FROM QUANTUM LOOP ALGEBRAS TO YANGIANS

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Abstract. The main purpose of this note is to give a proof of a statement of V. Drinfeld in [Dr1] regarding Yangians and quantum loop algebras, namely how the former can be constructed as limit forms of the latter. We also apply the same ideas to twisted quantum loop algebras to recover the (non-twisted) Yangians.

Introduction

Quantum loop algebras and Yangians are examples of affine quantum groups that representation theorists have all heard about since they have been well studied for over twenty-five years. It is also well-known that Yangians are, in some sense, limit forms of quantum loop algebras, but, as far as the authors know, a complete proof had never appeared until very recently [GaTL] [1], although a precise statement of this fact is formulated in [Dr1] and a proof is certainly known to some experts. The main purpose of this note is to give a detailed proof of how to realize Yangians as limits of quantum loop algebras since this may be useful to other mathematicians who may not be experts in the field (like the present authors). The precise statement is given in theorem 2.1 below. Some of the ideas in the proof were applied in [Gu1] to quantum toroidal algebras to obtain affine Yangians and deformed double current algebras. Those ideas were also used in [Gu2] to propose a generalization of deformed double current algebras to semisimple Lie algebras other than those of type A. We should also mention the articles [DHZ1, DHZ2] where a certain correspondence between quantum affine algebras and double Yangians is used to study representations of the latter. The authors believe that it is important that a proof of theorem 2.1 be published to fill a gap in the literature, hoping that it will give more visibility to theorem 2.1 whose statement is buried in a paragraph in [Dr1] instead of being stated in a more conspicuous way. Moreover, the proof is connected to the very important recent paper [GaTL] which provides, among other results, a strengthening of theorem 2.1. Indeed, the degeneration isomorphism π in the proof of theorem 2.1 is the inverse of the associated graded of the homomorphism Φ of geometric type in [GaTL]: see theorem 6.5 in *loc. cit.*

After proving theorem 2.1 for untwisted quantum loop algebras, we realized that the same ideas could be applied to twisted quantum loop algebras. These are examples of quantum Kac-Moody groups which have been studied in a few papers over the years by mathematicians (see for instance [CFS, ChMo, ChPr, He1, JiMi, Sz]) and by mathematical physicists [DGZ, GMW]. We show that the limit forms of these quantum groups that we consider yield the usual (non-twisted) Yangians associated to complex simple Lie algebras of type A, D, E (under one assumption).

We treat the untwisted and the twisted cases at the same time. The first section contains a definition of quantum loop algebras, of Yangians and of seemingly new algebras $Y(\mathfrak{g}, \sigma)$ (for σ a Dynkin diagram automorphism) which are proved to be isomorphic to $Y(\mathfrak{g})$. The second section starts with two versions of the main theorem and continues with its proof. In the last section, we explain briefly the relation between quantum toroidal algebras, affine Yangians and deformed double current algebras.

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[1] Most of this paper was written before [GaTL] was made public.

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1. Quantum loop algebras and Yangians

1.1. **Definitions.** In this subsection, after introducing the notation, we simply recall the definition of untwisted and twisted quantum loop algebras and of Yangians associated to a complex simple Lie algbra $\mathfrak{g}$. Let $C = (c_{ij})_{i,j\in I}$ be the Cartan matrix of $\mathfrak{g}$ and let $d_i, i \in I$, be coprime positive integers such that the matrix $(d_i c_{ij})_{i,j\in I}$ is symmetric.

Let σ be a Dynkin diagram automorphism, so $\sigma : I \to I$ be a bijection such that $c_{\sigma(i)\sigma(j)} = c_{ij}$ for all $i, j \in I$. Let m be the order of σ, so $m = 1, 2$ or 3. Denote also by σ the corresponding Lie algebra automorphism of $\mathfrak{g}$. The different possibilites for the pair $(\mathfrak{g}, \sigma)$ are given explicitly in [JiZh]: in particular, if $m \neq 1$, then $\mathfrak{g}$ is of type A, D or E.

Fix a primitive m^{th} root of unity $\omega \in \mathbb{C}^\times$. For $d \in \mathbb{Z}/m\mathbb{Z}$, let $\mathfrak{g}_d$ be the eigenspace of σ on $\mathfrak{g}$ with eigenvalue ω^d. Then, $\mathfrak{g} = \oplus_{d\in\mathbb{Z}/m\mathbb{Z}}\mathfrak{g}_d$ is a $\mathbb{Z}/m\mathbb{Z}$-gradation of $\mathfrak{g}$. The fixed point set $\mathfrak{g}_0$ of σ is a simple Lie algebra. The nodes of its Dynkin diagram are naturally indexed by I_σ, the set of σ-orbits in I. For $i \in I$, let $\bar{\imath}$ be the σ-orbit of i, so $\bar{\imath} \in I_\sigma$. Moreover, $\mathfrak{g}_d$ is an irreducible representation of $\mathfrak{g}_0$. The twisted affine Lie algebra $\widehat{\mathfrak{g}}^\sigma$ is the universal central extension (with one dimensional centre) of the twisted loop algebra

$$L(\mathfrak{g})^\sigma = \{f \in \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] | f(\omega t) = \sigma(f(t))\}.$$

$\widehat{\mathfrak{g}}^\sigma$ is a symmetrizable Kac-Moody algebra whose Dynkin diagram has nodes indexed by $\widehat{I}_\sigma = I_\sigma \coprod\{0\}$. Let $A^\sigma = (c^\sigma_{\bar{\imath},\bar{\jmath}})_{\bar{\imath},\bar{\jmath}\in\widehat{I}_\sigma}$ be the generalized Cartan matrix of $\widehat{\mathfrak{g}}^\sigma$, and let $\{d_{\bar{\imath}}\}_{\bar{\imath}\in\widehat{I}_\sigma}$ be the coprime positive integers such that the matrix $(d_{\bar{\imath}} c^\sigma_{\bar{\imath}\bar{\jmath}})$ is symmetric (except that, in the $A^{(2)}_{2n}$ case, one of the $d_{\bar{\imath}}$ equals $\frac{1}{2}$ - see [JiMi]). Let $(\cdot, \cdot)$ denote a symmetric, non-degenerate, bilinear form on the dual of a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ which is invariant under the Weyl group action and normalized so that $(\alpha, \alpha) = 2$ if α is a short root. Let $\{\alpha_i\}_{i\in I}$ be a set of simple roots of $\mathfrak{g}$ and let $\{\alpha_{\bar{\imath}}\}_{\bar{\imath}\in I_\sigma}$ be a set of simple roots of $\mathfrak{g}_0$. The entries c_{ij} of the Cartan matrix of $\mathfrak{g}$ are $c_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$ and we set $d_i = \frac{(\alpha_i,\alpha_i)}{2}$. For $i, j \in I$, we have $c_{\bar{\imath},\bar{\jmath}} = \frac{2\sum_{d=0}^{m-1} c_{i,\sigma^d(j)}}{\sum_{d=0}^{m-1} c_{i,\sigma^d(i)}} = \frac{2\sum_{d=0}^{m-1}(\alpha_i,\alpha_{\sigma^d(j)})}{\sum_{d=0}^{m-1}(\alpha_i,\alpha_{\sigma^d(i)})}$ and $d_{\bar{\imath}} = \frac{\sum_{d=0}^{m-1}(\alpha_i,\alpha_{\sigma^d(i)})}{2}$.

