Chapter 2
First Order Equations

It turns out that first-order differential equations express the behavior of several real-world problems mathematically. Since there is no hope to solve general first-order equations in closed form, we confine our study to equations for which a standard method has already been developed. In the last section of this chapter, we study the existence-uniqueness problem of the general first-order differential equations.

2.1 Linear first-order equations

2.1.1 General form

Perhaps the most important type of differential equations is linear equations. The general form of a linear first-order equation is

$$a(x)\,y' + b(x)y = c(x). \tag{2.1}$$

Note that the algebraic exponent of y and of y' is one. If $c(x)$ is identically zero, the equation is called *linear homogeneous*, otherwise, linear non-homogeneous equation. The *canonical form* of a linear first-order equation is obtained by dividing the equation by $a(x)$ as

$$y' + p(x)y = r(x). \tag{2.2}$$

The equation models the behavior of several real phenomena. For example, the population dynamics of a typical living species, the voltage across the capacitor in a RC circuit, the change in the height of a falling body, the growth of the money you deposited in financial institute, the carbon dating of an old fossil, and many other real-world problems. For all those example, the rate of change of the amount of some quantity $y(x)$ is proportional to the current amount of that quantity, i.e, $\frac{dy}{dx} \propto y(x)$, and an external factor $r(x)$. The linear differential equations is a special case of a general *non-linear* one, for example, it turns out that the rate of change of the population of some living species has the form

$$\frac{dp}{dt} = \alpha(x, p)p,$$

where $\alpha(y) = r\left(1 - \frac{p}{K}\right)$, where r, K can be constant or functions of x. That is a non-linear equation that in general can not be solved in closed form.

On the other hand, linear differential equations enjoys pretty nice properties and for this are very popular in different fields of sciences. If we rewrite a first-order equation in the following *operator form*

$$\left(\frac{d}{dx}+p(x)\right)[y]=r,$$

and denote the *differential operator* by T, we reach the compact form of the equation $T[y]=r$. With this notation, the operator T maps a function $y(x)$ to another function $r(x)$ in the *linear manner*, that is,

$$T[\alpha y_1+\beta y_2]=\alpha T[y_1]+\beta T[y_2],$$

for arbitrary functions $y_1(x), y_2(x)$ and constants α, β. Therefore, a linear differential equation refers to the linearity of its *differential operator*. A function $y_h(x)$ is called a *homogeneous solution* of linear mapping T if $T[y_h] = 0$, and moreover, $T[c y_h] = 0$ for arbitrary constant c. The homogeneous equation for $r\equiv 0$ is

$$\frac{dy}{dx}+p(x)\,y=0,$$

or equivalently in the differential form

$$\frac{1}{y}dy+p(x)\,dx=0,$$

that through integration yields

$$y_h(x)=c e^{-\int p(x)dx},$$

for arbitrary constant. On the other hand, if $y_p(x)$ is any *particular solution* of $T[y] = r$, then $y(x)=y_h(x)+y_p(x)$ is also solves the equation due to the following argument

$$T[y_h+y_p]=T[y_h]+T[y_p]=0+r.$$

2.1.2 General solution

To solve Eq.2.2, we multiply it by factor $\mu(x)=e^{\int p(x)dx}$ which is called an *integrating factor* that will be clear immediately. By multiplying, we reach

$$e^{\int p(x)dx}y'+p(x)\,e^{\int p(x)dx}y=r(x)\,e^{\int p(x)dx},$$

and according to the equality

$$e^{\int p(x)dx}y'+p(x)\,e^{\int p(x)dx}y=\frac{d}{dx}\left[e^{\int p(x)dx}y\right],$$

we can write

$$d\left[e^{\int p(x)dx}y\right]=r(x)\,e^{\int p(x)dx}\,dx.$$

The obtained equation is an *integrable equation* and for this $\mu(x)$ is called an integrating factor since it transforms the original equation to an integrable one. Now, by integrating the above equation, we reach the following solution:

$$y(x) = c e^{-\int p(x)dx} + e^{-\int p(x)dx} \int r(x)\, e^{\int p(x)dx}\, dx. \tag{2.3}$$

The above formula is called the *general solution* of the given linear-first order equation. Before we justify the terminology of the *general solution*, let us solve an example.

Example 2.1. For equation $y' + a y = r(x)$, where a is a constant., the integrating factor is $\mu = e^{ax}$, as thus

$$\underbrace{e^{ax} y' + a\, e^{ax} y}_{\frac{d}{dx}(e^{ax} y)} = e^{ax}\, r(x),$$

or equivalently,

$$d(e^{ax}\, y) = e^{ax}\, r(x)\, dx$$

that by integration yields

$$e^{ax} y = c + \int e^{ax}\, r(x)\, dx,$$

and finally

$$y(x) = c e^{-ax} + e^{-ax} \int e^{ax}\, r(x)\, dx.$$

Note that $y(x)$ consists two terms

$$y(x) = \underbrace{c e^{-ax}}_{\text{homogeneous solution}} + \underbrace{e^{-ax} \int e^{ax}\, r(x)\, dx}_{\text{solution associated to } r(x)}.$$

The first term $y_h = c e^{ax}$ which is called the *homogeneous solution*, is the solution of the homogeneous equation

$$y' + a y = 0,$$

and the second term which is called the *particular solution* is contributed by the external term $r(x)$.

2.1.3 Initial value problem

The general form of a linear first-order *initial value problem* is

$$\begin{cases} y' + p(x) y = r(x) \\ y(x_0) = y_0 \end{cases}, \tag{2.4}$$

where y_0 is called the *initial value* of the solution $y(x)$ at $x = x_0$. The point x_0 is usually chosen such that functions p, r are continuous in an open interval centered at x_0. There are two methods to solve the above problem: 1) definite integration, 2) substitution in the general solution. The formula for the definite integration for solving the above problem is as follows

$$y(x) = y_0\, e^{-\int_{x_0}^{x} p(t)dt} + e^{-\int_{x_0}^{x} p(t)dt} \int_{x_0}^{x} r(t)\, e^{\int_{x_0}^{t} p(s)ds}\, dt. \tag{2.5}$$

It is simply verified that $y(x_0) = y_0$, and furthermore

$$y'(x) = -p(x)\left[y_0\, e^{-\int_{x_0}^{x} p(t)\,dt} + e^{-\int_{x_0}^{x} p(t)\,dt} \int_{x_0}^{x} r(t)\, e^{\int_{x_0}^{t} p(s)\,ds}\, dt\right] + r(x),$$

or equivalently $y'(x) = -p(x)\, y(x) + r(x)$.

Example 2.2. Consider the following IVP

$$\begin{cases} x y' - y = x^2 \\ y(1) = -1 \end{cases}.$$

The canonical form of the equation is $y' - \frac{1}{x} y = x$, and the integrating factor is

$$\mu(x) = e^{\int_1^x -\frac{1}{t} dt} = \frac{1}{x}.$$

Multiplying by μ, we reach

$$d\left(\frac{1}{x}\, y\right) = 1 \Rightarrow \int_1^x d\left(\frac{1}{t}\, y\right) = \int_1^x dx \Rightarrow \frac{1}{x} y(x) - y_0 = x - 1.$$

Finally, the solution is obtained as $y(x) = x(x-2)$.

Example 2.3. (Cont.) Let us solve the above problem by using the *general solution*. Note that the general solution of the equation is

$$y(x) = c x + x^2,$$

and applying the initial condition $y = -1$ at $x = 1$ yields $c = -2$, and thus $y(x) = x(x-2)$as before.

2.1.4 Existence, uniqueness and extension

Assume that p, q are continuous function in an interval $I = (x_0 - a, x_0 + a)$ in the initial value problem (2.4). Then it can be proved that the solution (2.5) is the unique continuously differentiable solution on I. The uniqueness is justifies as follows. Assume $y_1(x)$, $y_2(x)$ are two solutions to the problem, thus for $y = y_1 - y_2$, we have

$$\begin{cases} \frac{d}{dx} y + p(x)\, y = 0 \\ y(x_0) = 0 \end{cases}.$$

We should prove that $y = 0$ is the unique solution to the equation. Consider the function $z(x) = y(x)\, e^{\int_{x_0}^{x} p(t)\,dt}$. The continuity of $p(x)$ guarantees that the term $e^{\int_{x_0}^{x} p(t)\,dt}$ never vanishes for $x \in I$. We have

$$\frac{dz}{dx} = y'\, e^{\int_{x_0}^{x} p(t)\,dt} + p(x)\, e^{\int_{x_0}^{x} p(t)\,dt}\, y(x) = 0,$$

and thus $z(x) = c$ a constant that implies

$$y(x) = c\, e^{-\int_{x_0}^{x} p(t)\,dt}.$$

The condition $y(x_0) = 0$ implies $c = 0$ and finally $y(x) = 0$. The existence and extension of the solution to I are directly concluded from solution (2.5).

Example 2.4. Consider the following problem

$$\begin{cases} \cos(x)\, y' + \sin(x)\, y = \frac{1}{2}\sin(2x) \\ y(0) = 0 \end{cases}. \tag{2.6}$$

Since the initial point is $x_0 = 0$ and the term $\cos(x)$ is non-zero in the interval $(-\pi/2, \pi/2)$, we can safely rewrite the equation in the standard form

$$y' + \tan(x)\, y = \sin(x),$$

for $x \in (-\pi/2, \pi/2)$. The general solution to the equation is

$$y(x) = c\cos(x) + \cos(x)\ln|\sec(x)|.$$

Applying initial condition determines c as $c = 0$ and thus $y = \cos(x)\ln|\sec(x)|$ for $x \in (-\pi/2, \pi/2)$. The graph of the solution is shown in Fig.(2.1). The solution is continuous beyond this domain, however, y' goes unbounded at $x = \pm\frac{\pi}{2}$. Remember that the solution to an ODE must be continuously differentiable, and the obtained solution is not a *classical solution* eventhough it is continuous everywhere.

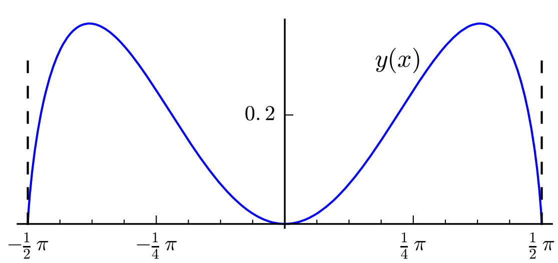


Figure 2.1.

2.1.5 Singular differential equations

Although, the differential equation in the above example fails at $x = 0$, the solution for the given initial condition extends on $(-\infty, \infty)$. However, this is not the case for all initial conditions. For example, the following problem does not have any solution

$$\begin{cases} x\,y' - y = x^2 \\ y(0) = -1 \end{cases},$$

while the problem

$$\begin{cases} x y' - y = x^2 \\ y(0) = 0 \end{cases},$$

has infinitely many solutions.

Example 2.5. Consider the following IVP

$$\begin{cases} x^2 y' - y = x^2 \\ y(1) = -1 \end{cases}.$$

The integrating factor is

$$\mu(x) = e^{\int_1^x -\frac{1}{t^2} dt} = e^{\frac{1}{x}},$$

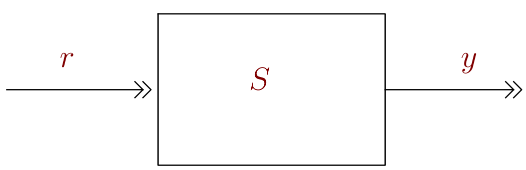
and thus

$$\int_1^x d\left(e^{\frac{1}{t}} y\right) = \int_1^x e^{\frac{1}{t}} dt \Rightarrow y(x) = -e^{1-\frac{1}{x}} + e^{-\frac{1}{x}} \int_1^x e^{\frac{1}{t}} dt.$$

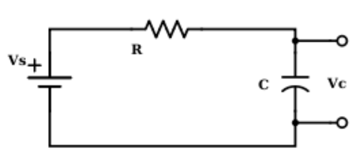
Obviously, $\lim_{x \to 0^-} y(x)$ does not exist and therefor $x = 0$ is a singular point for the solution, and hence the domain of the solution is $(0, \infty)$.

2.1.6 System interpretation

From applied science point of view, a system is a *transaction* that transforms an *input* to an *output*.



Several transactions in applied sciences can be described by differential equations. A system is called *linear* if its *response* to inputs is linear, that is, if its response to $\alpha r_1 + \beta r_2$ is equal to $\alpha y_1, + \beta y_2$ where y_1, y_2 are the responses to r_1, r_2 respectively. A linear differential equation that is described by the linear differential operator T is a linear system. For the initial value problem (2.4), the response consists two terms, the response to the initial condition $y(x_0)$ that can be considered as the internal energy of the system, and the response to the external factor $r(t)$. For example, consider the following LC circuit:



The input to the system is the voltage supply V_s and the output is V_c, the voltage across the capacitor. The differential equation that describes the system is as follows

$$\frac{dV_c}{dt} + \frac{1}{RC} V_c = \frac{1}{RC} V_s.$$

The response depends in addition to V_s on the initial voltage in the capacitor $V_c(0)$. This is a linear system with the transaction $T^{-1}[V_s] = V_c$ where

$$T := \left(\frac{d}{dt} + \frac{1}{Rc}\right).$$

It is simply seen that T^{-1} is defined by

$$T^{-1}[V_s] = \frac{1}{RC}\int_0^t V_s(s)\, e^{-\frac{(t-s)}{RC}} ds.$$

2.2 Linear piecewise equations

2.2.1 Jump discontinuities

If either function $p(x)$ or $r(x)$ in Eq.2.2 is *piecewise continuous*, the equation is called a piecewise or switching equation. A piecewise continuous functions is a function that is continuous everywhere except possibly at some finite points, and that its discontinuity points only are finite jumps. Assume that $r(x)$ in the following equation has a jump at x_0

$$y' + p(x)y = r(x).$$

This jump must be due to a jump in y' only. The reason is that if the jump is associated to $y(x)$, then $y'(x)$ will have an infinite jump at x_0 due to the definition of $y'(x_0)$

$$y'(x_0) = \lim_{h\to 0} \frac{y(x_0+h) - y(x_0)}{h}.$$

Therefore, $y(x)$ is continuous at x_0, and $y'(x)$ has finite jump at x_0.

Example 2.6. Let us solve the following initial value problem and draw the solution

$$\begin{cases} y' + y = r(x) \\ y(0) = 0 \end{cases},$$

where $r(x)$ is the following piecewise function.

$$\begin{cases} 1 & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}.$$

We solve the problem by two methods

Method I. In the domain $x < 1$, the equation reads

$$\begin{cases} y' + y = 0 \\ y(0) = 0 \end{cases},$$

and thus $y(x)=0$ for $x\in(-\infty,1)$. In the interval $x\in(1,2)$, the equation reads

$$y'+y=1.$$

Here we need an *appropriate* initial condition. Note that we can not use the initial condition $y(0)=0$ since the equation is defined for $1<x<2$ and not at $x=0$. We can assume that the solution $y(x)$ is *continuous* at $x=1$. According with that assumption, we can write the initial value problem as

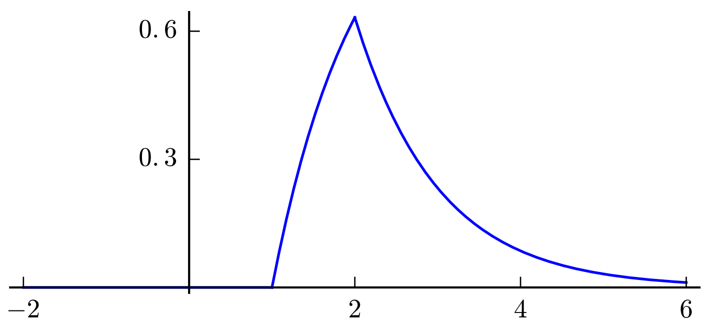
$$\begin{cases} y'+y=1 \\ y(1)=0 \end{cases},$$

and thus $y(x)=1-e^{1-x}$. For $x>0$, the problem reads

$$\begin{cases} y'+y=0 \\ y(2)=1-e^{-1} \end{cases},$$

which is solved for $y(x)=(e^2-e)\,e^{-x}$, and finally

$$y(x)=\begin{cases} 0 & x\le 1 \\ 1-e^{1-x} & 1\le x\le 2 \\ (e^2-e)\,e^{-x} & x\ge 2 \end{cases}$$



Method II. The general solution is

$$y(x)=ce^{-x}+e^{-x}\int_0^x r(t)\,e^x\,dx,$$

and by substituting $y(0)=0$, c is determined 0, and thus

$$y(x)=e^{-x}\int_0^x r(t)\,e^t\,dt$$

For $x<1$, $r(x)=0$ and thus $y(x)=0$ for $x<1$. For $x<2$, we have

$$y(x)=e^{-x}\left[\int_0^1 r(t)\,e^t\,dt+\int_1^x r(t)\,e^t\,dt\right]=e^{-x}\int_1^x e^t\,dt=1-e^{1-x}.$$

For $x>2$, the solution is

$$y(x)=e^{-x}\left[\int_0^1 r(t)\,e^t\,dt+\int_1^2 r(t)\,e^t\,dt+\int_2^x r(t)\,e^t\,dt\right]=e^{-x}\int_1^2 e^t\,dt=(e^2-e)\,e^{-x}$$

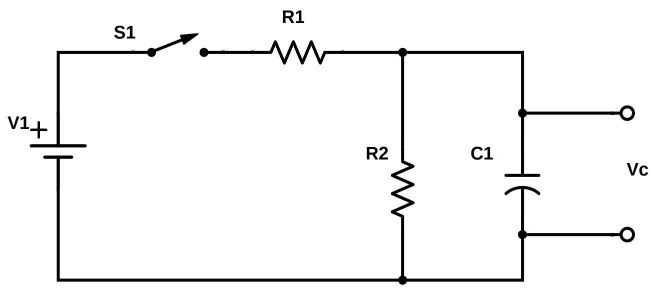
Finally

$$y(x)=\begin{cases}0 & x\leq 1\\ 1-e^{1-x} & 1<x<2\\ (e^2-e)\,e^{-x} & x\geq 2\end{cases}.$$

Remark. Remember that the solution to a differential equation is a smooth or continuously differentiable function. Therefore, the solution of piecewise equations is not classical in technical terms.

