obviously, we are unable to add up infinite terms of a series directly and calculate its value. Therefore, we should consider the *partial sums* of an infinite series

$$S_n(x) = \sum_{k=1}^{n} c_k (x - x_0)^k.$$

Therefore, we find a *function sequence* $(S_n(x))$ for $n = 0, 1, \cdots$, that we can study its convergence. In addition to the *pointwise convergence* of the sequence, that is,

$$\lim_{n\to\infty} S_n(a) = f(a),$$

for any a in the domain of convergence of f, we can define the notion of *uniform convergence* of the sequence. Let us first see an example.

Example 4.5. Consider function $f(x) = \frac{1}{1-x^2}$ in $(-1, 1)$ with the series expansion

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdots.$$

Now, consider the following partial sum

$$S_n(x) = 1 + x^2 + x^4 + \cdots + x^{2n} = \frac{1 - x^{2n+2}}{1 - x^2}.$$

Note that if $|x| < 1$ then $x^{2n+2} \xrightarrow{n\to\infty} 0$. Fig.4.1 shows the graph of function $f(x)$ and $S_n(x)$ for a few values of n. Also note that $f(x)$ goes unbounded at $x = \pm 1$ and no partial sums (of any terms) is able to catch up the function in adjacent of these two points. In other words, for any $n > 0$, there is some point $x_n \in (-1, 1)$ such that

$$|f(x_n) - S_n(x_n)| > 1.$$

On the other hand, let us restrict the domain to $\left[-1 + \frac{1}{m}, 1 - \frac{1}{m}\right]$ for any $m > 1$. Then for any $\varepsilon > 0$, there is n such that

$$\max_{x \in \left[-1+\frac{1}{m}, 1-\frac{1}{m}\right]} |f(x) - S_n(x)| < \varepsilon.$$

For example, for $[-0.99, 0.99]$ and $\varepsilon = 0.1$, we can choose as large as $n = 310$ to make sure the above inequality holds for all x in the *closed interval*.

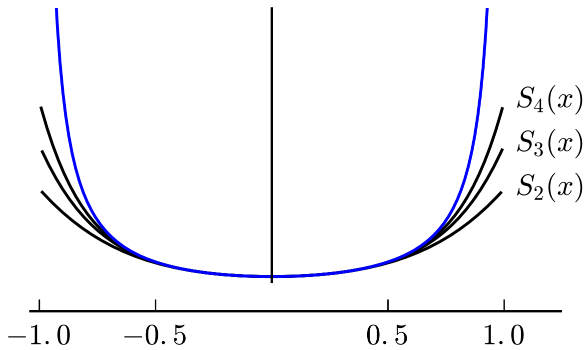

Figure 4.1.

Example 4.6. The function $\sin(x)$ is analytic everywhere and its power series representation centered at $x_0 = 0$ is

$$\sin(x) = x - \frac{1}{3!}x^3 + \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + \cdots.$$

Fig.4.2 shows $S_2(x)$, $S_3(x)$, $S_4(x)$ and $S_5(x)$ with the graph of the original function $\sin(x)$ in the range $[0, 2\pi]$ for the partial sum S_n

$$S_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1}. \tag{4.14}$$

Again, we can choose n sufficiently large such that $S_n(x)$ is very close to $f(x)$ in the given closed interval, however, there is no such n that be close enough to $f(x)$ in whole domain $(-\infty, \infty)$.

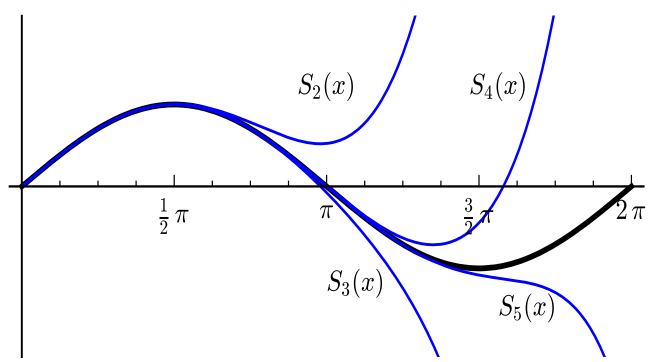

Figure 4.2. Graphs of $S_n(x)$ and $\sin x$.

Theorem 4.3. *Assume that a function f is analytic with the domain of convergence I, and $J \subset I$ is a closed subinterval. Then for any arbitrary $\varepsilon > 0$, there is $N(\varepsilon) > 0$ such that*

$$\max_{x \in J} |f(x) - S_N(x)| < \varepsilon,$$

where S_N is the partial sum of the expansion of $f(x)$ at x_0 up to order N.

Proof. For any $x \in [a, b]$, the series $S_n(x)$ converges to $f(x)$ and thus there is $N = N(x, \varepsilon)$ such that

$$|f(x) - S_N(x)| < \varepsilon.$$

Let $(x_n) \subset [a, b]$ be a sequence that maximize N, that is $N(x_{n+1}, \varepsilon) \geq N(x_n, \varepsilon)$. Since $[a, b]$ is closed, x_n converges to some $\bar{x} \in [a, b]$. On the other hand, $S_n(\bar{x})$ converges to $f(\bar{x})$ and thus there is $\bar{N}$ such that for all $n > \bar{N}$, we have

$$|f(\bar{x}) - S_n(\bar{x})| < \varepsilon,$$

and this completes the proof. □

Problems

Problem 4.2. Show that in any closed interval $[a, b] \subset (-1, 1)$, and any $\varepsilon > 0$, there is $N_0 = N_0(\varepsilon)$ such that

$$\max_{x \in [a,b]} |(1-x)^{-1} - S_n(x)| < \varepsilon, \forall n \geq N_0,$$

where $S_n(x) = 1 + \cdots + x^n$.

Problem 4.3. Consider the integral

$$I(x) = \int_0^x \frac{\sin(t)}{t}\, dt. \tag{4.15}$$

It is known that $I(x)$ can not be written in terms of elementary functions.

i. By substituting the series of $\sin(t)$ in the integral, find a power series for $I(x)$.

ii. Use the alternating series concept and find a partial sum $I_n(x)$ such that

$$|I(1) - I_n(1)| < 10^{-3}.$$

Repeat this for $x = 3$, that is $|I(3) - I_n(3)| < 10^{-3}$.

Problem 4.4. Use the power series of e^{-x} to approximate the following integral with the accuracy 10^{-4}

$$I = \int_1^2 \frac{e^{-x}}{x} dx.$$

Problem 4.5. Sketch the graph of each function and determine those that are analytic at $x_0 = 0$. For each analytic function, obtain the radius of convergence for the associated series.

i. $f(x) = e^{-|x|}$

ii. $f(x) = (4 - x^2)^{-1}$

iii. $f(x) = x^2|x|$

iv. $f(x) = \sin(|1 + x|)$

Problem 4.6. Find the radius of convergence of the following series

i.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} (x-1)^n$$

ii.

$$\sum_{n=1}^{\infty} \frac{3^n}{n+1}(2x+1)^n$$

iii.

$$\sum_{n=1}^{\infty} \frac{2^n}{n^n}(2x+1)^n$$

Problem 4.7. Show that the series

$$S(x) = x^2 - \frac{2^3}{4!}x^4 + \frac{2^5}{6!}x^6 - \cdots,$$

converges to the function $f(x) = \sin^2 x$. Find a power series expansion for $f(x) = \cos^2(x)$.

Problem 4.8. Here we give another proof for the formula (4.12).

a) Write f as

$$f(x) = \sum_{n=0}^{\infty} c_n x^n.$$

By the aid of binomial formula

$$(x+x_0)^n = \sum_{l=0}^{n} \binom{n}{l} x^l x_0^{n-l},$$

derive

$$f(x+x_0) = \sum_{n=0}^{\infty} c_n \sum_{l=0}^{n} \binom{n}{l} x_0^{n-l} x^l.$$

b) For $l=0$ calculate the series and show it is $f(x_0)$.

c) For $l=1$ calculate the series and show it is $f'(x_0)$.

d) For $l=2$ calculate the series and show it is $\frac{1}{2}f''(x_0)$.

Problem 4.9. If f, g are analytic functions, prove that $f \pm g$ and $f.g$ are analytic. Hint: may need the formula

$$\left(\sum_{n=0}^{\infty} f_n x^n\right)\left(\sum_{n=0}^{\infty} g_n x^n\right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} f_k g_{n-k}\right) x^n.$$

Problem 4.10. Plot the function $f(x) = e^{-1/|x|}$ for $-1 < x \leq 1$. Is it possible to find a series representation of f around $x_0 = 0$?

4.2 Linear differential equations: Analytic solutions

4.2.1 Equations with analytic coefficients

Consider the the initial value problem

$$\begin{cases} y'' + p(x)\, y' + q(x)\, y = 0 \\ y(x_0) = y_0,\, y'(x_0) = y_1 \end{cases}. \tag{4.16}$$

As we know, there is no general method to solve the problem in closed form, like an exponential, trigonometric or a polynomial function, but it does not mean that the solution can not be expressed in terms of a series.

Theorem 4.4. *Assume that functions p, q are analytic at x_0 with the radius of convergence L. Then problem (4.16) has an analytic solution at x_0 with the minimum radius of convergence L.*

The theorem states that the problem has a unique solution $y(x)$, and the solution can be expressed in terms of a power series as

$$y=\sum_{n=0}^{\infty} y_n\,(x-x_0)^n. \tag{4.17}$$

Here the coefficients y_n are unknown, and if we are able to determine them somehow, the true solution $y(x)$ can be at least approximated by a partial sum. The complete proof of the theorem needs some technical tools that is beyond the scope of this book. However, the following problem shows how the coefficients can be calculated.

Problem 4.11. Assume that p, q are analytic functions at x_0. Consider the problem

$$\begin{cases} y''+p(x)\,y'+q(x)\,y=0 \\ y(x_0)=y_0,\, y'(x_0)=y_1 \end{cases}.$$

We show that the coefficients of the series solution are derived by the formula

$$y_{n+2}=\frac{-1}{(n+1)(n+2)}\sum_{k=0}^{n}\,[(k+1)p_{n-k}\,y_{k+1}+q_{n-k}\,y_k],\quad n\geq 0, \tag{4.18}$$

where p_n, q_n are coefficients of the series of $p(x)$ and $q(x)$ respectively, i.e.,

$$\begin{cases} p(x)=p_0+p_1(x-x_0)+p_2(x-x_0)^2+\cdots \\ q(x)=q_0+q_1(x-x_0)+q_2(x-x_0)^2+\cdots \end{cases}. \tag{4.19}$$

i. By the relation

$$y^{(n+2)}=-(p(x)\,y')^{(n)}-(q(x)\,y)^{(n)},$$

conclude

$$y^{(n+2)}(x)=-\sum_{k=0}^{n}\binom{n}{k}[p^{(n-k)}(x)\,y^{(k+1)}(x)+q^{(n-k)}(x)\,y^{(k)}(x)].$$

ii. Now put $x=0$ in the above equation and conclude

$$(n+2)!\,y_{n+2}=-\sum_{k=0}^{n}\binom{n}{k}[(n-k)!\,(k+1)!\,p_{n-k}\,y_{k+1}+(n-k)!\,k!\,q_{n-k}\,y_k].$$

iii. Simplify the above identity and conclude (4.18).

