Chapter 3
Higher Order Equations

The dynamics of several physical systems are expressed in terms of second or higher-order ordinary differential equations. We study this class of equations in this chapter.

3.1 Introduction

1. **(General form)** The general form of a second-order ODE is

$$y'' = f(x, y, y'), \tag{3.1}$$

where $y'' = \frac{d^2y}{dx^2}$ is the second-order derivative of unknown function $y(x)$. A standard model of this equation in physics is the NEWTON's second law

$$m\frac{d^2x}{dt^2} = f(t, x, x'),$$

where $x = x(t)$ is the position function of a mass m, and f is the total force exercising on m. The general form of an *initial value problem* for second-order equation is as follows

$$\begin{cases} y'' = f(x, y, y') \\ y(x_0) = y_0 \\ y'(x_0) = y_1 \end{cases}.$$

To determine the location of a mass m at any instance of time, one needs in addition to the total force, the *initial state* of the mass which is the pair $(x(0), v(0))$ or equivalently $(x(0), x'(0))$. The general form of a n-order equation is

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}),$$

where $y^{(n)}$ stands for n^{th} order derivative $\frac{d^n y}{dx^n}$. The associated initial value problem contains n initial conditions of the form

$$y(x_0), y'(x_0), \ldots, y^{(n-1)}(x_0).$$

2. **(Linear equation)** If function f in Eq.3.1 is linear with respect to y and y', then the equation is called *linear*. The general form of a second-order linear equation is

$$y'' + p(x)y' + q(x)y = r(x). \tag{3.2}$$

The justification of the terminology will be clear if we think of differentiation as an *operator*. In fact, if we interpret $y' := \frac{dy}{dx}$ as the action of mapping $\frac{d}{dx}$ on function y and rewrite it as $\frac{d}{dx}[y]$ or simply by $D[y]$, where $D := \frac{d}{dx}$, then Eq.3.2 can be rewritten as

$$(D^2 + p(x)\,D + q(x))[y] = r(x), \tag{3.3}$$

where $D^2 := \frac{d^2}{dx^2}$. For the sake of simplicity, let us denote the *composite operator* $D^2 + p(x)\,D + q(x)$ by T, and rewrite equation in the following operator form

$$T[y] = r(x).$$

It is straightforward to verify that T is a *linear operator*, that is, for any two functions y_1, y_2 and any constants c_1, c_2, operator T satisfies the relation

$$T[c_1 y_1 + c_2\,y_2] = c_1\,T[y_1] + c_2\,T[y_2].$$

Remember the concept of *linear mappings* from linear algebra and compare it with the above concept of a *linear differential operator*.

3. **(Linear homogeneous equations)** If function $r(x)$ in Eq.3.2 is identically zero, the equation is called *linear homogeneous*, otherwise, linear non-homogeneous. The solutions of a linear homogeneous equation is called *homogeneous solution*. The general form of a linear homogeneous equation is

$$y'' + p(x)y' + q(x)y = 0,$$

or in the operator form $T[y] = 0$. Note that if $y_1(x)$, $y_2(x)$ are two homogeneous solutions, that is, $T[y_1] = T[y_2] = 0$, then any linear combination

$$y = c_1\,y_1(x) + c_2\,y_2(x),$$

is also a homogeneous solution. The claim is simply verified by the *linearity property* of T, that is,

$$T[y] = T[c_1\,y_1 + c_2\,y_2] = c_1\,T[y_1] + c_2\,T[y_2] = 0.$$

Remember the concept of *null space* of a linear mapping L: $\mathbb{R}^n \to \mathbb{R}^m$ in linear algebra. A vector $\vec{u}$ is in the null space of L if $L[\vec{u}] = 0$, and moreover, the null space of a linear mapping is a *vector subspace* of the domain space, that is, if $\vec{u}_1, \vec{u}_2$ are in null space of L, then for any constants c_1, c_2, the *linear combination* $c_1\,\vec{u}_1 + c_2\,\vec{u}_2$ is in the null space of L. The same property holds for the differential operator T and the linear homogeneous differential equations.

4. **(Non-homogeneous equations)** Consider non-homogeneous equation $T[y] = r(x)$. If $y_p(x)$ is a solution to the equation, that is, $T[y_p(x)] = r(x)$, then for any homogeneous solution y_h which is in the null space of T, $T[y_h] = 0$, function $y_h + y_p$ solves the non-homogeneous equation, that is, $T[y_h + y_p] = r(x)$. The claim is simply verified using the linearity property of T, that is,

$$T[y_h + y_p] = T[y_h] + T[y_p] = 0 + r(x).$$

5. **(Mass-spring system)** Consider the following figure where a mass m is connected to an ideal spring with stiffness k:

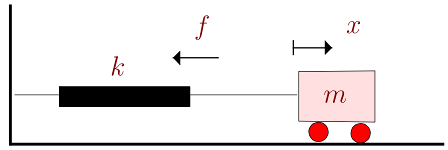


Figure 3.1.

In the figure, $x(t)$ represents the displacement of the mass with respect to its *resting position*, where the spring is not under stretch or contraction. The force exercising by the spring follows the HOOK's law $f=-kx(t)$ where $k>0$ denotes the stiffness of the spring. If m is not under any external force $f_{\text{ext}}(t)$, we reach the following equation for the displacement x

$$m\frac{d^2x}{dt^2}=-kx. \tag{3.4}$$

We have to solve the above equation in order to determine the position of m at any instance of time.

6. **(Initial states of the mass-spring)** The *state* of a mass-spring system is defined by pair $(x(t),\ x'(t))$, the position and the velocity, and thus the initial state is the pair $(x_0,\ v_0)$ where $v_0=x'(0)$ is the initial velocity of the mass. The following problem is the associated *initial value problem* of the mass-spring system

$$\begin{cases} m\frac{d^2x}{dt^2}=-kx \\ x(0)=x_0, x'(0)=v_0 \end{cases}. \tag{3.5}$$

It is simply verified that above problem is solved for the following function

$$x(t)=x_0\cos\left(\sqrt{\frac{k}{m}}\,t\right)+v_0\sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{k}{m}}t\right).$$

Note that if $x_0=v_0=0$, then $x(t)=0$ for all t, that means the system is in rest or equilibrium. Also note that differential equation

$$\frac{d^2x}{dt^2}+\frac{k}{m}x=0,$$

has two *linearly independent* solutions

$$x_1(t)=\cos\left(\sqrt{\frac{k}{m}}\,t\right),\quad x_2(t)=\sin\left(\sqrt{\frac{k}{m}}\,t\right).$$

It is clear from the solution that mass m moves back and forth following a trigonometric sine function.

7. **(External forces)** If an external engine is connected to m exercising a force $f = r(t)$, then Eq.(3.4) reads

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+kx=r(t). \tag{3.6}$$

Let us see the role of $r(t)$ in the solution of the equation. Consider the initial value problem

$$\begin{cases} m\frac{d^2x}{dt^2}+kx=r(t) \\ x(0)=0, x'(0)=0 \end{cases},$$

where $r(t)$ is as follows

$$r(t)=\begin{cases} 0 & t<1 \\ 1 & t>1 \end{cases}.$$

In fact, for $t<1$, the problem is

$$\begin{cases} m\frac{d^2x}{dt^2}+kx=0 \\ x(0)=0, x'(0)=0 \end{cases},$$

and the unique solution is $x(t)=0$ f or $t<1$. For $t>1$, the problem becomes

$$\begin{cases} m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+kx=1 \\ x(0)=0, x'(0)=0 \end{cases},$$

and the unique solution is $x(t)=\frac{m}{k}\left(1-\cos\left(\sqrt{\frac{k}{m}}\,t\right)\right)$. Therefore, the mass-spring systems starts motion only at $t=1$ due to the external force $r(t)$.

Remark. The general solution of a mass-spring system consists two terms: 1) the solution associated to the initial conditions, 2) the solution associated to the external force.

8. **(Energy interpretation)** Obviously if the mass-spring is not initially at the resting state, the the mass moves periodically, and the domain of motion depends on the internal energy of the system (in the absence of any external force). Multiply equation $mx''+kx=0$ by x', and rewrite it as follows

$$\frac{d}{dt}\left(\frac{1}{2}m\,|x'|^2+kx^2\right)=0.$$

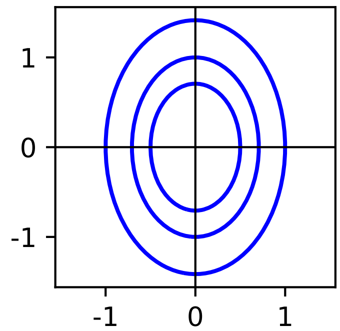
and therefore

$$E(t):=\frac{1}{2}m\,[x'(t)]^2+k\,[x(t)]^2=\text{const.}$$

Note that in the above formula, the first term is the *kinetic energy* of the mass, and the second term is the *potential energy* of the spring in accordance with the relation $f=-\frac{d}{dx}(kx^2)$. Therefore, the total energy of the mass is independent of t and thus is equal to its initial energy

$$E(t)=E_0:=\frac{1}{2}mv_0^2+kx_0^2.$$

The graph of the energy in the (x,x')-plane is very important for us. The following figure show three different level of energy for the system of $m=k=1$.



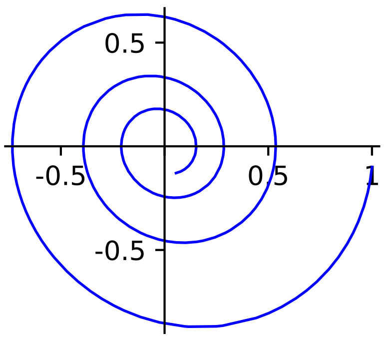
9. **(Damped mass-spring)** Now assume the friction force for the motion of the form $f_d=-2\xi v$ where ξ is a constant and v is the velocity of m. The differential equation in this case reads

$$m\frac{d^2x}{dt^2}=-kx-b\frac{dx}{dt}. \tag{3.7}$$

Note that in the presence of friction, the energy of the systems vanishes in time Accordingly, we expect that the motion vanishes in long term, that is,

$$\lim_{t\to\infty}x(t)=0.$$

The following figure shows the level of energy for $\xi=0.05$ of a system with $m=k=1$.



10. **(A nonlinear equation)** Now we consider a single pendulum that it mathematical model is a nonlinear differential equation. Consider the pendulum shown in Fig(3.2). The total force acting on m is $f=mg$, where the component $f_s=mg\cos(\theta)$ is balanced by the string tension. Therefore, $f_a=mg\sin(\theta)$ is the only force that causes the motion of m. Since $x=\ell\theta$, the Newton's second law reads

$$m\ell\frac{d^2\theta}{dt^2}=-mg\sin(\theta). \tag{3.8}$$

The negative sign enters because this force push the mass back to its *resting* position $\theta=0$. Canceling out m from both sides of the equation, we reach the following equation for $\theta(t)$

$$\frac{d^2\theta}{dt^2}+\frac{g}{\ell}\sin(\theta)=0. \tag{3.9}$$

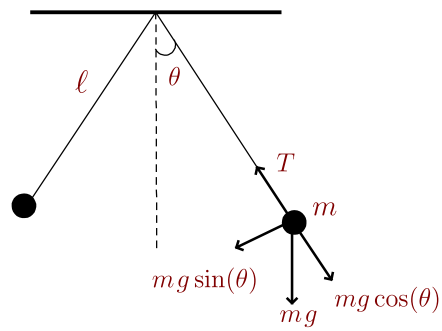


Figure 3.2.

Scientists usually tend to linearize the nonlinear equations, and thus by assuming $\sin\theta \sim \theta$ for small θ, they write the equation as follows

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0. \tag{3.10}$$

The reason is that the later equation is simply solved by an standard method, while the former one is a little complicate.

Problems

Problem 3.1. Consider the mass-spring system introduced in this section

a) Integrate the energy equality equation

$$\frac{1}{2}m\,[x'(t)]^2 + \frac{1}{2}k\,[x(t)]^2 = \frac{1}{2}m\,[x'(0)]^2 + \frac{1}{2}k\,[x(0)]^2$$

for $x'(0) = 0$ and $x(0) = x_0$, and find the displacement function $x(t)$ of

b) Now assume a drag force for the system. Show that the energy dissipate in long terms, that is,

$$\lim_{t\to\infty} E(t) = 0.$$

Problem 3.2. Consider the circuit shown in the figure (3.3)

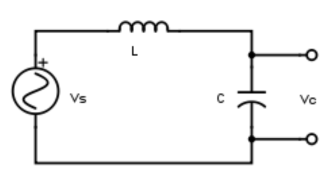


Figure 3.3.

Assume that V_c, the voltage across the capacitor C, is chosen as the response of the circuit to V_s, the power supply (input or forcing term).

a) Suppose that $V_s = 0$. Write down a differential equation for V_c. Recall that the voltage-current relationship for an inductor L is $V_L = L\frac{di}{dt}$ and for a capacitor C is $i = C\frac{dV_c}{dt}$.

b) Compare the derived differential equation with the mass-spring equation (3.4) and write down a relationship for k, m, C, L if two system have same response.

c) Verify that functions $V_c = \cos\left(\frac{1}{\sqrt{LC}}t\right)$ and $V_c = \sin\left(\frac{1}{\sqrt{LC}}t\right)$ solves the derived equation.

d) Write down the energy formulation of the circuit and determine initial conditions for which the circuit remain in its resting state (zero energy). With these initial conditions, connect the supply $V_s = 1$ to the circuit, that causes the circuit to operate. Verify that the response of circuit to this supply (when the initial conditions are zero) is

$$V_c(t) = 1 - \cos\left(\frac{t}{\sqrt{LC}}\right).$$

e) By using an electrical resistor, draw an electrical circuit that simulate the equation (3.7).

Problem 3.3. The following equation describes the motion of a vertical pendulum in terms of its angle θ with respect to the resting position

$$l\theta'' + g\sin\theta = 0.$$

a) Multiply the equation by θ' and derive the following energy equality

$$\frac{1}{2}l\,|\theta'|^2 + g(1 - \cos\theta) = \text{const}.$$

b) If $\theta'(0) = 0$ and $\theta(0) = \frac{\pi}{2}$, find time T when $\theta(T) = -\frac{\pi}{2}$. You need a numerical integration. Feel free to use any online or off-line integration software.

c) Determine an initial condition $\theta(0) = \theta_0$, $\theta'(0) = \theta_1$ such that

$$\lim_{t\to\infty} \theta(t) = -\pi.$$

Problem 3.4. The following equation is called the Duffing's equation

$$x'' + ax' + bx + cx^3 = 0.$$

a) If $a = 0$, and $b, c > 0$, show that the system conserves the energy.

b) If $a > 0$, and $b, c > 0$, show that $\frac{\mathrm{d}E}{\mathrm{d}t} < 0$.

c) If $b.c < 0$, find two non-zero equilibrium points of the equation by rewriting the equation as the following system

$$\begin{cases} x' = y \\ y' = -ay - bx - cx^3 \end{cases}.$$

In this case, $\bar{x}$ is an equilibrium for the system if the right-hand side of the above system is zero.

3.2 Theory of linear second-order equations

3.2.1 Existence and uniqueness problem

We first introduce a theorem on the existence of second-order initial value problem.

Theorem 3.1. (Existence) *Consider the following initial value problem*

$$\begin{cases} y'' = f(x, y, y') \\ y(x_0) = y_0, y'(x_0) = y_1 \end{cases},$$

and assume that there is a cube D centered at point (x_0, y_0, y_1) such that f is continuous on D. Then, there is at least one solution of the initial value problem.

Theorem 3.2. (Uniqueness) *In the above initial value problem, if f, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial y'}$ are continuous on D, then the problem has a* unique solution.

Although the proof of the uniqueness is similar to one done for the first-order problems, we should wait until we study the theory of first-order systems of differential equations in the last chapter. An immediate corollary is presented below.