Instead of working with the twisted quantum loop algebras for a fixed numerical value of the parameter $q \in \mathbb{C}^\times$, we will need to work with its h-adic complete version defined over the ring $\mathbb{C}[[h]]$. Set $q = e^{\frac{h}{2m}}$. We need some more notation. For $i, j \in I$, we define $d_{ij} \in \mathbb{Q}, P^\pm_{ij}(u_1, u_2), F^\pm_{ij}(u_1, u_2), G^\pm_{ij}(u_1, u_2) \in \mathbb{C}[[h]][u_1, u_2]$ as follows:

- If $\sigma(i) = i$, then $d_{ij} = \frac{d_i}{2}, P^\pm_{ij}(u_1, u_2) = 1$;
- if $c_{i\sigma(i)} = 0$ and $\sigma(j) \neq j$, then $d_{ij} = \frac{1}{2m}, P^\pm_{ij}(u_1, u_2) = 1$;
- if $c_{i\sigma(i)} = 0$ and $\sigma(j) = j$, then $d_{ij} = \frac{1}{2}, P^\pm_{ij}(u_1, u_2) = \frac{u_1^m q^{\pm 2m} - u_2^m}{u_1 q^{\pm 2} - u_2}$;
- if $c_{i\sigma(i)} = -1$, then $d_{ij} = \frac{1}{4}, P^\pm_{ij}(u_1, u_2) = u_1 q^{\pm 1} + u_2$;
- $F^\pm_{ij}(u_1, u_2) = \prod_{d\in\mathbb{Z}/m\mathbb{Z}}(u_1 - \omega^d q^{\pm d_i c_{i,\sigma^d(j)}} u_2)$;
- $G^\pm_{ij}(u_1, u_2) = \prod_{d\in\mathbb{Z}/m\mathbb{Z}}(u_1 q^{\pm d_i c_{i,\sigma^d(j)}} - \omega^d u_2)$.

In order to present a definition of quantum loop algebras which encompasses both the untwisted and the twisted cases, we use the definitions given in the original paper [Dr2] in theorems 3 and 4 with $q = e^{\frac{h}{2m}}$ and trivial central element ($c = 0$ in the notation of *loc. cit.*).

Definition 1.1. *The quantum loop algebra $\mathfrak{U}_h(L(\mathfrak{g})^\sigma)$ (or just $\mathfrak{U}_h(L(\mathfrak{g}))$ if σ is trivial) is the associative complete $\mathbb{C}[[h]]$-algebra topologically generated by elements $\mathcal{X}^\pm_{i,r}, \mathcal{H}_{i,r}$ for $i \in I, r \in \mathbb{Z}$, which satisfy the* *following relations for* $i, i_1, i_2 \in I$ *and* $r, k_1, k_2 \in \mathbb{Z}$*:*

$$\mathcal{X}^{\pm}_{\sigma(i),r} = \omega^r \mathcal{X}^{\pm}_{i,r}; \quad \mathcal{H}_{\sigma(i),r} = \omega^r \mathcal{H}_{i,r}; \quad \mathcal{H}_{i_1,r_1}\mathcal{H}_{i_2,r_2} = \mathcal{H}_{i_2,r_2}\mathcal{H}_{i_1,r_1}; \tag{1}$$

$$[\mathcal{H}_{i_1,0}, \mathcal{X}^{\pm}_{i_2,r}] = \pm\frac{1}{m}\left(\sum_{d\in\mathbb{Z}/m\mathbb{Z}} d_{i_1} c_{i_1,\sigma^d(i_2)}\right)\mathcal{X}^{\pm}_{i_2,r}; \tag{2}$$

$$[\mathcal{H}_{i_1,r_1}, \mathcal{X}^{\pm}_{i_2,r_2}] = \pm\left(\sum_{d\in\mathbb{Z}/m\mathbb{Z}} \frac{q^{r_1 d_{i_1} c_{i_1,\sigma^d(i_2)}} - q^{-r_1 d_{i_1} c_{i_1,\sigma^d(i_2)}}}{r_1 h}\omega^{dr_1}\right)\mathcal{X}^{\pm}_{i_2,r_1+r_2} \text{ if } r_1 \neq 0; \tag{3}$$

$$[\mathcal{X}^{+}_{i_1,r_1}, \mathcal{X}^{-}_{i_2,r_2}] = \sum_{d\in\mathbb{Z}/m\mathbb{Z}} \delta_{\sigma^d(i_1),i_2}\omega^{dr_2}\left(\frac{\Psi^{+}_{i_1,r_1+r_2} - \Psi^{-}_{i_1,r_1+r_2}}{h}\right), \tag{4}$$

where the $\Psi^{\pm}_{i,r}$ *are defined by*

$$\sum_{r=0}^{\infty}\Psi^{\pm}_{i,\pm r}u^{-r} = \exp\left(\pm\frac{h}{2}\mathcal{H}_{i,0}\right)\exp\left(\pm h\sum_{l\geq 1}\mathcal{H}_{i,\pm l}u^{-l}\right)$$

and $\Psi^{\pm}_{i,r} = 0$ *if* $\mp r > 0$*;*

$$F^{\pm}_{i_1 i_2}(u_1,u_2)\mathcal{X}^{\pm}_{i_1}(u_1)\mathcal{X}^{\pm}_{i_2}(u_2) = G^{\pm}_{i_1 i_2}(u_1,u_2)\mathcal{X}^{\pm}_{i_2}(u_2)\mathcal{X}^{\pm}_{i_1}(u_1), \quad \mathcal{X}^{\pm}_{i}(u) = \sum_{r\in\mathbb{Z}}\mathcal{X}^{\pm}_{i,r}u^{-r}; \tag{5}$$

$$\mathcal{X}^{\pm}_{i_1}(u_1)\mathcal{X}^{\pm}_{i_2}(u_2) = \mathcal{X}^{\pm}_{i_2}(u_2)\mathcal{X}^{\pm}_{i_1}(u_1) \text{ if } c_{i_1,i_2} = 0 = c_{i_1,\sigma^{\pm 1}(i_2)};$$

$$\begin{aligned}\mathrm{Sym}\Big\{P^{\pm}_{ii_3}(u_1,u_2)\big(\mathcal{X}^{\pm}_{i_3}(u_3)\mathcal{X}^{\pm}_{i}(u_1)\mathcal{X}^{\pm}_{i}(u_2) - (q^{2md_{ii_3}} + q^{-2md_{ii_3}})\mathcal{X}^{\pm}_{i}(u_1)\mathcal{X}^{\pm}_{i_3}(u_3)\mathcal{X}^{\pm}_{i}(u_2)\\ +\mathcal{X}^{\pm}_{i}(u_1)\mathcal{X}^{\pm}_{i}(u_2)\mathcal{X}^{\pm}_{i_3}(u_3)\big)\Big\} = 0\end{aligned} \tag{6}$$

if $c_{ii_3} = -1$ *and* $\sigma(i) \neq i_3$*, where* Sym *denotes symmetrization over* u_1 *and* u_2*;*

$$\mathrm{Sym}\Big\{\big(q^{\frac{3}{2}}u_1^{\mp 1} - (q^{\frac{1}{2}} + q^{-\frac{1}{2}})u_2^{\mp 1} + q^{-\frac{3}{2}}u_3^{\mp 1}\big)\mathcal{X}^{\pm}_{i}(u_1)\mathcal{X}^{\pm}_{i}(u_2)\mathcal{X}^{\pm}_{i}(u_3)\Big\} = 0 \tag{7}$$

and

$$\mathrm{Sym}\Big\{\big(q^{-\frac{3}{2}}u_1^{\pm 1} - (q^{\frac{1}{2}} + q^{-\frac{1}{2}})u_2^{\pm 1} + q^{\frac{3}{2}}u_3^{\pm 1}\big)\mathcal{X}^{\pm}_{i}(u_1)\mathcal{X}^{\pm}_{i}(u_2)\mathcal{X}^{\pm}_{i}(u_3)\Big\} = 0 \tag{8}$$

if $c_{i\sigma(i)} = -1$*, where* Sym *denotes symmetrization over* u_1, u_2 *and* u_3*;*
Finally, if $m = 1$*, the quantum Serre relations are:*

$$\sum_{\pi\in S_N}\sum_{k=0}^{N}(-1)^k\begin{bmatrix}N\\k\end{bmatrix}_q \mathcal{X}^{\pm}_{i,r_{\pi(1)}}\cdots\mathcal{X}^{\pm}_{i,r_{\pi(k)}}\mathcal{X}^{\pm}_{j,s}\mathcal{X}^{\pm}_{i,r_{\pi(k+1)}}\cdots\mathcal{X}^{\pm}_{i,r_{\pi(N)}} = 0 \tag{9}$$

$\forall\, i,j \in I$, $i \neq j$ *and for all sequences of integers* $r_1,\ldots,r_N \in \mathbb{Z}$ *where* $N = 1 - c_{ij}$ *and* $\begin{bmatrix}N\\k\end{bmatrix}_q$ *is the usual* q*-binomial coefficient.*

Remark 1.1. *The untwisted quantum loop algebra corresponds to the case when* $m = 1$ *and* σ *is trivial.*

We will need the following result [En].

