Chapter 1
Introduction

1.1 Motivational examples

To solve real-world problems, scientists formulate them in terms of mathematical expressions. Those expressions are usually in the form of either ordinary or partial differential equations (DEs). This book discusses the first type of equations, ordinary differential equations (ODEs).

1.1.1 Population growth

In 1798, T. MALTHUS presented a mathematical model for the growth of population. According to his model, the population increases exponentially that follows the following equation

$$\frac{dP}{dt} = rP.$$

In other words, the rate of change of the population function $P(t)$ at time t that is $\frac{dP}{dt}(t)$ is *proportional* to the present population $P(t)$, and the constant of the proportionality is a constant $r > 0$ that expresses the off-spring or reproduction rate. It is simply seen that $P(t)$ is exponential of the form $P(t) = P(0)\, e^{rt}$. On the other hand, food production (according to agricultural development) increases linearly, and thus, one leads to a pessimistic view of the future of humankind in starvation and misery. This was a shock at that time. In 1831, J. P. VERHULST published a paper and showed that the MALTHUS mathematical model is not accurate. He proposed a new model for $\frac{dP}{dt}$ as follows

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right).$$

$$(1.1)$$

where $K > 0$ is called *carrying capacity* and reflects the limitations of the population growth imposed by the environment. It is simply verified by the direct substitution that $P(t)$ is the following function if it follows the above differential equation

$$P(t) = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}},$$

where $P_0 = P(0)$ is the initial population at time $t = 0$.

1.1.2 A falling body

Consider a mass m falling from a height H. Assume that f_r, the air resistance against the body is proportional to its velocity $f_r = -kv$, where $k > 0$ is a constant (the negative sign enters because the resistance always acts against the direction of motion)

$$f = mg$$
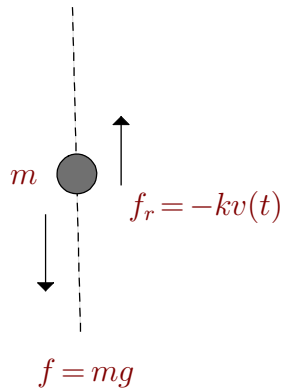
We would like to determine h, the height of the mass at any instance of time t. Applying the NEWTON'S second law $m\frac{dv}{dt} = f$ results to the following equation for $v(t)$, the velocity of the mass

$$m\frac{dv}{dt} = mg - kv. \tag{1.2}$$

The above equation is an *ordinary differential equation* that describes the rate of change of $v(t)$ in terms of t as well as $v(t)$ itself. If we solve the above equation and derive $v(t)$, then the $h(t)$, the height of the mass at any instance of time is simply determined by the following integral

$$h(t) = h_0 - \int_0^t v(s)\,ds,$$

where $h_0 = h(0)$ is the initial height of the mass. It is simply *verified* that the following function solves Eq.1.2 for arbitrary *parameter c*

$$v(t) = ce^{-\frac{kt}{m}} + \frac{mg}{k}.$$

In fact, by taking derivative, we have $\frac{dv}{dt} = -\frac{ck}{m}e^{-\frac{kt}{m}}$, and substituting this into the differential equation, transforms the equation to an *identity* in terms of t. The obtained solution that contains a parameter c is called the *general solution* of the equation. The parameter c is uniquely determined if an *initial condition* $v(0)$ is provided for the equation. For example, if $v(0) = 0$, then $c = -\frac{mg}{k}$. For this value of c, the solution becomes

$$v(t) = \frac{mg}{k}\left(1 - e^{-\frac{kt}{m}}\right).$$

This is called a *particular solution* to the equation. If the initial condition $v(0)$ is set to v_0, then

$$v(0) = c + \frac{mg}{k} = v_0,$$

and then the solution becomes

$$v(t) = \left(v_0 - \frac{mg}{k} \right) e^{-\frac{kt}{m}} + \frac{mg}{k}.$$

Problem 1.1. Solve the above differential equation by integration as follows

$$\int_{v_0}^{v} \frac{ds}{-\frac{k}{m}s + g} = \int_{0}^{t} ds,$$

where $v_0 = v(0)$ the initial velocity of the mass.

Problem 1.2. For high speed motions, the drag force acts as $f_r = -kv^2$ instead of linear term $-kv$. Assume a particle under the constant force f_0 is experiencing the drag force of form $-kv^2$. The differential equation in this case is as follows

$$m\frac{dv}{dt} = -kv^2 + f_0.$$

Rewrite the equation as

$$\frac{dv}{-\frac{k}{m}v^2 + \frac{f_0}{m}} = dt,$$

and integrate both sides of the equation to find the general solution $v(t)$. If $v(0) = 0$, find the particular solution.

1.1.3 The geometry of the equation

Consider again the equation (1.2) and rewrite it as

$$\frac{dv}{dt} = g - \frac{k}{m}v.$$

The physics of the equation states that the rate of change of the velocity of the falling body at any instance of time, $\frac{dv}{dt}(t)$, is equal to $g - \frac{k}{m}v(t)$. For example, at time $t = 0$, the change of the velocity is equal to

$$\frac{dv}{dt}(0) = g - \frac{k}{m}v(0),$$

and if $v(0) = v_0$ given, then $\frac{dv}{dt}(0) = g - \frac{k}{m}v_0$.

Remember that $\frac{dv}{dt}(t)$ represents geometrically the slope of tangent vector to the function $v = v(t)$ at time t. The following figure shows two *solution curves* of the equation for $v(0) = 10$ and $v(0) = 25$ for $m = 1, k = 0.5$. The tangent vectors are shown by arrows in the plane (t, v). The curves are tangent everywhere to the vectors according to the differential equation $v' = g - 0.5v$.
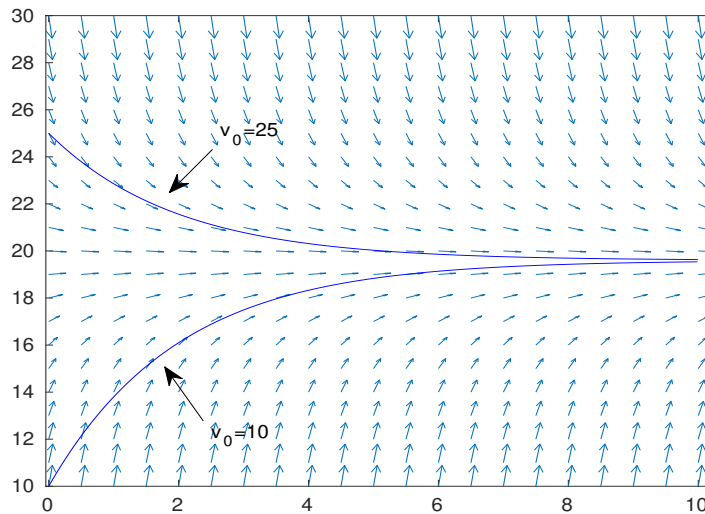
Observe that both solutions tends to the same value $2g$, that is,

$$\lim_{t\to\infty} v(t) = 20.$$

Problem 1.3. Use a computer software and draw the slope field of equation $\frac{dv}{dt} = -kv^2 + f_0$ for your choices of $k > 0$, and $f_0 > 0$. Find the *steady state solution* to the equation.