2.2.2 Switching circuits

As a simple example, consider the switching circuit shown below



Assume that the switch S_1 connects the circuit at time t_0. For $t>t_0$, let i_1 denote the electrical current in the resistor R_1. According to the KIRCHHOFF's nodal law, we can write $i_1=i_2+i_c$, where i_2, i_c are electrical currents in the resistor R_2 and the capacitor C respectively. For $t<t_0$, there is no current in R_1 and thus $i_2+i_c=0$. On the other hand, the KIRCHHOFF's mesh law states $V_{R_2}=V_c$, where V_{R_2}, V_c are the voltages across R_2 and C respectively. Now, we can obtain the equation for $t<t_0$ where the switch S_1 is off. Due to relations $i_c=C\frac{\mathrm{d}V_c}{\mathrm{d}t}$ and $V_{R_2}=i_2\,R_2$, we get $i_2=\frac{V_c}{R_2}$, and hence

$$\frac{V_c}{R_2}+C\frac{dV_c}{dt}=0.$$

Therefore, we derive the following equation for $t<t_0$:

$$\frac{dV_c}{dt}+\frac{1}{R_2\,C}V_c=0. \tag{2.7}$$

For $t>t_0$, we have

$$\frac{V_c}{R_2}+C\frac{dV_c}{dt}=i_1, \tag{2.8}$$

and by the KIRCHHOFF's mesh law $V_{R_1} + V_c = V_1$, we obtain $i_1 = \frac{V_1 - V_c}{R_1}$. Substituting i_1 into Eq.2.8 yields

$$\frac{V_c}{R_2} + C\frac{dV_c}{dt} = \frac{V_1 - V_c}{R_1},$$

and finally

$$\frac{dV_c}{dt} + \frac{1}{RC}V_c = \frac{1}{R_1C}V_1, \tag{2.9}$$

where $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. Two equations (2.7) and (2.9) can be rewritten in the compact form as follows

$$\frac{dV_c}{dt} + p(t)V_c = r(t), \tag{2.10}$$

where

$$p(t) = \begin{cases} \frac{1}{R_2C} & t < t_0 \\ \frac{1}{RC} & t > t_0 \end{cases}, r(t) = \begin{cases} 0 & t < t_0 \\ \frac{1}{R_1C} & t > t_0 \end{cases}.$$

Notice that in the above equation, functions p, r are piecewise functions. Let us solve the equation if the initial condition $V_c(0) = V_0$ is given. For $t < t_0$, the solution is

$$V_c(t) = V_0\, e^{-\frac{t}{R_2C}}.$$

For $t > t_0$, the appropriate initial condition is the value of $V_c(t_0)$. This value is provided by the solution for $t < t_0$, that is, $V_c(t_0) = V_0\, e^{-\frac{t_0}{R_2C}}$. To solve Eq.(2.9) with $V_c(t_0)$, we apply the formula for linear differential equations and obtain

$$V_c(t) = V_0\, e^{-\frac{t_0}{R_2C}}\, e^{-\frac{(t-t_0)}{RC}} + e^{-\frac{(t-t_0)}{RC}} \int_{t_0}^{t} e^{\frac{(x-t_0)}{RC}} \frac{V_1}{R_1C}\, dx.$$

If V_1 is a constant value, then V_c is after some straightforward simplifications

$$V_c(t) = V_0\, e^{\frac{t_0}{R_1C}}\, e^{-\frac{t}{RC}} + \frac{RV_1}{R_1}\Big(1 - e^{-\frac{(t-t_0)}{RC}}\Big).$$

Note that $V_c(t)$ is continuous at t_0 but not differentiable at this point. Observe also the property

$$\lim_{t\to\infty} V_c(t) = \frac{R}{R_1}V_1 = \frac{R_2}{R_1 + R_2}V_1,$$

that means capacitor C acts like an open circuit in long term.

2.3 BERNOULLI and RICCATI equations

These are two important *nonlinear* first-order equations that can be transformed to linear one after an algebraic transformation.

2.3.1 BERNOULLI equations

The standard form of a BERNOULLI's equation is

$$\begin{cases} y' + p(x)\, y = r(x)\, y^{\alpha} \\ y(x_0) = y_0 \end{cases}, \tag{2.11}$$

where $\alpha \neq 0, 1$. Note that the equation is linear for $\alpha = 0, 1$. Furthermore, if $y_0 = 0$, then $y(x) = 0$ is the solution to the equation. In fact, $y = 0$ is an equilibrium of the equation. Therefore, we assume $y_0 \neq 0$ as well.

The general method to solve the equation suggested by G. WILHELM LEIBNIZ is as follows. we divide the equation by y^{α} (remember that y_0 is non-zero and thus $y(x)$ is nonzero in an open interval around x_0) and obtain

$$y^{-\alpha}\, y' + p(x)\, y^{1-\alpha} = r(x). \tag{2.12}$$

Let $u(x)$ be the function $u = y^{1-\alpha}$, and thus $u' = (1-\alpha) y^{-\alpha} y'$. Therefore we obtain the following equation for u

$$\begin{cases} u' + (1-\alpha) p(x)\, u = (1-\alpha) r(x) \\ u(x_0) = y_0^{1-\alpha} \end{cases}. \tag{2.13}$$

This problem is linear in u and can be solved by the method presented above. Once $u(x)$ is determined, the true solution $y(x)$ is derived by the relation $y = u^{1/1-\alpha}$.

Example 2.7. Consider the equation

$$\begin{cases} y' - 4xy = 4x\sqrt{y} \\ y(0) = 4 \end{cases}.$$

Here $y(0) = 4 > 0$ and thus $y(x)$ remain positive in an interval around $x_0 = 0$. By dividing the equation by $y^{1/2}$ we obtain

$$y^{-1/2} y' - 4x y^{1/2} = 4x,$$

and for $u = y^{1/2}$ we obtain $u' = \frac{1}{2} y^{-1/2}\, y'$. Substituting u and u' into the equation leads

$$\begin{cases} u' - 2x u = 2x \\ u(0) = 2 \end{cases}.$$

The obtained equation is solved for $u = 3e^{x^2} - 1$. Since $y = u^2$, we obtain the solution $y = (3e^{x^2} - 1)^2$. Observe that the interval for the solution is $I = (-\infty, \infty)$.

Example 2.8. We solve the following initial value problem

$$\begin{cases} x y' - y = \ln(x)\, y^2 \\ y(1) = \frac{1}{2} \end{cases}.$$

First we write the equation in the standard form

$$y' - \frac{1}{x} y = \frac{1}{x} \ln(x)\, y^2.$$

Dividing the equation by y^2 leads to

$$y^{-2}y' - \frac{1}{x}y^{-1} = \frac{1}{x}\ln(x).$$

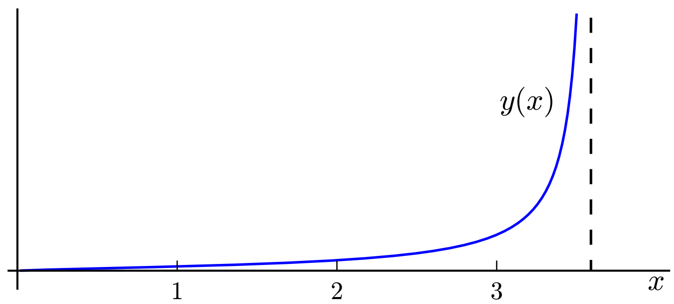
We take $u = y^{-1}$ to write the equation as

$$\begin{cases} u' + \frac{1}{x}u = -\frac{1}{x}\ln(x) \\ u(1) = 2 \end{cases},$$

which is a linear equation. The solution to the above problem is $u = x^{-1} - \ln(x) + 1$, and thus y is

$$y(x) = \frac{x}{x + 1 - x\ln(x)}.$$

The solution goes unbounded at $x \cong 3.59$ and thus the domain of the solution is $I = (0, 3.59)$. Although, the solution $y(x)$ is not defined at $x = 0$, it has limit at this point. The graph of the solution in the interval $(0, 3.59)$ is shown below.



2.3.2 Riccati equation

The general form of a RICCATI equation is

$$y' = a(x)y^2 + b(x)y + c(x). \tag{2.14}$$

The important point is that there is no standard method to solve the equation in general. However, if a solution to the equation is known, the general solution can be obtained by a simple trick. Assume that y_1 is a particular solution of the equation. Let us write the general solution as $y = y_1 + v(x)$ for an undetermined function $v(x)$. By substituting y into the equation, we reach the following equation which is a BERNOULLI equation

$$v' - (2a(x)y_1 + b(x))v = a(x)v^2. \tag{2.15}$$

Example 2.9. It is simply verified that the function $y_1 = x$ is a solution to the equation

$$y' = y^2 - 2xy + x^2 + 1.$$

Substitution $y = x + v(x)$ leads to $v' = v^2$ with the solution $v = -\frac{1}{x+c}$ and thus

$$y(x) = x - \frac{1}{x+c}.$$

Problems

Problem 2.1. Find the solution to each of the following problems and determine the domain of the definition for each solution:

i. $y' + y = 2xe^{-x},\ y(0) = 1$

ii. $xy' + 2y = e^{x^2},\ y(1) = 0$

iii. $\sin(x)y' + \frac{1}{2}\sin(2x)y = \frac{1}{3}\sin(3x),\ y(\pi/2) = 1$

iv. $y' + \frac{1}{x+1}y = e^x,\ y(0) = 0$

Problem 2.2. Write the general solution to the following equations

i. $y' + \tan(x)y = \cos^2(x)$.

ii. $y' + (\tan(x) + \cos(x))y = \cos^2(x)$.

iii. $xy' - 2y = x\ln(x)$.

iv. $xy' + 3y = 3x^2 - 2x$

v. $xy' - (x-3)y = \frac{x+1}{x^2}$

Problem 2.3. Consider the initial value problem the following problem

$$\begin{cases} (1+t^2)y' + 2ty = r(t) \\ y(0) = 1 \end{cases},$$

where $r(t) = \left\{\begin{smallmatrix} 0 & t<1 \\ 1 & t>1 \end{smallmatrix}\right.$. There is a jump in $y'(t)$ at $t = 1$, however, the solution is continuous. Find a continuous solution to the equation that is differentiable everywhere except at $t = 1$.

Problem 2.4. Solve the following initial value problem and draw the solution

$$\begin{cases} y' + y = r(x) \\ y(0) = 1 \end{cases},$$

where $r(x)$ is the following piecewise function.

$$\begin{cases} 1 & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}.$$

Problem 2.5. Solve the following piecewise problem and draw the solution:

$$\begin{cases} y' + p(x)\,y = -p(x) \\ y(0) = 1 \end{cases},$$

where

$$p(x) = \begin{cases} 0 & x \le 1 \\ 1 & x > 1 \end{cases}.$$

Problem 2.6. Solve the following initial value problem and draw the solution

$$\begin{cases} y' + p(x)y = r(x) \\ y(0) = 0 \end{cases}$$

where $p(x)$ is the function

$$p(x) = \begin{cases} 1 & x \le 3 \\ -1 & x > 3 \end{cases},$$

and $r(x)$ is

$$r(x) = \begin{cases} 1 & x > 1 \\ 0 & x \le 1 \end{cases}.$$

Problem 2.7. Find the solution to the following problems

i. $y' + y + y^2 = 0,\ y(0) = 2$

ii. $y' + y = x\sqrt{y},\ y(0) = 1$

iii. $y' - 2xy = xy^2,\ y(0) = -1$

iv. $2\sin(x)\, y' - \cos(x)\, y + \sin^2(x)\, y^3 = 0,\ y(\pi/2) = 1$

v. $2\cos(x) y y' - \sin(x) y^2 = \cos(x),\ y(0) = -1$

vi. $x^2 y' + 2xy - y^3 = 0,\ y(1) = 1$

Problem 2.8. Solve the following problem and draw the solution

$$\begin{cases} y' + y = r(x) y^2 \\ y(0) = 1 \end{cases},$$

where $r(x)$ is

$$r(x) = \begin{cases} 1 & x < 1 \\ 0 & x > 1 \end{cases}.$$

Problem 2.9. Solve the following problem and draw the solution

$$\begin{cases} y' + p(x) y = \sqrt{y} \\ y(1) = 1 \end{cases},$$

where $p(x)$ is

$$p(x) = \begin{cases} 1 & x < 2 \\ -1 & x > 2 \end{cases}.$$

Problem 2.10. (variation of parameter method) If $r \equiv 0$ in Eq2.2, then the solution of the equation is

$$y(x) = c e^{-\int p(x)}. \tag{2.16}$$

Now assume that r is not identically zero. We solve the equation by the method of variation of parameter. For this, assume that c is a function of x, i.e.,

$$y(x) = c(x)\, e^{-\int p(x)}.$$

Substitute $y(x)$ into the equation and find an expression for $c(x)$.

Problem 2.11. Rewrite the following equation as a linear equation with respect to $x = x(y)$

$$(p(y)\, x + q(y)) y' = r(y),$$

Use this idea to solve the following equations

i. $(x - y^2) y' = y$

ii. $(x^2 + e^y) y' = x$.

Problem 2.12. Consider the following problem

$$\begin{cases} y' - a y = e^{bx} \\ y(0) = y_0 \end{cases}.$$

i. Show that the problem has the following solution

$$y(x) = e^{ax} y_0 + \begin{cases} \frac{e^{bx} - e^{ax}}{b - a} & b \neq a \\ x e^{ax} & b = a \end{cases}. \tag{2.17}$$

ii. In the above solution, assume that b is a free parameter and let $b \to a$. Find the limit and show that the limit function is the solution when $b = a$, (the result implies that the solution is continuous with respect to b).

Problem 2.13. Consider the equation

$$\begin{cases} y' + ay = f(t) \\ y(0) = y_0 \end{cases},$$

where $a > 0$ is a constant and $f(t)$ is a continuous function and

$$\lim_{t \to \infty} f(t) = 0.$$

Show that regardless of y_0, the solution satisfies the relation

$$\lim_{t \to \infty} y(t) = 0$$

Problem 2.14. Suppose that $a > 0$, and f is a bounded function; that is, $\max_x |f(x)| \leq M$. Prove: there is a unique initial condition $y(0) = y_0$ such that the solution of the following IVP remain bounded

$$\begin{cases} y' - ay = f(t) \\ y(0) = y_0 \end{cases}.$$

hint: take the initial condition as follows and show the solution of the above IVP is bounded

$$y(0) = -\int_0^\infty e^{-at} f(t)\, \mathrm{d}t,$$

Show also that if f is periodic then the bounded solution is periodic.

Problem 2.15. Let $a > 0$ and $f(t) \xrightarrow{t \to \infty} b$. Prove that there is a unique bounded solution to the equation $ty' - ay = f(t)$ for $t \in (0, \infty)$.

2.4 General first-order equations

We consider general first-order equations of the following form

$$M(x, y)\, dx + N(x, y)\, dy = 0. \tag{2.18}$$

Note that the above differential form is equivalent to the derivative form $y' = f(x, y)$, where $f(x, y) = -\frac{M(x, y)}{N(x, y)}$.

2.4.1 Exact equations

The Eq.2.18 is called *exact* in an open domain $D \subset \mathbb{R}^2$ if the left-hand side of the equation is the total differential of some function $\phi(x, y)$ in D, i.e,

$$M(x, y)\, dx + N(x, y)\, dy = d\phi(x, y),$$

for all $(x, y) \in D$. Remember that the total differential of a scalar function $\phi(x, y)$ is defined as

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy,$$

and therefore, the form $Mdx + Ndy$ is exact in D if

$$M(x,y) = \frac{\partial\phi}{\partial x}(x,y), N(x,y) = \frac{\partial\phi}{\partial x}(x,y),$$

for all $(x,y) \in D$.

Remark. If we write the differential equation in the derivative form as

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)},$$

then we should assume $N(x,y)$ is non-zero in D, otherwise the differential equation does not make sense.

2.4.2 Solution of an exact equation

If we know the potential ϕ of an exact equation $Mdx + Ndy = 0$, then the solution is simply obtained by the integration of equation $d\phi = 0$ and thus

$$\phi(x,y) = \phi(x_0, y_0),$$

where $y(x_0) = y_0$ is an initial condition for the equation.

Example 2.10. Consider the following problem

$$\begin{cases} (2x + y^2)\,dx + (2xy + 1)\,dy = 0 \\ y(0) = 1 \end{cases}.$$

It is simply verified that $\phi = x^2 + xy^2 + y + c$ is a potential of the equation, and thus the solution is $\phi = \text{const.}$, and by applying the initial condition, we obtain the following solution

$$x^2 + xy^2 + y = 1.$$

Note that $N(x,y) = 2xy + 1$ is not zero at $(0,1)$ as well. The graph of the solution is a curve in the (x,y)-plane that is called the *integral curve* of the differential equation because it is obtained by the integration of the differential equation.

The solution of exact equations is usually in implicit form and needs to be verified that $y(x)$ can be defined explicitly in an open interval containing x_0. The following theorem gives a sufficient condition for the existence of such a function.