Proposition 1.1. $\mathfrak{U}_h(L(\mathfrak{g}))/h\mathfrak{U}_h(L(\mathfrak{g})) \cong \mathfrak{U}(L(\mathfrak{g}))$ *and* $\mathfrak{U}_h(L(\mathfrak{g}))$ *is isomorphic to* $\mathfrak{U}(L(\mathfrak{g}))[[h]]$ *as* $\mathbb{C}[[h]]$*-modules.*

Remark 1.2. *It is expected that proposition 1.1 holds in general for $\mathfrak{U}_h(L(\mathfrak{g})^\sigma)$ and this may even be known to be true to some experts, but the authors have not been able to locate any reference. Theorem 1.3 in* [En] *applies only to untwisted loop algebras; theorem 1.1 in loc. cit. is valid for symmetrizable quantum Kac-Moody algebras. It seems that the proof of the isomorphism in theorem 4.2 in* [JiZh] *uses the twisted analog of proposition 1.1, so it does not appear possible to combine both results to obtain an analog of theorem 1.3 in* [En] *for twisted quantum loop algebras.*

Definition 1.2. *The Yangian $Y(\mathfrak{g})$ is the algebra generated by $X^{\pm}_{i,r}, H_{i,r}, i \in I, r \in \mathbb{Z}_{\geq 0}$, which satisfy the following relations ($\forall\, i,j \in I$, $\forall\, r,s \in \mathbb{Z}_{\geq 0}$):*

$$[H_{i,r}, H_{j,s}] = 0, \quad [H_{i,0}, X^{\pm}_{j,s}] = \pm d_i c_{ij} X^{\pm}_{j,s}; \tag{10}$$

$$[H_{i,r+1}, X^{\pm}_{j,s}] - [H_{i,r}, X^{\pm}_{j,s+1}] = \pm \frac{d_i c_{ij}}{2}(H_{i,r}X^{\pm}_{j,s} + X^{\pm}_{j,s}H_{i,r}); \tag{11}$$

$$[X^{+}_{i,r}, X^{-}_{j,s}] = \delta_{ij} H_{i,r+s}; \tag{12}$$

$$[X^{\pm}_{i,r+1}, X^{\pm}_{j,s}] - [X^{\pm}_{i,r}, X^{\pm}_{j,s+1}] = \pm \frac{d_i c_{ij}}{2}(X^{\pm}_{i,r}X^{\pm}_{j,s} + X^{\pm}_{j,s}X^{\pm}_{i,r}); \tag{13}$$

$$\sum_{\pi \in S_N} \Big[X^{\pm}_{i,r_{\pi(1)}}, \big[\cdots, [X^{\pm}_{i,r_{\pi(N)}}, X^{\pm}_{j,s}]\cdots\big]\Big] = 0 \ \ \forall r_1, \ldots, r_N, s \geq 0 \ \textit{if } i,j \in I, i \neq j \tag{14}$$

where $N = 1 - c_{ij}$.

We now define an algebra $Y(\mathfrak{g},\sigma)$ which is seemingly different from the Yangian $Y(\mathfrak{g})$. The notation suggests that they are related to Yangians: actually, we will prove below in theorem 1.1 that $Y(\mathfrak{g},\sigma) \cong Y(\mathfrak{g})$.

As for twisted quantum loop algebras, we will need the following notation:

- If $\sigma(i) = i$, then $p_{ij}(v_1,v_2) = 1$;
- if $c_{i\sigma(i)} = 0$ and $\sigma(j) \neq j$, then $p_{ij}(v_1,v_2) = 1$;
- if $c_{i\sigma(i)} = 0$, $\sigma(j) = j$, then $p_{ij}(v_1,v_2) = v_1 + v_2 \in \mathbb{C}[v_1,v_2]/(v_1^2-1, v_2^2-1)$ if $m=2$ and $p_{ij}(v_1,v_2) = v_1^2 + v_1v_2 + v_2^2 \in \mathbb{C}[v_1,v_2]/(v_1^3-1, v_2^3-1)$ if $m=3$;
- if $c_{i\sigma(i)} = -1$, then $p_{ij}(v_1,v_2) = v_1 + v_2 \in \mathbb{C}[v_1,v_2]/(v_1^2-1, v_2^2-1)$;
- $F^{\pm}_{ij}(u_1,u_2,v_1,v_2) = u_1 - u_2 \mp \frac{1}{2}\sum_{d=0}^{m-1}\left(\sum_{e=0}^{m-1}\omega^{ed} d_i c_{i,\sigma^e(j)}\right) v_1^{m-d} v_2^d \in \mathbb{C}[u_1,u_2,v_1,v_2]/(v_1^m - 1, v_2^m - 1)$ and $G^{\pm}_{ij}(u_1,u_2,v_1,v_2) = -F^{\pm}_{ij}(u_2,u_1,v_1,v_2)$.

Definition 1.3. *Assume that σ is non-trivial. $Y(\mathfrak{g},\sigma)$ is the associative $\mathbb{C}$-algebra generated by elements $X^{\pm}_{i,r,k}, H_{i,r,k}$ with $i \in I, r \in \mathbb{Z}_{\geq 0}, 0 \leq k \leq m-1$, which satisfy the following relations:*

$$X^{\pm}_{\sigma(i),r,k} = \omega^k X^{\pm}_{i,r,k}, \quad H_{\sigma(i),r,k} = \omega^k H_{i,r,k}, \quad [H_{i,0,k_1}, X^{\pm}_{j,r,k_2}] = \pm\left(\sum_{d\in\mathbb{Z}/m\mathbb{Z}} \omega^{dk_1} d_i c_{i\sigma^d(j)}\right) X^{\pm}_{j,r,k_1+k_2}; \tag{15}$$

$$[H_{i_1,r_1,k_1}, H_{i_2,r_2,k_2}] = 0; \quad [X^{+}_{i_1,r_1,k_1}, X^{-}_{i_2,r_2,k_2}] = \sum_{d\in\mathbb{Z}/m\mathbb{Z}} \delta_{\sigma^d(i_1),i_2}\omega^{dk_2} H_{i_1,r_1+r_2,k_1+k_2}; \tag{16}$$

$$\partial_{u_1}\partial_{u_2} F^{\pm}_{i_1i_2}(u_1,u_2,v_1,v_2) X^{\pm}_{i_1}(u_1,v_1) X^{\pm}_{i_2}(u_2,v_2) = \partial_{u_1}\partial_{u_2} G^{\pm}_{i_1i_2}(u_1,u_2,v_1,v_2) X^{\pm}_{i_2}(u_2,v_2) X^{\pm}_{i_1}(u_1,v_1); \tag{17}$$

$$\partial_{u_1}\partial_{u_2} F^{\pm}_{i_1i_2}(u_1,u_2,v_1,v_2) H_{i_1}(u_1,v_1) X^{\pm}_{i_2}(u_2,v_2) = \partial_{u_1}\partial_{u_2} G^{\pm}_{i_1i_2}(u_1,u_2,v_1,v_2) X^{\pm}_{i_2}(u_2,v_2) H_{i_1}(u_1,v_1); \tag{18}$$

$$X^{\pm}_{i_1}(u_1,v_1) X^{\pm}_{i_2}(u_2,v_2) = X^{\pm}_{i_2}(u_2,v_2) X^{\pm}_{i_1}(u_1,v_1) \ \textit{if } c_{i_1,i_2} = 0 = c_{i_1,\sigma^{\pm 1}(i_2)}; \tag{19}$$

$$\begin{aligned} &\mathrm{Sym}\Big\{p_{ii_3}(v_1,v_2)\big(X^{\pm}_{i_3}(u_3,v_3)X^{\pm}_{i}(u_1,v_1)X^{\pm}_{i}(u_2,v_2) - 2X^{\pm}_{i}(u_1,v_1)X^{\pm}_{i_3}(u_3,v_3)X^{\pm}_{i}(u_2,v_2) \\ &\qquad + X^{\pm}_{i}(u_1,v_1)X^{\pm}_{i}(u_2,v_2)X^{\pm}_{i_3}(u_3,v_3)\big)\Big\} = 0 \end{aligned} \tag{20}$$

if $c_{ii_3} = -1$ *and* $\sigma(i) \neq i_3$*, where* Sym *denotes symmetrization over* (u_1, v_1) *and* (u_2, v_2)*;*

$$\mathrm{Sym}\Big\{\big(v_1 - 2v_2 + v_3\big)X_i^{\pm}(u_1, v_1)X_i^{\pm}(u_2, v_2)X_i^{\pm}(u_3, v_3)\Big\} = 0 \tag{21}$$

if $c_{i\sigma(i)} = -1$*, where* Sym *denotes symmetrization over* $(u_1, v_1), (u_2, v_2)$ *and* (u_3, v_3)*.*