Example 4.7. Consider the following initial value problem

$$\begin{cases} y''+(x+1)\,y'+x\,y=0 \\ y(0)=1,\, y'(0)=-1 \end{cases}. \tag{4.20}$$

It is simply seen that $y(x)=e^{-x}$ is the unique solution of the problem. Let us derive that solution by the series method. Since $x_0=0$, we write the solution as

$$y(x)=\sum_{n=0}^{\infty} y_n x^n.$$

Since $p_0=1, p_1=1$, and $p_n=0$ for $n\geq 2$, and $q_1=1$, $q_0=q_n=0$ for $n\geq 2$, the summation in formula (4.18) is nonzero only for $k=n-1,n$, and thus

$$y_{n+2}=\frac{-1}{(n+1)(n+2)}[(n+1)p_0\, y_{n+1}+n p_1\, y_n+q_1\, y_{n-1}],$$

and by substituting p_0, p_1, q_1, we obtain

$$y_{n+2}=-\frac{y_{n+1}}{n+2}-\frac{n\, y_n}{(n+1)(n+2)}-\frac{y_{n-1}}{(n+1)(n+2)}.$$

The above formula defines a *recursive formula* for the coefficients of the series solution of the given differential equation. Note that $y(0)=y_0=1$ and $y'(0)=y_1=-1$, and thus

$$y_2=\frac{-y_1}{2}=\frac{1}{2}, y_3=\frac{-1}{6}, y_4=\frac{1}{24},\cdots.$$

Therefore, the series solution has the form

$$y(x)=1-x+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{24}x^4-\cdots.$$

The above series is the representation of $y(x)=e^{-x}$ at $x_0=0$.

Example 4.8. (Cont.) Let us derive the recursive formula in the previous example by the direct calculation. Write the solution as the following series

$$y(x)=\sum_{n=0}^{\infty} y_n x^n.$$

Since $y(x)$ is analytic, then we can differentiate the series term by term and thus by formula (4.21) we can write

$$y'(x)=\sum_{n=1}^{\infty} n y_n x^{n-1},\ \text{and}\ y''(x)=\sum_{n=2}^{\infty} n(n-1) y_n x^{n-2}.$$

Note that the series of y' starts at $n=1$ and y'' starts at $n=2$. Substituting these formula into the equation, we reach

$$y''+(x+1)y'+x y=\sum_{n=2}^{\infty} n(n-1)y_n x^{n-2}+(x+1)\sum_{n=1}^{\infty} n y_n x^{n-1}+x\sum_{n=0}^{\infty} y_n x^n=0.$$

A simple algebraic simplification gives

$$\sum_{n=2}^{\infty} n(n-1)y_n x^{n-2}+\sum_{n=1}^{\infty} n y_n x^n+\sum_{n=1}^{\infty} n y_n x^{n-1}+\sum_{n=0}^{\infty} y_n x^{n+1}=0.$$

Here we have four summations. In order to merge these summations, we first make exponents of x in all summations equal. Taking $m=n-2$ in the first summation we reach

$$\sum_{n=2}^{\infty} n(n-1)y_n x^{n-2} = \sum_{m=0}^{\infty} (m+2)(m+1)y_{m+2} x^m.$$

In the second summation, we do not change the exponent and just take $m=n$. In the third summation, we take $m=n-1$ and then

$$\sum_{n=1}^{\infty} n y_n x^{n-1} = \sum_{m=0}^{\infty} (m+1)y_{n+1} x^m.$$

In the forth summation, we take $m=n+1$, we reach

$$\sum_{m=1}^{\infty} y_{m-1} x^m.$$

Therefore, we obtain

$$\sum_{m=0}^{\infty} (m+2)(m+1)y_{m+2} x^m + \sum_{m=1}^{\infty} m y_m x^m + \sum_{m=0}^{\infty} (m+1)y_{m+1} x^m + \sum_{m=1}^{\infty} y_{m-1} x^m = 0.$$

Now, first and third summations start from $m=0$ while second and fourth summation start from $m=1$. For this, we pull out one term from first and third summations and write

$$2y_2 + y_1 + \sum_{m=1}^{\infty} \left[(m+2)(m+1)y_{m+2} + m y_m + (m+1)y_{m+1} + y_{m-1}\right] x^m = 0.$$

Since the left hand side is identically zero for all x, we obtain the relations

$$2y_2 + y_1 = 0, \text{ and } (m+2)(m+1)y_{m+2} + m y_m + (m+1)y_{m+1} + y_{m-1} = 0.$$

From the first identity we obtain $y_2 = \frac{1}{2}$. From the second identity we obtain the following formula that is called a *recursive formula* for y_m

$$y_{m+2} = -\frac{y_{m+1}}{m+2} - \frac{m y_m}{(m+2)(m+1)} - \frac{y_{m-1}}{(m+2)(m+1)}.$$

Example 4.9. (Cont.) Instead of finding a recursive formula for the equation, we can calculate only a few terms of the coefficients by the following method. Write the solution as follows

$$y(x) = y_0 + y_1 x + y_2 x^2 + \cdots.$$

and substitute y into the equation. We have

$$y'(x) = y_1 + 2y_2 x + 3y_3 x^2 + \cdots, \text{ and } y''(x) = 2y_2 + 6y_3 x + 12 y_4 x^2 + \cdots, \tag{4.21}$$

and thus

$$(2y_2 + 6y_3 x + 12y_4 x^2 + \cdots) + (x+1)(y_1 + 2y_2 x + 3y_3 x^2 + \cdots) + x(y_0 + y_1 x + y_2 x^2 + \cdots) = 0.$$

Since the series in the left hand side is identically zero for all x, all coefficients of x^n for arbitrary $n \geq 0$ must be zero. We have

0-terms. (terms with x in power of zero)

$$2y_2 + y_1 = 0,$$

and thus $y_2 = -y_1/2 = \frac{1}{2}$.

1-terms. (coefficient of x)

$$6y_3 + y_1 + 2y_2 + y_0 = 0,$$

and thus $y_3 = \frac{-1}{6}$.

2-terms. (coefficients of x^2)

$$12y_4 + 2y_2 + 3y_3 + y_1 = 0,$$

and thus $y_4 = \frac{1}{24}$.

We can continue the calculation to find y_5, y_6, $\cdots$. As it is observed, the coefficients are as before.

Example 4.10. Find five terms of the series solution of the following problem

$$\begin{cases} y'' + e^x y' + \sin(x) y = 1 - 2x \\ y(0) = 1, y'(0) = -1 \end{cases}.$$

We substitute the series of y, y', y'' along with the series of e^x, $\sin(x)$ into the equation. With

$$\begin{cases} y(x) = y_0 + y_1 x + y_2 x^2 + \cdots \\ y'(x) = y_1 + 2y_2 x + 3y_3 x^2 + \cdots \\ y''(x) = 2y_2 + 6\, y_3 x + 12 y_4 x^2 + \cdots \end{cases},$$

and

$$\begin{cases} e^x = 1 + x + \frac{1}{2}x^2 + \cdots \\ \sin(x) = x - \frac{1}{6}x^3 + \cdots \end{cases},$$

we reach

$$(2y_2 + 6\, y_3 x + 12 y_4 x^2 + \cdots) + \left(1 + x + \frac{1}{2}x^2 + \cdots\right)(y_1 + 2y_2 x + 3y_3 x^2 + \cdots) + \\ + \left(x - \frac{1}{6}x^3 + \cdots\right)(y_0 + y_1 x + y_2 x^2 + \cdots) = 1 - 2x$$

0-terms. (terms with x in power of zero)

$$2y_2 + y_1 = 1 \Rightarrow y_2 = 1.$$

1-terms. (coefficient of x)

$$6y_3 + y_1 + 2y_2 + y_0 = -2 \Rightarrow y_3 = -\frac{2}{3}.$$

2-terms. (coefficients of x^2)

$$12y_4 + 3y_3 + 2y_2 + \frac{3}{2}y_1 = 0 \Rightarrow y_4 = \frac{1}{8}.$$

Therefore, the series solution is

$$y(x) = 1 - x + x^2 - \frac{2}{3}x^3 + \frac{1}{8}x^4 - \cdots.$$

Example 4.11. Find a recursive formula for the coefficients of the series solution of the following problem and verify that the series is convergent

$$\begin{cases} (1-x^2)y'' + 2y = 0 \\ y(0) = y_0,\, y'(0) = y_1 \end{cases}.$$

We first note that the coefficient of y'' that is $(1-x^2)$ goes zero at $x = \pm 1$. In other word, here $q(x) = \frac{2}{1-x^2}$ is analytic at $x_0 = 0$ with the radius of convergence $L = 1$. According to the theorem (4.4), the *minimum radius of convergence* of the series solution is L. We see how this domain of convergence shows itself in the series solution. By taking y as

$$y(x) = \sum_{n=0}^{\infty} y_n x^n,$$

we reach

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) y_n x^{n-2}.$$

Substituting into the equation, we reach

$$\sum_{n=2}^{\infty} n(n-1)y_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) y_n x^n + \sum_{n=0}^{\infty} 2y_n x^n = 0.$$

By taking $m = n - 2$, we reach

$$\sum_{n=2}^{\infty} n(n-1) y_n x^{n-2} = \sum_{m=0}^{\infty} (m+2)(m+1) y_{m+2} x^m.$$

Notice that without loss of generality we can assume that the the second summation starts from $n = 0$, that is,

$$\sum_{n=2}^{\infty} n(n-1) y_n x^n = \sum_{n=0}^{\infty} n(n-1) y_n x^n.$$

Putting altogether, we obtain

$$\sum_{m=0}^{\infty} [(m+2)(m+1) y_{m+2} - m(m-1) y_m + 2y_m]\, x^m = 0,$$

and therefore

$$y_{m+2} = \frac{m-2}{m+2} y_m, m \geq 0$$

By the above recursive formula, we calculate

$$y_2 = -y_0, y_3 = \frac{-1}{3} y_1, y_4 = 0, y_5 = \frac{1}{5} y_3, y_6 = 0, y_7 = \frac{3}{7} y_5, y_8 = 0, \cdots.$$

Therefore, the truncated solution is

$$y(x) \sim y_0(1-x^2) + y_1\left(x - \frac{1}{3}x^3 - \frac{1}{15}x^5 - \frac{3}{105}x^7\right) - \cdots.$$

Let us calculate the radius of convergence of the series by the aid of the recursive formula. Remember that the radius of convergence is determined by the formula

$$L := \lim_{n\to\infty} \left|\frac{y_n}{y_{n+1}}\right|.$$

The recursive of formula states

$$\frac{n+2}{n-2} = \frac{y_n}{y_{n+2}} = \frac{y_n}{y_{n+1}} \frac{y_{n+1}}{y_{n+2}},$$

and thus

$$L^2 = \lim_{n\to\infty} \left|\frac{y_n}{y_{n+1}}\right| \left|\frac{y_{n+1}}{y_{n+2}}\right| = \lim_{n\to\infty} \frac{n+2}{n-2} = 1,$$

and thus the series converges in the interval $x \in (-1, 1)$ with the radius of convergence $L = 1$, the same radius of convergence of $q(x) = \frac{2}{1-x^2}$. However, if $y_1 = 0$, then the series solution reduces to a polynomial solution $y(x) = y_0(1-x^2)$, and thus with the radius of convergence $L = \infty$. This justifies the claim why the radius of convergence of the solution is equal *at least* to the radius of convergence of p and q.

Let us summarize what we discussed in this section. We saw that if p, q are analytic functions, then the problem (4.16) has a unique analytic solution where its radius of convergence is at least equal to the radius of convergence of p, q. In addition, if $y'(x_0) = 0$, the recursive formula for coefficients y_n are only depends on y_0, and thus

$$y(x) = y_0(1 + c_1 x + c_2 x^2 + \cdots).$$

Similarly, if $y(x_0) = 0$, all coefficients will depends on y_1, and therefore

$$y(x) = y_1(x + d_2 x^2 + d_3 x^3 + \cdots).$$

4.2.2 Linear Equation with non-analytic coefficients

We solve here three equations with non-analytic coefficients to illustrate the difficulty that arises in this case.