Theorem 2.1. (Implicit function theorem) *Consider the implicit function $f(x,y) = 0$, and assume that $f(x_0, y_0) = 0$ for some (x_0, y_0). If f and its partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ are continuous in an open set D around (x_0, y_0), and furthermore $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$, then there is an open interval $I = (x_0 - \delta, x_0 + \delta)$ and a continuous function $y = y(x)$, such that $y(x_0) = y_0$, and $f(x, y(x)) = 0$ for all $x \in I$.*

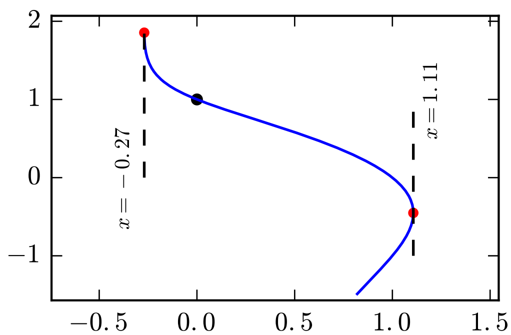
Example 2.11. (Continue) For the above example, let us f be

$$f(x,y) := x^2 + xy^2 + y - 1.$$

Obviously, f is well-defined on any neighborhood of $(0,1)$ in the (x,y)-plane, $f(0,1)=0$, and furthermore

$$\frac{\partial f}{\partial y}(0,1)=1\neq 0,$$

and thus the conditions of the implicit function theorem are satisfied. The graph of the solution is shown below. As it is seen, the acceptable domain for the solution is $I=(-0.27, 1.11)$ where the tangents at the end points go unbounded. The red points on the graph are points at which the coefficient term $N(x,y)$ become zero and thus $y'=-\frac{M}{N}$ goes unbounded. The black point is the initial value of the problem. Note that in this interval, y can be expressed as an *explicit function* $y=y(x)$. This explicit function is the solution to the initial value problem.



Remark. Since $\frac{\partial f}{\partial y}(x_0, y_0)=N(x_0, y_0)$, thus the condition of exactness and $N(x_0, y_0)\neq 0$ guarantees the existence of an *explicit solution* in an open interval of x_0.

2.4.3 Physical interpretation

Remember that a *force field* $\vec{f}$ is called *conservative* if there is a *potential functions* V such that $\vec{f}=-\nabla V$. Now, if we interpret $\vec{f}=-\begin{pmatrix} M(x,y) \\ N(x,y) \end{pmatrix}$ as a force field in the (x,y)-plane, and if $Mdx+Ndy$ is the differential of $\phi(x,y)$, then $\vec{f}=-\nabla\phi$. What is conserved in a conservative force field? If a mass-body with mass m is moving under the influence of $\vec{f}$ along path γ, then the total energy of the mass

$$E(t)=\frac{1}{2}m\,|v|^2+\phi(x,y),$$

is conserved *along* γ as shown simply by the calculation

$$\frac{dE}{dt}=v.\left(m\frac{dv}{dt}+\nabla\phi\right)=v.\left(m\frac{dv}{dt}-\vec{f}\right)=0.$$

If a conservative force filed $\vec{f}(x,y)$ is smooth in a *simple domain* $D\subset\mathbb{R}^2$, then its *line integral* is independent of the path of integration on all paths in D as the following computations show

$$\int_{t_0}^{T}\{M(\gamma(t))\,x'(t)+N(\gamma(t))\,y'(t)\}\,dt=-\int_{t_0}^{T}\phi'(x(t),y(t))\,dt=$$
$$=-\phi(x(T),y(T))+\phi(x(t_0),y(t_0)).$$

Remember that the line integral of a force field is called the *work* W done by that force, and for conservative force fields, it is equal to the changes in the potential. Therefore, the solution $\phi(x,y)=c$ of an exact equation $Mdx+Ndy=0$ defines equipotential curves of the equation.

2.4.4 A comment on the exactness

Theorem 2.2. *Assume that there is an open rectangle D of (x_0,y_0) in the (x,y)-plane such that functions M,N are continuously differentiable in D and furthermore*

$$\frac{\partial M}{\partial y}(x,y)=\frac{\partial N}{\partial x}(x,y),$$

for all $(x,y)\in D$, then there is a potential $\phi(x,y)$ for the form $Mdx+Ndy$.

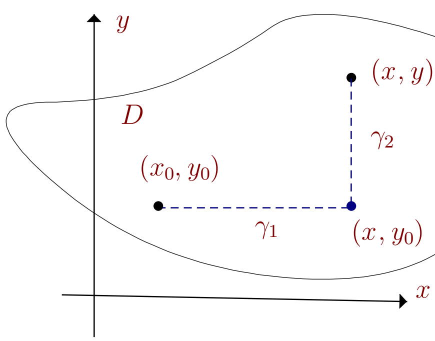
Proof. Let γ is any arbitrary closed curve in D and thus by GREEN's theorem, we have

$$\oint_\gamma Mdx+Ndy=\iint_\Omega\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dA=0,$$

where Ω is the domain inside of γ. Therefore, the integration is independent of the path of integration. Now, fix $(x_0,y_0)\in D$, and for arbitrary $(x,y)\in D$, we defined

$$\phi(x,y)=\int_{x_0}^{x}M(t,y_0)dt+\int_{y_0}^{y}N(x,t)dt.$$

Note that ϕ is the line integral of the form on the path γ_1: (t,y_0) for $t\in(x_0,x)$, and γ_2: (x,t) for $t\in(y_0,y)$. Because of the independence of path, ϕ is uniquely defined for all $(x,y)\in D$.



Note that $\frac{\partial\phi}{\partial y}=N(x,y)$, and

$$\frac{\partial\phi}{\partial x}=M(x,y_0)+\int_{y_0}^{y}\frac{\partial N}{\partial x}(x,t)\,dt.$$

By the relation

$$\frac{\partial N}{\partial x}(x,t)=\frac{\partial M}{\partial t}(x,t),$$

we obtain

$$\frac{\partial\phi}{\partial x}=M(x,y_0)+\int_{y_0}^{y}\frac{\partial N}{\partial x}(x,t)\,dt=M(x,y_0)+\int_{y_0}^{y}\frac{\partial M}{\partial t}(x,t)\,dt=M(x,y),$$

and this completes the proof. □

2.4.5 Two methods of solutions

If the equation $M dx + N dy = 0$ is exact, then there are two equivalent methods to determine the potential $\phi(x, y)$.

1. **(Definite integration)** We use the above theorem to find the potential

$$\phi(x, y) = \int_{x_0}^{x} M(t, y_0) dt + \int_{y_0}^{y} N(x, t) dt.$$

Example 2.12. Let us solve the following initial value problem

$$\begin{cases} (2x + y^2)\, dx + (2xy + 1)\, dy = 0 \\ y(0) = 1 \end{cases}.$$

The above formula gives

$$\phi = \int_0^x (2t + 1)\, dt + \int_1^y (2xt + 1)\, dt = x^2 + xy^2 + y - 1,$$

and thus the solution to the equation is

$$x^2 + xy^2 + y - 1 = 0.$$

Example 2.13. Let us solve the problem

$$\begin{cases} ye^{xy} + 1 + (xe^{xy} + \cos(y))y' = 0 \\ y(0) = 0 \end{cases}$$

The integral formula gives

$$\int_0^x dt + \int_0^y (xe^{xt} + \cos t)\, dt = x + e^{xy} - 1 + \sin(y) = 0.$$

The figure (2.2) shows the graph of the solution. The domain of the solution is shown by the blue line where the implicit solution can be solved explicitly for y. Note that at the boundary point of the blue line, the slope of the solution, y', goes unbounded.

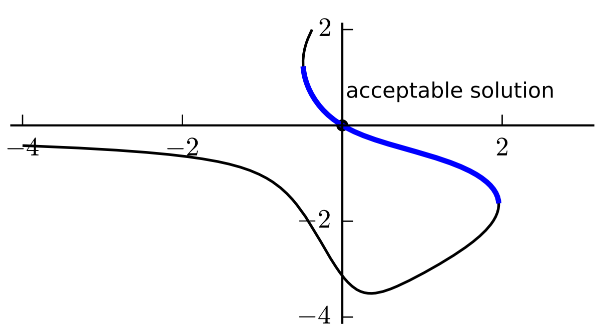


Figure 2.2.

2. **(Indefinite integral)** Let us illustrate this method by solving the following example

$$(2x+y^2)\,dx+(2xy+1)\,dy=0.$$

Let ϕ be the potential, therefore $\frac{\partial\phi}{\partial x}=2x+y^2$, and thus

$$\phi=\int(2x+y^2)\,dx=x^2+xy^2+g(y),$$

where $g(y)$ plays the role of the constant for the indefinite integral. Therefore,we need to determine $g(y)$ as well. By the relation $\frac{\partial\phi}{\partial y}=2xy+1$, we reach

$$2xy+g'(y)=2xy+1,$$

and thus $g'(y)=1$, or $g(y)=y+c$. Finally, the solution is

$$x^2+xy^2+y=c.$$

Example 2.14. Consider the following initial value problem

$$(ye^{xy}+1)\,dx+(xe^{xy}+\cos(y))\,dy=0 \tag{2.19}$$

Here M is $ye^{xy}+1$, and thus

$$\phi=\int(ye^{xy}+1)\,dx=e^{xy}+x+g(y).$$

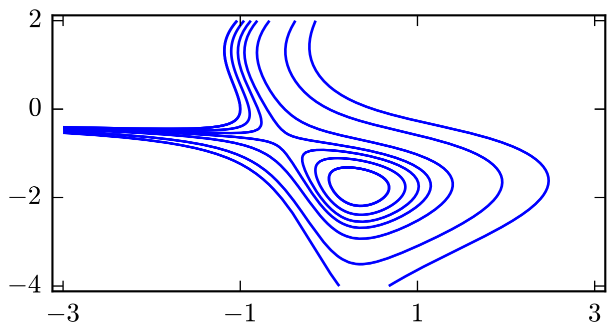
To determine $g(y)$, we use the relation

$$\frac{\partial}{\partial y}(e^{xy}+x+g(y))=N(x,y)=xe^{xy}+\cos(y),$$

and thus $g=\sin(y)+c$. Finally, the solution is

$$e^{xy}+x+\sin(y)=c.$$

The following figure shows few integral curves of this problem.



2.4.6 Two important types of exact equations

1. **(Separable equations)** The general form of a separable equation is as follows

$$N(y)y'+M(x)=0,$$

or equivalently in the differential form $N(y)dy + M(x)dx = 0$ where M, N are are continuous functions in an open domain in the (x, y) plane. Obviously, M, N satisfies the condition for exactness. Similarly, the initial value problem

$$\begin{cases} N(y)\, y' + M(x) = 0 \\ y(x_0) = y_0 \end{cases}, \tag{2.20}$$

is exact if and only if M, N are continuous in an open set D around (x_0, y_0), and $N(y)$ is nonzero in D. The solution to the above initial value is as follows

$$\int_{x_0}^{x} M(t)\, dt + \int_{y_0}^{y} N(t) dt = 0.$$

Example 2.15. Consider the following initial value problem

$$\begin{cases} y' = -\frac{y \tan x}{1+y} \\ y(0) = 1 \end{cases}.$$

Notice that $y(0) \neq 0$ and then it can be written in the separable form as follows

$$\frac{1+y}{y}\, y' = -\tan x.$$

Now, the solution is

$$\int_{1}^{y} \frac{1+t}{t}\, dt = -\int_{0}^{x} \tan(t)\, dt,$$

that gives

$$\ln|y| + y - 1 = \ln|\cos(x)|.$$

Since $y_0 = 1 > 0$ and $\cos(0) = 1 > 0$, it is safe to remove the absolute value sign and write the solution as $y + \ln(y \sec(x)) = 1$ in the domain $I = \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$. The graph of the solution is shown in the figure (2.3). Observe that the curve passes through the initial point $(0, 1)$ and that $y'(x)$ goes unbounded when $x \to \pm\frac{\pi}{2}$.

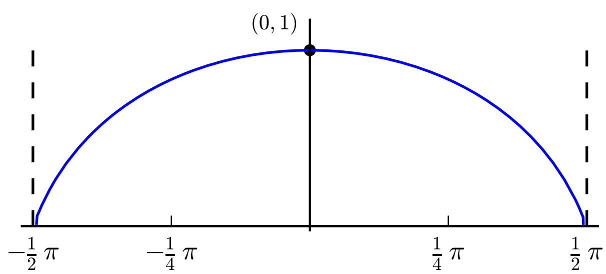


Figure 2.3.

Example. Consider the equation $xy' = 1 + y^2$.

a. Find the general solution to the equation.

b. Find the solution to the problem

$$\begin{cases} xy' = 1 + y^2 \\ y(1) = 1 \end{cases}.$$

To solve part a), we write the equation in the standard form

$$\frac{dy}{1+y^2} = \frac{dx}{x}.$$

Integrating both sides of the equation gives the general solution $\tan^{-1}(y) = \ln|x| + c$. To solve part b), we apply the initial condition that determines $c = \frac{\pi}{4}$. Since $x_0 = 1 > 0$, we can safely remove the absolute value sign and write

$$y = \tan\left(\frac{\pi}{4} - \ln(x)\right).$$

Example 2.16. Let us solve the following non-smooth problem

$$\begin{cases} y' = y\,|y| + 1 \\ y(0) = 1 \end{cases}.$$

Since $y_0 = 1 > 0$, we expect that the solution remains positive in an interval around $x_0 = 0$. Thus we assume first $y > 0$, and solve the following problem

$$\begin{cases} y' = y^2 + 1 \\ y(0) = 1 \end{cases}.$$

This is a separable equation with the solution $\phi_1(x) = \tan\left(\frac{\pi}{4} + x\right)$. Clearly, the solution remains positive in the interval $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$, (notice that the solution goes unbounded at $x = \frac{\pi}{4}$). Now, let us solve the problem for $x < \frac{-\pi}{4}$. Note that $y\left(-\frac{\pi}{4}\right) = 0$ and $y'\left(-\frac{\pi}{4}\right) = 1$. This implies that $y(x)$ is negative in an interval on the left side of $x = \frac{-\pi}{4}$. The equation in this interval reads

$$\begin{cases} y' = -y^2 + 1 \\ y\left(-\frac{\pi}{4}\right) = 0 \end{cases}.$$

The equation is solved for $\phi_2(x) = \frac{e^{2(x+\pi/4)} - 1}{e^{2(x+\pi/4)} + 1} = \tanh\left(x + \frac{\pi}{4}\right)$. Thus the solution to the given problem is

$$y = \begin{cases} \tan\left(x + \frac{\pi}{4}\right) & x \in \left(\frac{-\pi}{4}, \frac{\pi}{4}\right) \\ \tanh\left(x + \frac{\pi}{4}\right) & x < \frac{-\pi}{4} \end{cases}.$$

2. **(Homogeneous equations)** The general form is

$$y' = f\left(\frac{y}{x}\right),$$

for some continuous function. The equation is not exact in its original form, however, if we define a new variable $u=\frac{y}{x}$, then the equation reduces to the following one for $u(x)$

$$xu'=f(u)-u, \tag{2.21}$$

that is separable, and thus *integrable*.

Case 1. If u_0 is a root of $f(u)-u$, then $u=u_0$ is a trivial solution to (2.21), and therefore $y=u_0x$ is a solution to the original equation.

Case 2. If $f(u)-u\neq 0$, the general solution is derived by the integrating of the following equation

$$\frac{du}{f(u)-u}=\frac{dx}{x}.$$

Example 2.17. Solve the initial value problem

$$\begin{cases} y'=\frac{y(y+2x)}{x^2} \\ y(2)=2 \end{cases}.$$

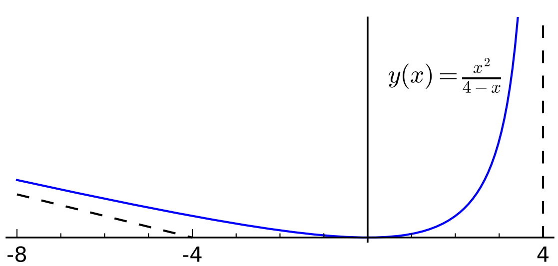
Observe that the equation is homogeneous. Taking $y=xu$ leads to

$$\begin{cases} xu'=u(u+1) \\ u(2)=1 \end{cases}.$$

Since the right hand side of the above equation is nonzero around the initial value $u=1$, we can rewrite the equation in the separable form

$$\frac{du}{u(u+1)}=\frac{dx}{x},$$

and thus $u(x)=\frac{x}{c-x}$, that yields $y=\frac{x^2}{c-x}$. Applying the initial condition determines $c=4$ and therefore $y=\frac{x^2}{4-x}$. Observe that the solution has the vertical asymptote $x=4$. Thus the interval of the solution, regarding the initial point $x_0=2$, is $I=(-\infty, 4)$. The solution has the inclined asymptote $y=-x-4$ when $x\to-\infty$.



Problems

Problem 2.16. For each of the following scalar functions, find the total differential at the given points

a) $\varphi=\sin(xy)+x^2+y^2$ at $(1,\pi)$

b) $\varphi = e^{x+y} + xy$ at $(0,1)$

Problem 2.17. For each of the following differential forms, determine whether it is exact or not. If it is exact find its general potential

a) $y\,\mathrm{d}x + (y+x)\,\mathrm{d}y$

b) $(x-y)\,\mathrm{d}x + (y+x)\,\mathrm{d}y$

c) $(2xy^2 + y)\,\mathrm{d}x + (2x^2y + x)\,\mathrm{d}y$

Problem 2.18. Determine the domain in which the following initial value problems are exact

a) $\left(\frac{1}{2}y^2 + \frac{1}{2}x^2 - 1\right)\mathrm{d}x + (xy-1)\,\mathrm{d}y = 0,\ y(1) = 2$

b) $(2xy + x^2 + 1)\,\mathrm{d}x + (x^2 + 4y^2 - 1)\,\mathrm{d}y = 0,\ y\left(\frac{1}{2}\right) = 0$

Problem 2.19. Consider the implicit function $y^2 + y(x^2-1) + x^2 = 0$.

a) Use the implicit differentiation and find the slope of tangent line at the point $(0,0)$ on the graph of the curve.

b) By the implicit function theorem, determine an interval around $x_0 = 0$ such that the implicit function can be solved for y in that interval (hint: find the equation for which the slope become infinity. Substitute the obtained equation into the implicit function and determine the interval around $x_0 = 0$ for which the slope is not infinity).