Here we use the notation $X_i^{\pm}(u, v) = \sum_{r\in\mathbb{Z}_{\geq 0}} \sum_{k\in\mathbb{Z}/m\mathbb{Z}} X_{i,r,k}^{\pm} u^{-r-1}v^{-k} \in Y(\mathfrak{g}, \sigma)[[u^{-1}]]\otimes_{\mathbb{C}}\mathbb{C}[v, v^{-1}]/(v^m - 1)$*, which we view as an element of* $Y(\mathfrak{g}, \sigma)[[u^{-1}]] \otimes_{\mathbb{C}} \mathbb{C}[v, v^{-1}]/(v^m - 1)$*.*

Remark 1.3. *We could extend definition 1.3 to the case when* σ *is trivial by adding relation* (14) *to it. However, it will be more convenient for us to refer to a separate definition in the proof of theorem 1.1 below.*

Theorem 1.1. *The algebras* $Y(\mathfrak{g}, \sigma)$ *and* $Y(\mathfrak{g})$ *are isomorphic.*

We will use the following notation: if $a, b \in A$ for some ring A, then $S(a, b) = ab + ba$.

Proof of theorem 1.1. We would like to construct an algebra homomorphism $\varphi : Y(\mathfrak{g}, \sigma) \longrightarrow Y(\mathfrak{g})$ by setting

$$\varphi(X_{i,r,k}^{\pm}) = m^r \sum_{s=0}^{m-1} \omega^{-sk} X_{\sigma^s(i),r}^{\pm}, \quad \varphi(H_{i,r,k}) = m^r \sum_{s=0}^{m-1} \omega^{-sk} H_{\sigma^s(i),r}. \tag{22}$$

We will now perform a few computations to check that φ is an algebra homomorphism. This will be enough since φ is surjective on generators, hence an epimorphism, and it is also injective: $Y(\mathfrak{g}, \sigma)$ is a filtered algebra with $X_{i,r,k}^{\pm}, H_{i,r,k}$ of degree r and that the associated graded homomorphism $gr(\varphi)$ is one-to-one follows from the PBW theorem for $Y(\mathfrak{g})$ [Le] which states that $gr(Y(\mathfrak{g})) \cong \mathfrak{U}\mathfrak{g}(\mathbb{C}[u])$. Alternatively, one could also verify that $X_{i,r}^{\pm} \mapsto \frac{1}{m^{r+1}} \sum_{k=0}^{m-1} X_{i,r,k}^{\pm}$ and $H_{i,r} \mapsto \frac{1}{m^{r+1}} \sum_{k=0}^{m-1} H_{i,r,k}$ define a homomorphism $Y(\mathfrak{g}) \longrightarrow Y(\mathfrak{g}, \sigma)$ which is the inverse of φ. We are assuming that σ is non-trivial, so $d_i = 1$ for the rest of the proof.

1. Relation (15)*:* The first two relations can be obtained from the definition of φ easily, so we show the last one. Using (10), we have

$$\begin{aligned}
[\varphi(H_{i,0,k_1}), \varphi(X_{j,r,k_2}^{\pm})] &= m^r \sum_{s,t=0}^{m-1} \omega^{-sk_1-tk_2}[H_{\sigma^s(i),0}, X_{\sigma^t(j),r}^{\pm}] \\
&= m^r \sum_{s,t=0}^{m-1} \pm\omega^{-sk_1-tk_2} c_{\sigma^s(i),\sigma^t(j)} X_{\sigma^t(j),r}^{\pm} \\
(\text{set } d = t - s) \quad &= \pm\left(\sum_{d=0}^{m-1} \omega^{dk_1} c_{i,\sigma^d(j)}\right) \sum_{t=0}^{m-1} m^r \omega^{-t(k_1+k_2)} X_{\sigma^t(j),r}^{\pm} \\
&= \pm\left(\sum_{d=0}^{m-1} \omega^{dk_1} c_{i,\sigma^d(j)}\right) \varphi(X_{j,r,k}^{\pm}),
\end{aligned}$$

which is exactly the rightmost relation in (15).

2. Relation (16)*:* The first one is trivial. Let us prove the second one. Using (12) we have

$$\begin{aligned}
[\varphi(X^+_{i_1,r_1,k_1}),\varphi(X^-_{i_2,r_2,k_2})] &= m^{r_1+r_2}\sum_{s_1=0}^{m-1}\sum_{s_2=0}^{m-1}\omega^{-s_1k_1}\omega^{-s_2k_2}[X^+_{\sigma^{s_1}(i_1),r_1},X^-_{\sigma^{s_2}(i_2),r_2}]\\
&= m^{r_1+r_2}\sum_{s_1=0}^{m-1}\sum_{s_2=0}^{m-1}\omega^{-s_1k_1}\omega^{-s_2k_2}\delta_{\sigma^{s_1}(i_1),\sigma^{s_2}(i_2)}H_{\sigma^{s_1}(i_1),r_1+r_2}\\
&= m^{r_1+r_2}\sum_{s_1=0}^{m-1}\sum_{s_2=0}^{m-1}\omega^{-s_1(k_1+k_2)}\omega^{(s_1-s_2)k_2}\delta_{\sigma^{s_1-s_2}(i_1),i_2}H_{\sigma^{s_1}(i_1),r_1+r_2}\\
&= \sum_{d=0}^{m-1}\delta_{\sigma^d(i_1),i_2}\omega^{dk_2}\varphi(H_{i_1,r_1+r_2,k_1+k_2}).
\end{aligned}$$

3. Relations (17)*,* (18)*:* Using (13) and the definition of φ, we have

$$\begin{aligned}
&[\varphi(X^\pm_{i_1,r_1+1,k_1}),\varphi(X^\pm_{i_2,r_2,k_2})]\\
&= m^{r_1+r_2+1}\sum_{s_1,s_2=0}^{m-1}\Big([\omega^{-s_1k_1}X^\pm_{\sigma^{s_1}(i_1),r_1},\omega^{-s_2k_2}X^\pm_{\sigma^{s_2}(i_2),r_2+1}]\Big)\\
&\quad\pm\frac{m^{r_1+r_2+1}}{2}\sum_{s_1,s_2=0}^{m-1}c_{\sigma^{s_1}(i_1),\sigma^{s_2}(i_2)}S(\omega^{-s_1k_1}X^\pm_{\sigma^{s_1}(i_1),r_1},\omega^{-s_2k_2}X^\pm_{\sigma^{s_2}(i_2),r_2})\\
&= [\varphi(X^\pm_{i_1,r_1,k_1}),\varphi(X^\pm_{i_2,r_2+1,k_2})]\pm R_m,
\end{aligned}\tag{23}$$

where

$$\begin{aligned}
R_2 &= 2^{r_1+r_2-1}(c_{i_1,i_2}+c_{i_1,\sigma(i_2)})S(X^\pm_{i_1,r_1}+(-1)^{k_1}X^\pm_{\sigma(i_1),r_1},X^\pm_{i_2,r_2}+(-1)^{k_2}X^\pm_{\sigma(i_2),r_2})\\
&\quad+2^{r_1+r_2-1}(c_{i_1,i_2}-c_{i_1,\sigma(i_2)})S(X^\pm_{i_1,r_1}-(-1)^{k_1}X^\pm_{\sigma(i_1),r_1},X^\pm_{i_2,r_2}-(-1)^{k_2}X^\pm_{\sigma(i_2),r_2})\\
&= \pm\frac{1}{2}\Big((c_{i_1,i_2}+c_{i_1,\sigma(i_2)})S\big(\varphi(X^\pm_{i_1,r_1,k_1}),\varphi(X^\pm_{i_2,r_2,k_2})\big)\\
&\quad+(c_{i_1,i_2}-c_{i_1,\sigma(i_2)})S\big(\varphi(X^\pm_{i_1,r_1,k_1-1}),\varphi(X^\pm_{i_2,r_2,k_2-1})\big)\Big),
\end{aligned}$$