1.1.4 A variable mass

Consider a raindrop in the shape of a ball falling straight down, and suppose it absorbs water in the form of the stem from the ambient space. Let us obtain the equation of the motion of such a raindrop. Assume the raindrop is in the shape of a ball with the initial mass m_0 and that its absorption rate is proportional to its surface area. In this case, the mass changes with time and the NEWTON's law reads as follows

$$\frac{d}{dt}(mv) = -mg, \tag{1.3}$$

where $m = m(t)$ changes with time. According to the assumption on the shape of the rain drop, and the formula $S = 4\pi r^2$, one can write

$$\frac{dm}{dt} = kr^2,$$

for some $k > 0$. Note that the mass of a ball of volume V and density ρ is $m = \rho V = \frac{4\pi\rho}{3} r^3$, and therefore $r^2 = \left(\frac{3}{4\pi\rho} m\right)^{\frac{2}{3}}$, that by the assumption $\frac{dm}{dt} = kr^2$ gives

$$\frac{dm}{dt} = k \left(\frac{3}{4\pi\rho}\right)^{\frac{2}{3}} m^{\frac{2}{3}}.$$

Therefore, we obtain the following system of equations

$$
\begin{cases}
m\frac{dv}{dt} + v\frac{dm}{dt} = -mg \\
\frac{dm}{dt} = k\left(\frac{3}{4\pi\rho}\right)^{\frac{2}{3}} m^{\frac{2}{3}}
\end{cases}.
$$

However, the equation (1.3) can be integrated as

$$
\int_0^t d(mv) = -g \int_0^t m(s)\, ds,
$$

that yields

$$
m(t)\, v(t) - m(0)v(0) = -g \int_0^t m(s)\, ds.
$$

Here we need $m(t)$ to calculate the right hand side of the above equality. We can rewrite the equation for $m(t)$ as follows

$$
m^{-\frac{2}{3}} dm = k\left(\frac{3}{4\pi\rho}\right)^{\frac{2}{3}} dt,
$$

and therefore

$$
d\left(3m^{\frac{1}{3}}\right) = k\left(\frac{3}{4\pi\rho}\right)^{\frac{2}{3}} dt,
$$

that is solved by integration as

$$
3\left[m(t)^{\frac{1}{3}} - m(0)^{\frac{1}{3}}\right] = k\left(\frac{3}{4\pi\rho}\right)^{\frac{2}{3}} t,
$$

or

$$
m(t) = \left(k_1 t + m_0^{\frac{1}{3}}\right)^3,
$$

where $k_1 = \frac{k}{3}\left(\frac{3}{4\pi\rho}\right)^{\frac{2}{3}}$. If $v(0)$, the initial velocity of the rain drop is zero, we obtain

$$
mv = -g\int_0^t \left(k_1 s + m_0^{\frac{1}{3}}\right)^3 ds = -\frac{g}{4k_1}\left[\left(k_1 t + m_0^{\frac{1}{3}}\right)^4 - m_0^{\frac{4}{3}}\right],
$$

and finally

$$
v(t) = -\frac{g}{4k_1}\left[m - \frac{m_0^{\frac{4}{3}}}{m}\right].
$$

Problem 1.4. The second Newton law reads $m\frac{d^2x}{dt^2} = f$ only if m is constant. When m is variable, the formula needs some modification. First, we write the Newton's second law in its original form $\frac{dp}{dt} = f$, where p is the momentum of the system.

    i. Consider a system consisting two masses: the mass m that moves with velocity v and mass δm that moves with velocity u. The momentum of the system is

$$
p(t) = mv(t) + \delta m\, u(t). \tag{1.4}
$$

Suppose that at time $t + \delta t$, the mass δm attaches to mass m and the combined mass moves with velocity $v(t) + \delta v$. Find $\frac{dp}{dt}(t)$ by the following limit

$$
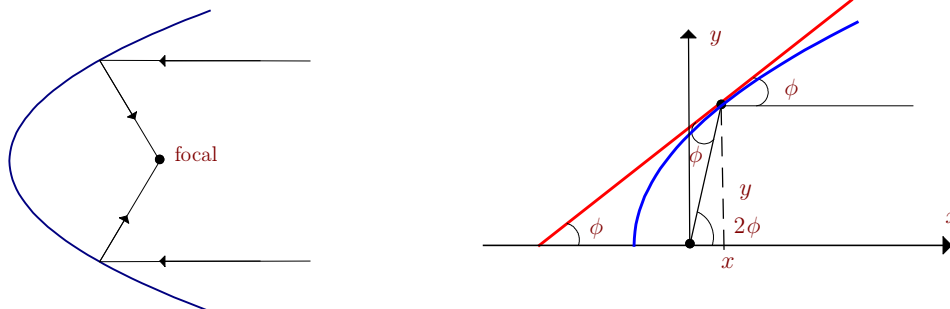\frac{dp}{dt}(t) = \lim_{\delta t \to 0} \frac{p(t + \delta t) - p(t)}{\delta t}.
$$

    ii. Conclude that the Newton's second law in this case read as follows

$$
\frac{d}{dt}(mv) - u\frac{dm}{dt} = f.
$$

iii. Use the above formula and derive $\frac{d}{dt}(mv) = -mg$ we used in the above example.

1.1.5 A geometrical problem

Example 1.1. We are going to make a mirror that collects light rays coming from a far source into a focal point.

To obtain the shape of the mirror, we have to make a differential equation as follows. Let $y(x)$ be the shape function of the mirror. By the above figure, we can write $\frac{dy}{dx} = \tan\phi$, and since $\tan(2\phi) = \frac{y}{x}$, we obtain by the trigonometric identity

$$\tan(2\phi) = \frac{2\tan\phi}{1 - \tan^2\phi} = \frac{2y'}{1 - y'^2},$$

and finally we reach the following equation for the shape of the mirror

$$(1 - y'^2)y = 2xy'.$$

The above equation is called the LAGRANGE equation. It is simply seen that function

$$x = \frac{y^2}{4c} - c,$$

solves the equation for any $c \neq 0$. Let us verify the claim. Taking derivative of the function with respect to x gives $1 = \frac{1}{2c}yy'$, and thus $y' = \frac{2c}{y}$. Substituting this into the differential equation results to $y^2 - 4c^2 = 4cx$, and substituting x from the solution we get an *identity*. Therefore, the obtained solution transforms the differential equation into an *identity*. Note that the shape of the mirror is a *parabola*.

Problem 1.5. Let us try to solve the above differential equation. If we denote y' by p, then y is represented by the parametric function $y = \frac{2xp}{1 - p^2}$. We need to find $x = x(p)$.

i. Find a differential equation for x in terms of p by calculating $\frac{dx}{dp}$. You should find it as follows

$$p\frac{dx}{dp} = \frac{-2}{1 - p^2}x.$$

ii. Solve the differential equation obtain in part (i) by integrating

$$\int \frac{dx}{x} = \int \frac{-2}{p(1 - p^2)}\,dp$$

and find $x(p)$

iii. Eliminate parameter p from system $x(p)$ and $y = \frac{2xp}{1-p^2}$ and derive the solution.

1.2 Preliminaries

1. **(Differential)** The *differential* of a single variable function $y = f(x)$ is defined by the relation $dy = f'(x)\,dx$, where dx is the differential of x and $f'(x)$ is the derivative function of f which is defined as follows

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

For example, if $y = e^{\alpha x}$, then $dy = \alpha e^{\alpha x} dx$. The *fundamental theorem of calculus* states a formula for the integration of differential, that is,

$$\int_{x_0}^{x} dy = \int_{x_0}^{x} f(x)\,dx = f(x) - f(x_0).$$

2. **(Total differential)** For a two-variable smooth functions $z = f(x, y)$, the differential dz is defined as

$$dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}dy,$$

where $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are *partial derivative functions* of f

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h},$$

$$\frac{\partial f}{\partial y}(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}.$$

For example, if $f(x, y) = x^2 + y^2 + \sin(xy)$ then

$$df = (2x + y\cos(xy))\,dx + (2y + x\cos(xy))dy.$$

The fundamental theorem of calculus or the integration over a path that connects points (x_0, y_0) and (x, y) is

$$\int_{(x_0, y_0)}^{(x, y)} dz = f(x, y) - f(x_0, y_0).$$

3. **(Classical differential equation)** In the period of the early development of Calculus by I. NEWTON in seventeen century, mathematicians studied equations of the following form

$$M(x, y)\,dx + N(x, y)\,dy = 0, \tag{1.5}$$

for given functions M, N. These types of equations called *differential equations* show themselves in the process of the mathematical modeling of real-world problems, in physics, biology, chemistry, engineering, and geometry. The goal is to find a *continuously differential* function $y = y(x)$ in an *open interval I* such that

$$M(x, y(x))\, dx + N(x, y(x))\, y'(x)\, dx = 0,$$

or equivalently

$$M(x, y(x)) + N(x, y(x))\, y'(x) = 0$$

for all $x \in I$. In several cases, the function $y(x)$ is obtained in the *implicit form* as
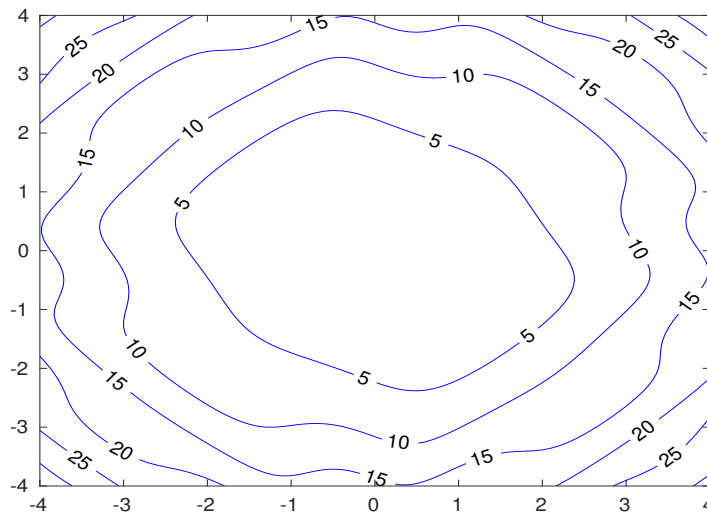
$$f(x, y) = \text{const.}$$