Problem 2.20. Write the general solution to the following equations

i. $(3y^3 e^{3xy} - 1)\,dx + e^{3xy}(2y + 3xy^2)\,dy = 0$

ii. $(y + x\sin(x))\,\mathrm{d}x + (e^y + x)\mathrm{d}y = 0$

iii. $(xe^{2x^2+y^2} + 2^{-x})\,dx + \left(\frac{1}{2}ye^{2x^2+y^2} + \sin^{-1}y\right)dy = 0$

iv. $(y\sin(xy) + 3^{-x})\,dx + \left(x\sin(xy) - \sqrt{9-4y^2}\right)dy = 0$

Problem 2.21. Solve the following problems

i. $\sin(y)\cos(x)\,dx + \sin(x)\cos(y)\,dy = 0,\ y(\pi/2) = \pi/4$

ii. $ye^{xy}\,dx + (1 + xe^{xy})\,dy = 0,\ \ y(0) = 1$

iii. $(1+x)e^x y\,dx + (xe^x + 2)dy = 0,\ y(0) = 1$

Problem 2.22. Assume that the function $f(x,y)$ satisfies the relation $f(\lambda x, \lambda y) = f(x,y)$. Show that there is a function g such that $f(x,y) = g(y/x)$.

Problem 2.23. Find the solution to the following problems and draw the solution curves in their domain of definition.

i. $\cos(x)y' + y^2 = 0,\ y(0) = 1$

ii. $y' - (1+y^2)xe^x = 0,\ y(0) = -1$

iii. $\ln(\cos(x))\,y' - \tan(x)\cot(y) = 0,\ y(\pi/4) = \pi/4$

iv. $(1+x^2)y' = 2xy\ln(y),\ y(0) = e$

Problem 2.24. Write the general solution to the following equations

i. $y' = (\tan(x) + \cos(x))\cot(y)$

ii. $3xyy' + 4y^2 = 1$

iii. $(1+x^2)\,yy' = (1+y^2)\,x$

Problem 2.25. Solve the following problem

$$\begin{cases} y' = \sqrt{1+|y|} \\ y(0) = 1 \end{cases}.$$

Problem 2.26. Find a transformation to reduce the equation

$$y' = f(ax + by + c),$$

to a separable one and then solve the following problems

i. $y' = (4x + y)^2$, $y(1/2) = -2$

ii. $y' = -1 + 2(2x + y) + (2x + y)^2$, $y(0) = 0$

iii. $y' = \tan^2(y - x)$, $y(0) = 0$

Problem 2.27. Find $\lim_{x\to\infty} y(x)$ if $y(x)$ is the solution to the following problem

$$\begin{cases} x^2 y' = (1 + y^2) \\ y(1) = 0 \end{cases}.$$

Problem 2.28. Solve the following problems

i. $xy' = x + y + \sqrt{xy}$, $y(1) = 1$

ii. $xy' = y + x\cos^2(\frac{y}{x})$, $y(1) = 0$.

iii. $xy' - y = x\sec(y/x)$, $y(1) = \frac{\pi}{2}$

iv. $xyy' = (x^2 + 2y^2)$, $y(1) = -1$

Problem 2.29. Determine values of y_0 for which the solution to the following problem approaches zero when $x \to \infty$

$$\begin{cases} xy' = y(\ln(y) - \ln(x) + 1) \\ y(1) = y_0 \end{cases}.$$

Problem 2.30. Solve the following equations by suggested substitutions:

i. $y' = (y - x)^2$, $v = y - x$

ii. $xy' = e^{-xy} - y$, $v = xy$

iii. $y' = (x + y - 1)^2 - (x + y + 1)^2$, $v = x + y$

Problem 2.31. Equations of the form

$$y' = f\left(\frac{ax + by + e_1}{cx + dy + e_2}\right),$$

for $ad - bc \neq 0$ can be transformed to a separable equation by a simple trick. Shift axes x, and y by $x = X + \alpha$ and $y = Y + \beta$ for some (unknown yet) constants α, β. Substitution into the equation gives

$$Y' = f\left(\frac{aX + bY + a\alpha + b\beta + e_1}{cX + dY + c\alpha + d\beta + e_2}\right). \tag{2.22}$$

Now determine α, β such that

$$\begin{cases} a\alpha + b\beta + e_1 = 0 \\ c\alpha + d\beta + e_2 = 0 \end{cases}.$$

Use this trick to solve the following problem

$$y' = \left(\frac{x + 2y}{2x}\right)^2, \quad \text{and} \quad y(1) = 1.$$

2.4.7 Non-exact equations and integrating factor

It frequently occurs that a given equation is not exact. In this case, we try to find an integrating factor to male the equation exact while the solution of the original equation is unchanged. For example, a first order linear equation

$$y' + p(x)y = r(x),$$

that can be rewritten as the differential form

$$dy + (p(x)y - r(x))\,dx = 0,$$

is not exact. Similarly, the BERNOULLI's equation

$$dy + y(p(x) - r(x)y^{\alpha-1})\,dx = 0,$$

is not exact. We saw before that to solve a linear equation, we should multiply it with the integrating factor $\mu(x) = e^{\int p(x)}$. In fact, if we do that, we obtain the equation

$$e^{\int p(x)}\,dy + \left(p(x)\,e^{\int p(x)}y - r(x)e^{\int p(x)}\right)dx = 0,$$

which is evidently exact. For a second example, consider the following equation

$$xydx + (x^2 + y)\,dy = 0.$$

Clearly the equation is not exact, but if we multiply it by $I(x, y) = y$, we reach the following equation which is exact

$$xy^2\,dx + (x^2y + y^2)\,dy = 0.$$

But, how in general can we solve the equation

$$M(x, y)\,dx + N(x, y)\,dy = 0,$$

if it is non-exact? In this section we answer that question. We start off by a definition.

Definition 2.1. *The function* $\mu(x, y)$ *is called an* integrating factor *for the problem*

$$\begin{cases} M(x, y)dx + N(x, y)dy = 0 \\ y(x_0) = y_0 \end{cases}, \tag{2.23}$$

if there exists a domain D *around* (x_0, y_0) *such that*

i. $\mu(x, y) \neq 0$ *for* $(x, y) \in D$

ii. and that the problem $\mu(x, y)M(x, y)\,dx + \mu(x, y)N(x, y)\,dy = 0$ *is exact in* D.

2.4.8 Derivation of the integrating factor

If the equation

$$\mu(x, y)M(x, y)\,dx + \mu(x, y)N(x, y)dy = 0,$$

is exact, then the following relation holds

$$M\frac{\partial\mu}{\partial y}+\mu\frac{\partial M}{\partial y}=N\frac{\partial\mu}{\partial x}+\mu\frac{\partial N}{\partial x}. \tag{2.24}$$

This is a partial differential equation for I which we study in great detail in the second volume of this book. In order to solve it here, we consider spacial cases where the solution to the partial differential equation are significantly simplified.

Case 1. If $\mu=\mu(y)$, then (2.24) reduces to the following ordinary equation

$$\frac{d\mu}{\mu}=\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dy. \tag{2.25}$$

Since the left hand side is a function of y, the assumption $\mu=\mu(y)$ is valid if and only if the right hand side of (2.25) is a function of y. For example, for the equation $xy\,dx+(x^2+y)\,dy=0$, we have $\frac{d\mu}{\mu}=\frac{dy}{y}$ and thus $\mu=y$ is an integrating factor of the equation.

Case 2. If $\mu=\mu(x)$, then (2.24) reduces to

$$\frac{d\mu}{\mu}=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)dx. \tag{2.26}$$

Since the left hand side is a function of x, the assumption $\mu=\mu(x)$ is valid if and only if the right hand side of (2.26) is a function of x. For example, for the equation $(xy+1)\,dx+x(x+y)dy=0$, we have $\frac{d\mu}{\mu}=-\frac{dx}{x}$ and thus $\mu=\frac{1}{x}$ is the integrating factor.

The above two cases are only simplest instances of more general forms. For example, the integrating factor may have types $\mu=\mu(xy)$, $\mu=\mu\left(\frac{x}{y}\right)$, $\mu=\mu(ax+by)$, and so on. For example, let us find a condition for the integrating factor of the form $\mu=\mu(xy)$. If we take $z=xy$, then

$$\frac{\partial\mu}{\partial x}=y\frac{d\mu}{dz},\frac{\partial\mu}{\partial y}=x\frac{d\mu}{dz},$$

and this transforms the equation (2.24) to the following one

$$\frac{d\mu}{\mu}=\frac{1}{xM-yN}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dz.$$

The assumption $\mu=\mu(xy)$ is true if and only if the expression in the right hand side is a function of $z=xy$.

Example 2.18. Let us solve the following equation by assuming $\mu=\mu(xy)$:

$$(2x\cos(y)-xy\sin(y))\,y'+2y\cos(y)=0.$$

It is simply seen that

$$\frac{d\mu}{\mu}=\frac{1}{z}dz,$$

for $z = xy$. Therefore $\mu = xy$ makes the equation exact.

2.4.9 LAGRANGE equations

The general form of the equation is a follows

$$y = x\,f(y') + g(y'). \tag{2.27}$$

As it is observed, it can not be put in the usual *differential form* $M dx + N dy = 0$.

Example 2.19. We look for the shape of a mirror with the following property. All light rays coming from a distant source are reflected to a focal point; see the figure (2.4).

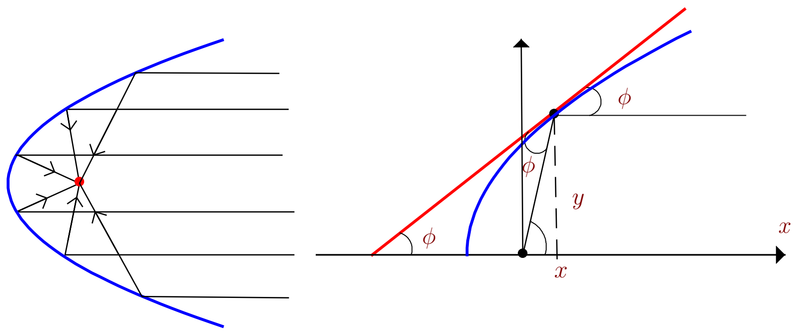


Figure 2.4.

It is simply seen from the figure in the right that $\tan(2\phi) = y\,/\,x$. From the relation $y' = \tan\phi$ and the identity $\tan(2\phi) = \frac{2\tan\phi}{1-\tan^2\phi}$, we obtain the following differential equation for the mirror:

$$y = x\frac{2\,y'}{1 - y'^2}. \tag{2.28}$$

This is a LAGRANGE equation.

To solve (2.27), we look for a solution of the parametric form

$$\begin{cases} x = x(p) \\ y = y(p) \end{cases}, \tag{2.29}$$

where p is a parameter. Remember that the solution to a differential equation is a family of planar curves, and a curves can be written as parametric as (2.29). If we take $p = y'$ for the parameter, we simply obtain

$$y(p) = x(p)\,f(p) + g(p). \tag{2.30}$$

The remaining job is to find $x = x(p)$. For this, we take derivative of (2.27) with respect to p and use the identity

$$\frac{dy}{dp} = \frac{dy}{dx}\frac{dx}{dp} = p\frac{dx}{dp}. \tag{2.31}$$

In this manner, we obtain

$$\frac{dy}{dp} = f(p)\,\frac{dx}{dp} + f'(p)\,x + g'(p), \tag{2.32}$$

that leads to the following linear first order equation for x

$$(p-f(p))\frac{dx}{dp}-f'(p)\,x=g'(p). \tag{2.33}$$

If $x=\varphi(p)$ is a solution to (2.33), then the parametric solution is

$$\begin{cases} x=\varphi(p) \\ y=\varphi(p)\,f(p)+g(p) \end{cases} \tag{2.34}$$

Example 2.20. To solve (2.28), let us take $y'=p$ as a free parameter and write the equation in the form $y=x\frac{2p}{1-p^2}$. The differential equation for $x(p)$ is

$$-p\,\frac{dx}{dp}=\frac{2x}{1-p^2}, \tag{2.35}$$

that is solved for $x=\frac{1-p^2}{p^2}$. The parametric solution to the equation is

$$\begin{cases} y(p)=\frac{2}{p}. \\ x(p)=\frac{1}{p^2}-1 \end{cases}. \tag{2.36}$$

Eliminating p from the above solution gives the algebraic shape of the mirror $x=\frac{1}{4}y^2-1$, that is a parabola.

Problems

Problem 2.32. Integrate the following equations with the given integrating factor

i. $(2xy-5y^2)+(1-5xy)\,y'=0$, $I=I(y)$
ii. $(x^2y+4)-x^2(y-x)\,y'=0$, $I=I(x)$
iii. $2xy\ln|y|+(x^2+y^2\sin(y^2))\,y'=0$, $I=I(y)$
iv. $y\ln(y)(1+x)+x(1+\ln(y))\,y'=0$, $I=I(x)$
v. $(xy+1)+x(x+y)y'=0$, $I=I(x)$
vi. $(xe^y-1)+(ye^{-x}+1)\,y'=0$, $I=I(x-y)$
vii. $(x^2+xy+1)+(y^2+xy-1)\,y'=0$, $I=I(x+y)$.

Problem 2.33. Consider the following equation

$$M(x+y^2)+2\,y\,y'=0$$

where $M(x+y^2)$ is a smooth function with respect to $x+y^2$. Prove or disprove: the above equation has an integrating factor of the type $I=I(x+y^2)$.

Problem 2.34.

i. For the following equation, obtain conditions under which the equation is exact:

$$M(x,y,z)\,\mathrm{d}x+N(x,y,z)\,\mathrm{d}y+P(x,y,z)\,\mathrm{d}z=0.$$

ii. Obtain conditions such that the above equation has an integration factor $I=I(x)$.
iii. Obtain conditions such that the above equation has an integration factor $I=I(y)$.

Problem 2.35. Integrate the following equations

i. $y = 1 + y'^2$, $y(0) = 1$

ii. $y = e^{yy'}$

iii. $y = xy' + \frac{1}{y'}$

iv. $y = xy'^2 + y'^2$

Problem 2.36. Show that the parametric solution to the equation $x = f(y')$ is

$$x = f(p), \qquad y = \int p f'(p)\, \mathrm{d}p.$$

Solve the following equation

$$x = y'^2(1 + y'^2).$$

Problem 2.37. A fisherman catches a fish at the distance $a = 10$ in a river.

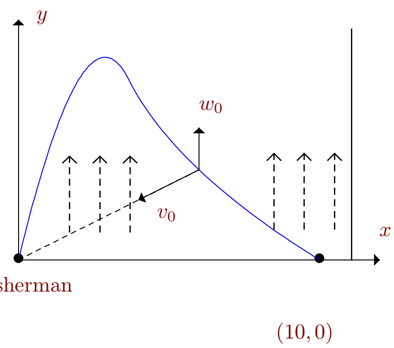


Figure 2.5.

If the fisherman pull the fish with the constant velocity $v_0 = 1$ toward himself, and the river velocity is $w_0 = 2$ in y-direction; see the figure (2.5), find the equation of the path the fish travels in xy-plane (the blue line in the figure).

Problem 2.38. In the above problem, assume that the river velocity is given by the function $w(x) = x(10 - x)$. Find the equation of path of the fish if $v_0 = 5$.

2.5 Theory of first order differential equations

In this section we briefly discuss the elements of the theory of first order differential equations. Our discussion is about the *existence-uniqueness* theorem and the PICARD's iteration formula. It is not trivial at all that a given initial value problem admits a solution. For example, it is simply seen that the following problem

$$\begin{cases} xy' = y \\ y(0) = 1 \end{cases},$$

admits no solution, while the problem

$$\begin{cases} xy' = y \\ y(0) = 0 \end{cases},$$

admits infinitely many solutions $y = cx$ for arbitrary c, and the problem

$$\begin{cases} xy' = y \\ y(1) = 1 \end{cases},$$

admits only one solution (unique) $y=x$.

2.5.1 Existence problem

The conditions under which an initial value problem admits *at least* one solution, is called the existence problem. We have the following theorem.

Theorem 2.3. (Existence) *Consider the following initial value problem*

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}, \tag{2.37}$$

and assume that there is an open rectangle D centered at (x_0, y_0),

$$D = \{(x,y); |x-x_0| \le a, |y-y_0| \le b\}, \tag{2.38}$$

such that $f(x,y)$ is continuous in D. Then there is at least one local solution to (2.37).

Recall that $y(x)$ is a solution to (2.37) if there is an open interval $I=(x_0-\delta, x_0+\delta)$ such that $y_0=y(x_0)$ and $y'(x)=f(x,y(x))$ for all $x\in I$. The rectangle D is called the *continuity rectangle* of the problem; see the figure (2.6).