Corollary. *A linear initial value problem*

$$\begin{cases} y'' + p(x)y' + q(x)y = 0 \\ y(x_0) = y_0, y'(x_0) = y_1 \end{cases},$$

has a unique solution if there is an open interval I centered at x_0 such that $p(x)$, $q(x)$ are continuous on I.

The corollary is simply verified. In fact, for $f(x, y, y') = -p(x)y' - q(x)y$, the continuity of f depends only on the continuity of $p(x), q(x)$ in an interval of x_0, and also

$$\frac{\partial f}{\partial y} = -q(x), \frac{\partial f}{\partial y'} = -p(x),$$

and thus the continuity of p, q guarantees the uniqueness of the initial value problem.

Example 3.1. Consider the following problem

$$\begin{cases} y'' = 30\, y^{\frac{2}{3}} \\ y(0) = y'(0) = 0 \end{cases}.$$

The equation passes the condition for the existence theorem, and thus possess one solution. Simply, the solution is $y(x) = 0$. However, there are other solutions as $y(x) = x^6$. Now, Consider the following problem

$$\begin{cases} x^2 y'' + 2x y' - 2y = 0 \\ y(0) = 0, y'(0) = 1 \end{cases}.$$

Obviously. the problem does not pass the condition even for the existence theorem, since $p(x) = \frac{2}{x}$, $q(x) = -\frac{2}{x^2}$ are not continuous in any interval around $x_0 = 0$. However, the equation has a unique solution $y(x) = x$. If the initial condition change to $y(0) = \alpha$ for any nonzero α, the no solution exists for the problem. Therefore, the existence and uniqueness theorems provide sufficient conditions for the existence and uniqueness of the solution of an IVP.

Problem 3.5. Assume p, q are continuous functions, T: $\frac{d^2}{dx^2} + p(x)\,\frac{d}{dx} + q(x)$, and $y = \phi(x)$ is the unique solution of the following IVP

$$\begin{cases} T[y] = 0 \\ y(x_0) = y_0, y'(x_0) = y_1 \end{cases}.$$

Show that $y = \alpha\phi(x)$ is the unique solution to the following problem

$$\begin{cases} T[y] = 0 \\ y(x_0) = \alpha y_0, y'(x_0) = \alpha y_1 \end{cases}.$$

In particular, the unique solution to the following problem is $y(x) = 0$

$$\begin{cases} T[y] = 0 \\ y(x_0) = 0, y'(x_0) = 0 \end{cases}.$$

Problem 3.6. Assume p, q are continuous functions, $T\colon \frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)$, and that y_1, y_2 are solutions of the following IVPs respectively

$$(1)\begin{cases} T[y] = 0 \\ y(x_0) = 1, y'(x_0) = 0 \end{cases}, (2)\begin{cases} T[y] = 0 \\ y(x_0) = 0, y'(x_0) = 1 \end{cases},$$

find the unique solution to the following IVP

$$\begin{cases} T[y] = 0 \\ y(x_0) = 1, y'(x_0) = 1 \end{cases}$$

3.2.2 Linear independence and Wronskian

We start with a definition.

Definition 3.1. *Two functions f, g are called linearly independent on an open interval I if the linear combination*

$$c_1 f(x) + c_2\, g(x) = 0, \forall x \in I$$

implies $c_1 = c_2 = 0$.

The above definition simply states that f can not be written in terms of function g or vice versa. For example, functions $f = \sin(x)$, $g = \cos(x)$ are linearly independent, while $f = \sin(x)$, $g = 0$ are linearly dependent. This is generalized for higher numbers of functions. Functions $f_1(x) = 1$, $f_2(x) = 2x + 1$, $f_3(x) = 3x - 1$ are linearly dependent in $\mathbb{R}$ due to equality $f_3 = \frac{3}{2} f_2 - \frac{5}{2} f_1$. Functions $f_1(x) = 1$, $f_2(x) = x$ and $f_3(x) = x^2$ are linearly independent in $\mathbb{R}$. In fact, identity $c_1 + c_2 x + c_3 x^2 \equiv 0$ implies $c_1 = c_2 = c_3 = 0$.

If f, g are continuously differentiable functions in I, then a linear combination

$$c_1 f + c_2\, g \equiv 0,$$

implies also

$$c_1 f' + c_2\, g' \equiv 0,$$

and thus, we can write them in the matrix form as

$$\begin{pmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.11}$$

Proposition 3.1. *If there is one point x_0 such that*

$$\det\begin{pmatrix} f(x_0) & g(x_0) \\ f'(x_0) & g'(x_0) \end{pmatrix} \neq 0, \tag{3.12}$$

then f, g are linearly independent.

Proof. If condition (3.12) holds, then matrix $\begin{pmatrix} f(x_0) & g(x_0) \\ f'(x_0) & g'(x_0) \end{pmatrix}$ is invertible, and thus Eq.3.11 is uniquely solved for $c_1 = c_2 = 0$, a condition for the linear independence of f, g. □

Remark 3.1. The reverse of the above proposition does not hold in general. That is, if

$$\det\begin{pmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{pmatrix} = 0,$$

for all x, then we can not conclude that f, g are *linearly dependent*. For example, two functions $f(x) = x^2$, $g(x) = x|x|$ are linearly independent on any interval $(-a, a)$ for $a > 0$, however, the determinant of their matrix is zero everywhere. It is left as an exercise to the reader to verify the claim.

Theorem 3.3. *Consider the following equation*

$$T[y] := y'' + p(x)y' + q(x)y = 0, \tag{3.13}$$

where p, q are assumed to be continuous on an open interval. If $y_1(x)$, $y_2(x)$ are two solutions of the equation, and if there is a point x_0 such that the Wronskian

$$W(y_1, y_2)(x_0) := \det\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix} = 0;$$

then $y_1(x), y_2(x)$ are linearly dependent. If there is a point x_0 such that $W(y_1, y_2)(x_0) \neq 0$, then $y_1(x), y_2(x)$ are linearly independent.

Proof. The second part is proved in the above proposition. We prove the first part. We assume that y_1, y_2 are not identically zero, since if so, they are obviously dependent. First assume that $y_1(x_0) \neq 0$ and $y_1'(x_0) \neq 0$. Then we can write

$$\frac{y_2(x_0)}{y_1(x_0)} = \frac{y_2'(x_0)}{y_1'(x_0)} = \alpha,$$

for some constant α. No consider the following IVP

$$\begin{cases} T[y] = 0 \\ y(x_0) = \alpha y_1(x_0) \\ y'(x_0) = \alpha y_1'(x_0) \end{cases}.$$

Clearly, functions $y = \alpha y_1(x)$ and $y = y_2(x)$ are both solutions of the above IVP, and according to the uniqueness theorem, it is possible only if $y_2(x) = \alpha y_1(x)$. If either one of $y_1(x_0)$ or $y_1'(x_0)$ is zero, the proof is similar and we left it as an exercise to the reader. □

The determinant $W(y_1, y_2)(x) := y_1(x)\, y_2'(x) - y_1'(x)\, y_2(x)$ is called Wronskian of y_1, y_2 after the Polish mathematician J. WRONSKI.

3.2.3 Number of core solutions

We prove that a linear second-order differential equations has exactly two *linearly independent* or core solutions if its coefficient functions are continuous.

Theorem 3.4. *Assume p, q are continuous functions, and $T := \frac{d^2}{dx^2} + p(x)\,\frac{d}{dx} + q(x)$. There are exactly two linearly independent solutions of equation $T[y]=0$.*

Proof. Consider the following IVPs

$$\begin{cases} T[y]=0 \\ y(x_0)=1, y'(x_0)=0 \end{cases}, \begin{cases} T[y]=0 \\ y(x_0)=0, y'(x_0)=1 \end{cases},$$

Clearly y_1, y_2 are linearly independent (why?). Assume $y=y_3(x)$ is a solution to the equation. Consider the following initial value problem

$$\begin{cases} T[y]=0 \\ y(x_0)=y_3(x_0), y'(x_0)=y_3'(x_0) \end{cases}.$$

It is simply seen that functions $y=y_3(x)$ and

$$y=y_3(x_0)\,y_1(x)+y_3'(x_0)\,y_2(x),$$

are the solutions of the given IVP, and thus according to the uniqueness

$$y_3(x)=y_3(x_0)\,y_1(x)+y_3'(x_0)\,y_2(x),$$

and this completes the proof. □

Corollary 3.1. *Assume that p, q are continuous functions, T: $\frac{d^2}{dx^2} + p(x)\,\frac{d}{dx} + q(x)$, and y_1, y_2 any two linearly independent solutions of equation $T[y]=0$. Then the general solution to the equation is*

$$y_h(x)=c_1\,y_1(x)+c_2\,y_2(x),$$

where c_1, c_2 are arbitrary constant. This means, for any initial value problem

$$\begin{cases} T[y]=0 \\ y(x_0)=\alpha, y'(x_0)=\beta \end{cases},$$

the solution can be written as a linear combination of y_1, y_2 with specified constants c_1, c_2.

Problem 3.7. Prove the corollary.

3.2.4 Abel's identity

There is a beautiful theorem about the Wronskian of the homogeneous solutions of a second-order linear ODE. This relation is called Abel's formula after the Norwegian mathematician N. H. Abel.

Theorem 3.5. (Abel) *Assume p, q are continuous functions, T: $\frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)$, and y_1, y_2 are two solutions (not necessarily independent) to equation $T[y]=0$. Then*

$$W(y_1, y_2)(x) = W(y_1, y_2)(x_0)\, e^{-\int_{x_0}^{x} p(s)ds}, \tag{3.14}$$

where x_0 is an arbitrary point in the domain of definitions of $y_1(x)$ and $y_2(x)$.

Proof. According to the derivative formula of matrices, we can write

$$\frac{d}{dx}W(y_1, y_2) = \det\begin{pmatrix} y_1' & y_2' \\ y_1' & y_2' \end{pmatrix} + \det\begin{pmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{pmatrix} = y_1\, y_2'' - y_2\, y_1''. \tag{3.15}$$

On the other hand, we have

$$y_1'' = -p(x)y_1 - q(x)y_1, \quad y_2'' = -p(x)y_2 - q(x)y_2,$$

and substituting them into (3.15) leads to the following differential equation for W

$$\frac{d}{dx}W(y_1, y_2) = y_1(-py_1' - qy_1) - y_2(-py_2' - qy_2) = -p(x)\, W(y_1, y_2), \tag{3.16}$$

which is solved for

$$W(y_1, y_2)(x) = W(y_1, y_2)(x_0)\, e^{-\int_{x_0}^{x} p(s)ds}, \tag{3.17}$$

and this completes the proof. □

Since $p(x)$ is continuous, function $e^{-\int_{x_0}^{x} p(s)ds}$ never vanishes, and thus $W(y_1, y_2) = 0$ if and only if $W(y_1, y_2)(x_0) = 0$ for some x_0.

3.2.5 Linear equations: extension of solutions

Theorem 3.6. *Assume that p, q are continuous functions in $(-\infty, \infty)$, and T: $\frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)$. Then the solution of the following initial value problem*

$$\begin{cases} T[y] = 0 \\ y(x_0) = \alpha, y'(x_0) = \beta \end{cases}, \tag{3.18}$$

extends in $(-\infty, \infty)$. for arbitrary x_0, y_0, y_1.

Proof. We need tow show that $|y(x)|$ does not blow up at any finite x. Define the following function

$$V(x) = y'^2(x) + y^2(x). \tag{3.19}$$

We have

$$\frac{dV}{dx} = 2y'\, y'' + 2\, yy' = -2y'\,(p(x)\, y' + q(x)\, y) + 2\, yy'. \tag{3.20}$$

Use the inequality

$$ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2, \tag{3.21}$$

and conclude that there is a continuous and positive functions $f(x)$ (depending on p, q) such that

$$\frac{dV}{dx} \leq f(x)\,V. \tag{3.22}$$

Let us multiply both sides of the above inequality by μ

$$\mu(x) = e^{-\int_0^x f(t)dt}, \tag{3.23}$$

and obtain

$$\frac{d}{dx}(\mu(x)V) \leq 0. \tag{3.24}$$

Since the above inequality holds for all x, we conclude

$$\mu(x)V(x) \leq V(0), \tag{3.25}$$

and thus

$$0 \leq V(x) \leq V(0)e^{\int_0^x f(t)dt} = (\alpha^2 + \beta^2)e^{\int_0^x f(t)dt} < \infty, \tag{3.26}$$

for all x. Since $|y(x)| \leq \sqrt{V(x)}$, we conclude $|y(x)| < \infty$ for all x. □

Problem 3.8. Repeat the argument for the non-homogeneous problem

$$\begin{cases} T[y] = r(x) \\ y(x_0) = \alpha, y'(x_0) = \beta \end{cases}$$

Exercise 3.1. Consider the equation $y' = f(y)$ such that f is continuous everywhere and $f(y) \leqslant y$. prove that the solution can be extend arbitrary.

Exercise 3.2. Show that the solution of the following problem can extends in (x_0, ∞)

$$\begin{cases} y' = x^3 - y^3 \\ y(x_0) = y_0 \end{cases}.$$

Problems

Problem 3.9.

i. Show that functions $y_1(x) = x^3$ and $y_2(x) = |x|^3$ are solutions to the problem

$$\begin{cases} x^2y'' - 6y = 0 \\ y(0) = 0, y'(0) = 0 \end{cases}.$$

ii. Show that $W(y_1, y_2) \equiv 0$.

iii. Show that y_1, y_2 are linearly independent. Does this results contradict what we proved in this section?

Problem 3.10. If y_1 and y_2 are two solutions to the equation

$$y'' + \cos(x)\, y' + e^x\, y = 0,$$

show that

$$W(y_1, y_2)(0) = W(y_1, y_2)(\pi).$$

Problem 3.11. If y_1 and y_2 are solutions to the equation

$$y'' + p(x)\, y' + q(x) y = 0,$$

show that

$$q(x) = -\frac{W(y_1', y_2')}{W(y_1, y_2)}.$$

Problem 3.12. Generalize the ABEL's formula for the following equation

$$y^{(n)} + a_1(x)\, y^{(n-1)} + \cdots + a_n(x) y = 0.$$

Problem 3.13. Let $y_1(x)$ and $y_2(x)$ be two core solutions to the homogeneous equation

$$y'' + p(x) y' + q(x) y = 0. \tag{3.27}$$

The solution to the initial value problem

$$\begin{cases} y'' + p(x) y' + q(x) y = 0 \\ y(x_0) = y_0,\, y'(x_0) = y_1 \end{cases}, \tag{3.28}$$

is obtained by the formula

$$y(x) = \frac{W(y, y_2)(x_0)}{W(y_1, y_2)(x_0)} y_1(x) - \frac{W(y, y_1)(x_0)}{W(y_1, y_2)(x_0)} y_2(x). \tag{3.29}$$

Problem 3.14. Show that functions $f(x) = e^{\lambda_1 x}$, $g(x) = e^{\lambda_2 x}$ for $\lambda_1 \neq \lambda_2$ are linearly independent functions. Repeat the argument for functions $f(x) = e^{\lambda x}$, $g(x) = x e^{\lambda x}$, and also for functions $f(x) = e^{\sigma x} \cos(\omega x)$, $g(x) = e^{\sigma x} \sin(\omega x)$.

Problem 3.15. If the determinant of $\begin{pmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{pmatrix}$ is zero at a point, it does not imply that functions f, g are linearly dependent. Even, the determinant may be zero everywhere, and even functions are linearly independent. Consider functions $f(x) = x^2$, $g(x) = x\,|x|$ defined on $(-1, 1)$. Show that the determinant of the associated matrix is zero in $(-1, 1)$ but functions are linearly independent.