For example, the equation

$$df = (2x + y \cos(xy))\, dx + (2y + x \cos(xy))dy = 0,$$

has the implicit solution

$$f(x, y) = x^2 + y^2 + \sin(xy) = c.$$

4. **(Implicit function theorem)** Remember that implicit functions may represents a true *explicit functions $y = y(x)$* or not depending on the *implicit function theorem*.

Theorem 1.1. *Given an implicit function $f(x, y) = 0$, assume*

    i. there is a point (x_0, y_0) such that $f(x_0, y_0) = 0$,

    ii. the partial derivative $\frac{\partial f}{\partial y}$ is a continuous at (x_0, y_0) and $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$,

then there is an open interval $I = (x_0 - \delta, x_0 + \delta)$ and a function $y \colon I \to \mathbb{R}$ such that $f(x, y(x)) = 0$ for all $x \in I$.

For example, consider the function $f := x^2 + y^2 + \sin(xy) = 1$. At $(x_0, y) = (1, 0)$, we have

$$\frac{\partial f}{\partial y}(1, 0) = 2x + x\cos(xy)|_{(1,0)} = 3,$$

and thus there is an interval $I = (1 - \delta, 1 + \delta)$ and a continuous function $y : I \to \mathbb{R}$ such that

$$x^2 + y(x)^2 + \sin(xy(x)) = 1, \forall x \in I.$$

5. **(Relation with physics)** Consider a *force field* $F = \left(\begin{smallmatrix} f(x, y) \\ g(x, y) \end{smallmatrix} \right)$ and a mass m moving under the influence of this field. The *path* or *trajectory* of the mass in the (x, y)-plane is usually represented by a *parametric curve* $\gamma(t)$, where t denote the time variable. The total energy of the mass is expressed by the following expression

$$E = \frac{1}{2}m\,|v|^2 + V(x, y),$$

where V is the potential of the force field

$$f = -\frac{\partial V}{\partial x}, g = -\frac{\partial V}{\partial y},$$

and v is the velocity of the mass. It is simply seen that the total energy of the mass *along* its trajectory $\gamma(t)$ is constant. In fact, we have

$$\frac{dE}{dt} = mv \cdot v' + \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} = mv \cdot v' - f\frac{dx}{dt} - g\frac{dy}{dt} = mv \cdot v' - F \cdot v = v \cdot (mv' - F),$$

where the last term is the immediate result of the chain rule $\frac{d}{dt}V(x(t), y(t))$ and the observation that the velocity vector v is equal to $\left(\begin{smallmatrix} x'(t) \\ y'(t) \end{smallmatrix} \right)$. According to the second NEWTON's law $mv' = F$, we obtain $\frac{dE}{dt} = 0$. On the other hand, we have

$$\frac{dE}{dt} = \frac{\partial E}{\partial x}\frac{dx}{dt} + \frac{\partial E}{\partial x}\frac{dx}{dt},$$

and thus we conclude

$$dE := \frac{\partial E}{\partial x}dx + \frac{\partial E}{\partial x}dy = 0,$$

in the (x, y)-plane.

6. **(General form of first-order equations)** It turns out in a while that there are several real-world problems whose differential equation can not be expressed in terms of the above form studied by NEWTON and others. For example, equation $y = \frac{2xy'}{1 - y'^2}$ that we studied in the previous section is of this form. For this reason, mathematicians usually defines the general form of a first-order ordinary differential equation as follows

$$G(x, y, y') = 0, \tag{1.6}$$

where $y' := \frac{dy}{dx}$, and G is a functional relation between x, y and y'. If the functional relation can be solved algebraically for y' as the explicit form $y' = f(x, y)$, then it can be put in the differential form $dy - f(x, y)dx = 0$.

7. **(Initial value problem)** An initial value problem associated to a general first-order differential equation is as follows

$$\begin{cases} G(x, y, y') = 0 \\ y(x_0) = y_0 \end{cases},$$

where (x_0, y_0) is called the *initial condition* for the solution of the equation. For example, consider the following equation that states the relation between the rate of change of the population $P(t)$ of a living species and the population itself

$$\frac{dP}{dt} = r P\left(1 - \frac{P}{K}\right),$$

where r is the off-spring rate of the species, and K is the carrying capacity of the environment or the ambient space. It is simply *verified* that any function of the form

$$P(t) = \frac{cK}{c + e^{-rt}},$$

solves the equation for arbitrary *parameter* c. However, if we know the *initial population* of the species at time $t = 0$, say $P(0) = P_0$, then the obtained solution must satisfies this initial condition as well. If we put $P = P_0$ for $t = 0$, we reach

$$\frac{cK}{1 + c} = P_0,$$

or equivalently, $c = \frac{P_0}{K - P_0}$, and thus the *particular solution* is

$$P(t) = \frac{\frac{P_0 K}{K - P_0}}{\frac{P_0}{K - P_0} + e^{-rt}}.$$

Problem 1.6. What happen if $P_0 = K$? State the physical implication of this condition.

8. **(System of equations)** While the equation $y' = f(t, y)$ states the relation between the *rate of change* of a quantity y in terms of the quantity y and time t, the real-world problems involve several factors that cause the change in a quantity. For example, the change of the population of a certain living species, say prey, depends in addition to its off-spring rate, the population of other species, say predator. A simple mathematical model that express the dynamics is as follows

$$\begin{cases} \frac{dx}{dt} = r_1 x - \alpha x^2 - k_1 x y \\ \frac{dy}{dt} = -r_2 y + k_2 x y \end{cases},$$

where $r_1, r_2, \alpha, k_1, k_2$ are positive constants, $x(t)$ is the population of prey, say rabbits, and $y(t)$ is the population of predator, say foxes. Observe that $\frac{d}{dt} x(t)$ depends on $y(t)$ in addition to $x(t)$, and similarly $\frac{d}{dt} y(t)$. For this reason, one cal above equations, a *system* of differential equations.

Problems

Problem 1.7. Let $y(x) = x e^x + 1$. Find dy as a function of dx and determine dy if $dx = 3$.

Problem 1.8. Consider a disk of radius r. If the rate of change of r is as follows

$$\frac{dr}{dt} = R - r,$$

for some $R > 0$, find $\lim_{t \to \infty} A(t)$ where A is the area of the disk.

Problem 1.9. Consider the falling body problem discussed in this section and assume that the mass is unit, $m = 1$, and the air resistance is in the form $f_r = -kv^2$ (in the case when speed is too high, the air resistance applies in the square of speed).

    a) Write down the initial value problem for v assuming $v(0) = 0$.

    b) Solve the equation and show that the solution is

$$v(t) = \sqrt{\frac{g}{k}} \frac{e^{2\sqrt{kg}\,t} - 1}{e^{2\sqrt{kg}\,t} + 1}.$$

    c) Find the terminal velocity of the body if the mass is released from a very high position

Problem 1.10. Consider a water tank in the shape of a cube with dimensions (L, W, H)
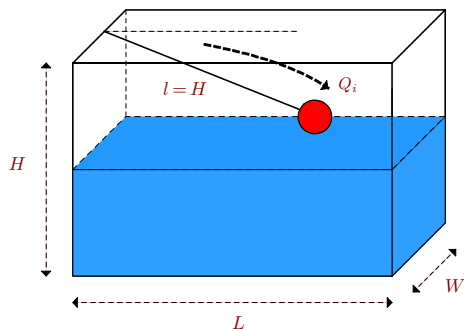
Let $h(t)$ denote the water level in the tank at time t. We would like to determine the water level $h(t)$ for $t > 0$ if a current of water run into the tank with the rate $Q(t) = k\,(H - h(t))\, m^3/s$, assuming $h(0) = 0$.

    a) If $V(t)$ denotes the volume of water in the tank at time t, then the *conservation principle* states

$$\frac{dV}{dt} = \text{input rate} - \text{output rate}.$$

    By the above relation, derive a the following differential equation for $h(t)$

$$\frac{dh}{dt}(t) = \frac{k\,(H - h(t))}{LW}.$$

    b) Solve the equation by integration and conclude that the water tank does not become full in any finite time.