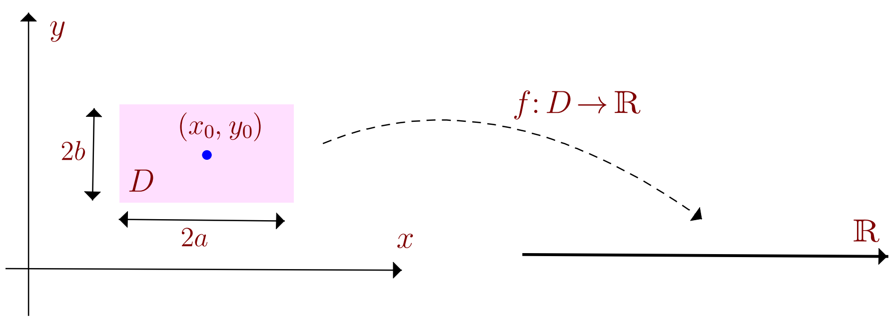


Figure 2.6.

The proof of the theorem is beyond the scope of this book and can be found in advanced textbooks on the theory of ordinary differential equations.

Example 2.21. The existence condition is satisfied for the problem

$$\begin{cases} y' = 2xy^2 \\ y(0) = 1 \end{cases}. \tag{2.39}$$

In fact, the function $f(x,y)=2xy^2$, is continuous everywhere in $\mathbb{R}^2$. It is simply seen that the function $y=\frac{1}{1-x^2}$ solves the problem. Note that the domain of the solution (regarding the initial point $x_0=0$) is $I=(-1,1)$. The existence condition is satisfied for the following problem as well

$$\begin{cases} y' = y^{\frac{2}{3}} \\ y(0) = 0 \end{cases}. \tag{2.40}$$

The function $f(x,y)=y^{\frac{2}{3}}$ is continuous everywhere and thus the existence of at least one solution is guaranteed for the problem. We will see that this problem has infinitely many solutions (see the problem set). The problem

$$\begin{cases} xy'=y \\ y(0)=1 \end{cases}, \tag{2.41}$$

does not satisfy the existence condition and it does not have a solution. In fact the function $f(x,y)=\frac{y}{x}$, is unbounded in any rectangle about the initial point $(0,1)$.

Remark 2.1. The theorem states only a sufficient condition for the existence of a solution. The condition may fail for a problem and even it admits a solution. For example, the problem

$$\begin{cases} xy'=y \\ y(0)=0 \end{cases}, \tag{2.42}$$

does not satisfy the condition of the existence theorem, however, it has infinitely many solutions $y=cx$ for all $c\in\mathbb{R}$.

2.5.2 EULER explicit method

Let us show how the existence theorem is justified by the EULER numerical method. For the sake of simplicity, we assume that $x_0=0$. Let x be an arbitrary point in the domain of definition of the solution. Divide the segment $[0,x]$ (if $x>0$) into n division with the length $h=\frac{x}{n}$, and points $x_1,x_2,\cdots,x_n$ where $x_k=kh$. Remember the *linear approximation* formula for a differentiable function $g(x)$ at x_0

$$g(t)\cong g(x_0)+g'(x_0)\,(t-x_0).$$

Applying the formula for the solution $y(x)$ at x_1, gives

$$y'(x_1)\cong y(0)+y'(0)x_1=y_0+f(0,y_0)h.$$

Let call this value y_1, that is, y_1 is an approximate value for true solution $y(x)$ at $x=x_1$. Repeating that process for x_2, we get

$$y(x_2)\cong y(x_1)+y'(x_1)(x_2-x_1)\cong y_1+f(x_1,y_1)h,$$

and generally for x_k we derive the general formula

$$y_k=y_{k-1}+f(x_{k-1},y_{k-1})h.$$

In particular for $k=n$ we have

$$y_n=y_{n-1}+f(x_{n-1},y_{n-1})h.$$

Note that y_n is the approximation of the true solution $y(x)$. The error between these two values depends of course to h

$$e(h)=|y(x)-y_n|.$$

Now, if $n\to\infty$, that is $h\to 0$, we have the following result

$$\lim_{n\to\infty}|y(x)-y_n|=0.$$

Example 2.22. Consider the initial value problem $y' = y$ and $y(0) = y_0$. Obviously, the problem has the solution $y = y_0 e^x$. Let us solve this problem by the EULER's method. Divide the segment $[0, x]$ into n sub-intervals with the length $h = \frac{x}{n}$. We have

$$y_1 = y_0 + y_0 \frac{x}{n} = y_0\left(1 + \frac{x}{n}\right). \tag{2.43}$$

repeating for $2h$ we have

$$y_2 = y_0\left(1 + \frac{x}{n}\right) + y_0\left(1 + \frac{x}{n}\right)\frac{x}{n} = y_0\left(1 + \frac{x}{n}\right)^2, \tag{2.44}$$

and for nh, we have

$$y_n = y_0\left(1 + \frac{x}{n}\right)^n. \tag{2.45}$$

Notice that $nh = x$ and if we let $n \to \infty$ we obtain

$$\lim_{n\to\infty} y_n = y_0 \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = y_0\, e^x = y(x). \tag{2.46}$$

Example 2.23. Let us solve the following problem by the EULER's method

$$\begin{cases} y' = \sin(xy) \\ y(0) = 1 \end{cases}. \tag{2.47}$$

For $h = 0.2$, we obtain

$$y_1 = y_0 + f(0, y_0)h = 1.$$

Likewise we have

$$y_2 = y_1 + f(0.2, y_1)h = 1.04.$$

The following figure shows the real solution $y = \phi(x)$ and the numerical one obtained by the EULER's method in the range $[0, 6]$.

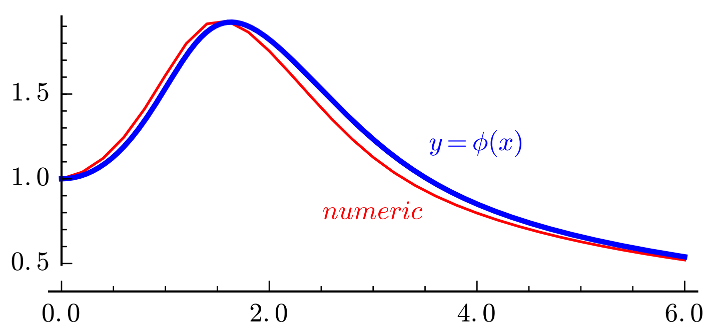


Figure 2.7.

2.5.3 Uniqueness problem

Theorem 2.4. (Uniqueness) *If in addition to the existence condition of the problem* (2.37), *the function* $\frac{\partial f}{\partial y}: D \to \mathbb{R}$ *is continuous, then the problem has a unique solution.*

Proof. The proof is based on the integral representation of the initial value problem

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,\mathrm{d}t. \tag{2.48}$$

It is straightforward to verify that the above integral equation is equivalent to the given initial value problem. Consider a continuity rectangle for the function f as

$$D := \{(x, y), |x - x_0| < a, |y - y_0| < b\}.$$

Fix $\bar{x}$ for which a solution exists in the interval $[x_0, \bar{x}]$. Assume that there is two solutions to the problem. Then for any arbitrary $x \in [x_0, \bar{x}]$, we have

$$|y_1(x) - y_2(x)| \leq \int_{x_0}^{\bar{x}} |f(t, y_1(t)) - f(t, y_2(t))|\,\mathrm{d}t. \tag{2.49}$$

Since f is continuously differentiable with respect to y, we can write

$$|f(t, y_1) - f(t, y_2)| \leq M\,|y_1 - y_2|, \tag{2.50}$$

for some $M > 0$ that is

$$M = \max_{y \in \bar{D}} \left|\frac{\partial f}{\partial y}(t, y)\right|. \tag{2.51}$$

This implies

$$|y_1(x) - y_2(x)| \leq M \int_{x_0}^{\bar{x}} |y_1(t) - y_2(t)|\mathrm{d}t \leq M\,\|y_1 - y_2\|\,(\bar{x} - x_0), \tag{2.52}$$

where $\|y_1 - y_2\| = \max_{c \in [x_0, \bar{x}]} |y_1(c) - y_2(c)|$. Since the above inequality holds for all $x \in [x_0, \bar{x}]$, we obtain

$$\max_{c \in [x_0, \bar{x}]} |y_1(c) - y_2(c)| \leq M\,\|y_1 - y_2\|\,(\bar{x} - x_0), \tag{2.53}$$

or equivalently

$$\|y_1 - y_2\| \leq M(\bar{x} - x_0)\|y_1 - y_2\|.$$

Now, we can choose $\bar{x} - x_0$ so small such that $M(\bar{x} - x_0) < 1$, and thus if $\|y_1 - y_2\| = 0$, and thus $y_1(x) = y_2(x)$ for all $x \in [x_0, \bar{x}]$.

□

Example 2.24. The uniqueness condition is satisfied for the problem (2.39). In fact, the function $f(x, y) = 2xy^2$ and $\partial_y f = 4xy$ are both continuous in $\mathbb{R}^2$ and thus the uniqueness is guaranteed. The uniqueness condition is not satisfied for the problem (2.40). In fact, we have $\partial_y f = \frac{2}{3} y^{-1/3}$ that is not continuous in any rectangle about the initial point $(0, 0)$. It is simply verified that functions $y = 0$ and $y = \frac{1}{27} x^3$ are both solutions to the problem.

Example 2.25. Consider the initial value problem

$$\begin{cases} y' = |y| + 1 \\ y(0) = 0 \end{cases}.$$

Clearly, the function $f(x, y) = |y| + 1$ is Lipschitz (with $L = 1$) but not differentiable at $y = 0$. This implies that the problem has a unique solution. Let us solve the problem directly and obtain the solution. If $y > 0$, the problem has the solution $y(x) = -1 + e^x$. Clearly $y > 0$ if $x > 0$. At $x = 0$, we have $y(x) = 0$ and since $y'(0) = 1$, we conclude that y is negative for x in some interval $(-a, 0)$. But if $y < 0$, the equation reads $y' = -y + 1$ and thus $y(x) = 1 - e^{-x}$. Observe that $y(x)$ remain negative for all $x < 0$. This implies that the solution to the problem is

$$\phi(x) = \begin{cases} -1 + e^x & x \geq 0 \\ 1 - e^{-x} & x \leq 0 \end{cases}. \tag{2.54}$$

Note that ϕ is $C^1(-\infty, \infty)$ and is a solution to the initial value problem; see the figure (2.8). Notice that $\phi''(0)$ does not exist.

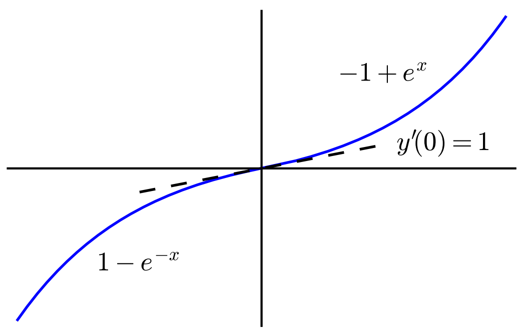


Figure 2.8.

Definition 2.2. *A function $f(y)$ is called* Lipschitz *in an interval I if there is a constant $L > 0$ such that the following inequality holds for all y_1 and y_2 in I*

$$|f(y_1) - f(y_2)| \leq L|y_1 - y_2|. \tag{2.55}$$

Example 2.26. Function $f(y) = |y|$ is Lipschitz in $\mathbb{R}$ but not differentiable at $y = 0$. In fact, we have

$$|f(y_1) - f(y_2)| = ||y_1| - |y_2|| \leq |y_1 - y_2|. \tag{2.56}$$

Here the Lipschitz constant L is 1. Function $f(y) = e^y$ is Lipschitz in any bounded interval. In fact, if $I = (-a, a)$ then

$$|f(y_1) - f(y_2)| = |e^{y_1} - e^{y_2}| \leq e^a |y_1 - y_2|. \tag{2.57}$$

Here $L = e^a$. Function $f(y) = \sqrt{|y|}$ is not Lipschitz in interval $I = (-a, a)$. In fact, for $y_2 = 0$ and $y_1 > 0$, we have

$$L \geq \frac{|f(y_1) - f(0)|}{y_1} = \frac{1}{\sqrt{y_1}}, \tag{2.58}$$

which is not bounded when $y_1 \downarrow 0$.

Proposition 2.1. *Let I be an open interval and assume that $f \in C^1(I)$, i.e., f' is continuous in I, then f is* LIPSCHITZ *in any interval $[a,b] \subset I$.*

For a multi-valued functions $f(x,y)$, the LIPSCHITZ condition is defined similarly. The function $f(x,y)$ is called LIPSCHITZ in the domain D with respect to y if there exists $L > 0$ such that for all (x, y_1) and (x, y_2) in D, the following inequality holds

$$|f(x,y_1) - f(x,y_2)| \leq L|y_1 - y_2|. \tag{2.59}$$

The condition in the uniqueness theorem (that $\partial_y f$ to be continuous is a neighborhood of (x_0, y_0)) can be relaxed according to the following theorem.

Theorem 2.5. (uniqueness) *Assume that there is a rectangle D centered at (x_0, y_0) such that the function $f: D \to R$ is continuous in D and is* LIPSCHITZ *with respect to y. Then the initial value problem*

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}, \tag{2.60}$$

has a unique solution.

An immediate consequence of the uniqueness theorem is the uniqueness for linear problems. In fact, if $p(x), r(x)$ are continuous functions, by the existence and uniqueness theorem, the following problem

$$\begin{cases} y' + p(x)y = r(x) \\ y(x_0) = y_0 \end{cases},$$

has a unique solution. It also can be shown that the solution is extended in the whole of $\mathbb{R}$. The uniqueness part is immediately verified by the observation

$$\frac{\partial}{\partial y}(r(x) - p(x)y) = p(x).$$

That the solution is extended in $\mathbb{R}$ is justified by the exponential form of the solution obtained in this chapter. By the aid of the uniqueness theorem, we can prove the following important fact which is left as an exercise.

Proposition 2.2. *Let y_e be an equilibrium for the equation $y' = f(y)$. If f is continuously differentiable in a neighborhood of y_0, then the solution to the problem*

$$\begin{cases} y' = f(y) \\ y(0) = y_0 \neq y_e \end{cases},$$

can not touch y_e for a finite x.

Problem 2.39. Prove proposition (2.1).

Problem 2.40. Prove proposition (2.2).

Problem 2.41. Verify the uniqueness theorem for the following problems

i.
$$\begin{cases} y' = |\sin(y)| \\ y(0) = 0 \end{cases}$$

ii.
$$\begin{cases} y' = (1+x)\, f(y) \\ y(0) = 0 \end{cases},$$

where
$$f(y) = \begin{cases} \sin(y) & y \geq 0 \\ 0 & y < 0 \end{cases}.$$

iii.
$$\begin{cases} y' = x + |\tan(y)| \\ y(0) = 0 \end{cases}$$

Problem 2.42. Verify the uniqueness condition for the following problem and find its solution

$$\begin{cases} y' + 2|x|y = x \\ y(0) = 0 \end{cases}$$

Problem 2.43. Verify the uniqueness theorem for the following problem and find its unique solution

$$\begin{cases} y' = 1 + y|y| \\ y(0) = 0 \end{cases}.$$

Problem 2.44. Verify the uniqueness theorem for the following problem and find its unique solution

$$\begin{cases} y' = \sqrt{1+|y|} \\ y(0) = 0 \end{cases}.$$

Problem 2.45. Verify the uniqueness theorem for the following problem and find its unique solution

$$\begin{cases} y' + y = y|y-1| \\ y(0) = 1 \end{cases}.$$

Problem 2.46. Prove that the following problem has a unique solution and then find its solution

$$\begin{cases} y' = |2x - y| \\ y(0) = 0 \end{cases}.$$

Problem 2.47. Integrate the following i.v.p.s using suitable transformations. Determine the domain of definition for each solution.

i. $\cos(y)\, y' + \sin(y) = 1, y(0) = \pi/2$

ii. $y' = y(x + \ln(y)), y(0) = 1$

iii. $\int_0^1 y(xt)\, dt = 2y - x, y(1) = -1/3$

Problem 2.48. Verify that the function

$$\phi(x) = \begin{cases} \frac{1}{4}x^2 & x \geq 0 \\ \frac{-1}{4}x^2 & x < 0 \end{cases},$$

is a solution to the problem

$$\begin{cases} y' = \sqrt{|y|} \\ y(0) = 0 \end{cases}.$$

In particular, you need to show that the given function is continuously differentiable at $x = 0$. The other solution is simply $y(x) = 0$. Can you construct other solutions to the problem?

2.5.4 Picard iteration method

Let us explain a method introduced by the French mathematician E. Picard to estimate the true solution to an initial value problem. Consider again the problem (2.37). It is simply sen that the problem is equivalent to the following *integral equation*

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,\mathrm{d}t. \tag{2.61}$$

In fact, we have $y(x_0) = y_0$, and furthermore, by fundamental theorem of calculus, we have

$$y'(x) = f(x, y(x)).$$

Then, if we could find a function $y(x)$ that satisfies Eq.(2.61), then $y(x)$ is the solution to the original problem (2.37). But how can we find $y(x)$? Picard suggested to make a sequence of functions $y_1(x), y_2(x), \cdots$, defined through the following recursive formula

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\,\mathrm{d}t,$$

where $y_0(x) = y_0$, the initial condition. It is shown that $y_n(x) \to y(x)$ when $n \to \infty$ if function f is continuously differentiable with respect to y.

Example 2.27. Consider the initial value problem

$$\begin{cases} y' = y \\ y(0) = y_0 \end{cases}. \tag{2.62}$$

As it is know, the solution is $y = y_0\, e^x$. Starting from y_0, we obtain

$$y_1(x) = y_0 + \int_0^x y_0\, ds = y_0(1 + x).$$

$$y_2(x) = y_0 + y_0 \int_0^x (1 + t)\,\mathrm{d}t = y_0\left(1 + x + \frac{1}{2}x^2\right),$$

and in the n^{th} step,

$$y_n(x) = y_0\left(1 + \frac{1}{2}x + \cdots + \frac{1}{n!}\,x^n\right). \tag{2.63}$$

Clearly, $y_n(x)$ is the first n terms of the series expansion of $y = y_0\, e^x$, i.e., $y_n(x) \to y_0\, e^x$.