$$\begin{aligned}
R_3 &= \pm\frac{3^{r_1+r_2+1}}{2}\Big(c_{i_1,i_2}\big(S(X^\pm_{i_1,r_1},X^\pm_{i_2,r_2})+S(\omega^{-k_1}X^\pm_{\sigma(i_1),r_1},\omega^{-k_2}X^\pm_{\sigma(i_2),r_2})+\\
&\quad S(\omega^{-2k_1}X^\pm_{\sigma^2(i_1),r_1},\omega^{-2k_2}X^\pm_{\sigma^2(i_2),r_2})\big)+c_{i_1,\sigma(i_2)}\big(S(X^\pm_{i_1,r_1},\omega^{-k_2}X^\pm_{\sigma(i_2),r_2})\\
&\quad+S(\omega^{-2k_1}X^\pm_{\sigma^2(i_1),r_1},X^\pm_{i_2,r_2})+S(\omega^{-k_1}X^\pm_{\sigma(i_1),r_1},\omega^{-2k_2}X^\pm_{\sigma^2(i_2),r_2})\big)\\
&\quad+c_{i_1,\sigma^2(i_2)}\big(S(\omega^{-k_1}X^\pm_{\sigma(i_1),r_1},X^\pm_{i_2,r_2})+S(X^\pm_{i_1,r_1},\omega^{-2k_2}X^\pm_{\sigma^2(i_2),r_2})\\
&\quad+S(\omega^{-2k_1}X^\pm_{\sigma^2(i_1),r_1},\omega^{-k_2}X^\pm_{\sigma(i_2),r_2})\big)\Big)\\
&= \pm\frac{1}{2}\Big((c_{i_1,i_2}+c_{i_1,\sigma(i_2)}+c_{i_1,\sigma^2(i_2)})S\big(\varphi(X^\pm_{i_1,r_1,k_1}),\varphi(X^\pm_{i_2,r_2,k_2})\big)+\\
&\quad(c_{i_1,i_2}+\omega c_{i_1,\sigma(i_2)}+\omega^2c_{i_1,\sigma^2(i_2)})S\big(\varphi(X^\pm_{i_1,r_1,k_1+2}),\varphi(X^\pm_{i_2,r_2,k_2+1})\big)+\\
&\quad(c_{i_1,i_2}+\omega^2c_{i_1,\sigma(i_2)}+\omega c_{i_1,\sigma^2(i_2)})S\big(\varphi(X^\pm_{i_1,r_1,k_1+1}),\varphi(X^\pm_{i_2,r_2,k_2+2})\big)\Big).
\end{aligned}$$

This shows that relation (17) holds. Similarly, one can show that relation (18) holds.

4. Relations (19)*,* (20)*,* (21)*:* φ respects relation (19) because of relation (14) in the case $c_{ij}=0$. As for the other two, they can be proved case-by-case: the details are in Appendix A.

From steps 1-4, we can deduce that φ is a homomorphism of algebras, hence an isomorphism (as explained before). □

2. Statement and proof of the main theorem

We have the following sequence of algebra homomorphisms:

$$\mathfrak{U}_h(L(\mathfrak{g})^\sigma) \twoheadrightarrow \mathfrak{U}_h(L(\mathfrak{g})^\sigma)/h\mathfrak{U}_h(L(\mathfrak{g})^\sigma) \longrightarrow \mathfrak{U}(L(\mathfrak{g})^\sigma) \overset{t\mapsto 1}{\twoheadrightarrow} \mathfrak{U}(\mathfrak{g}).$$

Let $\mathbf{K}$ be the kernel of this composite. Let $\widetilde{Y}(\mathfrak{g},\sigma)$ be the $\mathbb{C}[[h]]$-subalgebra of $\mathbb{C}((h)) \otimes_{\mathbb{C}[[h]]} \mathfrak{U}_h(L(\mathfrak{g})^\sigma)$ generated over $\mathbb{C}[[h]]$ by $\mathfrak{U}_h(L(\mathfrak{g})^\sigma)$ and $\frac{\mathbf{K}}{h}$. In the case when σ is trivial, the following theorem was asserted in [Dr1].

Theorem 2.1. *If proposition 1.1 holds true for $\mathfrak{U}_h(L(\mathfrak{g})^\sigma)$, then $\widetilde{Y}(\mathfrak{g},\sigma)/h\widetilde{Y}(\mathfrak{g},\sigma)$ is isomorphic to $Y(\mathfrak{g},\sigma)$.*

We rephrase this theorem in a slightly different form as suggested in [GaTL] (see theorem 6.4 when σ is trivial). We have a decreasing filtration on $\mathfrak{U}_h(L(\mathfrak{g})^\sigma)$ given by the powers of $\mathbf{K}$. The graded ring $gr_{\mathbf{K}}(\mathfrak{U}_h(L(\mathfrak{g})^\sigma))$ associated to this filtration is $\bigoplus_{n=0}^\infty \mathbf{K}^n/\mathbf{K}^{n+1}$. The algebras $Y(\mathfrak{g})$ and $Y(\mathfrak{g},\sigma)$ can be defined over the ring $\mathbb{C}[h]$ (and are then denoted $Y_h(\mathfrak{g})$ and $Y_h(\mathfrak{g},\sigma)$) if we add a factor of h in the right-hand side of relations (11) and (13) (respectively, if we set $F_{ij}^{\pm}(u_1,u_2,v_1,v_2) = u_1 - u_2 \mp \dfrac{h}{2}\sum_{d=0}^{m-1}\left(\sum_{e=0}^{m-1}\omega^{ed} d_i c_{i,\sigma^e(j)}\right) v_1^{m-d} v_2^d$). Whenever $c \in \mathbb{C}\setminus\{0\}$, we have isomorphisms between $Y_{h=c}(\mathfrak{g})$ and $Y_{h=1}(\mathfrak{g})$ and between $Y_{h=c}(\mathfrak{g},\sigma)$ and $Y_{h=1}(\mathfrak{g},\sigma)$. $Y_h(\mathfrak{g})$ and $Y_h(\mathfrak{g},\sigma)$ become graded algebras if we assign degree one to h. Theorem 1.1 is also true for $Y_h(\mathfrak{g})$ and $Y_h(\mathfrak{g},\sigma)$.

Theorem 2.2. *If proposition 1.1 holds true for $\mathfrak{U}_h(L(\mathfrak{g})^\sigma)$, then $Y_h(\mathfrak{g},\sigma)$ is isomorphic to $gr_{\mathbf{K}}(\mathfrak{U}_h(L(\mathfrak{g})^\sigma))$.*

Before proving this version of the main theorem below, let us state a lemma which will be necessary.

Lemma 2.1. *The following relation holds in $\mathfrak{U}_h(L(\mathfrak{g})^\sigma)$:*

$$F_{ij}^{\pm}(z,w)\Psi_i^+(z)\mathcal{X}_j^{\pm}(w) = G_{ij}^{\pm}(z,w)\mathcal{X}_j^{\pm}(w)\Psi_i^+(z), \tag{24}$$

where $\Psi_i^+(z) = \sum_{k\geq 0}\Psi_{i,k}^+ z^{-k}$ and $\mathcal{X}_j^{\pm}(w) = \sum_{l\in\mathbb{Z}}\mathcal{X}_{j,l}^{\pm} w^{-l}$.