Example 4.12. Consider the problem

$$\begin{cases} y'' + |x| y = 0 \\ y(0) = y_0, y'(0) = y_1 \end{cases}. \tag{4.22}$$

The existence and uniqueness are guaranteed for the problem since the coefficients are continuous. Note that the function $|x|$ is not analytic at $x_0 = 0$ and for this, the assumption on the analyticity of the solution may fail. Let us first find a series solution to the problem for $x > 0$. The equation in this domain reads

$$\begin{cases} y'' + x y = 0 \\ y(0) = y_0, y'(0) = y_1 \end{cases}.$$

The recursive formula for the coefficients of the series solution is

$$y_{n+2} = -\frac{1}{(n+2)(n+1)} y_{n-1}, n \geq 1,$$

and $y_2 = 0$. Let us denote this solution by $y_+(x)$

$$y_+(x) = y_0 + y_1 x - \frac{y_0}{6} x^3 - \frac{y_1}{2} x^4 + \frac{y_0}{180} x^6 + \frac{y_1}{84} x^7 - \cdots.$$

For $x < 0$, the problem reads

$$\begin{cases} y'' - x y = 0 \\ y(0) = y_0, y'(0) = y_1 \end{cases}$$

and the recursive formula changes to

$$y_{n+2} = \frac{1}{(n+2)(n+1)} y_{n-1}, n \geq 1.$$

and again $y_2 = 0$. Let us denote this solution by $y_-(x)$

$$y_-(x) = y_0 + y_1 x + \frac{y_0}{6} x^3 + \frac{y_1}{2} x^4 + \frac{y_0}{180} x^6 + \frac{y_1}{84} x^7 + \cdots.$$

Two solution y_+ and y_- connect at $x_0 = 0$ smoothly of order 2, i.e., $y_+^{(k)}(0) = y_-^{(k)}(0)$ for $k = 0, 1, 2$ but not for $k \geq 3$. Hence, the derived series solution is not analytic at x_0. The figure (4.3) shows the solution for $y_0 = 1$ and $y_1 = -1$. The dashed line is the tangent line at x_0.

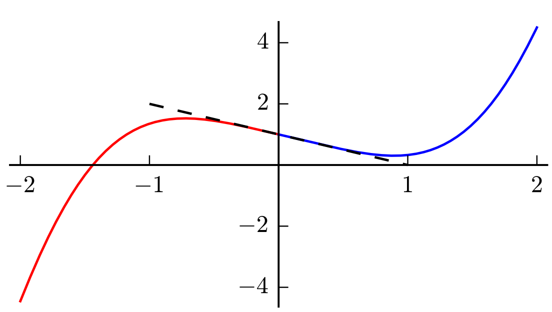

Figure 4.3.

Example 4.13. Consider the following problem:

$$\begin{cases} x^2\,y''+(x-x^2)\,y'-y=0 \\ y(0)=y_0,\, y'(0)=y_1 \end{cases}. \tag{4.23}$$

Apparently the problem does not satisfy the conditions of the existence theorem and therefore, there is no guarantee that the problem possesses a solution (analytic or non-analytic). Let us try to find a series solution and see the result. By substituting $y=\sum_{n=0}^{\infty} y_n\,x^n$ into the equation, we reach

$$\sum_{n=2}^{\infty} n(n-1)\,y_n\,x^n+\sum_{n=0}^{\infty} n\,y_n\,x^n-\sum_{n=0}^{\infty} n\,y_n\,x^{n+1}-\sum_{n=0}^{\infty} y_n\,x^n=0. \tag{4.24}$$

Merging summations leads to the following equation

$$-y_0+\sum_{n=2}^{\infty} (n^2-1)\,y_n-(n-1)y_{n-1})\,x^n=0, \tag{4.25}$$

and therefore $y_0=0$ and

$$y_n=\frac{1}{n+1}y_{n-1},\, n\geq 2. \tag{4.26}$$

Note that if $y_0\neq 0$, there is no series solution to the problem. To justify that the series converges to the true solution, we have to show that the obtained series has a non-zero radius of convergence. This is justified easily by the relation

$$R=\lim_{n\to\infty}\left|\frac{y_{n-1}}{y_n}\right|=\lim_{n\to\infty} n+1=\infty, \tag{4.27}$$

and thus this series converges to the true solution in $(-\infty,\infty)$. A few terms of the series are

$$y_2=\frac{1}{3}y_1,\;\; y_3=\frac{1}{4}y_2=\frac{1}{3\times 4}y_1,\;\; y_4=\frac{1}{5}y_3=\frac{1}{3\times 4\times 5}y_1,\cdots. \tag{4.28}$$

Observe that $y_n=\frac{2}{(n+1)!}\,y_1$ and thus

$$y=2y_1\left(\frac{1}{2!}x+\frac{1}{3!}x^2+\frac{1}{4!}x^3+\cdots\right). \tag{4.29}$$

Therefore we are able to find only one analytic solution to the equation

$$x^2\,y''+(x-x^2)\,y'-y=0,$$

at $x_0=0$, and it corresponds to $y_0=0$. It is simply verified that the series in the bracket is the expansion of the function

$$y(x)=\frac{e^x-1-x}{x}. \tag{4.30}$$

Observe that

$$\lim_{x\to 0} y(x)=0.$$

In order to find a second linearly independent solution, we can use the reduction of order method and obtain $y(x)=\frac{x+1}{x}$. Note that the new solution is unbounded at $x=0$.

Example 4.14. Consider the following equation

$$\begin{cases} x^3 y'' - y = 0 \\ y(0) = y_0, y'(0) = y_1 \end{cases}. \tag{4.31}$$

Like the second example, the problem does not satisfy the condition of the existence theorem and thus a solution is not guaranteed for the problem. Let us try to find a series solution to the problem. We have

$$\sum_{n=2}^{\infty} n(n-1)\, y_n\, x^{n+1} - \sum_{n=0}^{\infty} y_n\, x^n = 0. \tag{4.32}$$

A simplification gives

$$-y_0 - y_1 x - y_2 x^2 + \sum_{n=3}^{\infty} (\,(n-1)(n-2)\, y_{n-1} - y_n) x^n = 0, \tag{4.33}$$

and thus $y_0 = y_1 = y_2 = 0$. The recursive formula for $n \geq 3$ is

$$y_n = (n-1)(n-2) y_{n-1}. \tag{4.34}$$

According to the formula, we derive $y_n = 0$ for all $n \geq 3$ and thus $y_n = 0$ for all n. Therefore, the problem has a solution if and only if $y_0 = y_1 = 0$ and in this case, the only possible solution is the trivial one $y \equiv 0$.

Problems

Problem 4.12. By the mathematical induction prove the following formula

$$(fg)^{(n)} = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} f^{(k)}\, g^{(n-k)}.$$

Use this formula to find a recursive formula for the series solution to the equation

$$\begin{cases} y'' + xy = 0 \\ y(0) = 1, y'(0) = 0 \end{cases},$$

and determine the radius of convergence of the series.

Hint: Write

$$y^{(n)}(0) = -(xy(x))^{(n-2)}|_{x=0},$$

and use the above formula.

Problem 4.13. Consider the initial value problem

$$\begin{cases} y' + 2xy = 0 \\ y(0) = 1 \end{cases}.$$

i. Find the closed form solution to the problem

ii. Find a series solution and compare it with the closed form solution.

Problem 4.14. Consider the initial value problem

$$\begin{cases} xy' - y = 0 \\ y(1) = 1 \end{cases}.$$

i. Find the closed form solution to the problem

ii. Find a series solution and compare it with the closed form solution.

Problem 4.15. Find a series solution to the problem

$$\begin{cases} y'' + xy' + y = 0 \\ y(0) = 1, y'(0) = 0 \end{cases}.$$

Verify that the solution is the expansion of the solution $y = e^{-x^2/2}$.

Problem 4.16. Find a series solution to each of the following problems

i.
$$\begin{cases} y'' + 2y' + y = 0 \\ y(0) = 0, y'(0) = 1 \end{cases}$$

ii.
$$\begin{cases} xy'' + y = 0 \\ y(0) = 0, y'(0) = 1 \end{cases}$$

iii.
$$\begin{cases} xy'' + y = 0 \\ y(0) = 1, y'(0) = 0 \end{cases}$$

iv.
$$\begin{cases} x^2 y'' + 4xy' + 2y = 0 \\ y(0) = y_0, y'(0) = y_1 \end{cases}$$

Problem 4.17. Consider the problem

$$\begin{cases} x^2y'' - 2y = 0 \\ y(0) = y'(0) = 0 \end{cases}.$$

Clearly the problem has the trivial solution $y(x) \equiv 0$. Try to find a series solution to the problem. Why the problem has multiple solutions?

Problem 4.18. If x_0 is non-zero, it is convenient to shift it to zero. Consider the following problem:

$$\begin{cases} y'' - xy' - y = 0 \\ y(1) = 1, y'(1) = 0 \end{cases}. \tag{4.35}$$

a) Take $t = x - 1$ and write the equation in terms of t.

b) Find a power series solution to the new equation and then rewrite the solution in terms of x

Problem 4.19. The following equation is called a CAUCHY-EULER equation

$$x^2y'' + xy' - y = 0.$$

This equation has solutions $y_1 = x$ and $y_2 = x^{-1}$ which $y(0) = 0$ and $y_2(0)$ is unbounded at $x_0 = 0$.

i. Set appropriate initial conditions (at $x_0 = 0$) such that the problem has a bounded solution at $x_0 = 0$.

ii. Try to solve the problem (with the initial conditions you set for the equation) by the power series method.

Problem 4.20. Find four nonzero terms of the power series solution to each of the following equations.

i. $y'' + xy' + e^x y = 0, \qquad y(0) = 1, y'(0) = 1$

ii. $y'' - \sin(x)\, y = \cos(x), \qquad y(0) = 1, y'(0) = -1$

iii. $(1 - x^2)y'' - 2x y' + \sin(x) y = 0, \qquad y(0) = 1, y'(0) = 0.$

iv. $e^x y'' + 3x y' - \tan(x)\, y = \sec(x), \qquad y(0) = 0, y'(0) = 1.$

v. $\cos(x)\, y'' + e^x y' + y = \sin(x),\ y(0) = 1, y'(0) = -1.$

Problem 4.21. Consider the equation

$$\begin{cases} y'' + y = \sin(2x) \\ y(0) = 1, y'(0) = 0 \end{cases}.$$

i. This a linear equation with constant coefficients. Find a solution of the equation.

ii. Now, expand $\sin(2x)$ and find a power series solution to the equation and compare two solutions.

Problem 4.22. Consider the equation

$$\begin{cases} y'' + y = \tan^{-1}(x) \\ y(0) = 1, y'(0) = 0 \end{cases}.$$

Find a recursive formula for the series solution of the equation and calculate 5 non-zero terms.