Problem 3.16. Assume that three functions $f(x)$, $g(x)$ and $h(x)$ are linearly independent in an interval around x_0. Show that

$$\det\begin{pmatrix} f(x_0) & g(x_0) & h(x_0) \\ f'(x) & g'(x_0) & h'(x_0) \\ f''(x_0) & g''(x_0) & h''(x_0) \end{pmatrix} \neq 0.$$

Problem 3.17. Assume that two functions $y_1 = 3 e^{2x} + x e^{-x}$ and $y_2 = 2 e^{3x} - 3 x e^{-x}$ are solutions to the equation

$$y'' + p(x)\, y' + q(x) y = 0.$$

Which one of the following functions are solution to the given equation?

i. $y = 9 e^{2x} + 2 e^{3x}$

ii. $y = 6 e^{2x} + 2 e^{3x} - x e^{-x}$

iii. $y = 2 e^{3x} - 2 x e^{2x}$

Problem 3.18. Show that functions $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$ are linearly independent functions if $\lambda_1 \neq \lambda_2$. Repeat the argument for the functions $y_1 = e^{\lambda x}$, $y_2 = x e^{\lambda x}$; and also for the functions $y_1 = e^{\sigma x} \cos(\omega x)$, $y_2 = e^{\sigma x} \sin(\omega x)$.

Problem 3.19. This problem justifies why the superposition property does not hold for nonlinear equations. Consider the nonlinear equation

$$y'' + y' + y^2 = 0.$$

i. If $y(x)$ is a non-trivial solution to the equation, show that the function $cy(x)$, for a constant c is a solution to the equation if and only if $c = 0$ or $c = 1$.

ii. If y_1 and y_2 are two solutions to the equation, show that the function $y = y_2 - y_1$ is a solution to the equation if and only if $y_1 = y_2$.

Problem 3.20. Assume that p, q are continuous functions. Show that any nontrivial solution to the equation

$$y'' + p(x)\, y' + q(x) y = 0,$$

can not be tangent to the x-axis.

Problem 3.21. Assume ϕ_1 and ϕ_2 are respectively solutions to the following initial value problems

$$\begin{cases} y'' + p(x) y' + q(x) y = 0 \\ y(0) = a, y'(0) = b \end{cases}, \begin{cases} y'' + p(x) y' + q(x) y = 0 \\ y(0) = c, y'(0) = d \end{cases},$$

where p, q are continuous functions and $ad - bc = 0$. Show that ϕ_1, ϕ_2 are linearly dependent.

Problem 3.22. Assume p, q are continuous functions and y_1, y_2 are two solutions to the following equation

$$y'' + p(x)\, y' + q(x) y = 0.$$

Prove the following properties

i. if y_1 and y_2 have maximum at a same value x_0, then they are linearly dependent.

ii. if y_1 and y_2 vanishes at a same value x_0, then they are linearly dependent.

Problem 3.23. Assume that p, q are continuous functions and $p(x) \neq 0$. Suppose that y_1 and y_2 are two linearly independent solutions to the following equation

$$y'' + p(x)\, y' + q(x) y = 0.$$

Show that y_1 , y_2 can not have same inflection point.

Problem 3.24. Assume that f_1 and f_2 are two linearly independent functions in an interval I (they are not necessarily differentiable). Show that tow functions $g_1 = af_1 + bf_2$, $g_2 = cf_1 + df_2$ are linearly independent if and only if $ad - bc \neq 0$.

Problem 3.25. Assume that f is a nontrivial continuously differentiable function in $I = (-a, a)$ for some $a > 0$ such that $f(0) = f'(0) = 0$. Show functions f and g

$$g(x) = \begin{cases} f(x) & 0 \leq x \leq \pi \\ -f(x) & -\pi \leq x \leq 0 \end{cases},$$

are linearly independent while $W(f, g) \equiv 0$ on I.

Problem 3.26. Determine the linearly dependence or independence of the following set of functions in the given interval I:

i. $f(x) = e^{\sigma x} \cos(\omega x)$, $g(x) = e^{\sigma x} \sin(\omega x)$, $I = \mathbb{R}$.

ii. $f(x) = |x| \sin(x)$, $g(x) = x \sin(x)$, $I = (-\pi/2, \pi/2)$.

iii. $f(x) = |x|$, $g(x) = \begin{cases} 0 & x > 0 \\ x & x \leq 0 \end{cases}$, $h(x) = \begin{cases} 0 & x < 0 \\ x & x \geq 0 \end{cases}$, $I = \mathbb{R}$.

iv. $f(x) = 1$, $g(x) = x + 1$, $h(x) = 1 - x$, $I = \mathbb{R}$.

v. $f(x) = x$, $g(x) = \cos(\ln(x))$, $h(x) = \sin(\ln(x))$, $I = (0, \infty)$.

3.3 Solution to linear homogeneous equations

3.3.1 Equations with constant coefficients

Consider the following equation

$$y'' + a y' + b y = 0, \tag{3.30}$$

where a_1 and a_2 are constants. Let us assume that the equation has solutions of the exponential form $y = e^{\lambda x}$ for some unknown constant λ. Substituting this function into the equation leads to the following algebraic equation

$$p(\lambda) := \lambda^2 + a\lambda + b = 0,$$

which is called the *characteristic equation* of the given differential equation. There are three possibilities for the roots of the characteristic equation: 1) two distinct real roots $\lambda_1 \neq \lambda_2$, 2) two complex conjugate roots $\lambda_{1,2} = \sigma \pm i\omega$, and 3) one repeated root λ.

Case 1. Real distinct roots

If λ_1, λ_2 are two distinct roots, then the differential equation has two core solutions

$$y_1(x) = e^{\lambda_1 x}, \quad y_2(x) = e^{\lambda_2 x},$$

and thus the *general homogeneous solution* is $y_h(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$.

Example 3.2. Consider the initial value problem

$$\begin{cases} y'' + 3y' + 2y = 0 \\ y(0) = 1, y'(0) = 0 \end{cases}.$$

The characteristic equation is

$$\lambda^2 + 3\lambda + 2 = 0,$$

with two roots $\lambda_1 = -2$, and $\lambda_2 = -1$, and thus

$$y_h(x) = c_1 e^{-2x} + c_2 e^{-x}.$$

Applying the given initial conditions yields $c_1 = -1, c_2 = 2$, and finally

$$y(x) = 2 e^{-x} - e^{-2x}.$$

Remark. If we write the differential equation in the operator form as

$$(D^2 + a D + b)[y] = 0,$$

then by decomposing the operator as

$$D^2 + a D + b = (D - \lambda_1)(D - \lambda_2),$$

and thus the equation is reduced to the following $T_1 T_2[y] = 0$, where $T_1 := (D - \lambda_1)$, and $T_2 := (D - \lambda_2)$. Clearly, the equation also can be written as $T_2 T_1[y]$. In any case, we obtain two simple first-order equation $T_1[y] = 0$, which is solved for $y_1 = e^{\lambda_1 x}$, and $T_2[y] = 0$ with the solution $y_2 = e^{\lambda_2 x}$.

Case 2. Complex roots

If roots of $p(\lambda)$ are complex $\lambda = \sigma \pm i\omega$, functions $y_1 = e^{\sigma x}\, e^{i\omega x}$, and $y_2 = e^{\sigma x}\, e^{-i\omega x}$ solve the equation. Remember the EULER's formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta),$$

and therefore

$$y_1 = e^{\sigma x}(\cos(\omega x) + i\sin(\omega x)), \quad y_1 = e^{\sigma x}(\cos(\omega x) - i\sin(\omega x)).$$

Since linear combinations

$$\frac{y_1 + y_2}{2}, \frac{y_1 - y_2}{2i},$$

are also solutions of the equation, we obtain two *real solutions* $y_1 = e^{\sigma x}\cos(\omega x)$, and $y_2 = e^{\sigma x}\sin(\omega x)$ of the equation, and finally

$$y_h(x) = e^{\sigma x}(c_1\cos(\omega x) + c_2\sin(\omega x)).$$

Example 3.3. The characteristic equation of the equation

$$y'' + \omega^2 y = 0,$$

is $\lambda^2 + \omega^2 = 0$ and thus $\lambda = \pm i\omega$. Note that the real part of the characteristic root is zero. The equation has two core solutions $y_1 = \cos(\omega x)$, $y_2 = \sin(\omega x)$. Now, consider the equation

$$y'' + 2by' + \omega^2 y = 0,$$

where $b < \omega$. The roots of characteristic polynomial are $\lambda = -b \pm i\sqrt{\omega^2 - b^2}$ and thus the general solution is

$$y = e^{-bx}(c_1\cos(\tilde{\omega}\, x) + c_2\sin(\tilde{\omega}\, x)),$$

where $\tilde{\omega} = \sqrt{\omega^2 - b^2}$.

Remark. The differential operator in this case has complex roots as

$$D^2 + a\,D + b = (D - \sigma - i\omega)(D - \sigma + i w),$$

and therefore two complex conjugate solutions y_1, $\bar{y}_1$. Accordingly, $y_1 + \bar{y}_1 = 2\,\mathrm{Re}\{y_1\}$, and $y_1 - \bar{y}_2 = 2i\,\mathrm{Im}(y_1)$ provide us with desired real solutions.

Case 3. Repeated roots

If $\lambda = \lambda_1 = \lambda_2$, then $y_1 = e^{\lambda x}$ is one solution to the equation. Let us write the second solution y_2 as $y_2 = e^{\lambda x}\, v(v)$ for some unknown function $v(x)$. Substituting y_2 into the equation gives

$$(\lambda^2 + a\lambda + b)\; v(x) + (2\lambda + a)\; v(x) + v''(x) = 0.$$

Notice that terms $\lambda^2 + a\,\lambda + b$ and $2\lambda + a$ are zero (why?), and thus we reach $v''(x) = 0$ with the solution $v(x) = x$. Therefore, the second core solution is $y_2 = x\,e^{\lambda x}$.

Problem 3.27. The general solution of $v''=0$ is $v(x)=Ax+B$. Why did we choose only $v(x)=x$?

Example 3.4. Let us solve the equation

$$\begin{cases} y''+2y'+y=0 \\ y(0)=1, y'(0)=0 \end{cases}.$$

The characteristic equation is $\lambda^2+2\lambda+1=0$ and thus $\lambda=-1$ is the repeated root. The general solution to the equation is $y=e^{-x}(c_1+c_2x)$. By applying the initial conditions, we obtain $y=(1+x)e^{-x}$.

Remark. In the operator form, we have

$$\left(D+\frac{a}{2}\right)\left(D+\frac{a}{2}\right)[y]=0,$$

or just $T_1\,T_1[y]=0$. Definitely, one solution is $y=e^{-\frac{a}{2}}$ which is the solution of $T_1[y]=0$. The second solution can be obtained as follows. If $z=T_1[y]$, then $T_1[z]=0$. Definitely, $z=e^{-\frac{a}{2}x}$, and thus we reach $T_1[y]=e^{-\frac{a}{2}x}$, or $y'+\frac{a}{2}y=e^{-\frac{a}{2}}$ which is solved for

$$y=c_1\,e^{-\frac{a}{2}x}+c_2\,x\,e^{-\frac{a}{2}x}.$$

Problems.

Problem 3.28. Find two core solutions for each of the following equations and then write down the general solution:

i. $y''+4y'+2=0$

ii. $(y'-6y)'+9y=0$

iii. $(y'-1)'=1-y$

iv. $y''+7y'+12y=0$

Problem 3.29. Which one of the following equations does have a solution which remain bounded when $x\to\pm\infty$? Determine equations with the property that its all solutions remain bounded when $x\to+\infty$. Determine equations with the property that their all solutions remain bounded when $x\to-\infty$.

i. $y''+2y'+y=0$

ii. $y''-5y'+4y=0$

iii. $y''+3y'+2y=0$

iv. $y''-2y'=0$

v. $y''+9y=0$.

Problem 3.30. For each of the following equations, set an initial conditions such that the solution (non-trivial) to the corresponding initial value problem remain bounded when $x\to+\infty$.

i. $y''+y'-2y=0$

ii. $y''-5y'=0$

iii. $y''-4y=0$

iv. $y''-y'-12y=0$

Problem 3.31. Consider the equation

$$y'' + 2\lambda y' + \omega^2 y = 0,$$

where $\lambda^2 > \omega^2 > 0$. If $y(x)$ is a solution to the given equation, show that

$$\lim_{x\to\infty} |y(x)| = \begin{cases} 0 & \lambda > 0 \\ \infty & \lambda < 0 \end{cases}.$$

Problem 3.32. Solve the following problems:

i. $y'' + 3y + 2 = 0,\ y(0) = 1,\ y'(0) = -1$

ii. $y'' + 4y' + 13y = 0,\ y(0) = 0,\ y'(0) = 1$

iii. $y'' - 4y' + 4y = 0,\ y(1) = 0,\ y'(1) = 0$

iv. $y'' + 9y = 0,\ y(\pi/3) = 1,\ y'(\pi/3) = -1$

v. $y'' + 3y' = 0,\ y(0) = 1,\ y'(0) = 0$

Problem 3.33. Find y_0 such that the solution to the problem

$$\begin{cases} y'' - y' - 2y = 0 \\ y(0) = y_0, y'(0) = 2 \end{cases},$$

remains bounded when $x \to \infty$.

Problem 3.34. For each pair of the given functions, write down a differential equation having them as its core solutions.

i. $y_1 = 3e^{2x} + 2e^{3x},\ y_2 = e^{3x} - e^{2x}$

ii. $y_1 = (x-1)e^{-x},\ y_2 = (x+1)e^{-x}$

iii. $y_1 = \sin x + 2\cos x,\ y_2 = 2\sin x$

iv. $y_1 = 2e^{-x}\cos(2x),\ y_2 = e^{-x}(\sin(2x) - \cos(2x))$.

3.3.2 Equations with variable coefficients

Consider the equation

$$y'' + p(x)y'' + q(x)y = 0. \tag{3.31}$$

Although there is no general method to solve equations with variable coefficients, there are two important cases that they can be reduced to first-order linear equations, and thus to be solved by simple integration methods.

Case 1. Defective equations

If $q(x)$ in (3.31) is identically zero, the equation is called *defective*:

$$y'' + p(x)y' = 0, \tag{3.32}$$

By taking $y' = u(x)$, the equation is reduced to a linear first-order equation which is separable as well.

Example 3.5. Consider the initial value problem

$$\begin{cases} xy'' - y' = 0 \\ y(1) = 1, y'(1) = -1 \end{cases}.$$

Taking $u = y'$ transforms the equation to $x u' - u = 0$ and thus $u = c_1 x$. Therefore, we obtain $y = \frac{1}{2} c_1 x^2 + c_2$. Applying the given initial conditions yields $y(x) = -\frac{1}{2}x^2 + \frac{3}{2}$.

Case 2. Reduction of order by the variation of parameter

Suppose we know one solution y_1 of Eq.(3.31). If we take the second solution as $y_2 = y_1 v(x)$ for an unknown function $v(x)$ and substitute it into the equation, we reach

$$y_1 v'' + (2y_1' + p(x) y_1) v' + (y_1'' + p(x) y_1' + q(x) y_1) v = 0.$$

Since y_1 is a solution of the equation, the above equation reduces to the following defective one

$$y_1 v'' + (2y_1' + p(x) y_1) v' = 0.$$

By taking $v' = u(x)$, we obtain

$$y_1 u' + (2y_1' + p(x) y_1) u = 0,$$

which is solved for

$$u = \frac{1}{y_1^2} e^{-\int p(x) dx},$$

and finally

$$y_2 = y_1 \int \frac{e^{-\int p(x) dx}}{y_1^2(x)} dx. \tag{3.33}$$

Example 3.6. Consider the following equation

$$y'' - \frac{1}{x} y' + \frac{y}{x(x+1)} = 0.$$

a) Find a solution to the equation if we know it has a first order polynomial solution.

b) Find the solution to the following initial value problem

$$\begin{cases} y'' - \frac{1}{x} y' + \frac{1}{x(x+1)} y = 0 \\ y(1) = 1, y'(1) = \frac{1}{2} \end{cases}.$$

To solve part a), we substitute $y_1(x) = ax + b$ for unknown a and b into the equation and obtain $y_1 = a(x+1)$ for some constant a. Without loss of generality, we assume $a = 1$ and write $y_1 = x + 1$. For part b), We use formula (3.33) to determine y_2 as

$$y_2(x) = (x+1) \int \frac{x}{(x+1)^2} dx = (x+1) \log|x+1| + 1.$$

The general solution is then equal to

$$y_h(x) = c_1 (x+1) + c_2 (x+1) \ln|x+1| + c_2.$$

Applying the initial conditions determines $c_1 = \frac{1}{2}$ and $c_2 = 0$ and hence the solution is obtained as $y = \frac{1}{2}(x+1)$.