Proof. The proof in general is similar to the one found in section 3.2 of [He2] when σ is trivial. For details, see Appendix B. □

Denote by $O(h^\ell)$ an arbitrary element in $h^\ell\mathfrak{U}_h(L(\mathfrak{g})^\sigma)$. For $r,k \geq 0$, define elements $\mathbf{X}_{i,r,k}^{\pm}, \mathbf{H}_{i,r,k}$ in $\mathfrak{U}_h(L(\mathfrak{g})^\sigma)$ by:

$$\mathbf{X}_{i,r,k}^{\pm} = \sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}\mathcal{X}_{i,k+ms}^{\pm}, \quad \mathbf{H}_{i,r,k} = \sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}\frac{\Psi_{i,k+ms}^{+} - \Psi_{i,k+ms}^{-}}{h}.$$

Using $\binom{r+1}{s} = \binom{r}{s} + \binom{r}{s-1}$, we obtain the inductive relations

$$\mathbf{X}_{i,r+1,k}^{\pm} = \mathbf{X}_{i,r,k+m}^{\pm} - \mathbf{X}_{i,r,k}^{\pm}, \qquad \mathbf{H}_{i,r+1,k} = \mathbf{H}_{i,r,k+m} - \mathbf{H}_{i,r,k}. \tag{25}$$

Lemma 2.2. *For any $i \in I, r,k \geq 0$, we have $\mathbf{X}_{i,r,k}^{\pm}, \mathbf{H}_{i,r,k} \in \mathbf{K}^r$.*

Proof. The proof is by induction on r. We can assume that $\mathbf{K}^{\widetilde{r}}$ contains $\mathbf{X}^{\pm}_{i,\widetilde{r},k}$ for $0 \leq \widetilde{r} \leq r$ and all $k \in \mathbb{Z}$. (This is true at least when $\widetilde{r} = 0, 1$.) We have

$$\begin{aligned}
[\mathcal{H}_{i,m} - \mathcal{H}_{i,0}, \mathbf{X}^{\pm}_{i,r,k}] &= \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} \left([\mathcal{H}_{i,m}, \mathcal{X}^{\pm}_{i,k+ms}] - [\mathcal{H}_{i,0}, \mathcal{X}^{\pm}_{i,k+ms}] \right) \\
&= \pm \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} \sum_{d \in \mathbb{Z}/m\mathbb{Z}} \left(\frac{q^{md_i c_{i,\sigma^d(i)}} - q^{-md_i c_{i,\sigma^d(i)}}}{mh} \mathcal{X}^{\pm}_{i,k+m(s+1)} - \frac{d_i c_{i,\sigma^d(i)}}{m} \mathcal{X}^{\pm}_{i,k+ms} \right) \\
&= \pm \sum_{s=0}^{r} \frac{(-1)^{r-s}}{m} \binom{r}{s} \left(\sum_{d \in \mathbb{Z}/m\mathbb{Z}} d_i c_{i,\sigma^d(i)} \left(\mathcal{X}^{\pm}_{i,k+m(s+1)} - \mathcal{X}^{\pm}_{i,k+ms} \right) + O(h^2) \mathcal{X}^{\pm}_{i,k+m(s+1)} \right) \\
&= \pm \sum_{s=0}^{r+1} \frac{(-1)^{(r+1)-s}}{m} \left(\binom{r}{s-1} + \binom{r}{s} \right) \sum_{d \in \mathbb{Z}/m\mathbb{Z}} d_i c_{i,\sigma^d(i)} \mathcal{X}^{\pm}_{i,k+ms} \pm O(h^2) \mathbf{X}^{\pm}_{i,r,k+m} \\
&= \pm \frac{1}{m} \sum_{d \in \mathbb{Z}/m\mathbb{Z}} d_i c_{i,\sigma^d(i)} \mathbf{X}^{\pm}_{i,r+1,k} \pm O(h^2) \mathbf{X}^{\pm}_{i,r,k+m}
\end{aligned}$$

Since $\mathcal{H}_{i,m} - \mathcal{H}_{i,0} \in \mathbf{K}$ and $\mathbf{X}^{\pm}_{i,r,t} \in \mathbf{K}^r$ by assumption for any $t \in \mathbb{Z}_{\geq 0}$ (so that $O(h^2)\mathbf{X}^{\pm}_{i,r,t} \in \mathbf{K}^{r+2} \subset \mathbf{K}^{r+1}$), we deduce that $\mathbf{X}^{\pm}_{i,r+1,k} \in \mathbf{K}^{r+1}$. By induction, this must be true for any $r \in \mathbb{Z}_{\geq 0}$.

From equation (4), we get

$$\begin{aligned}
[\mathbf{X}^{+}_{i,r,k}, \mathbf{X}^{-}_{i,0,0}] &= \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} [\mathcal{X}^{+}_{i,k+ms}, \mathcal{X}^{-}_{i,0}] \\
&= \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} \sum_{d \in \mathbb{Z}/m\mathbb{Z}} \delta_{\sigma^d(i),i} \frac{\Psi^{+}_{i,k+ms} - \Psi^{-}_{i,k+ms}}{h} = \sum_{d \in \mathbb{Z}/m\mathbb{Z}} \delta_{\sigma^d(i),i} \mathbf{H}_{i,r,k},
\end{aligned}$$

which implies $\mathbf{H}_{i,r,k} \in \mathbf{K}^r$ for any $r, k \geq 0$ since $\sum_{d \in \mathbb{Z}/m\mathbb{Z}} \delta_{\sigma^d(i),i} \neq 0$. □

Given an element $X \in \mathbf{K}^r$, we denote by $\overline{X}$ its image in the quotient $\mathbf{K}^r/\mathbf{K}^{r+1}$.

Lemma 2.3. $\overline{\mathbf{X}}^{\pm}_{i,r,k+m} = \overline{\mathbf{X}}^{\pm}_{i,r,k}$ *and* $\overline{\mathbf{H}}_{i,r,k+m} = \overline{\mathbf{H}}_{i,r,k}$.

Proof. This is a corollary of lemma 2.2. □

Proof of theorem 2.2. Let m^+, m^- be two scalars such that $m^+ m^- = m$. Define

$$\pi : Y(\mathfrak{g}, \sigma) \longrightarrow gr_{\mathbf{K}}(\mathfrak{U}_h(L(\mathfrak{g})^{\sigma})) \quad \text{by} \quad X^{\pm}_{i,r,k} \mapsto m^{\pm} m^r \overline{\mathbf{X}}^{\pm}_{i,r,k}, \quad H_{i,r,k} \mapsto m^{r+1} \overline{\mathbf{H}}_{i,r,k}. \tag{26}$$

where $r \geq 0, k \in \mathbb{Z}/m\mathbb{Z}$ and $\pi(h) = \overline{h} \in \mathbf{K}/\mathbf{K}^2$. From lemma 2.2, we know that this map makes sense. Now we show that it is an algebra homomorphism, that is, it respects the relations in definition 1.3.

Remark 2.1. *To prove theorem 2.1, the homomorphism* $Y(\mathfrak{g}, \sigma) \longrightarrow \widetilde{Y}(\mathfrak{g}, \sigma)/h\widetilde{Y}(\mathfrak{g}, \sigma)$ *to consider is given by the same formula except that* $\mathbf{X}^{\pm}_{i,r,k}, \mathbf{H}_{i,r,k}$ *should be replaced by* $h^{-r}\mathbf{X}^{\pm}_{i,r,k}, h^{-r}\mathbf{H}_{i,r,k}$ *(and it should be proved that these elements of* $\mathbb{C}((h)) \otimes_{\mathbb{C}[[h]]} \mathfrak{U}_h(L(\mathfrak{g})^{\sigma})$ *belong to* $\widetilde{Y}(\mathfrak{g}, \sigma)$*).*

1. Relations (15), (16)*:* Relations (15) for $\pi(X^{\pm}_{i,r,k}), \pi(H_{i,r,k})$ follow from (1), (2), (3). The first one on line (16) is also a direct consequence of (1). Let us now verify that π respects the second relation on line (16), that is, let us see that $[\pi(X^+_{i_1,r_1,k_1}),\pi(X^-_{i_2,r_2,k_2})]=\sum_{d\in\mathbb{Z}/m\mathbb{Z}}\delta_{\sigma^d(i_1),i_2}\omega^{dk_2}\pi(H_{i_1,r_1+r_2,k_1+k_2})$.