Problem 1.11. Consider the water tank problem discussed in the previous problem. Assume that a hole is placed at the bottom of the tank that let water runs out with the rate $Q_o = \alpha \sqrt{h}$ where $\alpha > 0$ is a constant (We will see later on that this assumption makes sense due to the Torricelli law).

    a) Write down the differential equation describing the water level $h(t)$ in the tank.

    b) The terminal level or equilibrium is defined as the water level reaches to a certain level without any further change. Find the terminal level of the water in the tank and determine whether the tank become full or not.

Problem 1.12. In the water tank problem 1.10, assume $L = W = 2$, $H = 1$ and $Q_i = \sin^2 \phi$.

    i. Write the equation for the water level $h(t)$ and solve it to find $h(t)$ for $t > 0$ assuming $h(0) = 0$.

ii. Find the time when the tank become half full.

iii. Find the time when the tank become full.

Problem 1.13. Consider a water thank in the shape of a cylinder of radius $R = 1$ and height $H = 1$. There is a drain hole of area $A_o = 0.01m^2$ in the bottom of the tank that let water run out of he tank with the speed $Q_o = \sqrt{2g}\,A_o\,\sqrt{h}\,m^3/s$, where $h(t)$ is the level of the water in the tank and $g \cong 9.8$. If the tank is initially full, find the time when the water level drops to half of H.

Problem 1.14. Consider the electrical circuit shown in the following figure.
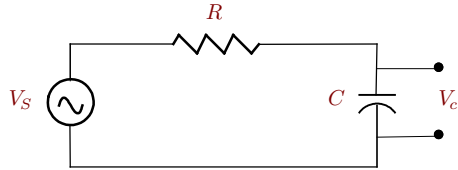
The main tool for analyzing electrical circuit is the KIRCHHOFF's laws. The mesh law for this circuit states $V_R + V_c = V_s$, where V_R, V_C are the voltage across the resistance R, and the capacitor C respectively, and V_s is the power supply of the circuit. The current-voltage relation across R obeys the law $V_R = Ri$, and the voltage across the capacitor is $i = C\frac{dV_c}{dt}$ where i is the electric current in the circuit.

a) Show that the differential equation for V_c is as follows

$$RC\frac{\mathrm{d}V_c}{\mathrm{d}t} + V_c = V_s.$$

b) Verify that the solution to the equation is as follows if V_s is constant and $V_c(0) = V_0$

$$V_c(t) = V_s(1 - e^{-t/RC}) + V_0 e^{-t/RC},$$

and find $\lim\limits_{t \to \infty} V_c(t)$.

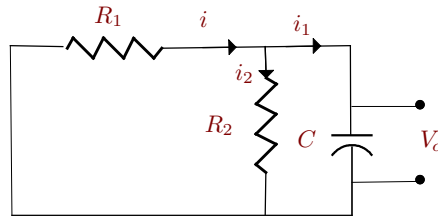**Problem 1.15.** Consider the electrical circuit shown in the figure (1.1).

Figure 1.1.

a) Show that the differential equation describing V_c in this case again is

$$\frac{dV_c}{dt} + \frac{1}{RC}V_c = 0,$$

where $R = \frac{R_1 R_2}{R_1 + R_2}$. (Hint: you need the KIRCHHOFF's law $i = i_1 + i_2$ in this case).

b) Verify that the steady state of V_c is zero regardless of the initial condition $V_c(0) = V_0$.

Problem 1.16. Consider a water tank that contains V_0 cubic meter of pure water at time $t = 0$. Suppose that a current of c_0 gr/lit salty water runs into the tank with speed v_0 lit/s. The content is kept thoroughly mixed, and simultaneously, the same amount of water runs out of the tank. We would like to determine the salt concentration in the tank at any instance of time $t > 0$.

a) Let $C(t)$ denote the total amount of salt in the tank at time t. The *conservation principle* states

$$\frac{dC}{dt} = \text{total salt going in} - \text{total salt going out.}$$

By the above relation, derive the following differential equation for $c(t)$

$$\frac{dc}{dt} = -\frac{v_0}{V_0}c + \frac{c_0\,v_0}{V_0}.$$

b) Show that the above equation is solved for the following function

$$c(t) = c_0(1 - e^{-v_0 t/V_0}).$$

Note that $c(0) = 0$.

Problem 1.17. Consider a water tank with initial V_0 lit of pure water. Assume that a current of c_0 gr/lit salty water runt into the tank with the speed v_0 lit/sec.

a) Show that the differential equation describing the salt concentration in the tank is

$$(v_0 t + V_0)\frac{dc}{dt} + v_0\,c = c_0\,v_0.$$

b) Find the solution to the equation.

Problem 1.18. A mass $m = 0.01\,$kg is released from the height h_0 with the initial velocity $v_0 = 0$. Assume that the air resistance against the mass is $f_r = -9.8 \times 10^{-6}\,v^2$.

   i. Write down the differential equation for the velocity function $v = v(t)$ of the mass (assume $g = 9.8\,m/s^2$).

   ii. Find the terminal velocity $\bar{v}$ (the velocity when $t \to \infty$)

   iii. Determine the time $t = T$ when $v(T) = 0.95\,\bar{v}$.

   iv. Determine h_0 such that the velocity of m is exactly equal to $0.95\bar{v}$ when it hits the ground.

Problem 1.19. We use the NEWTON's second law to derive a mathematical model of the motion of a mass m that is connected to a spring k. Consider the following figure
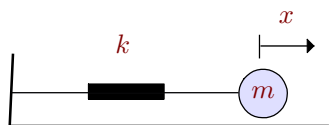
where x is the displacement of mass m with respect to its *resting position*. The NEWTON's second law states

$$m\frac{d^2x}{dt^2} = f,$$

where f is the force applying on the mass. The HOOK's law states that $f = -kx$, a linear relationship between the force and the contraction-stretching of the spring.

a) Derive the differential equation and verify that the obtained equation has two solutions $\phi_1 = \cos\!\left(\sqrt{\frac{k}{m}}\,t\right),\ \phi_2 = \sin\!\left(\sqrt{\frac{k}{m}}\,t\right)$.

b) Now assume that the ground is not frictions-less and it applies a force to m during its motion on the surface. The friction force is $f = -\alpha v$ for some constant α. Derive the equation in this case.

1.3 Vector field and the geometry of solution

1.3.1 Existence and uniqueness

Definition 1.1. *Consider the first-order equation*

$$y' = f(x, y).$$

A continuously differentiable function $y = y(x)$ is called a solution in an open interval I if

$$y'(x) = f(x, y(x)), \forall x \in I.$$

In addition, if an initial condition $y(x_0) = y_0$ is set to the equation, then $y(x)$ is called a solution to the associated initial value problem if there is an open interval $I = (t_0 - \delta, t_0 + \delta)$ if $y(x_0) = y_0$ and $y'(x) = f(x, y(x))$ for all $x \in I$.

It is not trivial that an initial value problem possess a solution and that solution is unique. The following example explains the point.

Example 1.2. Consider the following equation

$$\begin{cases} y' = \frac{y}{x} \\ y(0) = 1 \end{cases}.$$

It is seen that the equation does not possess any solution. In fact, the differential form of the equation is

$$\frac{dy}{y} = \frac{dx}{x},$$

and thus the *general solution* is $y(x) = cx$ for arbitrary constant c. Obviously, the obtained general solution does not satisfy the initial condition. Now, consider the following initial value problem

$$\begin{cases} y' = \frac{y}{x} \\ y(0) = 0 \end{cases}.$$

The equation has *infinitely* many solution $y(x) = cx$ for arbitrary c. The initial value problem

$$\begin{cases} y' = \frac{y}{x} \\ y(1) = 1 \end{cases},$$

has the *unique* solution $y(x) = x$.