3.3.3 Operator form of equations with variable coefficients

Perhaps one is tempted to write the operator for of the equation as

$$(D^2 + p(x)\,D + q(x))[y] = 0,$$

and try to decompose T into T_1, T_2 as we did for equations with constant coefficients. Let us assume the following decomposition for T:

$$(D^2 + a_1(x)D + a_2(x))\,[y] = (D - \lambda_2(x))\{(D - \lambda_1(x))[y]\}, \tag{3.34}$$

where λ_1, λ_2 are two unknown functions. According to the following relation

$$\begin{aligned}(D - \lambda_2(x))\{(D - \lambda_1(x))[y]\} = (D - \lambda_2(x))\{y' - \lambda_1(x)y\} = \\ = y'' - (\lambda_1 + \lambda_2)y' + (\lambda_1\lambda_2 - \lambda_1')y,\end{aligned}$$

we reach the following system for $\lambda_1(x), \lambda_2(x)$:

$$\begin{cases} \lambda_1(x) + \lambda_2(x) = -p(x) \\ \lambda_1(x)\lambda_2(x) - \lambda_1'(x) = q(x) \end{cases}. \tag{3.35}$$

If we eliminating λ_2 in the above system, we reach the following equation for λ_1

$$\lambda_1' + p(x)\lambda_1 = -q(x) - \lambda_1^2. \tag{3.36}$$

This is a RICCATI equation and it is known that there is no general method to solve it. Therefore, in contrast to equations with constant coefficients, it is not in general possible to write a decomposed form for equations with variable coefficients.

Problem

Problem 3.35. Solve the following initial value problems

i.

$$\begin{cases} y'' + \frac{1}{4}xy' = x \\ y(0) = -1, y'(0) = 4 \end{cases}.$$

ii.

$$\begin{cases} (1 + e^x)y'' - y' = e^{2x} \\ y(0) = 0, y'(0) = \frac{1}{2} \end{cases}.$$

iii.

$$\begin{cases} y'' + 2\tan x\, y' = 3 + \tan^2 x \\ y(0) = 0, y'(0) = 0 \end{cases}$$

Problem 3.36. Consider the following equation

$$xy'' + (x - 1)y' - y = 0.$$

a) There is a solution for the equation in the exponential form. Find this solution and call it $y_1(x)$.

b) Use reduction of order method and find the solution to the following initial value problem

$$\begin{cases} xy'' + (x-1)y' - y = 0 \\ y(1) = 1, y'(1) = 0 \end{cases}.$$

Problem 3.37. Consider the following equation

$$xy'' + 2(1-x)\,y' + (x-2)y = 0.$$

a) Find a solution to the equation in exponential form.

b) Now find the solution to the following initial value problem

$$\begin{cases} xy'' + 2(1-x)\,y' + (x-2)y = 0 \\ y(1) = 1, y'(1) = -1 \end{cases}.$$

Problem 3.38. Consider the following equation

$$xy'' - (x+1)y' + y = 0.$$

a) Find a solution to the equation in exponential form.

b) Use reduction of order method and find the solution to the following initial value problem

$$\begin{cases} xy'' - (x+1)y' + y = 0 \\ y(1) = 1, y'(1) = 0 \end{cases}.$$

Problem 3.39. Find the general solution to the following equation if we know that one of solutions of the equation is a first order polynomial

$$x^2\,y'' - x(x+2)y' + (x+2)\,y = 0.$$

Problem 3.40. If the equation $(x^2+1)y'' - 2y = 0$ has a polynomial solution of order 2, find its general solution.

Problem 3.41. Consider the equation

$$\cos(x)y'' + \sin(x)y' + \sec(x)y = 0.$$

i. Verify that $y_1 = \cos(x)$ is a solution to the equation.

ii. Find the general solution to the problem.

Problem 3.42. Consider the problem

$$\begin{cases} y'' + |y| = 0 \\ y(0) = 0, y'(0) = 1 \end{cases}.$$

a) State the result of existence and uniqueness theorem for the problem.

b) Solve the problem in the interval $(-\infty, \pi)$. This is a unique solution to the equation.

c) Does it contradicts the uniqueness theorem?

Problem 3.43. Consider the initial value problem

$$\begin{cases} xy'' - (x+1)y' + y = 0 \\ y(0) = 0, y'(0) = 0 \end{cases}. \tag{3.37}$$

a) State the result of the existence and uniqueness theorem for this problem.

b) Find a solution to the equation

$$xy'' - (x+1)y' + y = 0,$$

if we know it has a solution of the type a first order polynomial.

c) Find a second solution that is linearly independent to the first solution.

d) Find a non-trivial solution to the problem (3.37).

3.4 Linear non-homogeneous equations

3.4.1 General solution

Consider the non-homogeneous equation

$$T[y] = r(x), \tag{3.38}$$

where T is the differential operator $D^2 + p(x)D + q(x)$. Remember that T has a null space of dimension 2. If $y_p(x)$ is a *particular solution* to the above equation, that is, $T[y_p] = r(x)$, then the *general solution* of the equation is

$$y(x) = c_1\, y_1(x) + c_2\, y_2(x) + y_p(x),$$

where y_1, y_2 are two core homogeneous solutions of the equation. The fact that $y(x)$ solves the equation is straightforward. In fact, we have

$$T[c_1\, y_1 + c_2\, y_2 + y_p] = T[c_1\, y_1 + c_2\, y_2] + T[y_p] = 0 + r(x).$$

Theorem 3.7. *Consider the following initial value problem*

$$\begin{cases} T[y] = r(x) \\ y(x_0) = y_0 \\ y'(x_0) = y_1 \end{cases}.$$

There are unique c_1, c_2 such that the solution of the above IVP is

$$y(x) = c_1\, y_1(x) + c_2\, y_2(x) + y_p(x),$$

where y_1, y_2 are two core homogeneous solution and y_p is a particular solution of the equation.

Proof. Assume that ϕ is a solution to the above IVP. We have

$$T[\phi - y_p] = T[\phi] - T[y_p] = 0,$$

and thus $\phi - y_p \in \mathrm{Null}(T)$ and therefore, there are constants c_1, c_2 such that

$$\phi - y_p = c_1\, y_1 + c_2\, y_2.$$

We show c_1, c_2 are unique. We have

$$\begin{cases} y_0 - y_p(x_0) = c_1\, y_1(x_0) + c_2\, y_2(x_0) \\ y_1 - y_p'(x_0) = c_1\, y_1'(x_0) + c_2\, y_2'(x_0) \end{cases}.$$

In the matrix form, we have

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 - y_p(x_0) \\ y_1 - y_p'(x_0) \end{pmatrix},$$

and thus

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix}^{-1}\begin{pmatrix} y_0 - y_p(x_0) \\ y_1 - y_p'(x_0) \end{pmatrix},$$

and this completes the proof. □

Problem 3.44. Suppose that $y_1(x) = \sin(\pi \sin(x))$ and $y_2 = \cos(\pi\cos(x))$ are two solutions to the equation

$$y'' + p(x)y' + q(x)y = 0. \tag{3.39}$$

i. Find the solution to the problem

$$\begin{cases} y'' + p(x)y' + q(x)y = 0 \\ y(0) = 2, \, y'(0) = \pi \end{cases}.$$

ii. If $y_p(x) = \sin(\pi e^x)$ is a solution to the equation

$$y'' + p(x)y' + q(x)y = r(x), \tag{3.40}$$

find the solution to the problem

$$\begin{cases} y'' + p(x)y' + q(x)y = 2r(x) \\ y(0) = 2, \, y'(0) = \pi \end{cases}$$

Problem 3.45. If $y_1(x)$ and $y_2(x)$ are two solutions to the equation

$$y'' + \sin(x)y' + y = e^x,$$

which one of the following functions is a solution to the equation $y'' + \sin(x)y' + y = -e^x$?

i. $y_1(x) - y_2(x)$

ii. $2y_1(x) - y_2(x)$

iii. $y_1(x) - 2y_2(x)$

iv. $2y_2(x) - 2y_1(x)$

Problem 3.46. Assume that $y_1(x)$ is a solution to the equation

$$y'' + \sin(x)y' + e^x y = 0,$$

and $y_2(x)$ is a solution to the equation

$$y'' + \sin(x)y' + e^x y = \sin(x).$$

Which one of the following functions is a solution to the following equation?

$$y'' + \sin(x)y' + e^x y = -2\sin(x)$$

i. $-2y_1(x)$

ii. $-2y_1(x) + y_2(x)$

iii. $-2y_1(x) - 2y_2(x)$

iv. $y_1(x) - 2y_2(x)$

v. $2y_1(x) - 2y_2(x)$

Problem 3.47. Consider the problem

$$\begin{cases} L[y] = r \\ y(0) = y_0, y'(0) = y_1 \end{cases}.$$

Show that the solution to above problem is the summation of solutions to the sub-problems

$$\begin{cases} L[y] = r \\ y(0) = 0, y'(0) = 0 \end{cases}, \begin{cases} L[y] = 0 \\ y(0) = y_0, y'(0) = y_1 \end{cases}.$$

3.4.2 Variation of parameters method

Consider the following equation

$$y'' + p(x)y' + q(x)y = r(x), \tag{3.41}$$

and suppose that two linearly independent solutions y_1, y_2 of the associated homogeneous equation are known. We write the particular solution as follows

$$y_p = y_1 v_1(x) + y_2 v_2(x), \tag{3.42}$$

where functions $v_1(x)$, $v_2(x)$ should be determined such that y_p satisfies Eq.3.41. Let us substitute y_p into the equation. We have

$$y_p' = y_1' v_1 + y_1 v_1' + y_2' v_2 + y_2 v_2'.$$

Here, we assume

$$y_1 v_1' + y_2 v_2' = 0, \tag{3.43}$$

and continue the calculation:

$$y_p'' = y_1'' v_1 + y_1' v_1' + y_2'' v_2 + y_2' v_2'. \tag{3.44}$$

Substituting y_p, y_p', y_p'' into Eq.(3.41) results to

$$y_1' v_1' + y_2' v_2' = r(x). \tag{3.45}$$

In this way, we reach the following two equations in two unknowns for v_1' and v_2':

$$\begin{cases} y_1 v_1' + y_2 v_2' = 0 \\ y_1' v_1' + y_2' v_2' = r(x) \end{cases}. \tag{3.46}$$

Example 3.7. Consider the equation

$$y'' + y = \tan(x).$$

Since $y_1 = \cos(x)$, $y_2 = \sin(x)$ are two linearly independent solutions to the homogeneous equation, we write y_p as

$$y_p = \cos(x)\, v_1(x) + \sin(x)\, v_2(x),$$

where v_1, v_2 satisfy the following system

$$\begin{cases} \cos(x)\, v_1' + \sin(x)\, v_2' = 0 \\ -\sin(x)\, v_1' + \cos(x)\, v_2' = \tan(x) \end{cases}.$$

Eliminating v_2' gives $v_1' = -\sin(x)\tan(x)$ and

$$v_1(x) = \sin(x) - \ln|\tan(x) + \sec(x)|.$$

Eliminating v_1' gives $v_2' = \sin(x)$ and thus $v_2(x) = -\cos(x)$ and finally the particular solution is obtained as

$$y = -\cos(x)\ln|\tan(x) + \sec(x)|.$$

Example 3.8. Consider the following equation

$$(1-x)y'' + xy' - y = (1-x)^2.$$

It is simply verified that functions $y_1 = x$, $y_2 = e^x$ are linearly independent solutions to the homogeneous equation. For the particular solution, we solve the system

$$\begin{cases} x\, v_1' + e^x\, v_2' = 0 \\ v_1' + e^x\, v_2' = 1 - x \end{cases}.$$

Note that $r(x) = 1 - x$ after dividing by the coefficient of y''. Eliminating v_2' gives $v_1' = 1$, and $v_1(x) = x$. Eliminating v_1' gives $v_2' = -e^{-x}$ and thus $v_2(x) = e^{-x}$. Therefore, the particular solution is $y_p = x^2 + 1$.

Remark 3.2. The assumption (3.43) simplifies significantly our calculations. If instead of taking it equal zero, we take it any other function, like $f(x)$, makes our calculations unnecessarily complicated. For example, by assumption it equal $f(x)$, we reach the following system

$$\begin{cases} y_1\, v_1' + y_2\, v_2' = f(x) \\ y_1'\, v_1' + y_2'\, v_2' = r(x) - f'(x) - p(x)\, f(x) \end{cases}.$$

Remark 3.3. To determine v_1, v_2 in a unique way, we need to justify that

Remark 3.4. Why is it possible to solve system (3.46)? Let us rewrite the system in the following matrix form

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} v_1'(x) \\ v_2'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ r(x) \end{pmatrix},$$

and thus v_1', v_2' are determined by the following formula as long as the coefficient matrix is invertible

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ r(x) \end{pmatrix} = \frac{1}{W(y_1, y_2)(x)}\begin{pmatrix} -r(x)y_2 \\ r(x)y_1 \end{pmatrix}$$

We saw in the previous section that $W(y_1, y_2)(x)$ remain nonzero for all x in the domain of the definition of the equation as long as y_1, y_2 are linearly independent solutions of the homogeneous equation. Therefore, we obtain

$$v_1 = -\int \frac{r(x)\, y_2(x)}{W(y_1, y_2)(x)} dx, \quad v_2 = \int \frac{r(x)\, y_1(x)}{W(y_1, y_2)(x)} dx, \tag{3.47}$$

and finally

$$y_p = -y_1 \int \frac{r(x)\, y_2(x)}{W(y_1, y_2)(x)} dx + y_2 \int \frac{r(x)\, y_1(x)}{W(y_1, y_2)(x)} dx. \tag{3.48}$$

Example 3.9. Let us find the general solution to the following initial value problem

$$\begin{cases} y'' + 4y' + 4y = e^x \\ y(0) = \frac{1}{9}, y'(0) = 0 \end{cases}.$$

The homogeneous solutions are $y_1 = e^{-2x}$, $y_2 = x e^{-2x}$, and thus $W(y_1, y_2) = e^{-4x}$. We have

$$y_p = e^{-2x} \int \frac{-x e^{-2x} e^x}{e^{-4x}} dx + x e^{-2x} \int \frac{e^{-2x} e^x}{e^{-4x}} dx = \frac{1}{9} e^x.$$

Therefore,, the general solution is

$$y = (c_1 + c_2 x) e^{-2x} + \frac{1}{9} e^x.$$

Applying the given initial conditions yields $c_1 = 0$, $c_2 = -\frac{1}{9}$ and thus $y = \frac{1}{9}(e^x - x e^{-2x})$.

3.4.3 Undetermined coefficient method

In spite of variation of parameters method, the undetermined coefficient method is applied only to constant coefficient equations. Consider the equation

$$y'' + a y' + b y = r(x). \tag{3.49}$$

This method

i. applies only if a, b are constants,

ii. and if $r(x)$ has the following forms:

   a. an exponential,

   b. a polynomial,

   c. trigonometric cosine and sine functions.

1. Exponential functions.

If $r(x) = e^{\alpha x}$, and if $e^{\alpha x}$ is not a homogeneous solution of the equation, then $y_p(x) = A e^{\alpha x}$ for an undetermined A that should be determined by substitution y_p into the equation. For example, consider the equation

$$y'' + 3y' + 2y = 2e^{3x}.$$

The equation has two solutions $y_1 = e^{-2x}$, $y_2 = e^{-x}$, and thus $y_p = Ae^{3x}$. Substituting this into the equation gives $A = \frac{1}{10}$ and therefore $y_p = \frac{1}{10}e^{3x}$.