Example 2.28. Consider the following initial value problem

$$\begin{cases} y' = y^2, \\ y(0) = \frac{1}{2} \end{cases}. \tag{2.64}$$

It is simply verified that the solution is $y = \frac{1}{2 - x}$. The Picard method gives the sequence

$$y_1(x) = \frac{1}{2} + \frac{1}{4}x,\ y_2(x) = \frac{1}{2} + \frac{1}{4}x + \frac{1}{8}x^2 + \frac{1}{48}x^3, \cdots.$$

In the figure (2.9) the solution $y(x)$ and the sequence y_1, y_2, y_3 and y_4 are given

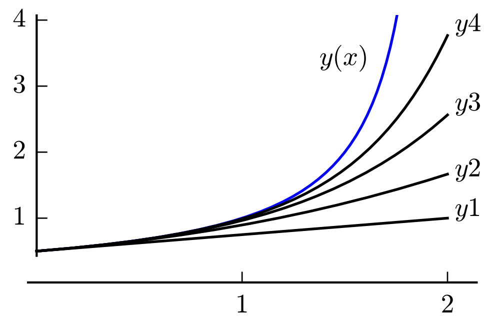


Figure 2.9.

2.5.5 Estimation of the domain of solution

As we learned in this chapter, the solution to an initial value problem is usually local, that is, a function $y(x)$ with the domain of definition an open interval $(x_0 - \delta, x_0 + \delta)$. In this subsection, we estimate the value δ for the domain of definition of the initial value problem (2.37). In Fig.(2.10), the continuity rectangle and an integral curve are represented. As it is seen, it is possible that the integral curves goes unbounded when x approaches $\pm\delta$ where $\delta < a$.

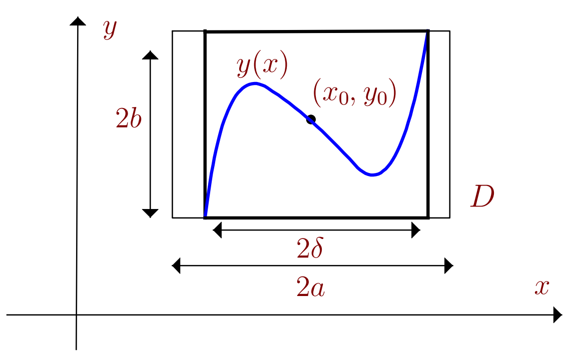


Figure 2.10.

We observe that

$$|y(x) - y_0| \leq b, \forall x \in I := (x_0 - \delta, x_0 + \delta). \tag{2.65}$$

By the fundamental theorem of calculus, we have

$$y(x) - y_0 = \int_{x_0}^{x} f(t, y(t))\, \mathrm{d}t, \tag{2.66}$$

and thus

$$|y(x) - y_0| \leq \int_{x_0}^{x} |f(t, y(t))|\, \mathrm{d}t \leq |x - x_0|\, M, \tag{2.67}$$

where

$$M = \max_{(x,y)\in\bar{D}} |f(x,y)|. \tag{2.68}$$

Note that the maximum exists because f is continuous in D. Since $|x - x_0| < \delta$, we obtain the inequality $\delta M \leq b$, and finally

$$\delta = \min\left(\frac{b}{M}, a\right).$$

Example 2.29. For the problem

$$\begin{cases} y' = 2xy^2 \\ y(0) = 1 \end{cases},$$

the function $f(x,y) = 2xy^2$ is continuous everywhere. If we choose D as

$$D = \{(x,y); |x| \leq a, |y-1| \leq b\}, \tag{2.69}$$

then we have

$$M = \max_{(x,y)\in\bar{D}} |f(x,y)| = 2a\,(1+b)^2. \tag{2.70}$$

Thus

$$\delta = \min\left(\frac{b}{2a(1+b)^2}, a\right).$$

Since $\frac{b}{(1+b)^2} \leq \frac{1}{4}$, then $\delta = \min\left(\frac{1}{8a}, a\right)$. To maximize δ, we put $a = \frac{1}{8a}$, which gives $\delta = a = \frac{1}{2\sqrt{2}}$.

Example 2.30. The solution to the problem

$$\begin{cases} y' = \frac{y}{1-x} \\ y(0) = 1 \end{cases} \tag{2.71}$$

is $y = \frac{1}{1-x}$ with the domain of definition $(-\infty, 1)$. Let us find δ without solving the problem. The function $f(x,y) = \frac{y}{1-x}$ is continuous in the box $D = \{(x,y); |x| \leq a, |y-1| \leq b\}$ where $0 < a < 1$ and b is arbitrary. Therefore $M = \frac{1+b}{1-a}$ and $\delta = \frac{(1-a)b}{1+b}$. Since $\frac{b}{1+b} \leq 1$ we have $\delta \leq 1-a$. Since $\delta \leq a$, the condition $a < 1$ implies $\delta < 1$. Thus the interval of the solution is $I_\delta = (-1, 1)$.

Problems

Problem 2.49. For each of the following problem, solve the equation and find an interval on which the solution can be extended. Then apply the method described in this section to estimate the interval of solution.

i. $y' = \frac{1}{2x}\,y, \quad y(1) = 1$

ii. $y' = 2y^2, \quad y(0) = 1$

iii. $y' = \sec(x+y) - 1, \quad y(0) = 0$

iv. $(1+x^2)y' - 2y = 0, \quad y(1) = 1$

v. $yy' = 1 + \frac{1}{2}y^2, \quad y(0) = 0$

Problem 2.50. Verify the existence and uniqueness conditions for the following problems:

i. $y' = (1 + \sin(x))\, y^{5/3}$, $y(0) = 0$.

ii. $y' = e^x\, |1 + y|$, $y(0) = 0$.

iii. $y' = \cos(x) \sin(y)$, $y(0) = 0$

iv. $y' = |y|^{1/2}$, $y(0) = 1$.

Problem 2.51. Consider the following initial value problem

$$\begin{cases} yy' = \cos(x)\sin(y) \\ y(\pi/2) = 0 \end{cases}.$$

What can you say about the existence and uniqueness of the solution to the problem?

Problem 2.52. Consider the initial value problem

$$\begin{cases} y' = \frac{y}{\sin(x)} \\ y(0) = 0 \end{cases}.$$

i. Show that the problem does not satisfies the condition of the existence theorem.

ii. Solve the equation in the problem and obtain a solution.

iii. In what sense the obtained solution to the equation is a solution to the given initial value problem?

Problem 2.53. Consider the initial value problem

$$\begin{cases} \cos(y) y' = 2x \\ y(0) = \frac{\pi}{2} \end{cases}.$$

Verify that the implicit function $\sin(y) - x^2 = 1$ satisfies the differential equation and the initial value. Is ϕ an acceptable solution to the problem?

Problem 2.54. Consider the following initial value problem

$$\begin{cases} (\cos y - \sin y) y' = e^x \\ y(0) = \frac{\pi}{4} \end{cases}.$$

Try to integrate the equation and find a solution. Is the obtained solution acceptable?

Problem 2.55. Consider the following problem

$$\begin{cases} y' = |y|^{1/2} \\ y(0) = 0 \end{cases}.$$

a) State the existence and uniqueness theorem for the given problem.

b) Integrate the equation and find a solution.

c) Verify that the problem has infinitely many solutions for arbitrary $c \geq 0$ given below

$$y(x) = \begin{cases} \frac{1}{4}(x-c)^2 & x \geqslant c \\ 0 & -c \leqslant x \leqslant c \\ \frac{-1}{4}(x+c)^2 & x \leqslant -c \end{cases}$$

Problem 2.56. Consider the initial value problem

$$\begin{cases} y' = y^{2/3} \\ y(0) = 0 \end{cases}.$$

Integrate the equation and find a solution. Verify that for every $c \geq 0$, the following is a solution

$$\phi(x) = \begin{cases} \frac{1}{27}(x-c)^3 & x \geq c \\ 0 & -c \leq x \leq c \\ \frac{1}{27}(x+c)^3 & x \leq -c \end{cases}$$

Problem 2.57. Consider the initial value problem

$$\begin{cases} y' = y^{1/3} \\ y(0) = 0 \end{cases}.$$

Show that the equation has infinitely many solutions.

Problem 2.58. The uniqueness of the solution to linear problems can be proved without appealing to the uniqueness theorem as follows

i. Prove that the problem

$$\begin{cases} y' + p(x)y = 0 \\ y(0) = 0 \end{cases},$$

has the unique solution $y \equiv 0$. Hint: If $y(x)$ is another solution show that the function $z = y(x)\, e^{-P(x)}$, where $P(x) = \int p(x)$ is a constant.

ii. By the aid of the above result, prove that the problem

$$\begin{cases} y' + p(x)y = r(x) \\ y(0) = y_0 \end{cases},$$

has a unique solution. (Hint: Assume y_1, y_2 are two solutions and show $y_1 = y_2$)

Problem 2.59. Consider the equation

$$\begin{cases} y' = y\,|y| + 1 \\ y(0) = 0 \end{cases}.$$

a) State the result of the existence-uniqueness theorem for this problem.

b) Solve the equation and obtain the solution.

Problem 2.60. Consider the equation

$$\begin{cases} y' = |y| + 1 \\ y(0) = 0 \end{cases}.$$

a) State the result of the existence-uniqueness theorem for this problem.

b) Solve the equation and obtain the solution. This solution is unique and this justify that the uniqueness theorem stated in this section is a sufficient condition only.

Problem 2.61. Prove the proposition (2.2)

Problem 2.62. Use the EULER's explicit method with $h = 0.05$ to obtain a numerical value for $y(1)$ of the following problems:

i. $y' = x + y^2,\ y(0) = 0.$

ii. $y' = \sin(y),\ y(0) = 1.$

iii. $y' = \cos(x + y),\ y(0) = 0.5.$

Problem 2.63. Apply the PICARD method to approximate the solution to the following problems up to order 3. Use a computer software to solve each equation numerically and plot both solutions in the same coordinate to compare them.

i. $y' + 2xy = 1,\ y(0) = 1.$

ii. $y' = x + y^2,\ y(0) = 0.$

iii. $y' + y = 1 + y^2,\ y(0) = 0.$

2.6 Applications of first-order equations

2.6.1 Exponential law

The exponential growth and decay is very popular in natural and man-made systems. Roughly speaking, a quantity $y(t)$ is subject to the exponential law if its rate of change $y'(t)$ is proportional to the present value of y, that is, $y'(t) \propto y(t)$. Here we discuss three applications, 1) Mortgage calculation, 2) carbon dating, and 3) electrical circuits.

1. Mortgage and interest.

A financial institute uses different methods to calculate the interest rates to loans and mortgages. One of them is the daily compounded formula

$$C_d = C_0(1 + k/365)^d,$$

where C_0 is the initial money or loan, k is the annual interest rate and d is the number of days after the loan. An alternative method is the continuous compounded method

$$C(t) = C_0\, e^{kt}.$$

The formula is followed directly by the differential equation

$$\frac{\mathrm{d}C}{\mathrm{d}t} = kC.$$

Notice that the unit of t is year here because k is the annual interest rate. For example, for $C_0 = 1000\$$ and $k = 0.04$, the total loan that should be paid off at $t = 2.5$ (in two and half year) is $C(2.5) = 1000\, e^{0.1} \cong 1105.17\$$.

Now let us calculate the monthly payment installments $\bar{C}$ for a mortgage C_0 that should be paid off at the maturity date T (in terms of months for example). Since the installments are in month, we divide k by 12. Now the problem reduces to find $\bar{C}$ in the following equation

$$\begin{cases} C'(t) = \frac{k}{12} C - \bar{C} \\ C(0) = C_0, C(T) = 0 \end{cases}. \tag{2.72}$$

Note that the unit of t here is month. In order to find $\bar{C}$, we write the solution of the equation as follows (assuming $\bar{C}$ is constant)

$$C(t) = \left(C_0 - \frac{12}{k}\bar{C}\right) e^{kt/12} + \frac{12}{k}\bar{C}. \tag{2.73}$$

Now, the condition $C(T) = 0$, determines $\bar{C}$ as

$$\bar{C} = \frac{k C_0\, e^{kT/12}}{12(e^{kT/12} - 1)}. \tag{2.74}$$

For example, for a mortgage $C_0 = 1000\$$ with the maturity date $T = 60$ months (5 years) borrowed with the annual interest rate $k=0.04$, the monthly payment is $\bar{C} \cong 18.39$. In this way, the total money paid in 5 tears is equal $1103\$$. Note that if the money is paid as lump sum in the maturity date, the money would be $1000 \times 1.04^5 \cong 1217\$$.

2. Radioactive decay and carbon dating.

Another example of exponential law is the decay of radioactive isotopes. As it is shown in physics, the decay rate of these isotopes is proportional to the substance, that is, the following equation holds for the quantity C at time t

$$C'(t) = -kC(t), \tag{2.75}$$

where $k>0$ is some constant depends on the type of isotope. Usually radioactive isotopes are classified based on their *half-life*, the time T when C_0 become half. For example, the half-life of Radium-226 is 1600 years and of Uranium-238 is 4.5 billion years, (radioactive isotopes last a long time in the nature and most of them are very hazardous for living species). It is simply seen that the solution to (2.75) can be rewritten based on the half-life as $C(t) = C_0\, 2^{-t/T}$.

Of special interest is the half-life of carbon-14 (C_{14}) because this radioisotope is used for carbon dating. It is known that C_{14} (consisting 6 protons and 8 neutrons) is unstable and transforms to C_{12} with half-life $T_{14} = 5730$ years. On the other hand, it is known that the ratio C_{14}/C_{12} is constant in all living bodies (according to the carbon exchange with the atmosphere) and is equal to $\alpha_0 \cong 1.3 \times 10^{-12}$ (equal to the same ratio of C_{14}/C_{12} in the atmosphere). After death, this ratio changes due to the transformation of C_{14} to C_{12}. One can obtain an estimate of the age of death by measuring the quantity C_{14}/C_{12} in a dead body. The procedure is as follows. If the age death is τ, then

$$C_{14}(\tau) = C_{14}(0)\, 2^{-\tau/T_{14}}. \tag{2.76}$$

According to the relation $C_{14}(0) = \alpha_0\, C_{12}(0)$ and also to $C_{14}(\tau) = \alpha_1\, C_{12}(\tau)$ for $\alpha_1 = C_{14}/C_{12}$, the equation (2.76) is rewritten as

$$\frac{C_{12}(\tau)}{C_{12}(0)} = \frac{\alpha_0}{\alpha_1}\, 2^{-\tau/T_{14}}. \tag{2.77}$$

On the other hand, the mass conservation of C_{12} implies (C_{14} transforms to C_{12})

$$C_{12}(\tau) = C_{12}(0) + C_{14}(0) - C_{14}(\tau), \tag{2.78}$$

and thus dividing by $C_{12}(0)$, we obtain

$$\frac{C_{12}(\tau)}{C_{12}(0)} = 1 + \alpha_0 - \alpha_1 \frac{C_{12}(\tau)}{C_{12}(0)}. \tag{2.79}$$

We can write the above relation as

$$\frac{C_{12}(\tau)}{C_{12}(0)} = \frac{1+\alpha_0}{1+\alpha_1}, \tag{2.80}$$

and substituting this into (2.77) gives

$$\frac{\alpha_1(1+\alpha_0)}{\alpha_0(1+\alpha_1)} = 2^{-\tau/T_{14}}. \tag{2.81}$$

In order to find τ, we need to solve the above algebraic relation. The above calculations are subject to several corrections for practical uses.

3. Electrical circuits.

Electrical circuit theory is another filed in which the exponential law comes into play. For a simple example, consider the RC circuit shown in the figure (2.11).

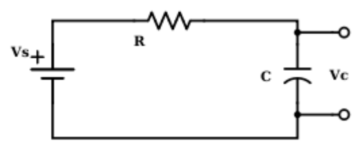


Figure 2.11.

In the figure, R and C stands for the electrical resistance and the capacitance respectively.

KIRCHHOFF's mesh law. This law states that the algebraic sum of voltages across elements in a closed mesh is zero. In the figure (2.12), if V_R and V_C denote respectively the voltage across the resistance R and the capacitance L, we have

$$V_R + V_C - V_s = 0, \tag{2.82}$$

where the negative sign of the supply V_s is due to its negative port encountered in the mesh.

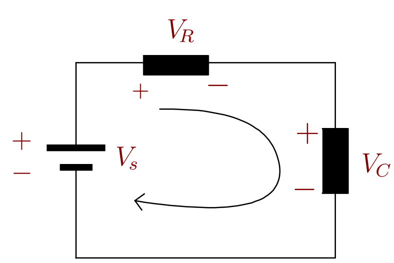


Figure 2.12.

According to the OHM's law, the voltage across the resistance is expressed by the formula

$$V_R(t) = Ri(t), \tag{2.83}$$

where $i(t)$ is the electrical current in the resistor. The voltage-current relationship in the capacitor is

$$i(t) = C\frac{dV_c}{dt}(t), \tag{2.84}$$

and thus, the KIRCHHOFF'S formula is expressed by the following differential equation

$$\frac{\mathrm{d}V_c}{\mathrm{d}t} + \frac{1}{RC}V_c = \frac{1}{RC}V_s. \tag{2.85}$$

The appropriate initial condition for the circuit has the form $V_c(0) = V_0$ for some constant V_0. If V_s is a constant supply, the voltage $V_c(t)$ is determined by the formula

$$V_c(t) = (V_0 - V_s)e^{-t/RC} + V_s \tag{2.86}$$

The figure (2.13) shows the graph of $V_c(t)$ with respect to time.