Problem 4.23. For each of the following problems, find the recursive formula for the power series solution and write down a series containing at least 5 nonzero terms.

i.

$$\begin{cases} y'' + (1 + x) y = 0 \\ y(0) = 1, y'(0) = 0 \end{cases}$$

ii.

$$\begin{cases} y'' + x y' + x^2 y = 0 \\ y(0) = 0, y'(0) = 1 \end{cases}$$

iii.

$$\begin{cases} y'' + x y' + y = 0 \\ y(0) = 1, y'(0) = -1 \end{cases}$$

Problem 4.24. For the equation

$$y'' + x y' + y = \frac{1}{1 - x}$$

a) Show that c_n, the coefficient of the power series solution of the equation around $x_0 = 0$, satisfies the following recursive formula for $n \geq 0$:

$$y_{n+2} = \frac{1}{(n+1)(n+2)} - \frac{y_n}{n+2}.$$

b) Calculate y_0 to y_4 for the initial conditions $y(0) = 0$, $y'(0) = 1$.

c) Show that the above formula is equivalent to

$$y_{n+3} = \frac{n+1}{n+3} y_{n+2} + \frac{n+1}{(n+2)(n+3)} y_n - \frac{1}{n+3} y_{n+1}.$$

d) Find the radius of convergence of the series solution.

Problem 4.25. The application of power series method for non-linear equations is limited. Consider the following IVP.

$$\begin{cases} y'' + x \sin(y) = 0 \\ y(0) = 1, y'(0) = 0 \end{cases}.$$

Linearize the equation around the working point and obtain 5 non-zero term of the power series solution. Use a computer software to compare the obtained solution with the numeric one given by the software.

4.3 Singular equations

Definition 4.3. *Consider the equation*

$$y'' + p(x)\, y' + q(x)\, y = 0. \tag{4.36}$$

If functions p, q are analytic at x_0, then x_0 is called a regular *point for the equation. If at least one of $p(x)$ or $q(x)$ are not analytic at x_0, the point is called a* singular point *for the equation. There are two kinds of singular points:*

- *If functions $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are analytic at x_0, then x_0 is a* regular-singular *point.*
- *If at least one of functions $(x - x_0)p(x)$ or $(x - x_0)^2 q(x)$ are non-analytic, the point is called an* essential singular *or* singular-singular *point.*

Example 4.15. The point $x_0 = 0$ is a regular point for the following equation

$$(1 + x^2)y'' + xy = 0, \tag{4.37}$$

since $p(x) = 0$ and $q(x) = \frac{x}{1+x^2}$ are both analytic at $x_0 = 0$. The point $x = -1$ is a regular-singular point for the following equation

$$(1 - x^2)y'' + \sin(1 - x)y' + (1 - x)y = 0. \tag{4.38}$$

In fact, for $p(x) = \frac{\sin(1-x)}{1-x^2}$, $q(x) = \frac{1-x}{1-x^2}$, the only singular point is $x = -1$. Note that both $p(x)$ and $q(x)$ have *removable* singularity at $x = 1$ as

$$\lim_{x\to 1} \frac{\sin(1-x)}{1-x^2} = \lim_{x\to 1} \frac{1-x}{1-x^2} = \frac{1}{2}.$$

Moreover,

$$(1 + x)\, p(x) = (1 + x)\frac{\sin(1-x)}{1-x^2} = \frac{\sin(1-x)}{1-x},$$

$$(1 + x)^2 q(x) = (1 + x)^2 \frac{1-x}{1-x^2} = \frac{1-x^2}{1-x},$$

are analytic at $x = -1$. The point $x_0 = 0$ is an essential singular point for the equation $y'' + |x|y = 0$. In fact, $x^2 q(x) = x^2|x|$ is not analytic at x_0.

4.3.1 CAUCHY-EULER equations

The reason for classifying a point will be clear when we discuss the CAUCHY-EULER equation. The general form of a homogeneous equidimensional or CAUCHY-EULER equation is

$$(x - x_0)^2 y'' + a(x - x_0)\, y' + by = 0, \tag{4.39}$$

where a and b are some constants. Here, we have

$$p(x) = \frac{a}{x - x_0}, \, q(x) = \frac{b}{(x - x_0)^2},$$

and thus x_0 is a regular-singular point for the equation. Fortunately, there is a simple transformation that converts the above equation to an equation with constant coefficients. Let $x - x_0 = e^t$ for $x > x_0$. We have

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}(x - x_0), \tag{4.40}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dx}(x - x_0)\right) = \left[\frac{d}{dx}\left(\frac{dy}{dx}(x - x_0)\right)\right](x - x_0) = \frac{d^2y}{dx^2}(x - x_0)^2 + \frac{dy}{dt}. \tag{4.41}$$

Substituting above formula into (4.39), yields

$$\frac{d^2y}{dt^2} + (a - 1)\frac{dy}{dt} + by = 0, \tag{4.42}$$

which is a second order equation with constant coefficients. The characteristic polynomial of the new equation is

$$f(s) = s^2 + (a - 1)s + b.$$

Case 1. If $f(s) = 0$ has two real distinct roots s_1, s_2, then the new equation has two solutions $e^{s_1 t}$, $e^{s_2 t}$, and by transformation $x - x_0 = e^t$, two solutions for the CAUCHY-EULER equation are obtained $y_1(x) = (x - x_0)^{s_1}, y_2(x) = (x - x_0)^{s_2}$.

Case 2. If $f(s) = 0$ has a repeated root $s_1 = s_2 = s$, then the equation has two solutions $e^{st}, t e^{st}$, and then $y_1(x) = (x - x_0)^s$, $y_2(x) = (x - x_0)^s \ln(x - x_0)$.

Case 3. If $f(s) = 0$ has complex roots $s = \sigma \pm i\omega$, the equation has two solutions $y_1(x) = (x - x_0)^\sigma \cos(\omega \ln(x - x_0))$, $y_2(x) = (x - x_0)^\sigma \sin(\omega \ln(x - x_0))$.

Example 4.16. The substitution $x = e^t$ transforms the equation

$$x^2 y'' + x y' - y = 0,$$

into equation $\frac{d^2y}{dt^2} - y = 0$ with solutions $y_1 = e^t$ and $y_2 = e^{-t}$, and thus $y_1(x) = x$ and $y_2(x) = x^{-1}$. Note that $y_2(x)$ goes unbounded when x approaches zero. Now, consider the following equation

$$x^2 y'' - x y' + y = 0.$$

The CAUCHY-EULER characteristic equation is

$$s^2 - 2s + 1 = 0,$$

and thus $y_1(t) = e^t$ and $y_2(t) = t\,e^t$ are two solutions of its transformed equation, and thus $y_1(x) = x$, $y_2(x) = x\ \ln(x)$ are two solutions to the above equation. Note that both solutions vanishes at $x = 0$. Consider the following equation

$$(x-1)^2 y'' + (x-1)y' + y = 0. \tag{4.43}$$

We take $x - 1 = e^t$ and obtain the following constant coefficients equation

$$\frac{d^2y}{dt^2} + y = 0. \tag{4.44}$$

It is simply seen that the original equation has following solutions

$$y_1(x) = \cos(\ln(x-1)),\ y_2(x) = \sin(\ln(x-1)).$$

Example 4.17. Consider the following equation

$$x^2\,y'' - 2x\,y' + 2y = x^3 e^x.$$

The homogeneous solutions to the equation are $y_1 = x$ and $y_2 = x^2$. The particular solution is obtained by the variation of parameters method

$$y_p(x) = -x\int\frac{x\,e^x\,x^2}{x^2} + x^2\int\frac{x^{\,2}e^x}{x^2} = x\,e^x.$$

The general solution is $y(x) = c_1 x + c_2\,x^2 + x\,e^x$.

4.3.2 Regular singular equations: I

Assume that x_0 is a regular-singular point for (4.36). If we multiply the equation by $(x - x_0)^2$, we derive the following one

$$(x-x_0)^2 y'' + (x-x_0)^2\,p(x)\,y' + (x-x_0)^2 q(x) y = 0.$$

Let us denote function $(x - x_0)p(x)$ by $a(x)$ and $(x - x_0)^2\,q(x)$ by $b(x)$. Then we can write the equation as follows

$$(x-x_0)^2 y'' + (x-x_0)\,a(x)\,y' + b(x) y = 0. \tag{4.45}$$

Observe that the obtained equation is very similar to a CAUCHY-EULER equation except that a, b are not constant in the present case. Since $a(x), b(x)$ are analytic, we can express them as

$$a(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n, \text{ and } \ b(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^n,$$

for some constants a_n, b_n. Comparing Eq.4.45 with the CAUCHY-EULER equation, we can write the solution as

$$y = (x-x_0)^s\,g(x),$$

for an analytic function $g(x)$, where s is a constant. The following equation is called the *characteristic polynomial* of Eq. (4.45):

$$f(s) = s^2 + (a_0 - 1)s + b_0, \tag{4.46}$$

where a_0, b_0 are first coefficient series representation of $a(x)$ and $b(x)$ respectively.

Theorem 4.5. *Assume that x_0 is a regular-singular point of (4.36) and that $f(s) = 0$ has two real roots $s_1 \geq s_2$. Then one solution of the equation is*

$$y_1(x) = (x - x_0)^{s_1} \sum_{n=0}^{\infty} c_n (x - x_0)^n, \tag{4.47}$$

The value c_0 is arbitrary and c_n are determined by the following recursive formula:

$$c_n = \frac{-1}{f(s_1 + n)} \sum_{k=0}^{n-1} ((k + s_1) a_{n-k} + b_{n-k}) c_k. \tag{4.48}$$

Proof. We prove the relation for $x_0 = 0$. First we rewrite the equation (4.36) as

$$x^2 y'' + x a(x) y' + b(x) y = 0. \tag{4.49}$$

This is very similar to a CAUCHY-EULER equation, and thus it is justifiable to assume the solution in the following form

$$y(x) = x^s g(x), \tag{4.50}$$

for some constant s and an analytic function g. Without loss of generality, we can assume that $g(0) \neq 0$, otherwise we can rewrite y as $y = x^{s+1} \tilde{g}(x)$ where $\tilde{g}(0) \neq 0$. Let us substitute (4.50) into (4.49). We have

$$y'(x) = s x^{s-1} g(x) + x^s g'(x),$$

$$y''(x) = s(s-1) x^{s-2} g(x) + 2s x^{s-1} g'(x) + x^s g''(x).$$

Substitution the above formula into the equation gives

$$x^s \{x^2 g'' + x(a(x) + 2s) g'(x) + F(s, x) g(x)\} = 0$$

where

$$F(s, x) = s^2 + (a(x) - 1)s + a(x).$$

In order to have the above identity valid in a neighborhood of x_0, it is necessary to have

$$x^2 g'' + x(a(x) + 2s) g'(x) + F(s, x) g(x) = 0. \tag{4.51}$$

Now, let $x \to 0$ and obtain $F(s, 0) g(0) = 0$ and thus $F(s, 0) = 0$. Note that $F(s, 0) = f(s)$, the characteristic polynomial of the equation. For a moment assume that roots of $f(s)$ are real (not necessarily distinct). We take the biggest root s_1 for s in (4.50). Now let us find a series expansion for $g(x)$. Write it as

$$g(x) = \sum_{n=0}^{\infty} c_n x^n,$$

where c_0 is arbitrary (we can take it equal to 1) and calculate the n order derivative of both sides of (4.51) at $x = 0$. We have

$$(x^2 g'')^{(n)}|_{x=0} = -(F(s_1, x) g)^{(n)}|_{x=0} - 2s_1 (x g')^{(n)}|_{x=0} - (x a(x) g')^{(n)}|_{x=0}. \tag{4.52}$$