The *existence and uniqueness* problem of ordinary differential equations are a deep question in the *theory of differential equations*. We introduce the topic only in an elementary level in this book.

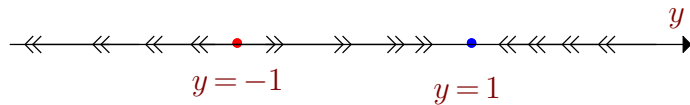### 1.3.2 Vector field of scalar equations

Consider the first-order equation

$$\frac{dy}{dx} = f(y),$$

for a smooth function $y = y(x)$. Equations that are not explicitly functions of time called *autonomous* equations. We can interpret $f(y)$ as a *vector field* over the y-axis. For example, consider the following equation

$$y' = 1 - y^2.$$

The following figure shows the vector field $f(y) = 1 - y^2$ over the y-axis

$$y = -1 \qquad\qquad y = 1$$

Two points $y = \pm 1$ where the vector field vanishes are called the *equilibrium* or *critical* points of the field. This filed enables us to determine the *behavior* of the solution *qualitatively*. If $y_0 > 1$ or $-1 < y_0 < 1$, then we expect

$$\lim_{x \to \infty} y(x) = 1.$$

In this case, the point $y = 1$ is called an *attractor* or asymptotically stable equilibrium for the equation. The point $y = -1$ is called repeller of the equation.

Definition 1.2. *An equilibrium $\bar{y}$ is called asymptotically stable if there is an interval $I = (\bar{y} - \varepsilon, \bar{y} + \varepsilon)$ for some $\varepsilon > 0$ such that the solution of the following initial value*

$$\begin{cases} y' = f(y) \\ y(0) = y_0 \in I \end{cases}, \tag{1.7}$$

approaches $\bar{y}$ in long term, that is,

$$\lim_{x \to \infty} y(x) = \bar{y}. \tag{1.8}$$

Theorem 1.2. *Let $\bar{y}$ be an equilibrium of the equation $y' = f(y)$ where f is a continuously differentiable function. The equilibrium is asymptotically stable if $f'(\bar{y}) < 0$, and it is unstable if $f'(\bar{y}) > 0$. The case $f'(\bar{y}) = 0$ is called* degenerate case *and $\bar{y}$ can be stable, unstable or none of them.*

See the following figure for a justification of the theorem. If $f'(\bar{y}) > 0$, there is an interval $(\bar{y} - \varepsilon, \bar{y} + \varepsilon)$ for which $y' > 0$ if $\bar{y} + \varepsilon > y > \bar{y}$ and $y' < 0$ if $\bar{y} - \varepsilon < y < \bar{y}$. Therefore, $y(x)$ is increasing in $\bar{y} + \varepsilon > y > \bar{y}$ and decreasing in $\bar{y} - \varepsilon < y < \bar{y}$. Therefore, $\bar{y}$ is unstable. A similar argument holds if $f'(\bar{y}) < 0$.
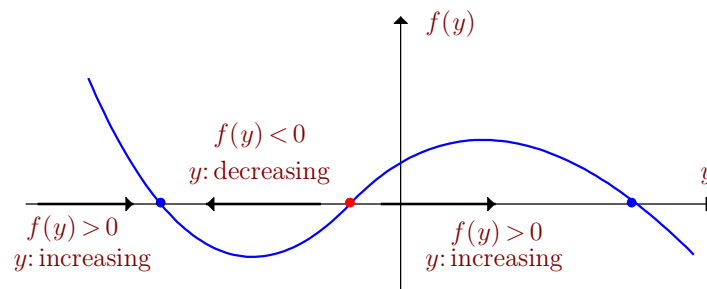

Figure 1.2.

Problem 1.20. The concavity of the solutions of an autonomous equation $y' = f(y)$ is determined by $\frac{d^2 y}{dt^2}$. Do the concavity analysis for the equation $y' = 1 - y^2$. Use the information and draw the graph of the solution of the equation for $y(0) = 0$.

Problem 1.21. By the aid of *slope* or *vector field* concept, we can calculate the solutions of differential equations *numerically*. Consider the following equation

$$\begin{cases} y' = 1 - y^2 \\ y(0) = 0 \end{cases}.$$

We want to find a numerical solution for $y(t)$ for $t \in [0, 1]$. Divide the interval into 10 segments. $x_0 = 0$, $x_1 = 0.1, ..., x_{10} = 1$.

    a) Use $y'(x_0)$ directly by the given equation.

    b) Use $y'(x_0)$ by the first order approximation

$$y(x_1) = y(x_0) + y'(x_0)(x_1 - x_0).$$

    c) Repeat the above argument and obtain $y(x_{10})$. Use MatLab or other software and draw the true solution of the equation and the obtained numerical solution.

1.3.3 Vector field of systems

Now consider the following system

$$\begin{cases} \frac{dx}{dt} = x - \alpha x^2 - 0.1 xy \\ \frac{dy}{dt} = -0.5 y + 0.1 xy \end{cases}. \tag{1.9}$$

The vector function $F = \begin{pmatrix} x - \alpha x^2 - 0.1 xy \\ -0.5 y + 0.1 xy \end{pmatrix}$ defines a vector field in the (x, y)-plane. This plane is called the *phase plane* of the system. The following figure shows the vector field for $\alpha = 0.1$.
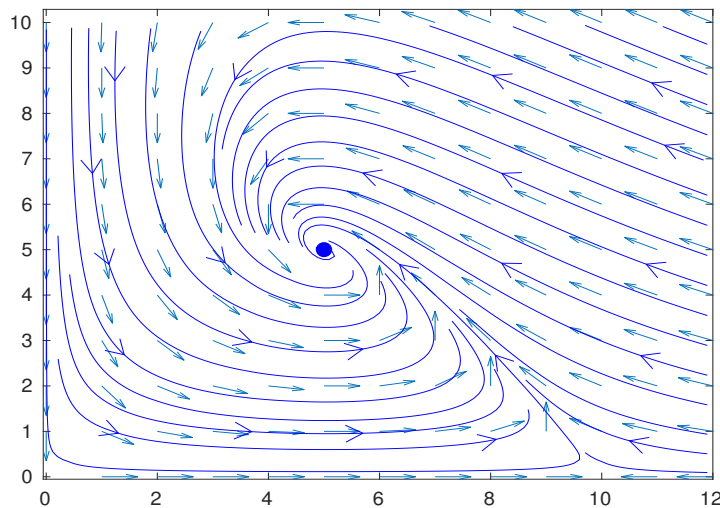
The equilibrium point is a point at which $\frac{dx}{dt} = 0, \frac{dy}{dt} = 0$. A simple calculation shows that there are two equilibrium for the system: $(0, 0), (5, 5)$. The solution curves approaches the second equilibrium in a spiral way. These curves are tangent at everywhere to the vector field F.

In general, the solution of a two dimensional *autonomous* system of the form

$$\begin{cases} \frac{dx}{dt} = f(x, y) \\ \frac{dy}{dt} = g(x, y) \end{cases},$$

is a *planar curve* $\gamma(t; p_0) \subset \mathbb{R}^2$ where $p_0 = (x_0, y_0)$ is the initial condition of the system $x(0) = x_0,\ y(0) = y_0$ such that $\gamma(0; p_0) = p_0$. In addition, the tangent vector $\gamma'(t; p_0)$ at any time t is equal to $\begin{pmatrix} f(\gamma(t)) \\ g(\gamma(t)) \end{pmatrix}$.

Problems

Problem 1.22. In the equation (1.9) , assume $\alpha = 0$. Use MatLab to draw the vector field and a few of trajectories of the system. If you are using MatLab, you can use the streamslice command for trajectories and quiver for the vector field. The code of the above figure is as follows

```
[x,y]=meshgrid(0:1:12,0:1:10);
dx=x-0.1*x.^2-0.1*x.*y;dy=-0.5*y+0.1*x.*y;
streamslice(x,y,dx,dy,0.5)
hold on
norm=sqrt(dx.^2+dy.^2);
quiver(x,y,dx./norm,dy./norm,0.5)
axis tight
```

Problem 1.23. Draw the vector field and a few of trajectories of the following system

$$\begin{cases} x' = y \\ y' = -\sin(x) \end{cases}$$

Problem 1.24. For the equation $y' = x - 2y$, determine regions in the plane (x, y) where the slope field is deceasing and increasing respectively. Draw few of the slopes in each region by hand.