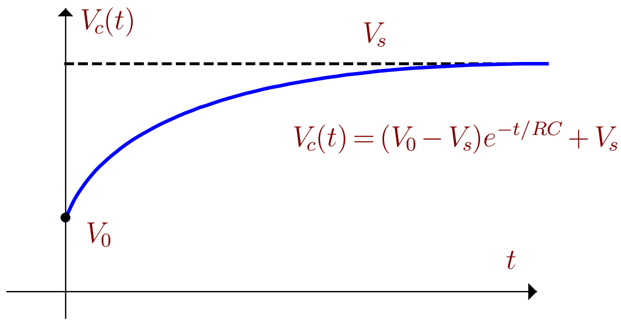


Figure 2.13.

Observe that

$$\lim_{t\to\infty} V_c(t) = V_s, \tag{2.87}$$

which means that the electric current $i(t)$ in the circuit goes zero in long term and the voltage across C will be equal to the voltage supply V_s.

Example 2.31. Consider the RC circuit shown in the figure (2.14).

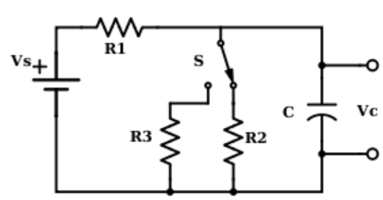


Figure 2.14.

Assume that the switch S connects to $R3$ at $t = t_0 > 0$. We would like to determine $V_c(t)$, the voltage across the capacitor C at any time $t > 0$ provided that $V_c(0) = 0$.

KIRCHHOFF's nodal law. This law states that the algebraic sum of electrical currents in a node is zero. In the figure (2.15), this law states that $i_1 + i_2 - i = 0$.

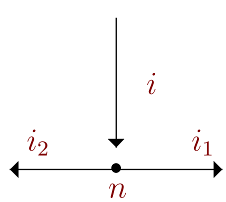


Figure 2.15.

In the figure (2.14), let us assume that i_1, i_2, i_3 are respectively the electric current in the resistances R_1, R_2 and R_3. The KIRCHHOFF's mesh law for the circuit is

$$R_1\, i_1(t) + V_c = V_s, \tag{2.88}$$

If i_c is the current passing through the capacitor C, then according to the KIRCHHOFF's nodal law, we have

$$i_1(t) = i_2(t) + i_c(t). \tag{2.89}$$

According to the voltage-current relation in a capacitor $i_c(t) = C\,\frac{dV_c}{dt}(t)$, we can write

$$R_1\left(i_2 + C\,\frac{dV_c}{dt}(t)\right) + V_c = V_s. \tag{2.90}$$

The KIRCHHOFF's mesh law for the resistor R_2 and the capacitor C is $R_2\, i_2(t) = V_c$, and substituting this into (2.90) yields the differential equation for $t < t_0$

$$\frac{\mathrm{d}V_c}{\mathrm{d}t} + \frac{R_{12}}{C} V_c = \frac{1}{R_1 C} V_s, \tag{2.91}$$

where $R_{12} = \frac{R_1 + R_2}{R_1 R_2}$. By $V_c(0) = 0$, the solution to the this equation is

$$V_c(t) = \frac{R_2}{R_1 + R_2} V_s(1 - e^{-R_{12}t/C}). \tag{2.92}$$

Since the switch connects to R_3 at $t = t_0$, the problem changes to the following for $t \geq t_0$

$$\begin{cases} \frac{\mathrm{d}V_c}{\mathrm{d}t} + \frac{R_{13}}{C} V_c = \frac{1}{R_1 C} V_s \\ V_c(t_0) = V_0 \end{cases}, \tag{2.93}$$

where $R_{13} = \frac{R_1 + R3}{R_1 R_3}$ and $V_0 = \frac{R_2}{R_1 + R_2} V_s(1 - e^{-R_{12}t_0/C})$. The solution for $t \geq t_0$ is

$$V_c(t) = \left(V_0 - \frac{R_3}{R_1 + R_3} V_s\right) e^{R_{13}t_0/C} e^{-R_{13}t/C} + \frac{R_3}{R_1 + R_3} V_s. \tag{2.94}$$

The figure (2.16) shows the output function $V_c(t)$ with respect to time. Observe that the V_c approaches to the limit $\frac{R_3}{R_1 + R_3}$ in long term. This means that the capacitor behaves like an open circuit in long term.

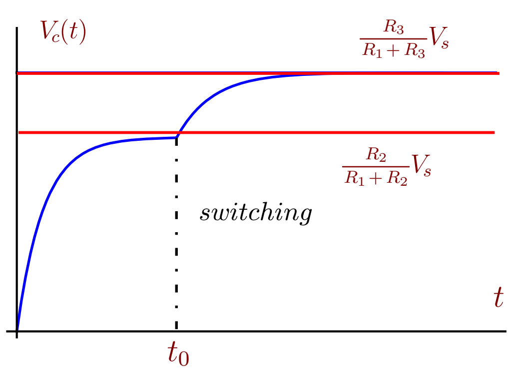


Figure 2.16.

Problem.

Problem 2.64. Find the maturity date T of a mortgage loan if $C_0 = 300,000\$$. Assume that the interest rate is $k = 0.035$ annually and that the borrower pays 1402$ monthly to pay off the mortgage at T.

Problem 2.65. Find the annual interest rate of a mortgage loan if the initial loan is $C_0 = 100,000\$$ and the borrower pays monthly $\bar{C} = 421\$$ to pay off the mortgage in 30 years. How much the borrower has to pay monthly if he/she wants to pay off the loan in 20 years?

Problem 2.66. Find the half life of a radio active substance with the decay rate $k = 2$.

Problem 2.67. Find the age date of a death body if the current C_{14} to C_{12} ratio is 10^{-12}.

Problem 2.68. By virtue of the Newton's law, the cooling rate of a body with temperature T in the air is proportional to $T - \bar{T}$ where $\bar{T}$ is the temperature of the air. If the initial temperature of the body is T_0 and if it drops to $(T_0 + \bar{T})/2$ in 1 hour, find the time when the temperature drops to $(T_0 + 3\bar{T})/4$.

Problem 2.69. Consider the RL circuit shown in the figure (2.17).

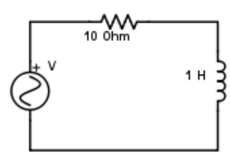


Figure 2.17.

Find the current function $i(t)$ in the inductor if $i(0) = 0$ and the voltage source V is as shown in the figure (2.18)

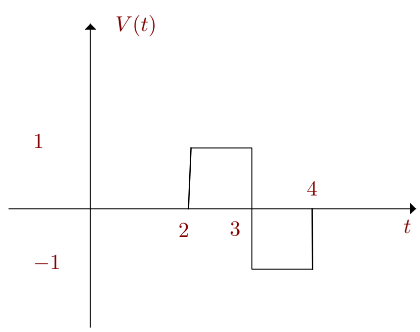


Figure 2.18.

Problem 2.70. Consider the circuit shown in the figure (2.19)

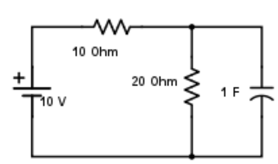


Figure 2.19.

Write down a differential equation describing V_c, the voltage across the capacitor C. Draw the solution $V_c(t)$ if $V_c(0) = 0$.

Problem 2.71. In the circuit shown in the figure (2.20), assume that the switch connects at $t=0$ and then disconnects from the voltage supply at $t=5$. Find the electric current function $i(t)$ if $i(0)=0$.

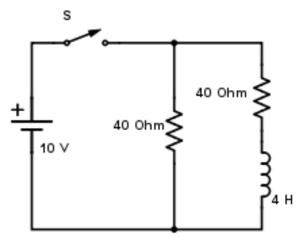


Figure 2.20.

Problem 2.72. Consider the circuit shown in figure (2.21). The switch connects the resistance 10 Ohm at $t=0$ and then changes to the other port at $t=5$. Find the electrical current function $i(t)$ in the inductor if $i(0)=0$.

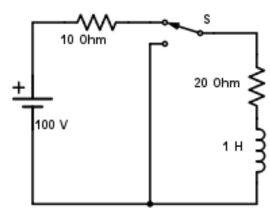


Figure 2.21.

2.6.2 Population dynamics

Logistic model.

In 1798, T. Malthus presented a mathematical formula of the form $P' = rP$ for the population growth and concluded that the population increases exponentially according to the solution $P(t) = P_0\, e^{rt}$. Regarding the linear growth of food production (according to the agricultural development), he led to a pessimistic view of the future of human kind in starvation. In 1831, J. P. Verhulst published a paper and showed that the model considered by Malthus is unrealistic. Based on some data collected from different sources, he considered the growth rate r as $r(P)=r_0\left(1-\frac{P}{K}\right)$ where r_0 is a constant and K is called the *carrying capacity* of the population. Finally, he suggested the following equation for the population dynamics

$$\begin{cases} P' = rP\left(1-\frac{P}{K}\right) \\ P(0)=P_0 \end{cases}. \tag{2.95}$$

Note that the equation is separable and is solved by the function

$$P(t) = \frac{P_0 K}{P_0 + (K-P_0)e^{-r_0 t}}. \tag{2.96}$$

VERHULST called the solution (2.96) a *logistic curve* and thus the equation (2.95) the logistic equation. Obviously, the equation has two equilibrium points $\bar{P}_1 = 0$ and $\bar{P}_2 = K$. Evidently for $f(P) = rP\big(1 - \frac{P}{K}\big)$, we have $f'(0) > 0$ and thus $\bar{P}_1$ is unstable. At $\bar{P}_2$, we have $f'(K) < 0$ and thus $\bar{P}_2$ is stable equilibrium. For the concavity analysis, we have

$$P'' = k^2(K - 2P)\left(1 - \frac{P}{K}\right)P, \tag{2.97}$$

and thus, $P_c = \frac{K}{2}$ is the inflection point of the solution curves $P(t; P_0)$. The typical solution curve is shown in the figure (2.22).

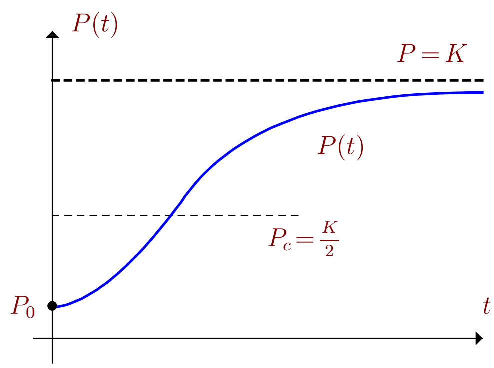


Figure 2.22.

Population growth with harvesting

There are different modifications of the logistic equation. Let us consider a harvesting term (constant or variable rate) as a source term in the logistic model (this is the case for example in a fish farm). If the harvesting rate is constant h_0 (daily, monthly or yearly), the harvesting logistic model reads

$$P' = kP\left(1 - \frac{P}{K}\right) + h_0. \tag{2.98}$$

Note that, this is a type of RICCATI equation if we rewrite it as follows

$$P' = -\frac{k}{K}\left(P^2 - \frac{1}{K}P - \frac{h_0 K}{k}\right). \tag{2.99}$$

We can also rewrite the equation as $P' = -\frac{k}{K}(P - r_1)(P - r_2)$ where $r_1 = \frac{1}{2K}\left(1 - \sqrt{1 + \frac{4h_0K^3}{k}}\right)$ and $r_2 = \frac{1}{2K}\left(1 + \sqrt{1 + \frac{4h_0K^3}{k}}\right)$. Therefore, the equation becomes a separable one if we rewrite it as

$$\frac{\mathrm{d}P}{(P - r_1)(P - r_2)} = -\frac{k}{K}\,\mathrm{d}t. \tag{2.100}$$

The solution is

$$\frac{P(t) - r_2}{P(t) - r_1} = \frac{P_0 - r_2}{P_0 - r_1}\, e^{k(r_2 - r_1)t/K}. \tag{2.101}$$

Time varying logistic model.

To let the growth rate factor (or decay) to be a function of time, we write the logistic equation in the following form (it is also called the BERNOULLI's equation)

$$P' - r(t)P = -k(t)\,P^2. \tag{2.102}$$

This equation is specially useful if the offspring rate of a living species varies with time. Of particular interest is when $r(t)$ or $k(t)$ is a periodic reflecting the periodic reproduction rate of the species. For example, let us write $r(t)$ as

$$r(t) = r_0(1 + \sin(\pi t/6)), \tag{2.103}$$

which is periodic with period 12 and consider $k(t) = k_0$. By the method we learned to solve a BERNOULLI's equation, we can rewrite the problem in the following linear one

$$\begin{cases} U' + r_0(1 + \sin(\pi t/6))\,U = k_0 \\ U(0) = P_0^{-1} \end{cases}, \tag{2.104}$$

where $U(t) = \frac{1}{P(t)}$. The equation (2.104) can not be integrated in terms of elementary functions. The figure (2.23) shows the numeric solution to the equation for some initial values P_0 and for $r_0 = 1$ and $k_0 = 0.1$. As it is observed from the figure, solution $P(t; P_0)$ converges to a periodic function with period 12. This solution is stable as it is observed from the slop field.

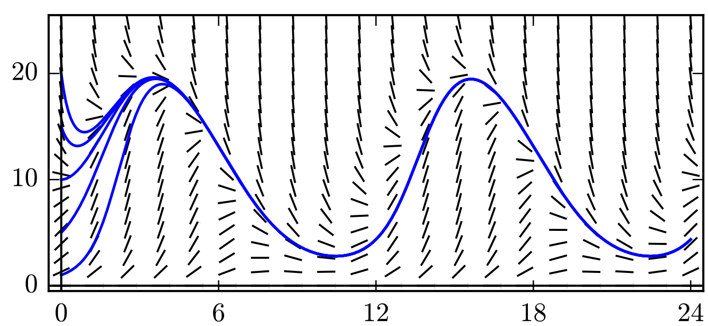


Figure 2.23.

BERTALANFFY's individual growth equation.

In his works on the individual growth model, Austrian biologist and general system theorist, L. VON BERTALANFFY suggested a mathematical equation describing the growth of the size of fish as

$$\frac{dL}{dt} = r\,(L_\infty - L), \tag{2.105}$$

where L_∞ is the ultimate size of the fish and r is a positive constant. The equation is solved for the length function $L(t)$ as

$$L(t) = L_\infty - (L_\infty - L_0)e^{-rt}, \tag{2.106}$$

where L_0 is the length at the start time $t = 0$. Note that $L \xrightarrow{t\to\infty} L_\infty$ and this means that a fish never stop increasing in length! Anyways, this equation presents a goal oriented behavior (the goal is L_∞) and BERTALANFFY used this notion for his theory of *general systems*.

Another model attributed to BERTALANFFY, is the weight equation of a living body. The change in the weight function $W(t)$ is positively proportional to anabolism (the building up and synthesis of complex molecules) and negatively to catabolism (the breaking down of complex molecules into simpler) which is written as

$$\frac{dW}{dt} = rW^{2/3} - kW, \tag{2.107}$$

where r (the anabolism rate) and k (the catabolism rate) are positive constants. It is simply seen that the equation has a non-trivial stable equilibrium $W_\infty = \left(\frac{r}{k}\right)^3$. It is seen as well that the weight function pursuits the ultimate value W_∞ and never reach this value in a finite time.

Problems.

Problem 2.73. Assume that the growth rate of a living species is $r = 0.002$. The harvesting rate k_0 is equal 5 for the first six months and then $k_0 = 0$ for the second six months. Assuming this pattern for all successive years, write down the differential equation for the population $P(t)$ and draw the solution if $P_0 = 1000$.

Problem 2.74. For the following models of population dynamics, do the stability analysis and draw some typical trajectories:

i.

$$P' = 2\left(1 - \frac{P}{3}\right)P.$$

ii.

$$P' = \left(1 - \frac{\sqrt{P}}{2}\right)P.$$

iii.

$$P' = 3(1 - P^{2/3})P.$$

Problem 2.75. Solve the following problem

$$P' = 2\left(1 - \frac{P}{3}\right)P + 1$$

Problem 2.76. For each of the following time varying logistic equations, use a computer software and draw the solution curves

i.

$$\begin{cases} P' = P(100 - 0.1(1 + \sin(\pi t))\,P) \\ P(0) = 10 \end{cases},$$

ii.

$$\begin{cases} P' = 3P(10 - 0.5(2 - \cos(t))\,P) \\ P(0) = 1 \end{cases}$$

iii.

$$\begin{cases} P' = P((3 + \sin(2\pi t)) - 0.1P) \\ P(0) = 10 \end{cases}$$

2.6.3 Water tank problems

Water level change in a tank.

Let us explain the method by solving an example.

Example 2.32. Consider a water tank in the shape of a cube $(\text{length}, \text{width}, \text{height}) = (L, W, H)$ and assume that a constant rate of water $Q_i m^3/s$ is running into the tank. Furthermore, assume that a small outlet is placed at the bottom of the tank that let water runs out; see the figure (2.24).

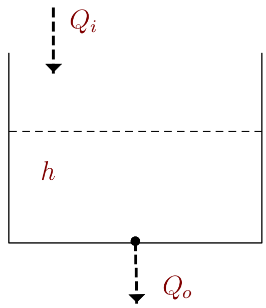


Figure 2.24.