Since

$$(x^2\, g'')^{(n)} = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) (x^2)^{(n-k)}\, g^{(k+2)},$$

is non-zero at $x=0$ only for $n-k=2$, we obtain

$$(x^2\, g'')^{(n)}|_{x=0} = n(n-1)\, g^{(n)}(0). \tag{4.53}$$

Note that $g^{(n)}(0) = n!c_n$ and then

$$(x^2\, g'')^{(n)}|_{x=0} = n(n-1)\, n!c_n.$$

Similarly we have

$$(x\, g')^{(n)}|_{x=0} = (n!\,)n\, c_n.$$

For the first term in the right hand side of (4.52), we have

$$(F(s_1, x)\, g)^{(n)} = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) F(s_1, x)^{(n-k)}\, g^{(k+2)}.$$

Note that for $k=n$, the expression $F(s_1, 0)$ is zero. For $0 \leq k \leq n-1$, we have

$$F(s_1, 0)^{(n-k)} = (n-k)!(s_1 a_{n-k} + b_{n-k}). \tag{4.54}$$

Therefore

$$(F(s_1, x)\, g)^{(n)}|_{x=0} = n! \sum_{k=0}^{n-1} (s_1\, a_{n-k} + b_{n-k})\, c_k. \tag{4.55}$$

For the last expression in (4.52), we have

$$(x\, a(x)\, g')^{(n)}|_{x=0} = n! \sum_{k=0}^{n-1} (k+1)\, a_{n-k-1}\, c_{k+1}. \tag{4.56}$$

If we take $k = k+1$, then we reach

$$(x\, a(x)\, g')^{(n)}|_{x=0} = n! \sum_{k=0}^{n} k\, a_{n-k}\, c_k. \tag{4.57}$$

Substitution all above formula into (4.52) gives the recursive formula (4.48). □

Example 4.18. Let us find a homogeneous solutions to the following equation

$$2x^2\, y'' - x(1+x)\, y' + y = 0, \tag{4.58}$$

for $x_0 = 0$. Note that zero is a regular-singular point of the equation and

$$a(x) = -\frac{1}{2} - \frac{1}{2}x, b(x) = \frac{1}{2}. \tag{4.59}$$

The characteristic polynomial is

$$f(s) = s^2 - \frac{3}{2}s + \frac{1}{2}, \tag{4.60}$$

with two roots $s_1 = 1$, $s_2 = \frac{1}{2}$. According to the theorem (4.5), the equation has one solution

$$y_1 = x(c_0 + c_1 x + c_2 x^2 + \ldots), \tag{4.61}$$

where $c_n, n \geq 1$ are obtained by (4.48). Note that $a_n = 0$ for $n \geq 2$, and $b_n = 0$ for $n \geq 1$ and then the summation is zero for $k \leq n - 2$. We obtain the following recursive formula for the coefficients

$$c_n = \frac{(s_1 + n - 1)}{2(s_1 + n - 1)\left(s_1 + n - \frac{1}{2}\right)} c_{n-1}. \tag{4.62}$$

Since $s_1 = 1$, the above formula reads

$$c_n = \frac{1}{2n + 1} c_{n-1}. \tag{4.63}$$

Since c_0 is arbitrary, we can safely take it equal to 1. Calculating few terms gives the following homogeneous solution to the equation

$$y_1(x) = x\left(1 + \frac{1}{3}x + \frac{1}{15}x^2 + \frac{1}{105}x^3 + \cdots\right). \tag{4.64}$$

Example 4.19. Consider the following equation

$$x^2 y'' - x(1 - x)\, y' + y = 0. \tag{4.65}$$

Again $x_0 = 0$ is a regular-singular point for the equation. With $a(x) = -1 + x$ and $b(x) = 1$, the characteristic polynomial is $f(s) = (s - 1)^2$ with repeated root $s_1 = 1$. Notice that $a_n = 0$ for $n \geq 2$ and $b_n = 0$ for $n \geq 1$. Thus the summation in (4.48) runs only for $k = n - 1$. The recursive formula (4.48) gives

$$c_n = -\frac{1}{n} c_{n-1}, \tag{4.66}$$

and thus for $c_0 = 1$ we obtain

$$y_1 = x\left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \cdots\right) = x e^{-x}. \tag{4.67}$$

Example 4.20. Consider the following equation

$$x^2 y'' + x(1 - x)\, y' - y = 0. \tag{4.68}$$

Here $a(x) = 1 - x$ and $b(x) = -1$ and $x_0 = 0$ is a regular-singular point of the equation. We have $f(s) = s^2 - 1$ and then $s_1 = 1$, $s_2 = -1$. The recursive formula is

$$c_n = \frac{1}{n + 2} c_{n-1}, \tag{4.69}$$

and thus

$$y_1 = x\left(1 + \frac{1}{3}x + \frac{1}{12}x^2 + \frac{1}{60}x^3 + \cdots\right). \tag{4.70}$$

It is seen that the obtained series is the expansion of the following function $y = 2\frac{e^x - 1 - x}{x}$.

4.3.3 Regular singular equations: II

The structure of a second linearly independent solution for a regular-singular point depends on the second root of $f(s)$. For our subsequent discussion, we need to rewrite (4.48) in the following form

$$c_n(s) = \frac{-1}{f(s+n)}\sum_{k=0}^{n-1} ((k+s)\, a_{n-k} + b_{n-k})c_k(s), \tag{4.71}$$

Here the formula emphasizes the dependence on s that can be any of two roots of the characteristic polynomial (4.46). Here $c_n(s_1)$ is the same c_n we used before. Again we assume that s_1, s_2, the roots of $f(s)$ are real

$s_1 - s_2$ is non-integer.

If $s_1 - s_2$ is non-integer, a second linearly independent solution y_2 is

$$y_2 = (x - x_0)^{s_2}\sum_{n=0}^{\infty} c_n(s_2)\,(x - x_0)^n. \tag{4.72}$$

Example 4.21. Consider the equation (4.58). Since $s_1 - s_2 = \frac{1}{2}$ is non-integer, the second solution is

$$y_2 = \sqrt{x}\left(c_0 + c_1\left(\frac{1}{2}\right)x + c_2\left(\frac{1}{2}\right)x^2 + \cdots\right).$$

The recursive formula for $c_n\left(\frac{1}{2}\right)$ is

$$c_n\left(\frac{1}{2}\right) = \frac{1}{2n}\, c_{n-1}\left(\frac{1}{2}\right). \tag{4.73}$$

Calculation of few terms gives

$$y_2 = \sqrt{x}\left(1 + \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{48}x^3 + \cdots\right). \tag{4.74}$$

Note that $y_2'(x)$ goes unbounded when x approaches 0.

$s_1 = s_2$.

If $f(s)$ has a repeated root s_1, the second solution y_2 is

$$y_2(x) = y_1(x)\ln(x - x_0) + (x - x_0)^{s_1}\sum_{n=1}^{\infty} h_n\,(x - x_0)^n, \tag{4.75}$$

where h_n are determined by the following formula

$$h_n = \frac{dc_n}{ds}(s_1). \tag{4.76}$$

Proposition 4.2. *A recursive formula for h_n is as follows*

$$h_n = -\frac{1}{n^2}\sum_{k=0}^{n-1} \{[(k+s_1)a_{n-k} + b_{n-k}]\, h_k + a_{n-k}\, c_k\} - \frac{2}{n}\, c_n. \tag{4.77}$$

Proof. By reduction of order method if $y_1(x)$ is a solution of the equation, we can write $y_2(x)$ the second solution as

$$y_2 = y_1(x) \int \frac{e^{-\int p(x)/x}}{y_1^2(x)} dx. \tag{4.78}$$

Notice that

$$\frac{e^{-\int p(x)/x}}{y_1^2(x)} = x^{-2s} \frac{x^{-p_0} e^{-\left(p_1 x + \frac{1}{2} p_2 x^2 + \cdots\right)}}{g^2(x)}. \tag{4.79}$$

Since $p_0 = 1 - 2s$ (note that $f(s) = 0$ has the repetitive root), we have $x^{-2s} x^{-p_0} = x^{-1}$ and then

$$\frac{e^{-\int p(x)/x}}{y_1^2(x)} = x^{-1}(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots), \tag{4.80}$$

for some sequence (α_n). Note that $\alpha_0 = \frac{1}{y_0^2} \neq 0$. The second solution $y_2(x)$ is then derived by

$$y_2 = y_1(x) \int x^{-1}(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots) = \alpha_0 \, y_1(x) \ln(x) + \\ + \alpha_1 \phi_1(x) x + \frac{1}{2} \alpha_2 \, \phi_2(x) \, x^2 + \cdots.$$

Let us take $\alpha_0 = 1$ for simplicity and write

$$y_2 = y_1(x) \ln(x) + x^s \left(\alpha_1 \, x \, g(x) + \frac{1}{2} \alpha_2 \, x^2 \, g(x) + \cdots \right). \tag{4.81}$$

Recall that $y_1 = x^s g(x)$. We write the series in the bracket as

$$h(x) = \sum_{k=1}^{\infty} h_n \, x^n, \tag{4.82}$$

and find h_n. Similar to the proof of the theorem (4.5), if we substitute

$$y_2 = x^s [g(x) \ln(x) + h(x)], \tag{4.83}$$

into the equation, we reach

$$2x \, g' + (2s + a(x) - 1) \, g + x^2 \, h'' + x(a(x) + 2s) h' + F(s, x) \, h = 0 \tag{4.84}$$

In order to find h_n for $n \geq 1$, we calculate the n order derivative of the above equation. We have

$$2(x \, g')^{(n)} + [(2s + a(x) - 1) \, g]^{(n)} + [x^2 \, h'' + x(a(x) + 2s) h' + F(s, x) \, h]^{(n)} = 0.$$

If we follow the proof of the theorem (4.5), we obtain

$$2n \, c_n + \sum_{k=0}^{n-1} a_{n-k} \, c_k + f(s+n) \, h_n + \sum_{k=0}^{n-1} [(k+s) a_{n-k} + b_{n-k}] \, h_k = 0.$$

The above recursive formula is the same as one given in the theorem after a straightforward simplification. □

Example 4.22. Consider the equation (4.65). Since $s_1 = s_2$, we can write the second solution as

$$y_2 = x e^{-x} \ln(x) + x(h_1 x + h_2 x^2 + \dots).$$

The coefficients h_n can be calculated by formula (4.76) or the recursive formula (4.77). If we sue (4.77), we get

$$h_n = -\frac{1}{n^2}(n h_{n-1} + c_{n-1}) - \frac{2}{n} c_n. \tag{4.85}$$

Here are some values of h_n

$$h_1 = 1, h_2 = -\frac{3}{4}, h_3 = \frac{11}{36}, \dots$$

and then

$$y_2 = x e^{-x} \ln(x) + x^2 \left(1 - \frac{3}{4} x + \frac{11}{36} x^2 - \cdots\right). \tag{4.86}$$

It is interesting to find the second solution by the aid of reduction of order method. Since $y_1 = x e^{-x}$, a linearly independent solution y_2 is

$$y_2 = y_1(x) \int \frac{\exp\left(\int \frac{1-x}{x}\right)}{y_1^2(x)} = x e^{-x} \int \frac{e^x}{x}.$$

If we write the argument in the integral as the series

$$\frac{e^x}{x} = \frac{1}{x} + 1 + \frac{1}{2} x + \cdots,$$

we reach

$$y_2 = x e^{-x} \ln(x) + x e^{-x} \left(x + \frac{1}{4} x^2 + \frac{1}{18} x^3 + \cdots\right).$$

Finally, if we expand e^{-x}, we obtain

$$e^{-x}\left(x + \frac{1}{4} x^2 + \frac{1}{18} x^3 + \cdots\right) = \left(1 - x + \frac{1}{2} x^2 + \cdots\right)\left(x + \frac{1}{4} x^2 + \frac{1}{18} x^3 + \cdots\right) =$$
$$= x - \frac{3}{4} x^2 + \frac{11}{36} x^3 + \cdots.$$

Putting all together we obtain the same solution we derived above by (4.77).