Problem 1.25. For a particle that moves under the equation $y = x - 2y$, find an approximate path in the plane (x, y) passing through the initial point $(0, 0.5)$. Take $h = 0.5$ and find point $y(0.5)$, $y(1)$ and $y(1.5)$. Connect these point and draw the path by hand or use a computer software.

Problem 1.26. Consider a particle moving along the y-axis that its motion obeys the equation $y' = 1 - y^2$.

a) What is the speed of the particle when $y = 3$? (note that the velocity is vector and speed is the length of this vector).

b) Use the linear approximation formula and find $f(h)$ for $h = 0.5$ if $y(0) = 3$. Why the result is wrong? Now assume h smaller, say $h = 0.2$. Find $y(0.2)$, and then use the obtained value to estimate $y(0.4)$.

c) If $y_0 = -0.9$, find $y(0.2)$ and $y(0.4)$ by the successive linear approximation.

d) Find $y(0.2)$ and $y(0.4)$ if $y_0 = -1.2$.

Problem 1.27. Use a computer software to draw the slope field of the following differential equations and draw some solution curves. (You can use online applications or use the free software wxMaxima for it. See the appendix of this chapter for a quick help on wxMaxima)

i. $y' = \sin(y)$

ii. $y' = \sin(xy)$

iii. $y' = x - e^{-y}$
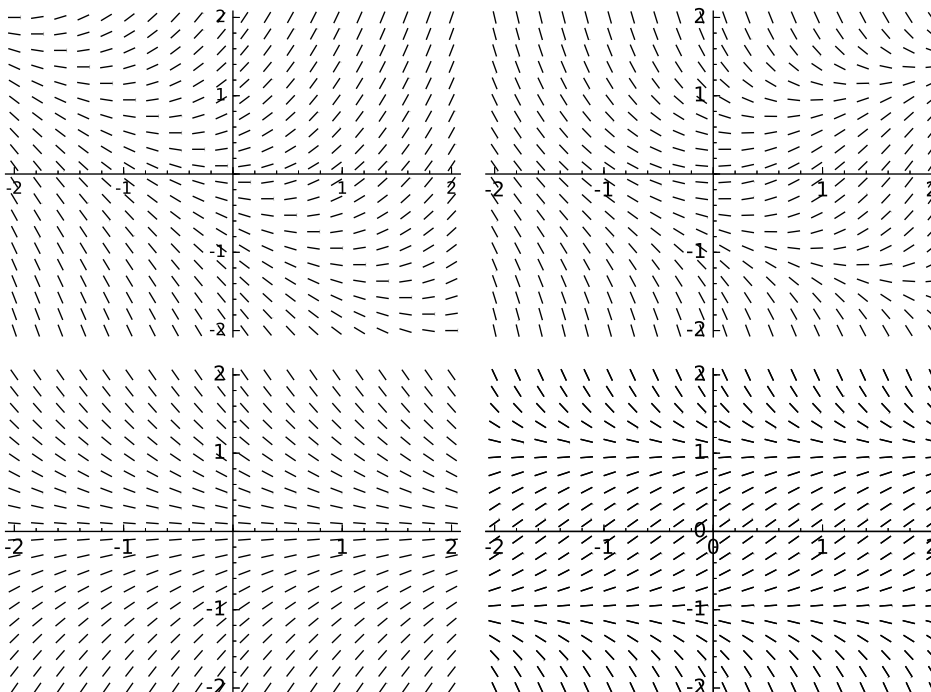
iv. $y' = e^{x-y}$

Problem 1.28. Match the following equations with the given slope fields in the figure (1.3)

   i. $y' = -y$

   ii. $y' = x + y$

   iii. $y' = x - y^2$

   iv. $y' = 1 - y^2$

Figure 1.3.

Problem 1.29. Consider the equation $y' = y - y^3$.

   i. Find equilibrium points of the equation.

   ii. Do stability analysis for obtained equilibrium points.

   iii. Draw the integral curve passing through $\left(0, \frac{1}{4}\right)$.

   iv. Verify that the function $\phi(x) = \frac{1}{\sqrt{1 + 15e^{-2x}}}$ is a solution to the equation satisfying the condition $y(0) = \frac{1}{4}$. Use a computer software to draw this function and compare it with the result of part (iii).

Problem 1.30. Consider the equation $y' = y^2 - y^3$.

   i. If $y(0) = \frac{1}{2}$ find $\lim\limits_{x \to \infty} y(t)$ and draw the solution,

   ii. If $y(0) = \frac{-1}{2}$ find $\lim\limits_{x \to \infty} y(t)$ and draw the solution,

   iii. If $y(0) = 2$ find $\lim\limits_{x \to \infty} y(t)$ and draw the solution.

Problem 1.31. For each of the following equations, find equilibrium points, and do stability analysis and draw the solution curve

   i. $y' = -y^2 + 3y - 2$, $y(0) = 3/2$

ii. $y' = \sin(1 - y)$, $y(0) = \pi/2$

iii. $y' = y(1 - y)(2 - y)$, $y(0) = 3$

iv. $y' = e^y y(1 - y)$, $y(0) = 1/2$

v. $y' = (1 - \sqrt{y}) y$, $y(0) = 1/2$

Problem 1.32. Consider the equation

$$y' + xy = x^2 + 1.$$

i. At what angel does the graph of the solution cut the y-axis?

ii. If the graph of the solution intersects t-axis, show that the intersection angle θ lies in the range $\left[\frac{\pi}{4}, \frac{\pi}{2}\right)$.

Problem 1.33. Consider the problem

$$\begin{cases} (1 + y^2)y' = 1 + x^2 \\ y(0) = -1 \end{cases}$$

i. Find the slope of the solution curve $y = \phi(x)$ at $x = 0$.

ii. Prove that the solution curve $y = \phi(x)$ crosses the x-axis.

iii. Prove that the solution curve $y = \phi(x)$ crosses the tx-axis with the angle greater than $\frac{\pi}{4}$.

Problem 1.34. Consider the initial value problem

$$\begin{cases} y' = \sin(x)\sin(y) + 1 \\ y(0) = -1 \end{cases}.$$

i. Show that the solution curve intersects the x-axis.

ii. Show that the solution curve intersects the x-axis only at one point.

iii. Find the angle of the intersection.

1.4 Higher order equations

1.4.1 The general form

The general form of a *second-order* differential equation for a function $y = y(x)$ is

$$F(x, y, y', y'') = 0, \tag{1.10}$$

where $y'' = \frac{d^2 y}{dx^2}$. An initial value problem for a second-order equation is of the following form

$$\begin{cases} F(x, y, y', y'') = 0 \\ y(x_0) = y_0 \\ y'(x_0) = y_1 \end{cases}.$$

A differential equation of order n is

$$F(x, y, y', ..., y^{(n)}) = 0, \tag{1.11}$$

where $y^{(n)} = \frac{d^n y}{dx^n}$. An initial value problem of order n is of the following form

$$\begin{cases} F(x, y, y', ..., y^{(n)}) = 0 \\ y(x_0) = y_0 \\ y'(x_0) = y_1 \\ \vdots \\ y^{(n-1)}(x_0) = y_{n-1} \end{cases}.$$

If function F is linear with respect to y and its derivatives, then the differential equation is called *linear*. The general form of a linear first order equation is

$$p(x)y' + q(x)y = r(x).$$

A second order linear equation has the following form

$$p(x)y'' + q(x)y' + z(x)\,y = r(x).$$

If $r(x)$ is identically zero, the equation is called *linear homogeneous* equation. A linear differential equation of order n is of the following general form

$$a_n(x)\,y^{(n)} + a_{n-1}\,y^{(n-1)} + \cdots + a_0(x)y = r(x).$$

1.4.2 Nonlinear equations

A differential equation that is not linear is called *nonlinear differential equation*. Nonlinear equation are extremely difficult to solve and sometime show *strange* or *chaotic* behavior. Consider the pendulum shown in the following figure
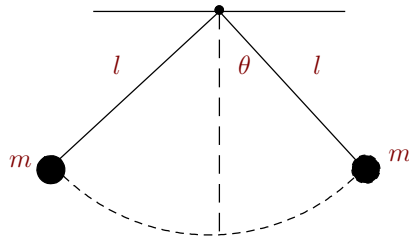
The motion of m follows the following equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin(\theta) = 0.$$