If $e^{\alpha x}$ is a homogeneous solution then $y_p = A x e^{\alpha x}$ as long as $x e^{\alpha x}$ is not a homogeneous solution. If $x e^{\alpha x}$ is a homogeneous solution then $y_p = A x^2 e^{\alpha x}$.

Example 3.10. Consider the equation

$$y'' + 3y' + 2y = 3e^{2x} - 2e^{-x}.$$

a) Find the general solution to the problem,

b) find the solution to the initial value problem

$$\begin{cases} y'' + 3y' + 2y = 3e^{2x} - 2e^{-x} \\ y(0) = \frac{1}{2}, y'(0) = -2 \end{cases}.$$

For part a), we observe that the roots of characteristic polynomial are $\lambda_1 = -1$, $\lambda_2 = -2$ and thus $y_1 = e^{-x}$ and $y_2 = e^{-2x}$ are two linearly independent solutions of the homogeneous equation. The right hand side consists two terms $r_1 = 3e^{2x}$, and $r_2 = -2e^{-x}$. Thanks to the linearity, we determine y_p one by one, that is y_{p_1} for the first term, and y_{p_2} for the second term and finally $y_p = y_{p_1} + y_{p_2}$. Associated to r_1, we consider $y_{p_1} = Ae^{2x}$, and substituting this into the equation $y'' + 3y' + 2y = 3e^{2x}$, determines $A = \frac{1}{4}$, and thus $y_{p1} = \frac{1}{4}e^{2x}$. Associated to r_2. we consider $y_{p_2} = A x e^{-x}$ (note that $\lambda = -1$ is a simple root of the characteristic polynomial) and we obtain $A = -2$, and hence $y_{p_2} = -2x e^{-x}$. Therefore, the particular solution is

$$y_p = y_{p1} + y_{p2} = \frac{1}{4}e^{2x} - 2x e^{-x}.$$

Finally, the general solution to the equation is

$$y = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{4}e^{2x} - 2x e^{-x},$$

for arbitrary constants c_1, c_2. For part b), applying given initial conditions determines $c_1 = 0$, and $c_2 = \frac{1}{4}$, and thus

$$y(x) = \frac{1}{4}e^{-2x} + \frac{1}{4}e^{2x} - 2xe^{-x}.$$

2. Polynomial functions.

If $r(x)$ is a polynomial of order n and if $\lambda = 0$ is not a root of $p(\lambda)$, then the particular solution is a polynomial of order n. In other word, if $r = p_0 + p_1 x + \ldots + p_n x^n$, then $y_p = q_0 + q_1 x + \ldots + q_n x^n$ if $y = 1$ is not a homogeneous solution. If $\lambda = 0$ is a simple root of $p(\lambda)$, the particular solution is of the form $y_p = x(q_0 + q_1 x + \ldots + q_n x^n)$. If $\lambda = 0$ is a repeated root of $p(\lambda)$, the particular solution is of the form $y_p = x^2(q_0 + q_1 x + \ldots + q_n x^n)$.

Example 3.11. Consider the equation

$$y'' + 3y' = 4e^x + 1 - 2x + e^{-3x}.$$

The roots of $p(\lambda) = \lambda^2 + 3\lambda$ are $\lambda = 0$ and $\lambda = -3$, and therefore

$$y_h(x) = c_1 + c_2\, e^{-3x}.$$

The particular solution associated to the forcing term $r_1 = e^x$ is $y_{p_1} = e^x$. Since $\lambda = 0$ is a root for $p(\lambda)$, the particular solution associated to the forcing terms $r_2 = 1 - 2x$ is $y_{p_2} = x(q_0 + q_1 x)$. Substituting y_{p_2} into the equation $y'' + 3y' = 1 - 2x$, determines $q_0 = \frac{5}{9}$ and $q_1 = \frac{-1}{3}$, and thus $y_{p_2} = \frac{5}{9}\, x - \frac{1}{3}x^2$. The particular solution associated to $r_3 = e^{-3x}$ is $y_{p_3} = x\, e^{-3x}$, and finally

$$y_p = e^x + \frac{5}{9}\, x - \frac{1}{3}x^2 + x\, e^{-3x}.$$

The general solution is

$$y = c_1 + c_2\, e^{-3x} + e^x + \frac{5}{9}\, x - \frac{1}{3}x^2 + x\, e^{-3x}.$$

3. Trigonometric functions.

If $r = \cos(\omega x)$ or $r = \sin(\omega x)$ and none of them is a homogeneous solution to the equation, then

$$y_p = A_1 \cos(\omega x) + A_2 \sin(\omega x), \tag{3.50}$$

where A_1, A_2 are undetermined coefficients. If they are homogeneous solutions then

$$y_p = x(A_1 \cos(\omega x) + A_2 \sin(\omega x)). \tag{3.51}$$

Example 3.12. Find the particular solution to the following equation.

$$y'' + 2y' + y = 2\sin(x) - 3e^{-x} + 1.$$

Functions $y_1 = e^{-x}$, $y_2 = x\, e^{-x}$ are homogeneous solutions to the equation. The particular solution associated to the terms $1 - 3e^{-x}$ is $1 - \frac{3}{2}x^2\, e^{-x}$. Regrading the term $2\sin(x)$, the particular solution is

$$y_{p1} = A_1 \cos(x) + A_2 \sin(x).$$

Substituting y_{p1} into the equation

$$y'' + 2y' + y = 2\sin(x),$$

determines $A_1 = -1$ and $A_2 = 0$ and thus

$$y_p = 1 - \frac{3}{2}\, x^2\, e^{-x} - \cos(x).$$

4. Multiplication of source terms.

In previous examples we saw how to find particular solution by undetermined coefficient methods for exponential, polynomials and sine and cosine functions, and also for the summations of them. Here we discuss the particular solution for the multiplication of these forms.

Case 1. If $r(x) = e^{\sigma x}\cos(\omega x)$ or $r(x) = e^{\sigma x}\sin(\omega x)$ and $\lambda = \sigma \pm i\omega$ are not roots of $p(\lambda)$, then the particular solution has the same form

$$y_p = A_1 e^{\sigma x}\cos(\omega x) + A_2 e^{\sigma x}\sin(\omega x). \tag{3.52}$$

If $\lambda = \sigma \pm i\omega$ are roots of $p(\lambda)$, the particular solution has the form

$$y_p = x(A_1 e^{\sigma x}\cos(\omega x) + A_2 e^{s\sigma x}\sin(\omega x)). \tag{3.53}$$

Example 3.13. Consider the equation

$$y'' + 2y' + 2y = e^{-x}\sin x.$$

a) Find the general solution to the equation.

b) Solve the following initial value problem

$$\begin{cases} y'' + 2y' + 2y = e^{-x}\sin x \\ y(0) = 0, y'(0) = -1 \end{cases}.$$

For part a), The characteristic polynomial has roots $\lambda = -1 \pm i$ and thus the homogeneous equation has two core solutions $y_1 = e^{-x}\cos x$ and $y_2 = e^{-x}\sin x$. Since the The particular solution is of the form

$$y_p = A_1 x e^{-x}\cos x + A_2 x e^{-x}\sin x.$$

Substituting y_p into the equation determines $A_1 = -\frac{1}{2}$ and $A_2 = 0$ and thus

$$y_p = -\frac{1}{2} x e^{-x}\cos x.$$

This implies that the general solution to the equation is

$$y = c_1 e^{-x}\cos x + c_2 e^{-x}\sin x - \frac{1}{2} x e^{-x}\cos x,$$

for arbitrary constants c_1, c_2. For part b), it is enough to apply the initial conditions and obtain $c_1 = 0$ and $c_2 = \frac{-1}{2}$, and thus

$$y = -\frac{1}{2} e^{-x}(\sin x + x\cos x).$$

Case 2. If $r = p_n(x)e^{\alpha x}$ ($p_n(x)$ a polynomial of order n) and $e^{\alpha x}$ is not a homogeneous solution, then the particular solution is of the form $y_p = q_n(x)\, e^{\alpha x}$ where $q_n(x)$ is a polynomial of order n. If $e^{\alpha x}$ is a homogeneous solution then y_p is of the form $y_p = x\, q_n(x)\, e^{\alpha x}$, and $x e^{\alpha x}$ is a homogeneous solution then $y_p = x^2 q_n(x)\, e^{\alpha x}$.

Example 3.14. Consider the following equation

$$y'' + 3y' + 2y = x e^{-x} + 2e^{-x}\sin x.$$

Functions $y_1 = e^{-x}$, $y_2 = e^{-2x}$ are homogeneous solution, then the particular solution associated to $r_1 = x e^{-x}$ is

$$y_{p_1} = x(q_0 + q_1 x)e^{-x}.$$

Substituting y_{p_1} into the equation gives $q_0 = -1$, $q_1 = \frac{1}{2}$ and then $y_{p_1} = \left(\frac{1}{2}x^2 - x\right)e^{-x}$. The particular solution associated to $r_2 = 2e^{-x}\sin x$ is $y_p = -e^{-x}\cos x - e^{-x}\sin x$, and finally

$$y_p = e^{-x}\left(\frac{1}{2}x^2 - x - \cos x - \sin x\right).$$

Example 3.15. Consider the equation $y'' + y' = x e^x + x e^{-x} + x$. The homogeneous solutions are $y_1 = 1$, $y_2 = e^{-x}$. The particular solution associated to the first term is $y_{p_1} = (A_1 x + A_2)\, e^x$ and to the second term is $y_{p_2} = x(A_3 x + A_4)e^{-x}$. Note the x multiplication in y_{p_2}. This is because $A_4 e^{-x}$ is a homogeneous solution. The particular solution associated to the last term is $y_{p_3} = x(A_5 x + A_6)$.

Case 3. If $r = p_n(x)\cos(\omega x)$ or $r = p_n(x)\sin(\omega x)$ and $\{\cos(\omega x), \sin(\omega x)\}$ are not homogeneous solutions, then

$$y_p = P_n(x)\cos(\omega x) + Q_n(x)\sin(\omega x), \tag{3.54}$$

where P_n and Q_n are polynomials of order n. If $\{\cos(\omega x), \sin(\omega x)\}$ are homogeneous solutions, then

$$y_p = x(P_n(x)\cos(\omega x) + Q_n(x)\sin(\omega x)). \tag{3.55}$$

Example 3.16. Find a particular solution to the equation

$$y'' + y = (x+1)\sin x.$$

Since $\{\cos(x), \sin(x)\}$ are homogeneous solutions, the particular solution has the form

$$y_p = x(p_0 + p_1 x)\cos(x) + x(q_0 + q_1 x)\sin(x).$$

Substituting y_p into the equation gives

$$y_p = -\left(\frac{1}{4}x^2 + \frac{1}{2}x\right)\cos(x) + \frac{1}{4}x\sin(x).$$

Problems

Problem 3.48. Solve the following initial value problems

i. $y'' + 3y' + 2y = e^x + e^{2x}$, $y(0) = 0$, $y'(0) = 0$

ii. $y'' + 4y' + 4y = e^{-2x}$, $y(0) = 1$, $y'(0) = 0$

iii. $y'' + 4y = 2\sin(2x)$, $y(0) = 0$, $y'(0) = -1$

Problem 3.49. Use undetermined coefficient method to find a particular solution for each of the following equations.

i. $y'' - 5y' + 4y = 2e^{3x} - 3e^{4x}$

ii. $y'' + 9y = \sin(2x) + x + 1$

iii. $y'' + 3y' + 2y = x^2 + 2e^{-2x}$

iv. $y'' + 3y' + 2y = (x-1)e^{-2x}$

v. $y'' + 4y = e^x\sin(2x)$

Problem 3.50. Use undetermined coefficient method to find the solution to each following problems:

i. $y'' + y = \sin(2x), \quad y(0) = 1, y'(0) = 0.$

ii. $y'' - 3y' + 2y = e^x + \sin(x), \quad y(0) = 0, y'(0) = 1.$

iii. $y'' + 4y' + 4y = x + 2e^{-x}, \quad y(0) = 1, y'(0) = -1.$

iv. $y'' + 4y' + 5y = x + 2e^{-x}, \quad y(0) = 1, y'(0) = -1.$

Problem 3.51. Use undetermined coefficient method to find the correct form of the particular solution to each of the following equations. DO NOT calculate the coefficients.

i. $y'' + 2y' + y = 3(x+1)\, e^{-x},$

ii. $y'' + 3y' + 2y = e^{-2x}\cos(x) - x^2 e^{-x}$

iii. $y'' + y = x\sin(x) + x\, e^{-x}$

iv. $y'' + y = e^x\,(x-3) + \cos(2x)$

v. $y'' + 3y' + 2y = x\, e^{2x} + x\, e^{-x}$

Problem 3.52. Use the variation of parameters method to find solution to each of the following initial value problems

i. $y'' + y = \sec(x), y(0) = y'(0) = 0$

ii. $y'' + y = \tan(x),\ y(1) = 0,\ y'(1) = -1$

iii. $y'' + 3y' + 2y = \sin(e^x),\ y(0) = y'(0) = 1$

iv. $y'' - 2y' + 2y = \sec(x)\, e^x,\ y(0) = y'(0) = 0$

v. $y'' + 3y' + 2y = (1 + e^x)^{-1},\ y(0) = 2,\ y'(0) = 0$

vi. $y'' - y = \frac{e^x}{1 + e^{2x}},\ y(0) = 0,\ y'(0) = 0.$

vii. $y'' - 3y' + 2y = \sin(1 + e^{-x}),\ y(0) = y'(0) = 0.$

Problem 3.53. Consider the problem

$$\begin{cases} y'' + y = r(x). \\ y(0) = y'(0) = 0 \end{cases}$$

i. Verify that the function ϕ defined as

$$y(x) = \int_0^x \sin(x-t)\, r(t)\, dt,$$

is the solution to the given problem.

ii. Obtain this formula from the variation of parameter formula.

Problem 3.54. Consider the following equation

$$x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 0.$$

i. Verify that the function $y_1(x) = \sin(x)/\sqrt{x}$ is a solution to the equation.

ii. Obtain the second linearly independent solution to the equation.

iii. Write down the general solution to the following equation

$$x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = x\sqrt{x}.$$

Problem 3.55. Verify that $y_1 = e^x$ is a solution to the following equation

$$x y'' + 2(1-x)\, y' + (x-2) y = 0,$$

and then find the solution to the following problem

$$\begin{cases} x y'' + 2(1-x)\, y' + (x-2) y = 1 \\ y(1) = 0,\, y'(1) = 0 \end{cases}.$$

Problem 3.56. Find the general solution to the following equations if we know they have a first order polynomial solution to the homogeneous equation

i. $x y'' - (x+1)\, y' + y = x^2$,

ii. $x^2\, y'' - x(x+2) y' + (x+2)\, y = x^3$

Problem 3.57. Consider the equation

$$\cos(x) y'' + \sin(x) y' + \sec(x) y = 0.$$

i. Verify that $y_1 = \cos(x)$ is a solution to the equation.

ii. Find the general solution to the equation

$$\cos(x) y'' + \sin(x) y' + \sec(x) y = 1.$$

Problem 3.58. Consider the equation

$$y'' + \tan(x) y' = \cos(x).$$

It is simply verified that $y_1 = 1$ is a homogeneous solution to the homogeneous equation.

i. Use reduction of order method to find the second solution $y_2(x)$ to the homogeneous equation.

ii. Use variation of parameters formula to derive the general solution to the equation.

iii. If we rewrite the equation as a system of two first order differential equation, we reach

$$\begin{cases} y' = p \\ p' + \tan(x) p = \cos(x) \end{cases}.$$

Solve this system and prove that the solution is equal to the solution obtained in part (ii).