$s_1 - s_2 = m$ an integer.

Let the roots of $f(s)$ are an integer, that is, $s_1 - s_2 = m \in \mathbb{Z}$. Then, a second linearly independent solution is

$$y_2 = \alpha y_1(x) \ln(x - x_0) + (x - x_0)^{s_2} \sum_{n=0}^{\infty} e_n (x - x_0)^n, \tag{4.87}$$

where the constant α and coefficients e_n are determined by the following formula

$$\alpha = \lim_{s \to s_2} (s - s_2) c_m(s), \quad \text{where} \quad m = s_1 - s_2, \tag{4.88}$$

$$e_n = \frac{d}{ds}(s - s_2) c_n(s)|_{s = s_2}. \tag{4.89}$$

4.3.4 Regular singular equations: III

If $f(s)$ has complex roots $s_{1,2}=\sigma \pm i\omega$ then $s_1 - s_2$ is non integer and thus we can write

$$y_1(x)=x^\sigma\, x^{i\omega}\sum_{n=0}^{\infty} c_n(s_1)\, x^n,\, y_2(x)=x^\sigma\, x^{i\omega}\sum_{n=0}^{\infty} c_n(s_2)\, x^n. \tag{4.90}$$

It is seen that $c_n(s_2)=\overline{c_n(s_1)}$, and thus

$$y_2(x)=x^\sigma\, x^{-i\omega}\sum_{n=0}^{\infty} \overline{c_n(s_1)}\, x^n. \tag{4.91}$$

According to the superposition property, we can derive two real solutions $\frac{1}{2}(y_1+y_2)$ and $\frac{1}{2i}(y_1-y_2)$ for the equation. It is simply verifies that two real solutions are as follows

$$y_1(x)=x^\sigma\sum_{n=0}^{\infty} \{\mathrm{Re}\, c_n(s_1)\, \cos(\omega \ln x) - \mathrm{Im}\, c_n(s_1)\, \sin(\omega \ln x)\}x^n \tag{4.92}$$

and

$$y_2(x)=x^\sigma\sum_{n=0}^{\infty} \{\mathrm{Re}\, c_n(s_1) \sin(\omega \ln x) + \mathrm{Im}\, c_n(s_1) \cos(\omega \ln x)\}\, x^n \tag{4.93}$$

Example 4.23. The following equation

$$x^2y''+x(1+x)y'+y=0,$$

has the index equation

$$s^2+1=0,$$

and thus $s_{1,2}=\pm i$. The formula for $c_n(i)$ is

$$c_n(i)=-\frac{n-1+i}{n(n+2i)}c_{n-1}(i).$$

and by assuming $c_0=1$, we obtain

$$c_1(i)=-0.4-0.2i,\, c_2(i)=0.1+0.05i,\, c_3(i)=-0.022-0.008i,\dots$$

and thus

$$y_1(x)=1+[-0.4\cos(\ln x)+0.2\sin(\ln x)]x+[0.1\cos(\ln x)-0.05\sin(\ln x)]x^2+\cdots$$

$$y_2(x)=1+[-0.4\sin(\ln x)-0.2\cos(\ln x)]x+[0.1\sin(\ln x)+0.05\cos(\ln x)]x^2+\cdots.$$

Problems

Problem 4.26. Classify the singular points of each of the following equations

i. $x^3\, y''-x\sin(x)\, y'+(1-\cos(x))\, y=0$,

ii. $x(1+x)\, y''+y'-y=0$,

iii. $(1-x^2)\, y''+xy'-e^x\, y=0$,

iv. $\sin^2(x)y''+y'+xy=0$.

Problem 4.27. Find the general solution to the following CAUCHY-EULER equations:

i. $x^2 y'' + 6xy' + 6y = 0$.

ii. $x^2 y'' + 7xy' + 9y = 0$.

iii. $x^2 y'' + xy' + 4y = 0$.

iv. $x^2 y'' + 3xy' + 2y = x\cos(\ln(x))$.

v. $(2x+1)^2 y'' + 2(2x+1)y' - 4y = x$.

Problem 4.28. Consider the following equation

$$xy'' + 2(1-x)y' + (x-2)y = 0$$

a) Find a series solution for the equation around $x_0 = 0$. Verify that the obtained series is the expansion of $\phi_1 = e^x$.

b) Use the reduction of order method to find $\phi_2(x)$, the second solution of the equation.

c) Use variation of parameters method to find the general solution to the following equation:

$$xy'' + 2(1-x)y' + (x-2)y = xe^x$$

Problem 4.29. Consider the following equation:

$$xy'' - (1-x)y' + y = 0.$$

a) Use FROBENIUS method to obtain a solution to the problem.

b) Verify that $y(x) = x^2 e^{-x}$ is a solution to the problem

$$\begin{cases} xy'' - (1-x)y' + y = 0 \\ y(0) = y'(0) = 0 \end{cases}.$$

This implies that the above problem has multiple solution. Why?

c) Obtain 5 nonzero terms of the second solution.

Problem 4.30. Consider the equation

$$x^2 y'' + x(1+x)y' - (1-2x)y = 0.$$

i. Use FROBENIUS method and show one solution is $y(x) = xe^{-x}$.

ii. Use reduction of order and conclude that the second solution is

$$z(x) = xe^{-x}\int \frac{e^x}{x^3}\,dx.$$

iii. Expand the integral and calculate few terms of the second solution.

iv. Calculate the second solution by the method described in this section and compare two solutions.

Problem 4.31. Consider the equation

$$x^2 y'' - x(1-x)y' + y = 0.$$

i. Use power series method and conclude that the first solution is $y(x) = xe^{-x}$.

ii. Use reduction of order to obtain the second solution as

$$z(x) = xe^{-x}\int \frac{e^x}{x}\,dx.$$

iii. Expand the e^x and calculate few terms of the above solutions.

iv. Calculate the second solution by the method described in this section and compare two solutions.

Problem 4.32. For each of the following equation, try to find two solutions. For each solution calculate few terms.

i. $4x^2 y'' - 2x(1+x)\, y' + 2y = 0$.

ii. $x^2 y'' + x y' + (x^2 - 1)\, y = 0$.

iii. $x^2 y'' + x(1+2x)\, y' + (x-2) y = 0$.

iv. $4x^2 y'' + 4x(1+2x)\, y' - y = 0$.

v. $x^2 y'' - x y' + (1-x) y = 0$.

vi. $x^2 y'' + x y' - (x + \frac{1}{9})\, y = 0$

vii. $x^2 y'' + x(x^2+1) y' - \frac{1}{4} y = 0$

viii. $x^2 y'' + x(1-2x) y' + (x - \frac{2}{9}) y = 0$

ix. $x^2 y'' + x(1+x)\, y' + (\frac{4}{3}x - \frac{1}{9}) y = 0$

Problem 4.33. For the equation

$$x(1-x)\, y'' + y' + (1-x)\, y = 0, \quad x > 0$$

a) Find two roots of the characteristic equation for $x_0 = 0$.

b) Show that y_n the coefficients of first series solution $y(x)$ is obtained by the following recursive formula for $y_1 = -y_0$ and for $n \geq 1$

$$y_{n+1} = \frac{n^2 - n - 1}{(n+1)^2}\, y_n + \frac{1}{(n+1)^2} y_{n-1}$$

c) Obtain the interval of convergence for $y(x)$ using the recursive formula.

d) Show that for the second solution $z(x)$, the coefficients (d_n) of the series solution are obtained through the following recursive formula

$$d_n = -\frac{2}{n} c_n - \frac{1}{n^2} \sum_{k=0}^{n-1} c_k - \frac{1}{n^2} \left\{ \sum_{k=0}^{n-1} k d_k + d_{n-1} \right\}$$

Problem 4.34. Here we obtain the power series solution associated with the complex roots of characteristic equation. For the equation

$$x^2 y'' + x(1-x)\, y' + y = 0, \quad x > 0$$

i. Show that $s_1 = i, s_2 = -i$ are roots of the characteristic equation for $x_0 = 0$.

ii. Show that the coefficients of power series solution at $x_0 = 0$ are obtained through the formula

$$y_n(i) = \left(\frac{n^2 - n + 2}{n(n^2+4)} + i \frac{2-n}{n(n^2+4)} \right) y_{n-1}(i) = \overline{y_n(-i)}$$

iii. Find four non-zero terms of each series solution.

4.4 Differential equations of mathematical physics

In this section, we study a few equations that are frequently used in mathematical physics. They appear again the second part of this book where we study partial differential equations.

4.4.1 Hermite equation

The general form of Hermite equation is

$$y'' - 2x\,y' + \lambda y = 0, \tag{4.94}$$

where $\lambda \in \mathbb{R}$ is a parameter. The point $x_0 = 0$ is a regular point of the equation and the recursive formula of the coefficients is

$$y_{n+2} = \frac{2n - \lambda}{(n+2)(n+1)}\, y_n, n \geq 0. \tag{4.95}$$

Here the values y_0 and y_1 are arbitrary and thus the equation admits two linearly independent analytic solutions. Observe that for $\lambda = 2n$, the coefficient y_{n+2} is zero and then $y_{n+2k} = 0$ for any k. Therefore, one of the solutions is a polynomial of order n which is denoted by $H_n(x)$ and is called the Hermite polynomial after the French Mathematician Charles Hermite (1822 – 1901).

Example 4.24. Consider the following initial value problem

$$\begin{cases} y'' - 2xy' + 10y = 0 \\ y(0) = y_0, y'(0) = y_1 \end{cases}. \tag{4.96}$$

The recursive formula (4.95) implies $y_7 = y_9 = y_{11} = \cdots = 0$, and thus

$$y = y_0\left(1 - 5x^2 + \frac{5}{2}x^4 + \frac{1}{6}x^6 + \cdots\right) + y_1\left(x - \frac{4}{3}x^3 + \frac{4}{15}x^5\right). \tag{4.97}$$

Now, if $y_0 = 0$, we obtain the polynomial solution

$$H_5(x) = x - \frac{4}{3}x^3 + \frac{4}{15}x^5. \tag{4.98}$$

Proposition 4.3. (Rodrigues formula) *The polynomial solution to the* Hermite *equation is obtained by the following formula*

$$H_n(x) = (-1)^n\, e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \tag{4.99}$$

Proof. We verify that H_n satisfies (4.94). For the sake of simplicity, let us use the notation $D^n = \frac{d^n}{dx^n}$. By substituting H_n into the equation, we reach

$$D^{n+2}\, e^{-x^2} + 2x\, D^{n+1}\, e^{-x^2} + 2(n+1) D^n\, e^{-x^2} = 0. \tag{4.100}$$

We claim that the above identity is true. We have

$$D^{n+2}\, e^{-x^2} = -2D^{n+1}(x\, e^{-x^2}) = -2\sum_{k=0}^{n+1} \begin{pmatrix} n+1 \\ k \end{pmatrix} D^k x\, D^{n+1-k}\, (e^{-x^2}),$$

and by simplifying Simplification the right hand side, we get

$$D^{n+2}\, e^{-x^2} = -2\,x\, D^{n+1}\, e^{-x^2} - 2(n+1) D^n\, e^{-x^2}, \tag{4.101}$$

and thus the claim. □

Proposition 4.4. *$H_n(x)$ is even function if n is even and odd if n is odd.*

Proof. According to the formula (4.99) and the fact that the derivatives of even polynomials are odd and vice versa, the function

$$D^n e^{-x^2} = D(D^{n-1} e^{-x^2}), \tag{4.102}$$

is even if $D^{n-1} e^{-x^2}$ is odd and odd if $D^{n-1} e^{-x^2}$ is even. But, $H_0 = 1$ is even and $H_1 = 2x$ is odd and this justifies the claim. □

Remark 4.1. The HERMITE equation finds its application in quantum mechanics. Usually, the HERMITE equation is written in the following *eigenvalue problem* form

$$\frac{d}{dx}[e^{-x^2} y'] = -\lambda e^{-x^2} y. \tag{4.103}$$

Here λ is called an *eigenvalue* and a non-zero solution of the equation is called an *eigenfunction* $\phi(x)$. Physicists are interested in eigenfunctions with the bounded energy, that is,

$$E[\phi] = \int_0^{\infty} e^{-x^2} \phi^2(x)\, dx < \infty. \tag{4.104}$$

It is seen that the above integral diverge if $\lambda \neq 2n$ and converges if $\lambda = 2n$, and thus $\phi(x) = H_n(x)$ are only acceptable eigenfunctions.