$$\begin{aligned}
[m^+m^{r_1}\mathbf{X}^+_{i_1,r_1,k_1},m^-m^{r_2}\mathbf{X}^-_{i_2,r_2,k_2}] &= m^{r_1+r_2+1}\left[\sum_{a=0}^{r_1}(-1)^{r_1-a}\binom{r_1}{a}\mathcal{X}^+_{i_1,k_1+am},\sum_{b=0}^{r_2}(-1)^{r_2-b}\binom{r_2}{b}\mathcal{X}^-_{i_2,k_2+bm}\right]\\
&= m^{r_1+r_2+1}\sum_{a=0}^{r_1}\sum_{b=0}^{r_2}(-1)^{r_1+r_2-a-b}\binom{r_1}{a}\binom{r_2}{b}[\mathcal{X}^+_{i_1,k_1+am},\mathcal{X}^-_{i_2,k_2+mb}]\\
&= m^{r_1+r_2+1}\sum_{d\in\mathbb{Z}/m\mathbb{Z}}\delta_{\sigma^d(i_1),i_2}\omega^{dk_2}\sum_{e=0}^{r_1+r_2}\sum_{a=0}^{e}(-1)^{r_1+r_2-e}\binom{r_1}{a}\binom{r_2}{e-a}\frac{\Psi^+_{i_1,k_1+k_2+em}-\Psi^-_{i_1,k_1+k_2+em}}{h}\\
&= m^{r_1+r_2+1}\sum_{d\in\mathbb{Z}/m\mathbb{Z}}\delta_{\sigma^d(i_1),i_2}\omega^{dk_2}\sum_{e=0}^{r_1+r_2}(-1)^{r_1+r_2-e}\binom{r_1+r_2}{e}\frac{\Psi^+_{i,k_1+k_2+em}-\Psi^-_{i,k_1+k_2+em}}{h}\\
&= m^{r_1+r_2+1}\sum_{d\in\mathbb{Z}/m\mathbb{Z}}\delta_{\sigma^d(i_1),i_2}\omega^{dk_2}\mathbf{H}_{i_1,r_1+r_2,k_1+k_2}.
\end{aligned}$$

2.1. Relation (17)*: $m=1$ case.* We will give full details in the case $m=1$ and just a few computations when $m=2$ or $m=3$. Taking the coefficients of $u^{-k_1}u^{-k_2}$ in (5), rearranging the terms and simplifying yields:

$$\begin{aligned}
&[\mathcal{X}^\pm_{i_1,k_1+1}-\mathcal{X}^\pm_{i_1,k_1},\mathcal{X}^\pm_{i_2,k_2}]+(1-q^{\pm d_{i_1}c_{i_1i_2}})\mathcal{X}^\pm_{i_2,k_2}\mathcal{X}^\pm_{i_1,k_1+1}\\
&\qquad=[\mathcal{X}^\pm_{i_1,k_1},\mathcal{X}^\pm_{i_2,k_2+1}-\mathcal{X}^\pm_{i_2,k_2}]+(q^{\pm d_{i_1}c_{i_1i_2}}-1)\mathcal{X}^\pm_{i_1,k_1}\mathcal{X}^\pm_{i_2,k_2+1}.
\end{aligned}\tag{27}$$

Assume by induction that we know that

$$\begin{aligned}
&[\mathbf{X}^\pm_{i_1,r_1+1,k_1},\mathcal{X}^\pm_{i_2,k_2}]+(1-q^{\pm d_{i_1}c_{i_1i_2}})\mathcal{X}^\pm_{i_2,k_2}\mathbf{X}^\pm_{i_1,r_1,k_1+1}\\
&\qquad=\left[\mathbf{X}^\pm_{i_1,r_1,k_1},\mathcal{X}^\pm_{i_2,k_2+1}-\mathcal{X}^\pm_{i_2,k_2}\right]+(q^{\pm d_{i_1}c_{i_1i_2}}-1)\mathbf{X}^\pm_{i_1,r_1,k_1}\mathcal{X}^\pm_{i_2,k_2+1}.
\end{aligned}\tag{28}$$

We have already shown this to be true when $r=0$ and all $k_1,k_2\in\mathbb{Z}$ by (27). Using the inductive relation (25), we can conclude by induction that (28) is true for any $k_1,k_2\in\mathbb{Z}$, $r_1\in\mathbb{Z}_{\geq 0}$.

Now, let us do the same for i_2,k_2 instead of i_1,k_1. Arguing again by induction, we conclude that the following equality holds for any $r_1,r_2\in\mathbb{Z}_{\geq0},k_1,k_2\in\mathbb{Z}$:

$$\begin{aligned}
&[\mathbf{X}^\pm_{i_1,r_1+1,k_1},\mathbf{X}^\pm_{i_2,r_2,k_2}]+(1-q^{\pm d_{i_1}c_{i_1i_2}})\mathbf{X}^\pm_{i_2,r_2,k_2}\mathbf{X}^\pm_{i_1,r_1,k_1+1}\\
&\qquad=[\mathbf{X}^\pm_{i_1,r_1,k_1},\mathbf{X}^\pm_{i_2,r_2+1,k_2}]+(q^{\pm d_{i_1}c_{i_1i_2}}-1)\mathbf{X}^\pm_{i_1,r_1,k_1}\mathbf{X}^\pm_{i_2,r_2,k_2+1}.
\end{aligned}$$

Therefore, in $\mathbf{K}^{r+1}/\mathbf{K}^r$, we obtain the desired relation:

$$\left[\overline{\mathbf{X}}^\pm_{i_1,r_1+1,k_1},\overline{\mathbf{X}}^\pm_{i_2,r_2,k_2}\right]-\left[\overline{\mathbf{X}}^\pm_{i_1,r_1,k_1},\overline{\mathbf{X}}^\pm_{i_2,r_2+1,k_2}\right]=\pm\frac{d_{i_1}c_{i_1i_2}\overline{h}}{2}\left(\overline{\mathbf{X}}^\pm_{i_2,r_2,k_2}\overline{\mathbf{X}}^\pm_{i_1,r_1,k_1+1}+\overline{\mathbf{X}}^\pm_{i_1,r_1,k_1}\overline{\mathbf{X}}^\pm_{i_2,r_2,k_2+1}\right).$$

This is true in particular when $k_1=0=k_2$, so we deduce that $\pi(X^\pm_{i_1,r_1})$ and $\pi(X^\pm_{i_2,r_2})$ satisfy relation (13).

2.2. Relation (17)*: $m=2$ case.* (5) can be rewritten as:

$$\begin{aligned}
[\mathcal{X}^\pm_{i_1,k_1+2}-\mathcal{X}^\pm_{i_1,k_1},\mathcal{X}^\pm_{i_2,k_2}]-[\mathcal{X}^\pm_{i_1,k_1},\mathcal{X}^\pm_{i_2,k_2+2}-\mathcal{X}^\pm_{i_2,k_2}] &= (q^{\pm c_{i_1i_2}\pm c_{i_1,\sigma(i_2)}}-1)(\mathcal{X}^\pm_{i_2,k_2}\mathcal{X}^\pm_{i_1,k_1+2}\\
&\quad+\mathcal{X}^\pm_{i_1,k_1}\mathcal{X}^\pm_{i_2,k_2+2})+(q^{\pm c_{i_1i_2}}-q^{\pm c_{i_1,\sigma(i_2)}})S(\mathcal{X}^\pm_{i_2,k_2+1},\mathcal{X}^\pm_{i_1,k_1+1}).
\end{aligned}$$

Using the induction relation (25) as in the case $m=1$, we can deduce by induction that

$$\begin{aligned}
&[\mathbf{X}^\pm_{i_1,r_1+1,k_1},\mathbf{X}^\pm_{i_2,r_2,k_2}]-[\mathbf{X}^\pm_{i_1,r_1,t_1},\mathbf{X}^\pm_{i_2,r_2+1,k_2}]\\
&\quad=(q^{\pm c_{i_1i_2}\pm c_{i_1,\sigma(i_2)}}-1)(\mathbf{X}^\pm_{i_2,r_2,k_2}\mathbf{X}^\pm_{i_1,r_1,k_1+2}+\mathbf{X}^\pm_{i_1,r_1,k_1}\mathbf{X}^\pm_{i_2,r_2,k_2+2})\\
&\quad\;+(q^{\pm c_{i_1i_2}}-q^{\pm c_{i_1,\sigma(i_2)}})S(\mathbf{X}^\pm_{i_2,r_2,k_2+1},\mathbf{X}^\pm_{i_1,r_1,k_1+1})
\end{aligned}\tag{29}$$

From Lemma 2.3, after passing to the quotient $\mathbf{K}^{r+1}/\mathbf{K}^r$, equation (29) becomes (17) (up to a factor of $\frac{1}{m}$ on the right-hand side, but it is accounted for in the definition of π).