Problem 3.59. There is a chance to solve autonomous second order equations that have the general form $y'' = f(y, y')$. The method is as follows. Take $p = y'$ and write

$$y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = \frac{dp}{dy} p. \tag{3.56}$$

Substituting the above relation into the equation gives the following first order equation in terms of $p = p(y)$

$$\frac{dp}{dy} p = f(y, p).$$

For the solution $p = \psi(y)$ of the last equation we reach another first order equation of the form $y' = \psi(y)$. Solution of the later equation is the desired solution. Use the described method to solve the following equations

i. $y'' + e^{2y} y'^3 = 0, \quad y(0) = 0, y'(0) = -2.$

ii. $y'' + (y+1) y' = 0, \quad y(0) = 0, y'(0) = -\frac{1}{2}.$

iii. $e^{-y} y'' = y'^3, \quad y(0) = 0, y'(0) = -1.$

Problem 3.60. Here we justify the structure of the particular solution $y_p = y_1 v_1(x) + y_2\, v_2(x)$ for second order equations with constant coefficients. Let us find the solution to the following problem

$$y'' + a y' + b y = r(x).,$$

where a, b are constants. Assume $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$ are two core solutions of the homogeneous equation. We can rewrite the equation in the following form

$$(D - \lambda_2)\{(D - \lambda_1)[y] = r(x).$$

If we take $Y(x)=(D-\lambda_1)[y]$, then we reach the following system

$$\begin{cases} y'-\lambda_1 y = Y(x) \\ Y'-\lambda_2 Y = r(x) \end{cases}.$$

i. If $\lambda_1 \neq \lambda_2$ show that the solution is

$$y_p(x) = e^{\lambda_2 x}\int \frac{r(x)\,e^{-\lambda_2 x}}{\lambda_2-\lambda_1}dx - e^{\lambda_1 x}\int \frac{r(x)e^{-\lambda_1 x}}{\lambda_2-\lambda_1}\,dx,$$

and conclude the formula is the variation of parameters formula (3.48).

ii. If $\lambda_1=\lambda_2=\lambda$, show that the solution is

$$y_p(x) = x e^{\lambda_x}\int r(x)\,e^{-\lambda x}\,dx - e^{\lambda x}\int x\,r(x)e^{-\lambda x}\,dx,$$

and conclude the formula is (3.48).

iii. If $\lambda_{1,2}=\sigma+i\omega$ show that the solution is

$$y_p(x) = e^{\sigma x}\cos(\omega x)\int \frac{-r(x)\,e^{-\sigma x}\sin(\omega x)}{\omega}dx + e^{\sigma x}\sin(\omega x)\int \frac{r(x)\,e^{-\sigma x}\cos(\omega x)}{\omega}\,dx,$$

and conclude the formula is (3.48).

3.5 Higher order equations

3.5.1 Homogeneous equations

The method of characteristics works equally well for homogeneous higher order equations with constant coefficients. Consider the following homogeneous equation

$$y^{(n)} + a_1\,y^{(n-1)} + \cdots + a_n\,y = 0, \tag{3.57}$$

where $a_1, \ldots, a_n$ are some constants. The characteristic polynomial $p(\lambda)$ of this equation is obtained by substituting $e^{\lambda x}$ into the equation.This gives

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n. \tag{3.58}$$

Proposition 3.2. *If λ is a simple real root of $p(\lambda)$ (i.e., the multiplicity of λ is one), then $y=e^{\lambda x}$ is a solution to the differential equation (3.57). If λ is a repeated root of multiplicity r, then all the functions $x^{k-1}e^{\lambda x}$, for $k=1,\ldots,r$ are solutions to the equation. If $\lambda=\sigma+i\omega$ is a simple complex root of $p(\lambda)$ then functions $e^{\sigma x}\cos(\omega x)$, and $e^{\sigma x}\sin(\omega x)$ are two solutions to the equation and if $\lambda=\sigma+i\omega$ is a complex root with the multiplicity r, then all functions $x^{k-1}e^{\sigma x}\cos(\omega x)$, and $x^{k-1}e^{\sigma x}\sin(\omega x)$ are solutions to the equation.*

Example 3.17. Consider the following equation

$$y''' - 6y'' + 11y' - 6y = 0.$$

The characteristic polynomial is

$$p(\lambda) = (\lambda-1)(\lambda-2)(\lambda-3),$$

and thus functions $y_1 = e^x$, $y_2 = e^{2x}$ and $y_3 = e^{3x}$ are solutions to the equation. The general homogeneous solution is

$$y = c_1\, e^x + c_2\, e^{2x} + c_3\, e^{3x}.$$

The equation

$$y''' + y'' + y' + y = 0,$$

has the characteristic polynomial $p(\lambda) = (\lambda + 1)(\lambda^2 + 1)$, and functions $y_1 = e^{-x}$, $y_2 = \cos(x)$ and $y_3 = \sin(x)$. The characteristic polynomial of the equation

$$y^{(4)} + 2y'' + y = 0,$$

is $p(\lambda) = (\lambda^2 + 1)^2$ with the repeated complex root $\lambda = i$. In this case, functions $y_1 = \cos(x)$, $y_2 = x\cos(x)$, $y_3 = \sin(x)$ and $y_4 = x\sin(x)$ are core solutions to the equation.

Example 3.18. Consider the following equation

$$y^{(4)} - y = 0.$$

The characteristic polynomial is

$$p(\lambda) = \lambda^4 - 1.$$

The roots of $p(\lambda)$ are $\lambda_1 = 1, \lambda_2 = -1, \lambda_{3,4} = \pm i$ and thus

$$y = c_1\, e^x + c_2\, e^{-x} + c_3 \cos(x) + c_4 \sin(x) x.$$

3.5.2 Particular solutions

The method of undetermined coefficients works equally well for the linear equations of higher order with constant coefficients. Since it is completely similar to the second order equation, we do not repeat it here. Let us formulate the variation of parameters method for third order equations here. Consider the following equation

$$y''' + a_1(x)\, y'' + a_2(x)\, y' + a_3(x)\, y = r(x), \tag{3.59}$$

and assume that y_1, y_2, y_3, the three linearly independent homogeneous solutions. Define

$$y_p = c_1(x) y_1 + c_2(x)\, y_2 + c_3(x)\, y_3. \tag{3.60}$$

To determine functions $c_k(x)$, we set

$$y_1\, c_1' + y_2\, c_2' + y_3\, c_3' = 0 \quad \text{and} \quad y_1'\, c_1' + y_2'\, c_2' + y_3'\, c_3' = 0, \tag{3.61}$$

and substitute y_p into the equation to obtain

$$y_1''\, c_1' + y_2''\, c_2' + y_3''\, c_3' = r(x). \tag{3.62}$$

We obtain a system to determine c_1', c_2', and c_3' as follows

$$\begin{cases} y_1\, c_1' + y_2\, c_2' + y_3\, c_3' = 0 \\ y_1'\, c_1' + y_2'\, c_2' + y_3'\, c_3' = 0 \\ y_1''\, c_1' + y_2''\, c_2' + y_3''\, c_3' = r(x) \end{cases}.$$

Example 3.19. Let us solve the following problem

$$y''' - 6y'' + 11y' - 6y = \frac{2e^{4x}}{1+e^x}.$$

The core solutions of the homogeneous equation are $y_1 = e^x$, $y_2 = e^{2x}$ and $y_3 = e^{3x}$. We reach the following system for c_1', c_2', c_3'

$$\begin{cases} e^x\, c_1' + e^{2x}\, c_2' + e^{3x}\, c_3' = 0 \\ e^x\, c_1' + 2e^{2x}\, c_2' + 3e^{3x}\, c_3' = 0 \\ e^x\, c_1' + 4e^{2x}\, c_2' + 9e^{3x}\, c_3' = \frac{2e^{4x}}{1+e^x} \end{cases}.$$

We obtain

$$c_1' = \frac{1}{2}e^{-x} r(x), c_2' = -r(x)e^{-2x}, c_3' = \frac{1}{2} r(x)e^{-3x}.$$

Therefore

$$c_1 = \frac{1}{2}\int \frac{e^{3x}}{1+e^x}dx = \frac{1}{2}e^{2x} - e^x + \ln(1+e^x),$$

$$c_2 = -\int \frac{e^{2x}}{1+e^x}dx = 2\ln(1+e^x) - 2e^x,$$

$$c_3 = \frac{1}{2}\int \frac{e^x}{1+e^x}dx = \ln(1+e^x).$$

The particular solution after simplification is determined as

$$y_p = e^x(1+e^x)^2 \ln(1+e^x).$$

Remark 3.5. The above three equations is put in the matrix form and gives

$$\begin{pmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \\ c_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ r(x) \end{pmatrix}, \tag{3.63}$$

and thus

$$c_1(x) = \int \frac{W(y_2, y_3)(x)\, r(x)}{W(y_1, y_2, y_3)(x)}\, dx, \tag{3.64}$$

$$c_2(x) = -\int \frac{W(y_1, y_3)(x)\, r(x)}{W(y_1, y_2, y_3)(x)} dx, \tag{3.65}$$

$$c_3(x) = \int \frac{W(y_1, y_2)(x)\, r(x)}{W(y_1, y_2, y_3)(x)}\, dx. \tag{3.66}$$

Here $W(y_1, y_2, y_3)$ is the determinant of the matrix in the left hand side of (3.63).

Example 3.20. Let us solve the following problem

$$\begin{cases} y''' - y'' - y' + y = 4e^x \sin(2x) \\ y(0) = y'(0) = y''(0) = 0 \end{cases}.$$

The core solutions of the homogeneous equation are $y_1 = e^x$, $y_2 = x\,e^x$ and $y_3 = e^{-x}$. We have $W(y_1, y_2, y_3)(x) = 4e^x$ and

$$W(y_1, y_2)(x) = e^{2x}, W(y_1, y_3)(x) = -2, W(y_2, y_3)(x) = -2x - 1. \tag{3.67}$$

Therefore

$$c_1 = \int_0^x (-2s-1)\sin(2s) = \frac{1}{2}(2x+1)\cos(2x) - \frac{1}{2}\sin(2x) - \frac{1}{2},$$
$$c_2 = 2\int_0^x \sin(2s)\,ds = -\cos(2x) + 1,$$
$$c_3 = \int_0^x e^{2s}\sin(2s)\,ds = \frac{e^{2x}}{4}(\sin(2x) - \cos(2x)) + \frac{1}{4}.$$

The solution after simplification is

$$y = \frac{e^x}{4}[\cos(2x) - \sin(2x)] - \frac{1}{2}e^x + x\,e^x + \frac{1}{4}e^{-x}. \tag{3.68}$$

Problems

Problem 3.61. Solve the following higher order problems

i. $y''' - 2y'' + y' - 2y = 0,\ y(0) = 1,\ y'(0) = y''(0) = 0$

ii. $y''' - y'' + y' - y = 0,\ y'(0) = 0,\ y'(0) = 1,\ y''(0) = 0$

iii. $y''' + y'' - 2y = 0,\ y(0) = y'(0) = 0,\ y''(0) = 1$

iv. $y^{(4)} - 3y'' + 2y = 0,\ y(0) = y'(0) = y''(0) = 0,\ y'''(0) = 1$

v. $y^{(5)} + 32y = 0,\ y(0) = 1,\ y^{(k)}(0) = 0$ for $k = 1, ..., 4$.

Problem 3.62. For each pair or triple of the given functions, write down a differential equation having them as its core solutions.

i. $y_1 = 1,\ y_2 = 2e^{-x},\ y_3 = -e^{2x}$

ii. $y_1 = 1,\ y_2 = x,\ y_3 = e^x$

Problem 3.63. Undetermined coefficients method works equally for higher order linear problems with constant coefficients. Use this method method to find the general solution to each following higher order equation:

i. $y''' - y' = x\,e^x + x\,e^{2x} + x$.

ii. $y''' - 2y'' + y' - 2y = e^{-x}$.

iii. $y^{(5)} + 32y = x^6 + x^4 + x^2 + 1$.

iv. $y^{(4)} + 4y'' + 4y = 3\cos(2x)$.

Problem 3.64. Consider the following equation

$$y''' - y'' + y' - y = r(x).$$

Use undetermined coefficient method to determine the correct form of a particular solution to the equation. You don't need to calculate the coefficients.

i. $r(x) = 2e^{-x} + \sin(x)$.

ii. $r(x) = e^x - x\cos(x)$.

iii. $r(x) = e^x\sin(x) + x$.

iv. $r(x) = e^x \sin(x) + e^x$

3.6 Applications

The following simple equation models the behavior of several physical systems mathematically

$$y'' + \omega_0^2\, y = 0. \tag{3.69}$$

Interpreting the equation as a *harmonic oscillator*, ω_0 is called the *natural frequency* of the oscillation. If we multiply the equation by y', we obtain

$$y'y'' + \omega_0^2 y'y = 0,$$

and integrate it, we reach the following equality representing the *energy* of the oscillator

$$\frac{1}{2}(y')^2 + \frac{1}{2}\omega_0^2 y^2 = E. \tag{3.70}$$

3.6.1 Conservation principle

Conservation of energy, mass, and momentum is a principal part of mathematical modeling of physical systems. Consider a particle of mass m moving along a path γ in the space. The *kinetic energy* E_k of the particle is defined as follows

$$K = \frac{1}{2}m\,|v|^2, \tag{3.71}$$

where $v(t)$ is the velocity of the particle. If this particle is under the influence of a *conservative force* f, the *potential energy* V of the mass is as follows

$$-\nabla V = f, \tag{3.72}$$

where ∇ (read nabla) is the gradient operator. The conservation of energy states that the total energy $K + V$ of the particle *along* its path γ is constant. In fact, the derivative of $K + V$ along γ is zero as the following calculation shows

$$\frac{d}{dt}\left\{\frac{1}{2}m\,|v|^2 + V\right\} = m\,v.\frac{dv}{dt} + \nabla V\,\frac{d\gamma}{dt},$$

where $\frac{d\gamma}{dt}$ is equal to velocity v of the particle. Therefore, we reach the equality

$$v.\left(m\frac{dv}{dt} + \nabla V\right) = v.\left(m\frac{dv}{dt} - f\right) = 0.$$

The expression in the bracket is related to the NEWTON's second law

$$m\frac{dv}{dt} = f,$$

where derivative are taken along the path of the motion of mass m.

3.6.2 Mass-spring system

Let us apply the conservation of energy to formulate the equation of a mass-spring system. The HOOK's law states the relationship between the displacement x (with respect to its resting length) and the force exerted on it; see figure (3.4)

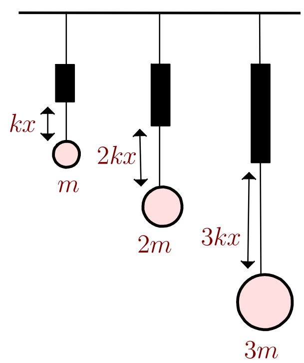


Figure 3.4.

The mass m causes the spring to be stretched kx unit. Replacing m with $2m$, causes the spring to be stretched $2kx$ and so on. This implies that the relationship between the force and the stretch is linear $f = kx$, where k is a positive constant called stiffness of the spring. This law is called the HOOK's law after the British physicist R. HOOK. The potential energy stored in a stretched (or contracted) spring is $V = \frac{1}{2}kx^2$ by Eq.3.72. Consider the mass-spring system shown in the figure (3.5).

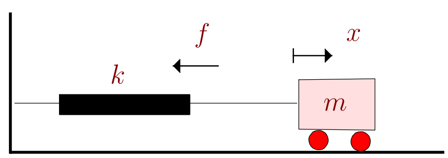


Figure 3.5.