The equation is non-linear due to the non-linear term $\sin(\theta)$. But for θ small, $\sin(\theta) \approx \theta$, and physicists tend to rewrite the equation as the following linear one

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{l}\theta = 0.$$

The general solution of the above linear equation is

$$\theta(t) = c_1 \cos\left(\sqrt{\frac{g}{l}}\,t\right) + c_2 \sin\left(\sqrt{\frac{g}{l}}\,t\right),$$

while the solution to the original equation is complicated. Let us continue the example and apply an external force on the mass of the form $f = \sin(2t)$. The original equation with this external force reads

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin(\theta) = \sin(2t).$$

The following figure shows the solution of both equations in the (θ, θ')-plane (this is called the *phase plane*). The left figure is for the nonlinear one, while the right figure is for the linear one. As it is observed, the motion of the nonlinear forced pendulum is *chaotic*, while the solution of linear equation is very regular.
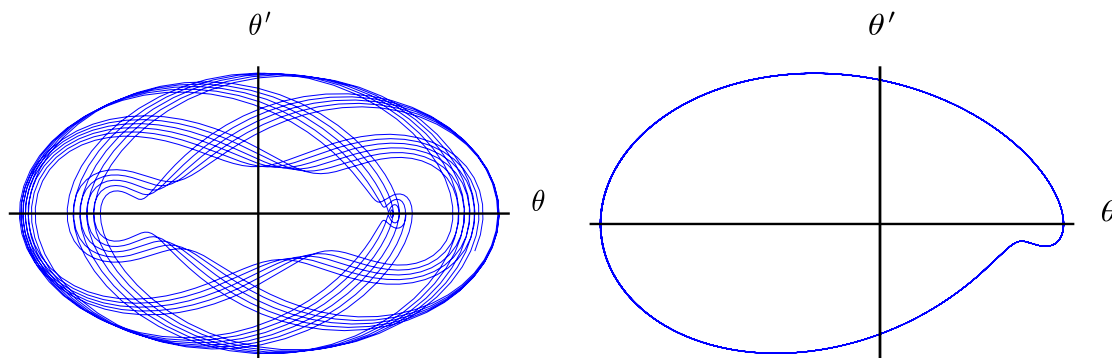
Problems

Problem 1.35. For each of the following equations, determine the order of equation, if it is linear or non-linear and also if it is autonomous or non-autonomous:

    i. $y' + xy = e^y$

    ii. $y'' + yy' = 0$

    iii. $y^{(4)} + y\,(y''')^5 = \sin(t)$

    iv. $y'' + \sin(y) = 0$

    v. $x^2 y'' + 2xy' + 2y = e^x$

Problem 1.36. Consider the linear equation $y' - \tan(x)y = 2x\sec(x)$.

    a) Verify that the function $y = x^2 \sec(x)$ is a solution to the equation.

    b) Now consider the initial condition $y(0) = 1$. Obviously, the solution $y = x^2 \sec(x)$ does not satisfies the initial condition. Verify that the function $y_h = c\sec(x)$ solves the equation $y' - \tan(x)\,y = 0$ for arbitrary constant c, and thus conclude that the function $y = c\sec(x) + x^2 \sec(x)$ satisfies also the original equation.

    c) Now, find a solution y that satisfies the equation and the initial condition $y(0) = 1$.

Problem 1.37. Consider the linear equation $y' + 2xy = x$. Obviously $y = \frac{1}{2}$ is a solution to the equation, however, the has other solutions too.

    a) Show that the function $y_h = ce^{x^2}$ solves the homogeneous equation $y' + 2xy = 0$ and conclude that the function $y = ce^{x^2} + \frac{1}{2}$ solves the original equation.

    b) Find and equation that satisfies $y(0) = 0$.

Problem 1.38. Consider the linear equation $y' + p(x)y = 0$.

a) Verify that if y is a solution to the linear equation then $y_h = cy(x)$ is also the solution for all constants c.

b) This property does not hold for non-linear equations in general. Consider the equation $y' + y^2 = 0$. Verify that $y = \frac{1}{x}$ is a solution to the equation. Is the function $y = \frac{c}{x}$ the solution to the equation for all c?

Problem 1.39. Consider the linear equation $y'' - 2y' + y = 0$.

a) Verify that functions $y_1 = e^x$, $y_2 = x e^x$ are solutions to the equation.

b) Verify that for any constant c_1, c_2, the function $y = c_1 y_1 + c_2 y_2$ is a solution to the equation as well.

Problem 1.40. Verify that the function $y = c_1 x + c_2 x^2$ is a solution to the equation

$$x^2 y'' - 2xy' + 2y = 0.$$

Find the solution to the equation satisfying initial conditions $y(1) = 1$, $y'(1) = 0$. Is it possible to find a solution to the equation satisfying the condition $y(0) = 1$, $y'(0) = 0$?

Problem 1.41. Verify that the function $y = c_1 \cosh(x) + c_2 \sinh(x)$ is a solution to the equation $y'' - y = 0$. Find the solution to the equation satisfying initial conditions $y(0) = y'(0) = 1$.

Problem 1.42. Consider the equation $y' = y - y^3$:

i. Verify that the function $y^2 - \frac{1}{1 + ce^{-2x}} = 0$ for arbitrary constant c is a solution to the equation.

ii. Find an explicit solution $y = \phi(x)$ for the equation satisfying the initial condition $y(0) = 1$. Find $\lim_{x \to \infty} y(x)$ and use a computer software to draw the solution in its domain of definition.

iii. Find an explicit solution $y = \phi(x)$ to the equation satisfying the initial condition $y(0) = -1$. Find $\lim_{x \to \infty} y(x)$ and draw the solution in its domain of definition.

iv. Find an explicit solution to the equation satisfying the initial condition $y(0) = 2$ and find the domain of the definition for the solution.

v. Find the solution to the equation satisfying the initial condition $y(0) = 1$.

Problem 1.43. Consider the equation $(1 - x^2)y' - 2xy = 0$.

a) Verify that the function $y = \frac{c}{1 - x^2}$ is a solution to the equation for arbitrary constant c.

b) Determine c if $y(0) = 1$ and obtain the maximum interval that this solution extends.

c) Repeat part b) if $y(2) = 1$.

Problem 1.44. Here we see a simple initial value problem without any possible solution. Consider the equation $(x + 1)y' = 1 + y$. Verify that the function $y = c(x + 1) - 1$ solves the equation for all constant c, and conclude that there is no solution to the initial value problem $(1 + x)y' = 1 + y$, $y(-1) = 1$.

Problem 1.45. Consider the equation $y' = \cos(y)$.

a) Verify that the implicit function $\sec(y) + \tan(y) = ce^x$ solves the equation for arbitrary constant c.

b) Determine c if $y(0) = 0$ and use the implicit function theorem to show that there is an explicit solution for the equation. Can you determine the possible interval the solution extend?

Problem 1.46. Find a solution to the problem

$$\begin{cases} x \cos(xy)\, y' + \cos(xy)\, y = e^x \\ y(1) = \frac{\pi}{2} \end{cases},$$

and determine the domain of definition for the solution. Draw the solution in its domain. (Hint: the left hand side is the derivative of the function $\sin(xy)$.)

Problem 1.47. Try to find a solution to the following equation

$$\begin{cases} (\cos(y) - y\sin(y))\, y' = 2x \\ y(0) = y_0 \end{cases}.$$

Draw integral curves for $y_0 = -2, -1, -0.5, 0.5$ in the interval $-3 < x < 3$, $-6 \le y \le 4$. Find the solution that passes through the point $(0,1)$.