4.4.2 CHEBYSHEV equation

The general form of the CHEBYSHEV equation is

$$(1 - x^2) y'' - x y' + \lambda y = 0, \tag{4.105}$$

where $\lambda \in \mathbb{R}$ is a parameter. The point $x_0 = 0$ is a regular point for the equation and the convergence interval of the series solution is $(-1, 1)$. The recursive formula for the coefficients of the series solution is

$$y_{n+2} = \frac{n^2 - \lambda}{(n+2)(n+1)} y_n, n \geq 0 \tag{4.106}$$

with y_0 and y_1 arbitrary and thus two linearly independent analytic solutions. Observe that if $\lambda = n^2$ then $y_{n+2} = 0$ and therefore $y_{n+2k} = 0$ for all $k \geq 0$. This implies that one solution to the equation is a polynomial of order n. This polynomial is denoted by $T_n(x)$, and is called the CHEBYSHEV polynomial after the Russian mathematician PAFNUTY CHEBYSHEV (1821-1894).

Example 4.25. Consider the following initial value problem

$$\begin{cases} (1 - x^2) y'' - x y + 4y = 0 \\ y(0) = y_0, y'(0) = y_1 \end{cases}. \tag{4.107}$$

The recursive formula implies $y_2 = -2y_0$ and that $y_{2k} = 0$ for all $k \geq 2$. The solution is

$$y = y_0(1 - 2x^2) + y_1\left(x - \frac{1}{2}x^3 - \frac{1}{8}x^5 - \cdots\right). \tag{4.108}$$

If $y_1 = 0$, the above series reduces to the polynomial $1 - 2x^2$.

Proposition 4.5. *The polynomial solution of* (4.105) *is derived by the formula*

$$T_n(x) = \cos(n\cos^{-1}x). \tag{4.109}$$

Proof. If we take $x = \cos(\theta)$ in (4.105), the equation is transformed to the form

$$\frac{d^2y}{d\theta^2} + \lambda y = 0. \tag{4.110}$$

Clearly, the above equation has one solution $y = \cos(\sqrt{\lambda}\theta)$. Assuming $\lambda = n^2$, $n \in \mathbb{Z}$, we obtain $y = \cos(n\cos^{-1}x)$. We show that this solution is a polynomial of order n. By the formula

$$\cos(n\theta) = \frac{1}{2}[(e^{i\theta})^n + (e^{-i\theta})^n] = \frac{1}{2}[(\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n], \tag{4.111}$$

and the binomial formula

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k, \tag{4.112}$$

we derive

$$\cos(n\theta) = \sum_{k:\text{even}}^{n} \binom{n}{k} (-1)^{k/2} \cos^{n-k}\theta \sin^k\theta. \tag{4.113}$$

Now replace $\sin^k\theta = (1 - \cos^2\theta)^{k/2}$ for k even, and $x = \cos\theta$, to obtain

$$\cos(n\cos^{-1}x) = \sum_{k:\text{even}}^{n} \binom{n}{k} x^{n-k}(1 - x^2)^{k/2}, \tag{4.114}$$

which is a polynomial of order n. □

Properties of Chebyshev polynomial.

$T_n(x)$ have important properties and are extensively used in the approximation of functions, and also in mathematical physics. We discuss some of its properties below.

1. The solution of a Chebyshev equation goes unbounded at $x = \pm 1$ if $\lambda \neq n^2$. The only possibility that the solution remain bounded is the case $\lambda = n^2$ for n, an integer.

2. $T_n(x)$ is an even function for n even and an odd function for n odd.

3. $T_n(x)$ satisfies the following recursive formula

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \tag{4.115}$$

where $T_0(x) = 1$ and $T_1(x) = x$. In fact, the above formula is another form of the familiar identity

$$\cos((n+1)\theta) + \cos((n-1)\theta) = 2\cos\theta\cos(n\theta). \tag{4.116}$$

The figure (4.4) shows a few of the CHEBYSHEV polynomials.

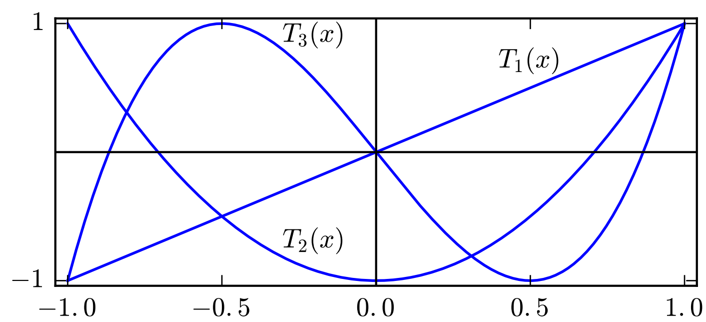

Figure 4.4. The graphs of some CHEBYSHEV polynomials.

4. Since every continuous function can be approximated by polynomials, we can approximate a continuous function defined in $-1 \le x \le 1$ by $T_n(x)$, that is,

$$f(x) \cong c_0 T_0 + c_1 T_1(x) + \cdots + c_n T_N(x). \tag{4.117}$$

The advantage of expanding f in terms of T_n comes from the following *orthogonal* property:

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_n(x)\, T_m(x) dx = \begin{cases} 0 & n \neq m \\ \pi & n = m = 0 \\ \frac{\pi}{2} & n = m \neq 0 \end{cases}. \tag{4.118}$$

The proof is left as an exercise. This property let us to determine constants c_k in (4.117) as follows. For c_0, we multiply (4.117) by $\frac{1}{\sqrt{1-x^2}} T_0$ and integrate in the interval $(-1,1)$. Since $T_0 = 1$, according to the orthogonality property, we obtain

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x) dx = c_0 \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} dx = c_0 \pi, \tag{4.119}$$

and thus

$$c_0 = \frac{1}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x)\, dx. \tag{4.120}$$

Repeating the calculation for $k \ge 1$, we reach

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x)\, T_k(x)\, dx = c_k \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_k^2(x) dx.$$

But by trigonometric substituting, we have

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_k^2(x) dx = \int_{0}^{\pi} \cos^2(k\theta)\, d\theta = \frac{\pi}{2},$$

and therefore

$$c_k = \frac{2}{\pi}\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, f(x)\, T_k(x). \qquad (4.121)$$

In the problem set, we asked the reader to approximate a continuous function with some polynomial and compare the results.

4.4.3 LEGENDRE equation

The general form of LEGENDRE equations is

$$(1-x^2)y'' - 2x y' + \lambda y = 0, \qquad (4.122)$$

where λ is a real value. The point $x_0 = 0$ is regular and the equation is defined in the interval $-1 < x < 1$. The recursive formula for the coefficients of the series solution is

$$y_{n+2} = \frac{n(n+1) - \lambda}{(n+2)(n+1)}\, y_n. \qquad (4.123)$$

The radius of convergence of the series is $L = 1$. If $\lambda = n(n+1)$ for some positive integer value n, then one solution becomes a polynomial which is denoted by $P_n(x)$ and is called the LEGENDRE polynomial after the French mathematician ADRIEN MARIE LEGENDRE (1752-1833).

Example 4.26. Consider the problem

$$\begin{cases} (1-x^2)y'' - 2xy' + 6y = 0 \\ y(0) = y_0,\ y'(0) = y_1 \end{cases}. \qquad (4.124)$$

Here $\lambda = 6 = 2 \times 3$ and then the equation have a polynomial solution. By the recursive formula, we have $y_2 = -3\,y_0$, and $y_{2k} = 0$ for $k \geq 2$. Therefore, the general solution is

$$y = y_0(1-3x^2) + y_1\left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5 + \cdots\right). \qquad (4.125)$$

If $y_1 = 0$, then $y = 1 - 3x^2$ is the LEGENDRE polynomial solution.

Proposition 4.6. (RODRIGUES) $P_n(x)$ *are derived by the formula*

$$P_n(x) = \frac{(-1)^n}{2^n\, n!}\frac{d^n}{dx^n}(1-x^2)^n. \qquad (4.126)$$

Proof. The factor in the front of the derivative is just to normalize the polynomials. First, we have

$$D^{n+1}[(1-x^2)\, D(1-x^2)^n] = \sum_{k=0}^{n+1} \binom{n+1}{k} D^k(1-x^2)\, D^{n+2-k}\,(1-x^2)^n. \qquad (4.127)$$

Simplifying the above formula gives

$$D^{n+1}[(1-x^2)\,D(1-x^2)^n] = (1-x^2)\,D^{n+2}(1-x^2)^n - 2x(n+1)\,D^{n+1}(1-x^2)^n - \\ -n(n+1)D^n(1-x^2)^n.$$

On the other hand, we have

$$D^{n+1}[(1-x^2)\,D(1-x^2)^n] = -2n\,D^{n+1}[x\,(1-x^2)^n] = \\ -2nx\,D^{n+1}(1-x^2)^n - 2n(n+1)\,D^n(1-x^2)^n.$$

Equating two above identities, gives

$$(1-x^2)\,D^2\,P_n(x) - 2x\,D\,P_n(x) + n(n+1)\,P_n(x) = 0, \tag{4.128}$$

which is the LEGENDRE equation. □

Properties of LEGENDRE polynomials.

We use LEGENDRE equation frequently in the second part of this book. Here, we introduce some of its important properties.

1. The solution of LEGENDRE equation goes unbounded at $x=\pm 1$ except the polynomial solution for $\lambda = n(n+1)$. In this case, the equation has a polynomial solution $P_n(x)$.
2. It is verified immediately from the RODRIGUES formula that $P_n(x)$ is even function for n even and odd for n odd. The figure (4.5) shows the graphs of some LEGENDRE polynomials:

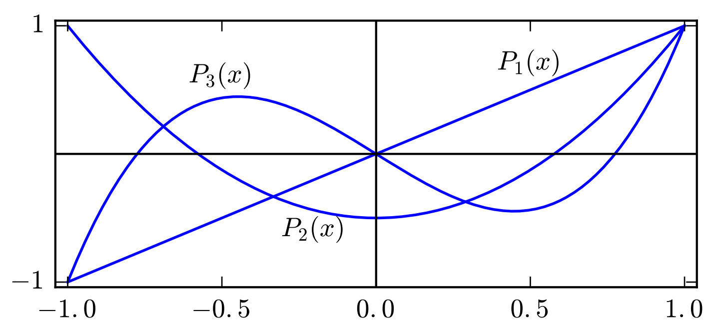

Figure 4.5. The graphs of some LEGENDRE polynomials.