The conservation of energy for this system reads

$$\frac{1}{2}mv^2(t) + \frac{1}{2}kx^2(t) = E. \tag{3.73}$$

Since E is constant, the value of E is determined by the initial state

$$E = \frac{1}{2}mv_0^2 + \frac{1}{2}kx_0^2, \tag{3.74}$$

where x_0 and v_0 are respectively the initial displacement (with resting to the resting position) and the initial velocity of the mass m. The equation (3.73) is solved for v by

$$v = \pm\sqrt{\frac{2E - kx^2(t)}{m}}. \tag{3.75}$$

Since $v = x'$, we obtain the following separable equation for $x = x(t)$:

$$x' = \pm\sqrt{\frac{2E - kx^2(t)}{m}}. \tag{3.76}$$

The function $x(t)$ is derived by the following integration

$$\int_{x_0}^{x} \frac{ds}{\sqrt{2E - ks^2}} = \frac{1}{\sqrt{m}} t, \tag{3.77}$$

and thus

$$x(t) = \sqrt{\frac{2E}{k}} \sin\left(\sqrt{\frac{k}{m}}\, t + \phi_0\right). \tag{3.78}$$

Here ϕ_0 is the constant

$$\phi_0 = \sin^{-1}\left(\sqrt{\frac{k}{2E}}\, x_0\right). \tag{3.79}$$

Note that the equivalent second order equation formulation of this system is

$$\begin{cases} x'' + \frac{k}{m} x = 0 \\ x(0) = x_0, x'(0) = v_0 \end{cases}.$$

It is simply verified that the solution to this equation is (3.78). In fact, applying given initial conditions to the general homogeneous solution

$$x = c_1 \cos\left(\sqrt{\frac{k}{m}} t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}} t\right),$$

gives $c_1 = x_0$ and $c_2 = \sqrt{\frac{m}{k}}\, v_0$. By simple trigonometric simplifications, one can derives Eq.(3.78).

Example 3.21. Suppose that a spring with the stiffness $k = 10^4 N/m$ is connected to a mass $m = 40\text{kg}$ (the mass-spring is under the gravity force) and assume that the system is at rest. If the spring is stretched for $x_0 = 0.1m$ and released (the initial velocity $v_0 = 0$), we obtain the position function $x(t)$. To do that, we calculate first the total energy of the system

$$E = \frac{1}{2} k x_0^2 = 50\,\text{joule}.$$

We also have

$$\sin(\phi_0) = \sqrt{\frac{k x_0^2}{2E}} = 1,$$

that gives $\phi_0 = \frac{\pi}{2}$. This implies that

$$x(t) = 0.1 \cos(5\sqrt{10}\, t).$$

The xv-plane for this system is called the *state space* or *phase plane*. For a numeric example, assume $k = 10^{-3} N/m$ and $m = 1$kg. If the initial displacement data are $x_0 = 10^{-1} m$, $v_0 = 0 m/s$, then the energy level of the system is $E = 5 \times 10^{-6}$ Jul. The figure (3.6) shows the graph of the solution.

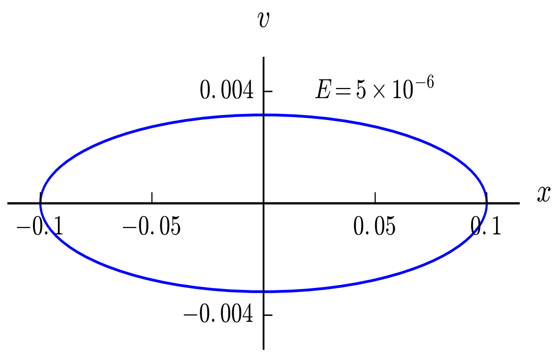


Figure 3.6.

Observe that the curve is closed. This implies that the functions $x = x(t)$ and $v = v(t)$ are periodic with respect to time. In fact, since the parametric curve $\gamma(t) = (x(t), v(t))$ is closed, there exists $T_0 > 0$ such that $\gamma(t + T_0) = \gamma(t)$, and therefore both $x(t)$ and $v(t)$ are periodic with the period T_0.

Now, assume a drag force acting on the mass of the form $f_d = -bx'$, where $b > 0$ is a constant. The equation of motion in this case is

$$mx'' + bx' + kx = 0.$$

Multiplying the equation by x' and integrating results to

$$E(t) = \frac{1}{2} m x'^2(t) + \frac{1}{2} k x^2(t) = E(0) - b \int_0^t x'^2(s)\, ds.$$

Note that the energy is dissipating in this case as shown in the figure (3.7).

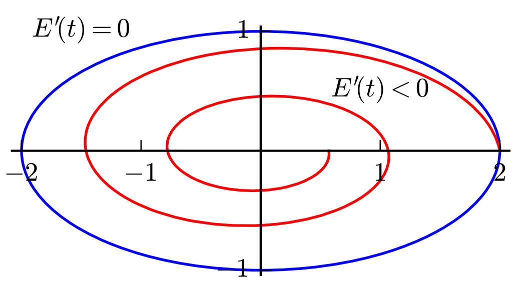


Figure 3.7.

3.6.3 Pendulums

The motion of a pendulum can also be derived by the conservation of energy. Consider the pendulum shown in the figure (3.8).

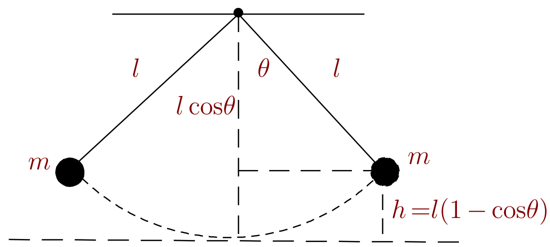


Figure 3.8.

Since the potential energy of the mass at height h is $V = mgh$, by the relation $h = l(1-\cos\theta)$, we obtain

$$V(\theta) = mgl(1-\cos\theta). \tag{3.80}$$

Note that $x = l\theta$ and consequently we obtain $v = x' = l\theta'$. By the conservation of energy, we have

$$\frac{1}{2}ml^2\theta'^2 + mgl(1-\cos\theta) = E. \tag{3.81}$$

If the mass is initially at $\theta = \theta_0$ and the initial velocity is $v_0 = 0$, total energy is

$$E = mgl(1-\cos\theta_0). \tag{3.82}$$

Consequently, we derive the following differential equation for the motion of the mass

$$\frac{d\theta}{dt} = \pm\sqrt{\frac{2g}{l}}\,\sqrt{\cos(\theta)-\cos(\theta_0)}. \tag{3.83}$$

Integrating the above equation gives

$$\sqrt{\frac{2g}{l}}\,t = \int_\theta^{\theta_0}\frac{d\theta}{\sqrt{\cos(\theta)-\cos(\theta_0)}}. \tag{3.84}$$

If θ_0 is small enough, we can write

$$\cos(\theta)-\cos(\theta_0) \cong \frac{1}{2}(\theta_0^2-\theta^2), \tag{3.85}$$

and then the integration gives

$$\sqrt{\frac{g}{l}}\,t \cong \frac{\pi}{2} - \sin^{-1}\left(\frac{\theta}{\theta_0}\right). \tag{3.86}$$

This gives θ as

$$\theta(t) \cong \theta_0\cos\left(\sqrt{\frac{g}{l}}\,t\right). \tag{3.87}$$

Note that the period of the oscillation is

$$T = 2\pi\sqrt{\frac{l}{g}}, \tag{3.88}$$

which is independent of the mass m and the initial angel θ_0.

The equation (3.81) is the energy formulation of the second order equation

$$\theta'' + \frac{g}{l}\sin\theta = 0, \tag{3.89}$$

which is derived in the beginning of this chapter. Note that $l\theta' = v$ and $\frac{1}{2}ml\theta'^2 = \frac{1}{2}mv^2$, the kinetic energy of the system. The figure (3.9) shows some energy levels of the pendulum.

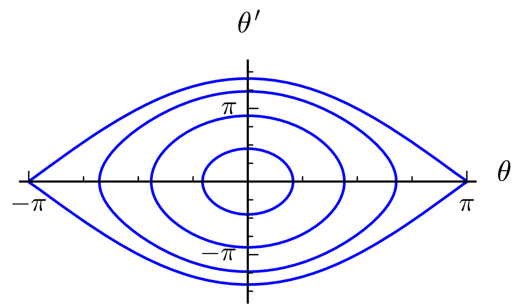


Figure 3.9.

3.6.4 Electrical circuits

Consider the LC circuit shown in the figure (3.10).

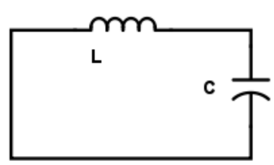


Figure 3.10.

As we observed in the first section, the differential equation describing the voltage across the capacitor C is

$$\frac{\mathrm{d}^2 V_c}{\mathrm{d}t^2} + \frac{1}{LC}V_c = 0. \tag{3.90}$$

The quantity $\omega_0 = \frac{1}{\sqrt{LC}}$ is the natural frequency of the circuit. We observe that a LC circuit conserves its energy. The energy formulation of the above circuit is simply derived by multiplying the equation by V_c' and integrating it

$$\frac{1}{2}L\,(V_c')^2 + \frac{1}{2C}V_c^2 = E.$$

Since $V_c' = i$, the electrical current in the inductor L, we obtain

$$\frac{1}{2}Li^2 + \frac{1}{2}CV_c^2 = E.$$

If the inductor L is not ideal, having a resistance R, the equation reads

$$\frac{\mathrm{d}^2 V_c}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}V_c}{\mathrm{d}t} + \frac{1}{LC}V_c = 0.$$

Observe the similarity between the behavior of an electrical circuit, a mechanical mass-spring systems and a pendulum. They follow same energy formulation and have same form of differential equations.

Now let us consider a switching circuit. Consider the figure (3.11).

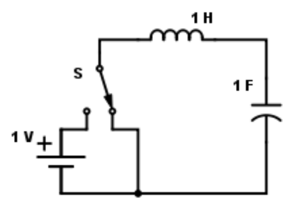


Figure 3.11.

Assume that the switch S connects at time $t=1$ to the voltage supply $V_s = 1$ Volt and disconnects from the supply at $t=2$. Furthermore, assume that $V_c(0)=1$ and $i(0)=0$. The mathematical model of this circuit is

$$\begin{cases} V_c'' + V_c = r(t) \\ V_c(0) = 1, V_c'(0) = 0 \end{cases}, \tag{3.91}$$

where $r = \begin{cases} 1 & 1<t<2 \\ 0 & \text{else} \end{cases}$ is a unit pulse. To solve this problem, we split up the problem into 3 sub-problems.

For $t<1$. The problem for $t<1$ reads

$$\begin{cases} V_c'' + V_c = 0 \\ V_c(0) = 1, V_c'(0) = 0 \end{cases}. \tag{3.92}$$

Note that in this case, the dynamic of the system is influenced by initial conditions. The solution is $V_c(t) = \cos(t)$ for $t \le 1$.

For $1<t<2$. The problem in this interval reads

$$\begin{cases} V_c'' + V_c = 1 \\ V_c(1) = \cos(1), V_c'(1) = -\sin(1) \end{cases}.$$

Note that the initial conditions of the problem set such that the solution is continuous at $t=1$. The solution to this problem is $V(t) = \cos(t) + 1 - \cos(t-1)$ for $1 \le t \le 2$.

For $t>2$. The problem reads

$$\begin{cases} V_c'' + V_c = 0 \\ V_c(2) = \cos(2) + 1 - \cos(1), V_c'(2) = \sin(1) - \sin(2) \end{cases}.$$

Again the initial conditions of the problem are set at $t = 2$ and are provided by the solution for $t < 2$. The solution to the problem for $t \geq 2$ is

$$V_c(t) = \cos(t) - \cos(t-1) + \cos(t-2).$$

The figure (3.12) shows the solution in the interval $[0, 8]$. We solve this type of problems by simpler method in next chapters when we introduce Laplace transform method.

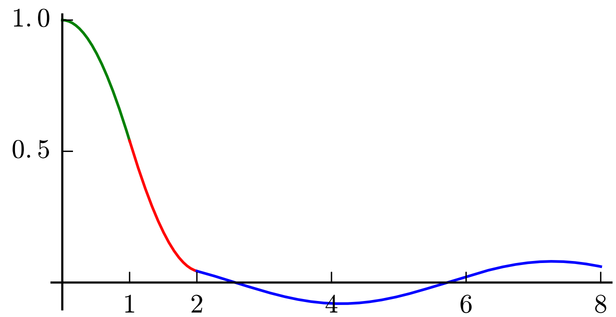


Figure 3.12.

3.6.5 Classification of damped oscillators

Engineers usually use a damper to control oscillations such that they vanish in long term. A damper like friction acts always against the direction of motion and causes the total energy of a system to be dissipated. For example, a dashpot is used to control the vibration in a mechanical mass-spring system. Let us write the general form of a damped oscillator as

$$y'' - 2\sigma y' + \omega_0^2\, y = 0, \tag{3.93}$$

where $\sigma > 0$ is a constant. The characteristic polynomial $p(\lambda)$ of the equation has roots

$$\lambda_{1,2} = -\sigma \pm \sqrt{\sigma^2 - \omega_0^2}. \tag{3.94}$$

There are three different cases for the solutions based on the term $\sigma^2 - \omega_0^2$.

Case 1. (under-damped) If $\sigma < \omega_0$ we have two complex roots

$$\lambda_{1,2} = -\sigma \pm i\omega, \tag{3.95}$$

where $\omega = \sqrt{\omega_0^2 - \sigma^2}$ and the solution can be written as

$$y = A_0\, e^{-\sigma t} \sin(\omega t + \varphi_0), \tag{3.96}$$

for some constants A_0 and φ_0. Observe that the solution is oscillatory due to the trigonometric term $\sin(\omega t + \varphi_0)$ and vanishes in long term due to the factor $e^{-\sigma t}$ (note that $\sigma > 0$). Consider the following example

$$\begin{cases} y'' + 2\sigma y' + 4y = 0 \\ y(0) = 0, y'(0) = 1 \end{cases}. \tag{3.97}$$

Clearly, for $0<\sigma<2$, the solution is under-damped. For a numerical example, let us assume $\sigma=0.5$. The solution in this case is

$$y(t)=\frac{2}{\sqrt{15}}e^{-t/2}\sin(\sqrt{15}\,t/2).$$

The figure (3.13) shows the graph of the solution.

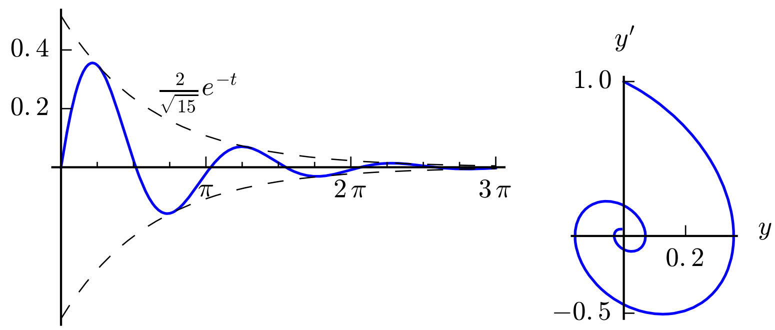


Figure 3.13.

Although the function is not periodic in the usual sense, we define $T=\frac{2\pi}{\omega}$ as the *quasi-period* T of the solution.

Case 2. (critically damped) If $\sigma=\omega_0$ then $\lambda_1=\lambda_2=-\sigma$ and then

$$y=(c_1+c_2t)\,e^{-\sigma t}. \tag{3.98}$$

For the problem (3.97), the solution is critically damped if $\sigma=2$. In this case the solution is $y=t\,e^{-2t}$. Note that $y(t)\to0$ when $t\to\infty$. The figure (3.14) shows the time and phase behavior of the solution.

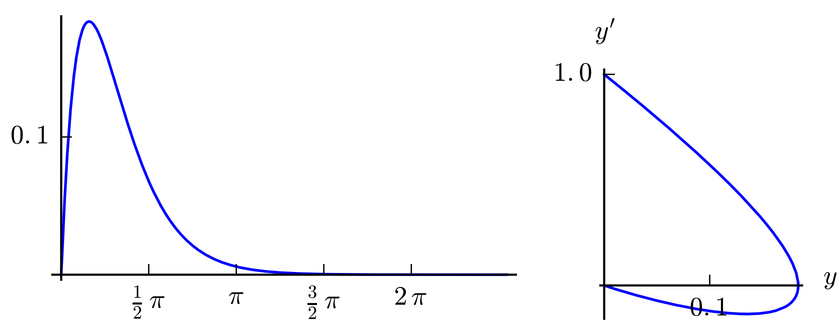


Figure 3.14.