Problem 1.48. Verify that the following functions solve the heat equation defined above.

$$u = e^{-D\omega t}\cos(\sqrt{\omega}\, x),\, u = e^{-D\omega t}\sin(\sqrt{\omega}\, x).$$

Problem 1.49. The partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

is called the *harmonic* equation for a function $u = u(x,y)$. Verify that the following functions are solution to the harmonic equation

$$u = e^x\cos(y),\, y = \cosh(x)\sin(y),\, y = x^2 - y^2.$$

Appendix A
Basic commands in wxMaxima

We review some basic commands in wxMaxima math software. The complete explanation can be found in the tutorial of WxMaxima.

To integrate a function $f(x)$

```
integrate(f,x);
```

For definite integral in the range $[a, b]$

```
integrate(f,x,a,b);
```

The following simple code plots the function f from $x = x_0$ to $x = x_1$:

```
plot2d(f,[x,x0,x1]);
```

Let us integrate the function $f(x) = x \sin(nx)$ in the range $x \in [0, \pi]$ for n nonzero integers

```
declare(n,integer);
integrate(x*sin(n*x),x,0,%pi]);
```

The summation `sum`:

```
sum(n,n,1,10);
```

adds numbers from 1 to 10. Similarly the command

```
sum(2*(1-cos(n*%pi))*sin(n*%pi*x)/(n*%pi),n,1,5)
```

adds the given function from 1 to 5.

For the plot of a function $f(x)$ in the range $[a, b]$

```
plot2d(f,[x,a,b]);
```

For example for $f = \sin(x)$ from $x = -\pi$ to π we can write

```
plot2d(sin(x),[x,%pi,%pi]);
```

You can save the output in a ps, pdf or png as follows

```
plot2d(f,[x,x0,x1],[ps_file,"filename.ps"]);
plot2d(f,[x,x0,x1],[pdf_file,"filename.pdf"]);
plot2d(f,[x,x0,x1],[png_file,"filename.png"]);
```

For multiple functions $f_1, f_2, \cdots$ we can draw them in a same coordinate

```
plot2d([f1,f2,...],[x,x0,x1]);
```

For example to draw $f_1 = e^{0.5x}$, $f_2 = \log(|x|)$ and $f_3 = 0.5e^x \cos(2x)$ in $-1 \le x \le 4$, write

```
plot2d([%e^(0.5*x),log(abs(x)),0.5*%e^x*cos(2*x)],[x,-1,4]);
```

If you want to hide the legend, just use the command

```
plot2d([%e^(0.5*x),log(abs(x)),0.5*%e^x*cos(2*x)],[x,-1,4],
                        [legend,false]);
```

To draw a parametric curve $(x(t), y(t))$, from $t = t_0$ to $t = t_1$ use

```
plot2d([parametric,x(t),y(t),[t,t0,t1]]);
```

For example, the circle $(\cos(t), \sin(t))$ for $t = 0$ to $t = 2\pi$ can be plotted by

```
plot2d([parametric,cos(t),sin(t),[t,0,2*%pi]]);
```

To make the axes ratio same, use the command `[yx_ratio,1]`. For example
```
plot2d([parametric,cos(t),sin(t),[t,0,2*%pi]],[yx_ratio,1]);
```
To draw a list `data` us the following command
```
plot2d([discrete, data]);
```
For example to plot the list $[[-3,9],[-2,4],[-1,1],[0,0],[1,1],[2,4],[3,9]]$ use
```
data:makellist([i,i^2],i,-3,3);
plot2d([discrete, data]);
```
To draw 3D plots
```
plot3d(f,[x,x0,x1],[y,y0,y1]);
```
For example
```
plot3d(%e^(x+y)*sin(x*y),[x,0,%pi],[y,0,%pi]);
```
To draw an implicit function $f(x,y)=0$, load first the package as
```
load(implicit_plot);
```
and then draw by
```
implicit_plot(f(x,y)=0,[x,x0,x1],[y,y0,y1]);
```
For example the following command draw a circle with radius 2:
```
implicit_plot(x^2+y^2=4,[x,-2,2],[y,-2,2]);
```
To draw the slope field (or vector field) of a differential equation, you should first load the package *plotdf* as
```
load(plotdf);
```
and then to draw the slope field. For example to draw the field of the equation $y'=\sin(y)$ just write
```
plotdf(sin(y),[x,x0,x1],[y,y0,y1]);
```
When you click on the plot page, a trajectory is appeared passing through the click point. In order to draw the vector field $[f(x,y),g(x,y)]$ just write
```
plotdf([f,g],[x,x0,x1],[y,y0,y1]);
```
For example the vector field $V=(-y,x)$ in $-1\le x\le 1$ and $-1\le y\le 1$ can be plotted as
```
plotdf([-y,x],[x,-1,1],[y,-1,1]);
```
To draw a trajectory starting at, say $(0.5,0.5)$ write
```
plotdf([-y,x],[x,-1,1],[y,-1,1],[trajectory_at,0.5,0.5]);
```
To solve an initial value problem numerically, use the Runge-Kutta method. The command
```
rk(f(x,y),y,y0,[x,x0,x1,xstep]);
```
solves the i.v.p. $y'=f(x,y)$, $y(x_0)=y_0$ for $x_0\le x\le x_1$ for steps xstep.
For example, the i.v.p. $y'=\cos(2x)+ye^{-y}$, $y(0)=1$ is solved numerically for $0\le x\le 5$ as
```
rk(cos(2*x)+y*%e^(-y),y,1,[x,0,5,0.1]);
```
To draw the integral curve of the above i.v.p. use the following command
```
data:rk(cos(2*x)+y*%e^(-y),y,1,[x,0,5,0.1]);
plot2d([discrete,data]);
```
For a system of equations $y'=f(x,y,z)$, $z'=g(x,y,z)$ do as follows.
```
data:rk([f,g],[y,z],[y0,z0],[x,x0,x1,xstep]);
plot2d([discrete,makelist([p[1],p[2]],p,data)]);
```
that above command plot $y(x)$. To draw $z(x)$ use
```
plot2d([discrete,makelist([p[1],p[3]],p,data)]);
```
and to draw the phase portrait $(y(x),z(x))$ use

```
plot2d([discrete,makelist([p[2],p[3]],p,data)]);
```

For constructing an approximate integral curve for a given vector field $V(-y, x)$, you can do as follows. Fist define the vector field

```
V(x,y):=(-y,x);
```

Then initialize the step size, say for $h = 0.02$

```
h:0.2;
```

and the initial point, say for $p_0 = (1, 0)$

```
p[0]:[1,0];
```

Define the recursive formula

```
p[n]:=p[n-1]+h*V(p[n-1][1],p[n-1][2]);
```

and then make a list, for example for 200 entries

```
data:makelist(p[n],n,1,200);
```

Now you can draw it as

```
plot2d([discrete,data]);
```

Let us solve the following i.v.p.

$$\begin{cases} y'' + \cos(x)\, y' + x\, y = e^x \\ y(0) = 1,\, y'(0) = -1 \end{cases}.$$

Step 1. First we transform the equation into a system of two first order equations as follows. If we take $y = y_1$ then the following system is equivalent to the equation:

$$\begin{cases} y_1' = y_2 \\ y_2' = e^x - x\, y_1 - \cos(x)\, y_2 \end{cases}.$$

Step 2. The initial conditions:

$$y_1(0) = 1,\, y_2(0) = -1$$

Step 3. Use the Runge-Kutta built in method in WxMaxima. The general form is

```
rk([system],[state variable],[initial conditions],[x,0,x1,step-size])
```

For our system and $x_1 = 2$ with the step size 0.1 it will be

```
rk([y2,-exp(x)-x*y1-cos(x)*y2],[y1,y2],[1,-1],[x,0,2,0.1])
```

Note that for smaller step size the solution is more accurate.

The output is a list of three entries as

$$[[0, y_1(0), y_2(0)], [0.1, y_1(0.1), y_2(0.1)], [0.2, y_1(0.2), y_2(0.2)], ...]$$

In order to plot the output list $[..., [x, y_1(x)], ...]$ we do as follows. First, let us assign a handle to the solution

```
sol: rk([y2,-exp(x)-x*y1-cos(x)*y2],[y1,y2],[1,-1],[x,0,2,0.1]);
```

Then make a list as follows

```
list:makelist([p[1],p[2]],p,sol);
```

Finally plot by the command plot2d in WxMaxima

```
plot2d([discrete,list]);
```

If you want plot the output an a function $f(x)$ is a same coordinate you can do as

```
plot2d([f(x),[discrete,list]],[x,0,2]);
```