If $V(t)$ denotes the volume of water at time t, then we can write

$$\frac{\mathrm{d}V}{\mathrm{d}t}(t) = Q_i - Q_o, \tag{2.108}$$

where Q_i, Q_o are the inflow and outflow rate of water at time t. According to the TORRICELLI's law, Q_o is expressed by the formula

$$Q_o = kA_o\sqrt{2gh}, \tag{2.109}$$

where k is a constant (depending on the liquid) and A_o is the area of the hole, we can rewrite Eq.(2.108) as

$$\frac{dh}{dt} = \frac{1}{A}(Q_i - \alpha\sqrt{h}), \tag{2.110}$$

where $\alpha = kA_o\sqrt{2g}$ and $A = LW$ is the cross section area of the tank. Note that the equilibrium level is $\bar{h} = \frac{1}{\alpha^2}Q_i^2$ and it is stable according to the relation

$$\frac{\mathrm{d}}{\mathrm{d}h}(Q_i - \alpha\sqrt{h}) = -\frac{\alpha}{2\sqrt{h}} < 0. \tag{2.111}$$

The equation (2.110) is separable and is solved by the formula

$$(Q_i - \alpha\sqrt{h}) - Q_i \ln|Q_i - \alpha\sqrt{h}| = \frac{\alpha^2}{2A}t. \tag{2.112}$$

Example 2.33. Consider the water tank in the shape of a cone shown in the figure (2.25).

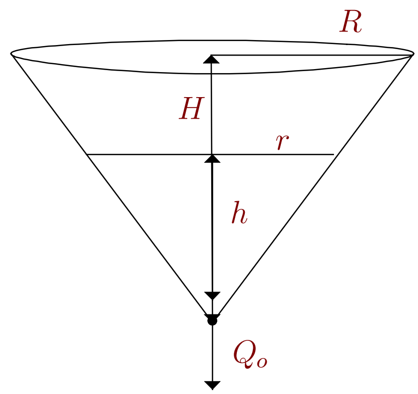


Figure 2.25. A cone water tank

Assume that $R = H = 1$ and the area of outlet is $A_o = 0.01m^2$. We would like to find time T when the tank become empty if the tank is initially full. To find T, we find first a differential equation describing $h(t)$. Note that

$$V(t) = \frac{1}{3}\pi r^2(t)\, h(t), \tag{2.113}$$

and since $r = \frac{R}{H}h = h$, we obtain $V(t) = \frac{\pi}{3}h^3(t)$. According to (2.108), we have

$$\pi h^2 \frac{\mathrm{d}h}{\mathrm{d}t} = -Q_o = -\alpha\sqrt{h} \cong -0.027\sqrt{h}. \tag{2.114}$$

Solving the above differential equation gives

$$\frac{2}{5}h^{5/2} = -\frac{0.027}{\pi}t + c, \tag{2.115}$$

and since $h(0) = 1$ we obtain

$$h^{5/2}(t) \cong -0.02t + 1. \tag{2.116}$$

It is seen that it takes about $T \cong 50\,\mathrm{sec}$that the tank become empty .

Chemical solutions.

The mixture of chemicals in fluids is another problem that bring differential equations into play. Let us explain this by solving an example.

Example 2.34. Consider the water tank shown in the figure (2.26). Assume that a constant rate of $10^{-3}\ m^3/s$ pure water runs into the container and that a hole of the area $A_o = 3\times 10^{-4}m^2$ is placed at the bottom. If $h_0 = 1m$ and the water is salty of the concentration %5 (5 gram salt in $1m^3$) at time $t = 0$, we would like to obtain the salt concentration when the water level is $h_1 = 1.3$.

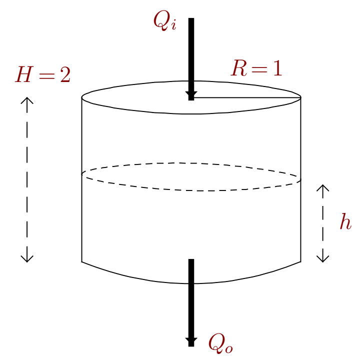


Figure 2.26.

Let $c(t)$ denote the amount of salt in the tank at time t. We derive a differential equation for $c(t)$ as follows. For small δt we can write

$$c(t+\delta t) \cong c(t) - \rho(t) Q_o \delta t, \tag{2.117}$$

where $\rho(t) = \frac{c(t)}{V(t)}$. This leads to the equation

$$\frac{\mathrm{d}c}{\mathrm{d}t} = -\rho Q_o. \tag{2.118}$$

If we replace Q_o from (2.109) and $V(t) = \pi R^2 h(t)$ into the above equation, we obtain

$$\frac{\mathrm{d}c}{\mathrm{d}t} = -\frac{\alpha c}{\pi R^2 \sqrt{h}}. \tag{2.119}$$

On the other hand, the equation for $h(t)$ is

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{\pi R^2}(Q_i - \alpha\sqrt{h}). \tag{2.120}$$

Note that the water level equilibrium is

$$\bar{h} = \frac{1}{\alpha^2} Q_i^2 \cong 1.57m. \tag{2.121}$$

From equations (2.119) and (2.120) we derive

$$\frac{\mathrm{d}c}{\mathrm{d}h} = \frac{-\alpha c}{\sqrt{h}(Q_i - \alpha\sqrt{h})}. \tag{2.122}$$

The equation (2.122) is solved by

$$c(h) = K\,(Q_i - \alpha\sqrt{h})^2, \tag{2.123}$$

where K is a constant determined by the initial condition. At time $t=0$, the water level is $h=1m$ and the salt is $c_0 = 5\pi\mathrm{gr}$. Substituting these amounts into the obtained solution determines $K \cong 3.8 \times 10^8$. Hence we obtain $c(h)$ as

$$c(h) \cong 380\,(1 - 0.797\sqrt{h})^2. \tag{2.124}$$

When $h = 1.3m$, we derive $c(1.3) \cong 3.17$ and thus $\rho = \frac{3.17}{1.3\pi} \cong \%0.8$.

Problems

Problem 2.77. Consider the water tank in the shape of cube shown in the figure (2.27) with the unit dimensions width, length and height.

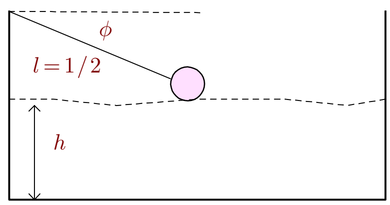


Figure 2.27.

Assume that the rate of inflow water into the tank is $Q_i = \sin(2\phi)$ and that $h(0) = 0$.

i. Find the time when $h = \frac{3}{4}m$.

ii. If there is a hole in the bottom of the tank with the area $A_o = 0.01$, find the final water level. Assume that $k = 0.5$.

Problem 2.78. For the water tank shown in the figure (2.27), let $L = W = 1m$ and $H = 2m$ and that $Q_i = k\sin(\phi)$. Assume that an outlet with the area $A_o = 0.1m^2$ is placed at the height $h = 1m$. For the simplicity, let us assume $\alpha\sqrt{2g} = 2.5$.

i. Find k such that the water level function has an equilibrium at $\bar{h} = 3/2m$

ii. With this value of k solve the equation and find $h(t)$ if tank is initially empty.

Problem 2.79. Consider a water tank in the shape of a cylinder with the height $H = 5m$ and the radius $R = 1m$ as shown in the figure (2.28)

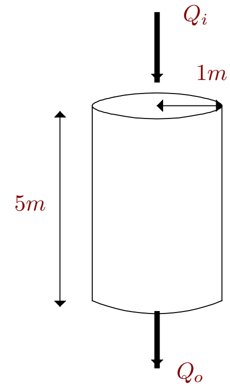


Figure 2.28.

Let a current of $Q_i = 0.5m^3/s$ water runs into the tank. Furthermore, assume that a hole of the area $A_o = 0.1m^2$ is placed at the bottom of the tank. According to the Torricelli's law, we know a flow of $Q_o = \alpha A_o\sqrt{h}\, m^3/s$ runs out of the tank. For the simplicity, take $\alpha = 2.65$.

i. Find the final water level in the tank.

ii. Find the time when the water level reaches half of the final level if the tank is initially empty.

Problem 2.80. A 10 liter container is filled with the %5 salty water (5gr salt per litter). Calculate the time when the salt concentration decrease to %1 if a constant rate 0.1 litter/s of pure water runs into the container and the same amount is simultaneously runs out the container.

Problem 2.81. In the above problem, assume that the inflow water is %1 salty. How long does it take that the total amount of salt reduces to the half of its initial value?

Problem 2.82. Consider a water tank in the shape of cube with sides $L = W = 1m$ and $H = 100m$ and assume the tank contains initially a volume of $10m^3$ salty water of %5 concentration. Let a %1 salty water with the constant rate $1m^3/h$ runs into the container. If a small hole of radius $r = 0.05m$ is placed at the bottom of the container, draw $c(t)$, the salt concentration of the water in the tank. Calculate the time when the concentration reduces to $c = \%3$. What is the concentration when the volume reaches $V = 15m^3$?

Problem 2.83. Repeat the problem for a cylinder tank with radius $R = 2m$ and $H = 8m$.

Problem 2.84. Assume that $V_0 m^3$ of a certain liquid is saturated with P_0 gr of a solid substance. This means that the saturation level of the liquid is $\bar{c} = \frac{P_0}{V_0}$. For $c(t)$, the concentration of the substance, the dissolution rate is proportional to $\bar{c} - c(t)$ with a positive proportionality factor $k > 0$. If you put P_0 gr of the substance into the $V_0\, m^3$ of the liquid, write down the differential equation describing $P(t)$, the amount of the substance dissolved into the liquid.

2.6.4 Geometric curves

Techniques of differential equations are employed to derive equations of curves with some required properties. Let us solve a few examples.

Example 2.35. Find a curves passing through the point $(0,1)$ with the following property: the projection of the tangent line segment on the x-axis has the fixed length k; see the figure (2.29).

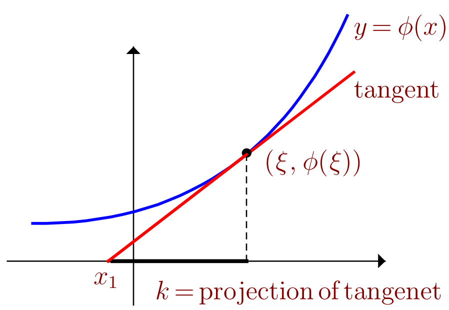


Figure 2.29.

Let $y = \phi(x)$ be the desired curve. The equation of tangent line at arbitrary point $(\xi, \phi(\xi))$ is

$$y - \phi(\xi) = \phi'(\xi)(x - \xi). \tag{2.125}$$

The x-intercept x_1 of the tangent line satisfies the equation

$$-\phi(\xi) = \phi'(\xi)(x_1 - \xi). \tag{2.126}$$

Since $|x_1 - \xi| = k$, we derive the differential equation $-\phi(\xi) = \pm k\phi'(\xi)$ for ϕ. This in turn gives $\phi(\xi) = ce^{\pm\xi/k}$. Since the curve passes through $(0,\ 1)$, the constant c is 1 and thus $\phi(x) = e^{\pm x/k}$.

Example 2.36. Find the equation of a curve passing through the origin that satisfies the following property: the area under the curve in the segment $[0, x]$ is equal to $\frac{1}{3}$ of the area of the rectangle constructed on the points $(0,0)$, $(x,0)$, (x,y) and $(0,y)$; see the figure (2.30)

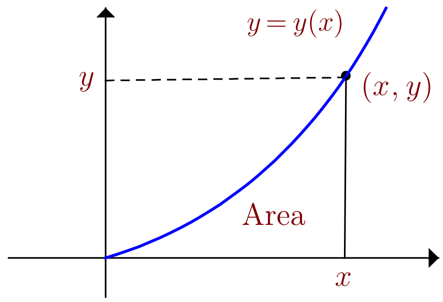


Figure 2.30.

For the desired curve $y=\phi(x)$, we have

$$\int_0^x \phi(s)\,ds = \frac{1}{3}\,x\,\phi(x). \tag{2.127}$$

Differentiating the above equation leads to the equation

$$3\phi(x) = \phi(x) + x\phi'(x), \tag{2.128}$$

and thus the solution is $\phi = cx^2$ for arbitrary constant c. Notice that the obtained solution is a family of parabola passing through the origin.

Differential equation of family of curves.

As we saw, the general solutions to a first order equation is a one parameter family of curves like $\psi(x,y)=c$. Conversely, if a one-parameter family of curves $\psi(x,y)=c$ is given, we can derive a first order differential equation having $\psi=c$ as its solution. The procedure is as follows. We eliminate the parameter c from the equations $\psi=c$ and its derivative, i.e.,

$$\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\,y' = 0. \tag{2.129}$$

Note that the later equation is just the implicit derivative of $\psi=c$. Let us show the method by solving following examples.

Example 2.37. Let us construct a first order differential equation for the family of ellipses

$$x^2 + c^2\,y^2 = 1.$$

For this, we eliminate the parameter c from the equation and its implicit derivative

$$x + c^2\,yy' = 0.$$

By this, we obtain the desired differential equation

$$(1-x^2)\,y' + x\,y = 0.$$

Example 2.38. Let us find a differential equation describing the family of circles in the first quadrant that are tangent to both x and y axis, see the figure (2.31)

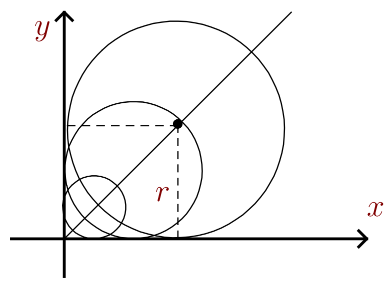


Figure 2.31.

Clearly, the family of curves has the following algebraic equation

$$(x-r)^2+(y-r)^2=r^2, \tag{2.130}$$

for a parameter r. Implicit derivative of the equation is

$$(x-r)+(y-r)y'=0, \tag{2.131}$$

and thus eliminating r from above equation gives the desired differential equation as

$$(x+yy')^2=(1+y'^2)(y-x)^2. \tag{2.132}$$

Orthogonal trajectories.

Assume $\phi(x, y) = c_1$ is a given family of curves. We would like to determine a family of curves $\psi(x, y)=c_2$ such that $\phi=c_1$ and $\psi=c_2$ are orthogonal at all their intersection points. Recall that the angel between two curves is defined by the angle between their tangent lines at the intersection point. If two curves $f(x, y)=0$ and $g(x, y)=0$ intersect at $p_0=(x_0, y_0)$, their slopes at p_0 are respectively $m_1=-\frac{\partial_x f(x_0, y_0)}{\partial_y f(x_0, y_0)}$ and $m_2=-\frac{\partial_x g(x_0, y_0)}{\partial_y g(x_0, y_0)}$, and thus the angle φ between these two curves is

$$\varphi=\tan^{-1}(m_1)-\tan^{-1}(m_2)=\tan^{-1}\left(\frac{m_1-m_2}{1+m_1m_2}\right). \tag{2.133}$$

Note that if $m_1m_2=-1$ then $\varphi=\frac{\pi}{2}$ which means f and g are orthogonal at the intersection point p_0.

The procedure for finding an orthogonal trajectories $\psi=c_2$ for a given family of curves $\phi=c_1$ is as follows.

i. Find the differential equation $y'=f(x, y)$ describing the family $\phi(x, y)=c_1$.

ii. Since ψ is orthogonal to ϕ, then the differential equation describing ψ must have the form $y'=-\frac{1}{f(x, y)}$.

iii. Solving the obtained differential equation determines the desired orthogonal trajectories $\psi(x, y)=c_2$.

Example 2.39. Let us find the orthogonal trajectories of the parabola $y = c_1 x^2$. The describing differential equation of the given parabola is $y' = \frac{2y}{x}$. The slope of the normal trajectories is $y' = -\frac{x}{2y}$. The solution to the latter equation is

$$y^2 + \frac{1}{2}x^2 = c_2^2. \tag{2.134}$$

Observe that the obtained curves are ellipses. The figure (2.32) shows two family of curves in a same coordinate.

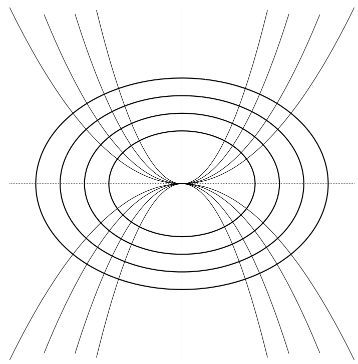


Figure 2.32.

Problems

Problem 2.85. Find the differential equation describing the following family of curves

i. $y = \sin(cx)$

ii. $y = (x - c)^2$

iii. $c_1 x^2 + c_2 y^2 = 1$. (Hint: you need second derivative in addition to the first derivative of the given family of curves)

Problem 2.86. Find the differential equation describing the family of circle contained in the sector $0 \le y \le x$ which are tangent to both lines $y = 0$ and $y = x$.

Problem 2.87. Obtain the differential equation describing the family of curves $y = cx^n$, n a positive integer. Conclude that the initial value problem $xy' - ny = 0$ with $y(0) \ne 0$ has no solution and with $y'(0) = 0$ has infinitely many solutions.

Problem 2.88. Find orthogonal trajectories of the family of curves $y = ce^x$.

Problem 2.89. Find the orthogonal trajectories of the ellipses $x^2 + c^2 y^2 = 1$.

Problem 2.90. Find orthogonal trajectories of the family of curves $(x - c)^2 + y^2 = c^2$.

Problem 2.91. Find the equation of curves that make angel $\phi = \pi/4$ with the curves $y = cx^4$.

Problem 2.92. Find the equation of a curve possessing the property: all normal lines to the curve pass through a common point.

Problem 2.93. Find a family of curves with the property that the y-axes bisects the tangent segment between the x-axes and the point (x, y) on the curves.