3. Polynomials $P_n(x)$ satisfies the following orthogonality property

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \begin{cases} 0 & n \neq m \\ \frac{2}{2n+1} & n = m \end{cases}. \tag{4.129}$$

We can approximate continuous functions by $P_n(x)$ in the interval $-1 \leq x \leq 1$. That is, if f is continuously defined in $[-1,1]$, then

$$f(x) \cong c_0\,P_0(x) + \cdots + c_n P_n(x), \tag{4.130}$$

where by (4.129), the coefficients are determined by the formula

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x)\, P_n(x)\, dx. \tag{4.131}$$

4.4.4 BESSEL equation

The general form of the BESSEL equation is

$$x^2 y'' + x y' + (x^2 - \lambda^2) y = 0, \tag{4.132}$$

where $\lambda \in \mathbb{R}$ is a parameter. The point $x_0 = 0$ is a regular-singular point for the equation. In order to keep the solution, we must impose the following initial conditions

$$\lim_{x \to 0} y(x)\text{: bounded}, \quad \text{and}, \quad \lim_{x \to 0} y'(x)\text{: bounded}. \tag{4.133}$$

Note that $a(x) = 1$ and $b(x) = x^2 - \lambda^2$ and the characteristic polynomial is $f(s) = s^2 - \lambda^2$, with roots $s_1 = \lambda$ and $s_2 = -\lambda$. Therefore, one solution is

$$y(x) = x^{\lambda} \sum_{n=0}^{\infty} c_n x^n,$$

where c_n are calculated from formula (4.48). A direct simplification yields

$$\begin{cases} c_n = -\frac{1}{n(n+2\lambda)} c_{n-2} \\ c_0 = \text{arbitrary} \\ c_1 = 0 \end{cases}. \tag{4.134}$$

Note that $c_n = 0$ for $n = 2k+1$ and for $n = 2k$, we have

$$c_{2k} = -\frac{1}{2^2\, k(k+\lambda)} c_{2k-2}, k = 1, 2, \cdots.$$

In particular, if $\lambda = m$ an integer, then

$$c_{2k} = \frac{(-1)^k\, m!}{2^{2k}\, k!\, (k+m)!},$$

and thus

$$y(x) = c_0 \sum_{k=0}^{\infty} \frac{(-1)^k\, m!}{2^{2k}\, k!\, (k+m)!} x^{2k+m}.$$

For $c_0 = \frac{1}{m! 2^m}$, we obtain the BESSEL function of the *first type*

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+m)!} \left(\frac{x}{2}\right)^{2k+m}.$$

The second solution of the BESSEL equation can be derived by the method outlined in this chapter. Since $s_1 - s_2 = 2\lambda$, if 2λ is not an integer, the second solution is

$$Y_{\lambda}(x) = x^{-\lambda} \sum_{n=0}^{\infty} c_n(-\lambda)\, x^n, \tag{4.135}$$

where $c_n(-\lambda)$ can be calculated by formula (4.71). If $2\lambda = n$ is an integer, the second solution is determined by the method we explained in the previous section. The second solution $Y_\lambda(x)$ is called the BESSEL function of *second type*. The following figure shows $J_{1/2}(x)$ and $J_2(x)$ in the same coordinate.

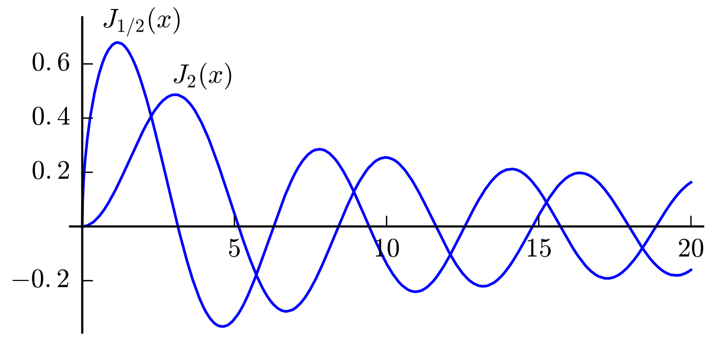

Figure 4.6.

Observe the quasi-periodicity of the BESSEL functions. We can justify this by the following argument. By the substitution $u = \sqrt{x}\ y$, the equation (4.132) becomes (see the problem set)

$$u'' + \left(1 - \frac{\lambda^2 - \frac{1}{4}}{x^2}\right) u = 0. \tag{4.136}$$

When $x \to \infty$, the equation (4.136) looks like a harmonic oscillator with the solutions $u = A_0 \sin(x + \varphi_0)$. This justifies the fact that for x sufficiently large, the solution $y(x)$ is

$$y \sim \frac{A_0}{\sqrt{x}} \sin(x + \varphi_0). \tag{4.137}$$

4.4.5 GAUSS hyper-geometric equation

The general form of GAUSS equation is

$$x(x-1)\, y'' + [(\alpha + \beta + 1)x - \gamma] y' + \alpha \beta y = 0, \tag{4.138}$$

where $\alpha,\ \beta,\ \gamma$ are constants. Although, the form of the equation seems somehow far reaching, the reader is asked to verify that all equations we studied above are specific instances of this general equation. Note that the GAUSS equation has tow regular-singular points $x_0 = 0$ and $x_1 = 1$. At $x_0 = 0$, we have

$$p(x) = \frac{(\alpha + \beta + 1)x - \gamma}{x - 1}, \text{ and } q(x) = \frac{\alpha \beta x}{x - 1}, \tag{4.139}$$

and the characteristic polynomial is

$$f(s) = s^2 + (\gamma - 1)s, \tag{4.140}$$

with roots $s=0$ and $s=1-\gamma$. If γ is not an integer, then there are two independent solutions

$$y_1(x)=\sum_{n=0}^{\infty} y_n(0)\,x^n,\ \text{and}\ y_2(x)=x^{1-\gamma}\sum_{n=0}^{\infty} y_n(1-\gamma)\,x^n. \tag{4.141}$$

If we substitute $y_1(x)$ and $y_2(x)$ into the Gauss equation, we get the following recursive formula for the coefficients

$$c_n(0)=\frac{(\alpha+n-1)(\beta+n-1)}{n(n-1+\gamma)}\,c_{n-1}(0), \tag{4.142}$$

and

$$c_n(1-\gamma)=\frac{(n+\alpha-\gamma)(n+\beta-\gamma)}{n(n+1-\gamma)}\,c_n(1-\gamma). \tag{4.143}$$

A straightforward calculation gives the following formula if $y_0=1$:

$$c_n(0)=\frac{\prod_{k=1}^{n}(\alpha+k-1)(\beta+k-1)}{n!\prod_{k=1}^{n}(\gamma+k-1)}, \tag{4.144}$$

$$c_n(1-\gamma)=\frac{\prod_{k=1}^{n}(k+\alpha-\gamma)(k+\beta-\gamma)}{n!\prod_{k=1}^{n}(k+1-\gamma)}. \tag{4.145}$$

The series with the coefficients $c_n(0)$ is denoted by $F(\alpha,\beta,\gamma;x)$ and are called hyper-geometric functions. Accordingly, the series with the coefficients $c_n(1-\gamma)$ is written $F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;x)$. We conclude that for γ a non-integer, the solution to (4.138) is

$$\phi(x)=c_1\,F(\alpha,\beta,\gamma;x)+c_2\,x^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;x). \tag{4.146}$$

For the solution at $x_1=1$, we take the substitution $t=1-x$ to rewrite the equation as

$$t(t-1)y_t''+[(\alpha+\beta+1)\,t-\gamma']\,y_t'+\alpha\beta y_t=0, \tag{4.147}$$

where $\gamma'=\alpha+\beta+1-\gamma$. Thus, for γ a non-integer, the solution at $x_1=1$ can be written as

$$\phi(x)=c_1\,F(\alpha,\beta,\gamma';1-x)+c_2\,(1-x)^{1-\gamma'}F(\gamma-\beta,\gamma-\alpha,2-\gamma';1-x). \tag{4.148}$$

For γ an integer, the second solution is determined by the method we presented in the previous section.

Problems

Problem 4.35. Verify that the radius of convergence of the series generated by (4.95) is infinity.

Problem 4.36. Find a polynomial solution for the following equation:

$$y''-2x\,y'+8y=0,$$

and compare it with $H_4(x)$.

Problem 4.37. Show that the $H_n(x)$ satisfy the following recursive formula

$$H_{n+1}-2x\,H_n+2n\,H_{n-1}=0.$$

Problem 4.38. Show the following relation for H_n

$$\frac{d}{dx}H_n = 2n\,H_{n-1}.$$

Problem 4.39. Find the radius of the convergence of the series generated by (4.106).

Problem 4.40. Use mathematical induction to prove that $\cos(n\theta)$ is a polynomial in terms of $\cos(\theta)$ and conclude that one solution of the CHEBYSHEV equation is a polynomial.

Problem 4.41. Show that $T_n(x)$ even if n is even number and odd if n is odd.

Problem 4.42. Show that the CHEBYSHEV polynomial satisfies the following recursive formula

$$T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x).$$

Use the above formula to calculate $T_n(x)$ for $n = 0, 1, 2, 3$.

Problem 4.43. Prove the orthogonality property (4.118).

Problem 4.44. Find two solutions of the following equations

i. $(1 - x^2)y'' - x\,y + 9y = 0.$

ii. $(1 - x^2)y'' - x\,y + 16y = 0.$

Problem 4.45. Find an approximation of the following analytic functions of in the interval $(-1, 1)$ in terms of polynomials $(T_0, \ldots, T_4)$ and compare them with the approximation by $(1, x, \ldots, x^4)$:

i. $f(x) = e^x$,

ii. $f(x) = \sin(x)$.

Problem 4.46. Solve the following equation

$$(1 - x^2)y'' - 2x\,y' + 2y = 0.$$

Problem 4.47. Show that the substitution $u = \sqrt{x}\ y$ transforms the BESSEL equation into the following equation

$$u'' + \left(1 - \frac{\lambda^2 - \frac{1}{4}}{x^2}\right)u = 0.$$

Since the above equation looks like a simple harmonic oscillator for large x, it justifies that $y(x) \to 0$ when $x \to \infty$.

Problem 4.48. Consider the following equation

$$y'' + c x^m\,y = 0,$$

for $c > 0$ and $m \neq -2$.

i. Apply the substitution $y = \sqrt{x}\,u$ to obtain

$$x^2 u'' + x\,u' + (c x^{m+2} - \frac{1}{4})u = 0.$$

ii. Now apply the substitution

$$t = \frac{2\sqrt{c}\,x^{\frac{m+2}{2}}}{m+2},$$

to obtain

$$t^2\,u'' + t\,u' + (t^2 - \frac{1}{(m+2)^2})u = 0.$$

iii. Now solve the following equation using the above substitution:

$$y'' + 9x^3\,y = 0.$$

Problem 4.49. Show that the substitution $t=2e^{x/2}$ transforms the following equation into a Bessel equation

$$y''+(e^x-m^2)y=0.$$

Use the above substitution to solve the following equation

$$y''+(e^x-4)y=0.$$

Problem 4.50. Show that the equation

$$(x-r_1)(x-r_2)\,y''+a(x-r_3)\,y'+by=0$$

can be transformed to the Gauss equation through the substitution

$$x=(r_2-r_1)z+r_1.$$

Use the above transformation to solve the following equation

$$3x(x-2)\,y''-(x+3)y'+y=0$$