Case 3. (over-damped) If $\sigma>\omega_0$, the characteristic $p(\lambda)$ has two real distinct roots

$$\lambda_{1,2}=-\sigma\pm\sqrt{\sigma^2-\omega_0^2}, \tag{3.99}$$

which are both negative, i.e., $\lambda_1,\lambda_2<0$. The solution in this case is

$$y(x)=c_1\,e^{\lambda_1t}+c_2\,e^{\lambda_2t}, \tag{3.100}$$

and therefore $y(t) \to 0$ in long term. For example, if $\sigma = 3$ in the problem (3.97), then $\lambda_{1,2} = -3 \pm \sqrt{5}$ and thus the solution is

$$y(t) = \frac{e^{(-3+\sqrt{5})t}}{2\sqrt{5}} - \frac{e^{(-3-\sqrt{5})t}}{2\sqrt{5}}.$$

The graph of the solution is shown in the figure (3.15).

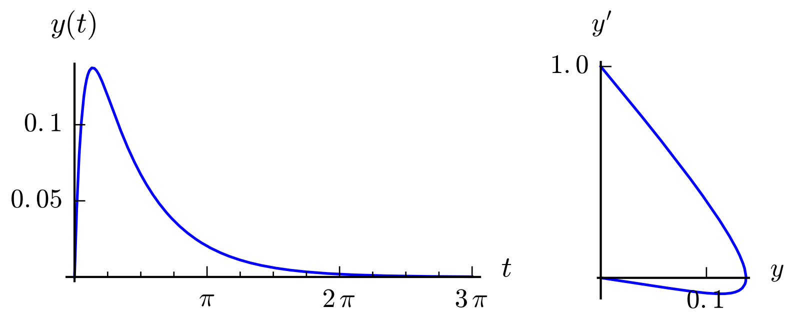


Figure 3.15.

3.6.6 Resonance

A harmonic oscillator shows the resonance behavior if it is stimulated by a forcing term having same frequency as the natural frequency ω_0. Consider the equation

$$y'' + \omega_0^2 y = \sin(\omega t). \tag{3.101}$$

Clearly if $\omega \neq \omega_0$, the function

$$y = A(\omega)\sin(\omega t), \tag{3.102}$$

where $A(\omega) = \frac{1}{\omega_0^2 - \omega^2}$ solves the equation. Observe that the magnitude of the solution, $A(\omega)$, depends on the difference $\omega_0 - \omega$. If $|\omega - \omega_0|$ is small, then $A(\omega)$ is very large. For example if $\omega_0 = 1$, then $A(0.9) = 5.26$, $A(0.95) = 10.26$, $A(0.98) = 25.25$ and $A(0.99) = 50.25$. As we know, the structure of the solution changes when $\omega = \omega_0$. In fact, if $\omega = \omega_0$, the solution is

$$y(t) = -\frac{1}{2}\omega_0\, t\sin(\omega_0 t). \tag{3.103}$$

This phenomena is called *resonance*. Consider the circuit shown in the figure (3.16).

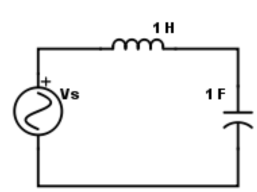


Figure 3.16.

The natural frequency of this circuit is $\omega_0 = 1$. If $V_s = \sin(t)$, the solution is $V_c = -\frac{1}{2}t\sin(t)$. The magnitude of the voltage across C increases unbounded with respect to time. The graph of the solution is shown in the figure (3.17). Clearly, no capacitor can endure this increase in magnitude and will eventually burn.

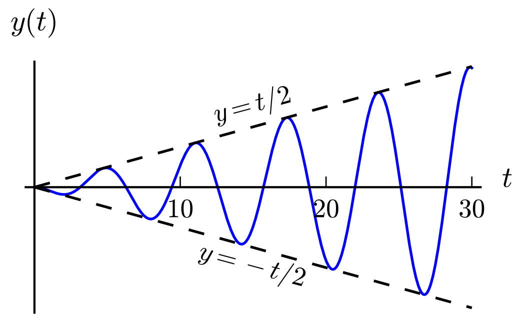


Figure 3.17.

3.6.7 Moving objects with variable masses

The familiar form of the NEWTON's second law $f = ma$ holds only if the mass remains unchanged during its motion. In several applications this assumption fails and the mass changes duration the motion. For this, we should write the law in its original form

$$\frac{dP}{dt} = f, \tag{3.104}$$

where $P(t) = m(t)\, v(t)$ is the momentum of the system at time t and f is the total forces acting on the mass $m(t)$ at that time. For a simple illustration, let us consider a system consisting two masses: the mass m that moves with the velocity v and mass δm that moves with the velocity u. The momentum of the system by definition is

$$P(t) = mv(t) + \delta m\, u(t). \tag{3.105}$$

Suppose that at time $t + \delta t$, the mass δm attaches to the mass m and the combined mass moves with the velocity $v(t + \delta t)$. The momentum at $t + \delta t$ is

$$P(t + \delta t) = (m + \delta m)\, v(t + \delta t). \tag{3.106}$$

According to (3.104), we can write

$$\lim_{\delta t \to 0} \frac{(m + \delta m)v(t + \delta t) - mv(t) - \delta m\, u(t)}{\delta t} = f. \tag{3.107}$$

The limit in the left hand side is equal to

$$\lim_{\delta t \to 0} \frac{(m + \delta m)v(t + \delta t) - mv(t) - \delta m\, u(t)}{\delta t} = m\frac{dv}{dt} + \frac{dm}{dt}v - \frac{dm}{dt}u. \tag{3.108}$$

Thus the correct form of the second NEWTON's law is

$$\frac{d}{dt}(mv) - u\frac{dm}{dt} = f. \tag{3.109}$$

Note that if m is constant, the above relation has the familiar form $f = ma$. The case when m is loosing the quantity is similar. In fact, at time $t + \delta t$ we have

$$P(t+\delta t) = m(t+\delta t)\, v(t+\delta t) - u\delta m\,, \tag{3.110}$$

which leads to the equation (3.109) as well.

A space rocket problem.

As an application of the variable mass formula (3.109), let us find the equation of the motion of a space rocket. Suppose that the mass of the rocket is m_1 and it has a fuel container of mass m_2; which is shown schematically in the figure (3.18). Furthermore assume that the rocket burns during its motion the fuel with the constant rate a and ejects the produced gas backward with the constant velocity c (relative to the rocket). In other word, if the velocity of the rocket is $v(t)$, then the produced gas will be emitted with the velocity $u(t) = -c + v(t)$.

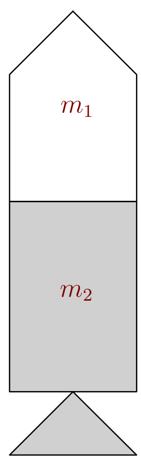


Figure 3.18. A simple model of space rocket.

If we ignore the air resistance against the rocket, formula (3.109) reads

$$m\frac{dv}{dt} + c\frac{dm}{dt} = -mg. \tag{3.111}$$

We can rewrite the above equation as

$$d(v + c\,\text{log}m) = -g dt. \tag{3.112}$$

Assuming $v(0) = 0$, gives $v(t)$ as

$$v = c\log\frac{m(0)}{m(t)} - gt. \tag{3.113}$$

Notice that $m(t) = -at + m_1 + m_2$ and then for $T = \frac{m_2}{a}$, the whole amount of the fuel in the container is consumed. At this moment the velocity will be

$$v(T) = c \log\left(1 + \frac{m_2}{m_1}\right) - \frac{g m_2}{a}. \tag{3.114}$$

The height function $h(t)$ is obtained by the integration of $v(t)$ as

$$h(t) = -\frac{1}{2} g t^2 + c t + \frac{c}{a} m(t) \log\left(\frac{m(t)}{m(0)}\right). \tag{3.115}$$

A rain drop

Consider a rain drop falling straight down and it absorbs water (in the form of steam) from air. Let us obtain the equation of the motion of the rain drop. Assume that the drop is in the shape of a ball with the initial mass m_0 and that it is absorbing water with the rate proportional to its surface are. According to the assumptions about the shape of the drop and the in-take rate, we can write

$$\frac{dm}{dt}(t) = k_1 r^2(t), \tag{3.116}$$

where k_1 is a positive constant. Since $m = k_2 r^3(t)$ for some constant k_2, we obtain the following differential equation for m

$$\frac{dm}{dt} = k_3 m^{2/3}, \tag{3.117}$$

where k_3 is some positive constant. Solving the above equation gives $m(t)$:

$$m = (k_3 t + m_0^{1/3})^3. \tag{3.118}$$

We assume that the velocity of the steam in the environment is zero, that is, $u = 0$. Thus the formula (3.109) es expressed by

$$\frac{d}{dt}(m v) = -m g. \tag{3.119}$$

Integrating the above equation gives

$$m(t)\, v(t) = -g \int_0^t m(s)\, ds = -\frac{g}{4 k_3} m^{4/3} + \frac{g}{4 k_3} m_0^{4/3}. \tag{3.120}$$

Finally, $v(t)$, the velocity of the drop is obtained

$$v(t) = -\frac{g}{4 k_3} m^{1/3} + \frac{g}{4 k_3} \frac{m_0^{4/3}}{m}. \tag{3.121}$$

Problems

Problem 3.65. Consider the problem

$$\begin{cases} y'' + 4y = r(t) \\ y(0) = 1, y'(0) = -1 \end{cases},$$

where $r(t) = \begin{cases} 1 & t<1 \\ 2 & t>1 \end{cases}$. Draw the solution in the interval $[0,6]$.

Problem 3.66. Consider the problem

$$\begin{cases} y'' + p(x)y' + y = 0 \\ y(0) = 1, y'(0) = -1 \end{cases},$$

where $p(x) = \begin{cases} 2 & x<\pi \\ 0 & x>\pi \end{cases}$. Find the solution to the problem and use a computer software to draw the solution in the interval $(0, 2\pi)$.

Problem 3.67. Consider a damped mass-spring system with a unilateral dashpot shown in the figure (3.19). Assuming $m = 0.1\text{kg}$, $K = 0.5N/m$ and $D = \begin{cases} 0.2v & x>0 \\ 0 & x<0 \end{cases}$, write the differential equation describing the position function $x(t)$ (with respect to the resting state) if initial conditions are $x(0) = 0$, $v(0) = 2$. Draw the solution in the interval $t = [0, \pi]$.

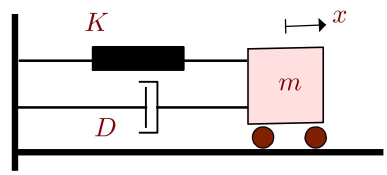


Figure 3.19.

Problem 3.68. Consider the mass-spring system shown in the figure (3.20). If $x(t)$ denotes the displacement of the mass m with respect to the resting position, write down the equation for $x(t)$.

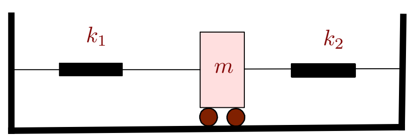


Figure 3.20.

Problem 3.69. Consider the forced oscillator shown in the figure (3.21)

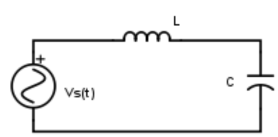


Figure 3.21.

where the supply V_s is

$$V_s(t) = \begin{cases} Lt & 0 \le t \le \frac{\pi}{\omega} \\ \frac{L\pi}{\omega} & t \ge \frac{\pi}{\omega} \end{cases},$$

for $\omega^2 = \frac{1}{LC}$. Write down the differential equation for $V_c(t)$, the voltage across the capacitor and solve it assuming $V_c(0) = i(0) = 0$ where $i(0)$ is the initial eclectic current in the inductor L.

Problem 3.70. Write down the differential equation of the parallel RCL circuit shown in the figure (3.22) and discus the type of solutions based on the values of R, L and C.

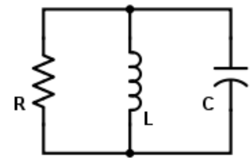


Figure 3.22.

Problem 3.71. Consider the switching circuit shown in the figure (3.23). If the switch S connects at $t=0$ and disconnects at $t=\frac{\pi}{2}$, find the value $V_c(\pi)$ if $V_c(0)=0$ and $V_c'(0)=2$.

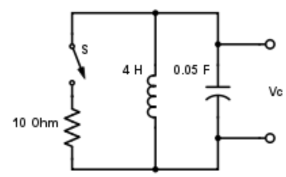
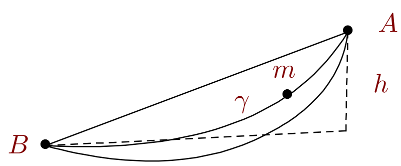


Figure 3.23.

Problem 3.72. Assume that the mass m starts off the point A with the initial velocity $v_0=0$ and reaches to the point B along the path γ. Show that its velocity at the point B is independent of the path γ and is equal to $v_B=\sqrt{2gh}$ where h is the relative height of the point A with respect to B.



Problem 3.73. For a body moving with the initial velocity V_0, assume the air resistance is proportional to the velocity. If the velocity of the body reduces to $V_0/2$ after T, find the time when the velocity reduces to $V_0/5$.

Problem 3.74. The water resistance against a moving raft with the mass $m=10\text{kg}$ is given by

$$F=-v-0.1v^3,$$

where v is the velocity of the raft. If the initial velocity of the raft is $v_0=100m/s$, find the time when the velocity drops to $1m/s$.

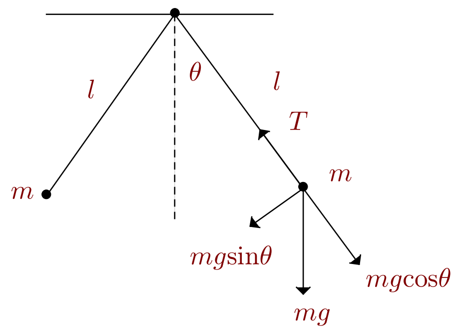
Problem 3.75. Assume that the air resistance against a falling body with the unit mass is proportional to its velocity. If the wind is blowing with the angel 45 degree with the constant force $1N$, write down the system of equations and propose a method to solve it.

Problem 3.76. Deduce the conservation of energy for conservative force fields using the Newton's law $m\dot{v}=F$.

Problem 3.77. Assume a spring displaces 0.05 meter if a 50kg mass hang to its end. Draw the displacement function function if a 25kg mass hang to the spring with the initial conditions $x_0=0$, $v_0=1m/s$.

Problem 3.78. Obtain the displacement of a mass-spring system $m=4\text{kg}$, $k=10^4$ with the initial conditions $x_0=0.1m$, $v_0=0$.

Problem 3.79. Consider the pendulum shown in the following figure.



According to the figure, the only force acting on the mass m is $F = mg\sin(\theta)$. Remember that the gradient operator ∇ in the polar coordinate has the form

$$\nabla V = \frac{1}{r}\frac{\partial V}{\partial \theta} + \frac{\partial V}{\partial r}.$$

Use this information to prove that

$$V(\theta) = mgl\,(1 - \cos(\theta)).$$

Problem 3.80. For the space rocket described in this chapter, assume as=n air resistance of the form $f = -kv$ for a constant $k > 0$. Obtain the differential equation and solve it.

Problem 3.81. For the rain drop problem described in this chapter, assume an air resistance of the form $f = -kv$ for a constant $k > 0$. Obtain the differential equation and solve it.

Problem 3.82. Assume a piece of ice in the shape of a ball with the initial radius r_0 is connected to a pendulum with the length l. If the ice loses its mass according to its surface area, write the equation of motion. Use a computer software to draw some trajectories of the equation.

Problem 3.83. In contrast to the constant acceleration g for a free falling body, prove that the acceleration of a rain drop explained in this chapter is

$$\tilde{g}(t) = -\frac{g}{4}\left(1 + 3\left(\frac{m_0}{m}\right)^{4/3}\right).$$