2.3. Relation (17)*: $m=3$ case.* (5) can be rewritten as ($d_{i_1}=1$ when $m=3$):

$$\begin{aligned}\left[\mathcal{X}^{\pm}_{i_1,k_1+3}-\mathcal{X}^{\pm}_{i_1,k_1},\mathcal{X}^{\pm}_{i_2,k_2}\right]-\left[\mathcal{X}^{\pm}_{i_1,k_1},\mathcal{X}^{\pm}_{i_2,k_2+3}-\mathcal{X}^{\pm}_{i_2,k_2}\right]&\\ =\pm\frac{h}{2m}\Bigg(\left(\sum_{d=0}^{2}c_{i_1,\sigma^d(i_2)}+O(h)\right)&(\mathcal{X}^{\pm}_{i_2,k_2}\mathcal{X}^{\pm}_{i_1,k_1+3}+\mathcal{X}^{\pm}_{i_1,k_1}\mathcal{X}^{\pm}_{i_2,k_2+3})\\ +\left(\sum_{d=0}^{2}c_{i_1,\sigma^d(i_2)}\omega^d+O(h)\right)&S(\mathcal{X}^{\pm}_{i_2,k_2+1},\mathcal{X}^{\pm}_{i_1,k_1+2})\\ +\left(\sum_{d=0}^{2}c_{i_1,\sigma^d(i_2)}\omega^{2d}+O(h)\right)&S(\mathcal{X}^{\pm}_{i_2,k_2+2},\mathcal{X}^{\pm}_{i_1,k_1+1})\Bigg)\end{aligned}$$

Using relation (25) and induction, we get:

$$\begin{aligned}&[\overline{\mathbf{X}}^{\pm}_{i_1,r_1+1,k_1},\overline{\mathbf{X}}^{\pm}_{i_2,r_2,k_2}]-[\overline{\mathbf{X}}^{\pm}_{i_1,r_1,k_1},\overline{\mathbf{X}}^{\pm}_{i_2,r_2+1,k_2}]\\ &=\pm\frac{\overline{h}}{6}\Big((c_{i_1i_2}+c_{i_1\sigma(i_2)}+c_{i_1\sigma^2(i_2)})\left(\overline{\mathbf{X}}^{\pm}_{i_2,r_2,k_2}\overline{\mathbf{X}}^{\pm}_{i_1,r_1,k_1+3}+\overline{\mathbf{X}}^{\pm}_{i_1,r_1,k_1}\overline{\mathbf{X}}^{\pm}_{i_2,r_2,k_2+3}\right)\\ &+(c_{i_1i_2}+c_{i_1\sigma(i_2)}\omega+c_{i_1\sigma^2(i_2)}\omega^2)S(\overline{\mathbf{X}}^{\pm}_{i_2,r_2,k_2+1},\overline{\mathbf{X}}^{\pm}_{i_1,r_1,k_1+2})\\ &+(c_{i_1i_2}+c_{i_1\sigma(i_2)}\omega^2+c_{i_1\sigma^2(i_2)}\omega)S(\overline{\mathbf{X}}^{\pm}_{i_2,r_2,k_2+2},\overline{\mathbf{X}}^{\pm}_{i_1,r_1,k_1+1})\Big).\end{aligned}\tag{30}$$

From Lemma (2.3), equation (30) shows that $\pi(X^{\pm}_{i_1,r_1,k_1}),\pi(X^{\pm}_{i_2,r_2,k_2})$ with $r_1,r_2\in\mathbb{Z}_{\geq 0},\ k_1,k_2=0,1,2,$ satisfy equation (17).

3. Relation (18)*:* This is quite similar to the previous case, starting this time with relation (24), so we give a few explanations only when $m=2$. $d_{i_1}=1$ when $m=2$ and (18) can be written as

$$\begin{aligned}[H_{i_1,r_1+1,k_1},X^{\pm}_{i_2,r_2,k_2}]-[H_{i_1,r_1,k_1},X^{\pm}_{i_2,r_2+1,k_2}]=&\pm\frac{h(c_{i_1i_2}+c_{i_1\sigma(i_2)})}{2}S(H_{i_1,r_1,k_1},X^{\pm}_{i_2,r_2,k_2})\\ &\pm\frac{h(c_{i_1i_2}-c_{i_1\sigma(i_2)})}{2}S(H_{i_1,r_1,k_1+1},X^{\pm}_{i_2,r_2,k_2+1}).\end{aligned}$$

In order to show that this equation is satisfied when $H_{i,r,k},X^{\pm}_{i,r,k}$ are replaced by $\pi(H_{i,r,k}),\pi(X^{\pm}_{i,r,k})$, we need the following equation:

$$\begin{aligned}&[\mathbf{H}_{i_1,r_1+1,k_1},\mathbf{X}^{\pm}_{i_2,r_2,k_2}]-[\mathbf{H}_{i_1,r_1,k_1},\mathbf{X}^{\pm}_{i_2,r_2+1,k_2}]\\ &=(q^{\pm c_{i_1i_2}\pm c_{i_1,\sigma(i_2)}}-1)(\mathbf{X}^{\pm}_{i_2,r_2,k_2}\mathbf{H}_{i_1,r_1,k_1+2}+\mathbf{H}_{i_1,r_1,k_1}\mathbf{X}^{\pm}_{i_2,r_2,k_2+2})\\ &+(q^{\pm c_{i_1i_2}}-q^{\pm c_{i_1,\sigma(i_2)}})S(\mathbf{H}_{i_1,r_1,k_1+1},\mathbf{X}^{\pm}_{i_2,r_2,k_2+1}).\end{aligned}\tag{31}$$

Notice that relation (24) can be written as

$$\begin{aligned}&\left[\Psi^{+}_{i_1,k_1+2}-\Psi^{+}_{i_1,k_1},\mathcal{X}^{\pm}_{i_2,k_2}\right]-\left[\Psi^{+}_{i_1,k_1},\mathcal{X}^{\pm}_{i_2,k_2+2}-\mathcal{X}^{\pm}_{i_2,k_2}\right]\\ &=(q^{\pm c_{i_1i_2}\pm c_{i_1,\sigma(i_2)}}-1)\left(\mathcal{X}^{\pm}_{i_2,k_2}\Psi^{+}_{i_1,k_1+2}+\Psi^{+}_{i_1,k_1}\mathcal{X}^{\pm}_{i_2,k_2+2}\right)\\ &+(q^{\pm c_{i_1i_2}}-q^{\pm c_{i_1,\sigma(i_2)}})S\left(\Psi^{+}_{i_1,k_1+1},\mathcal{X}^{\pm}_{i_2,k_2+1}\right)\end{aligned}\tag{32}$$

Using (32),(25) and induction, it is possible to establish (31) when $k_1>0$.

To establish (31), when $k_1 = 0$, rewrite (32) as:

$$\begin{aligned}\left[\Psi^+_{i_1,2} - \Psi^+_{i_1,0} + \Psi^-_{i_1,0}, \mathcal{X}^{\pm}_{i_2,k_2}\right] - \left[\Psi^+_{i_1,0} - \Psi^-_{i_1,0}, \mathcal{X}^{\pm}_{i_2,k_2+2} - \mathcal{X}^{\pm}_{i_2,k_2}\right] \\ = (q^{\pm c_{i_1 i_2} \pm c_{i_1,\sigma(i_2)}} - 1)\left(\mathcal{X}^{\pm}_{i_2,k_2}\Psi^+_{i_1,2} + \Psi^+_{i_1,0}\mathcal{X}^{\pm}_{i_2,k_2+2}\right) \\ + (q^{\pm c_{i_1 i_2}} - q^{\pm c_{i_1,\sigma(i_2)}}) S\left(\Psi^+_{i_1,1}, \mathcal{X}^{\pm}_{i_2,k_2+1}\right) + \left[\Psi^-_{i_1,0}, \mathcal{X}^{\pm}_{i_2,k_2+2}\right]\end{aligned} \tag{33}$$

Using $[\Psi^-_{i_1,0}, \mathcal{X}^{\pm}_{i_2,k_2+2}] = (1 - q^{\pm c_{i_1 i_2} \pm c_{i_1,\sigma(i_2)}})\Psi^-_{i_1,0}\mathcal{X}^{\pm}_{i_2,k_2+2}$ and dividing by h, we obtain relation (31) when $k_1 = 0$ and $r_1 = 0 = r_2$. The general case follows again by induction using (25).

4. Relations (20), (21)*:* Let us see why π respects relation (20), (21) being rather similar. We start with equation (6). We can expand that relation as

$$\sum_{k_1,k_2,k_3} P_{ii_3^{\pm}}(u_1,u_2) C_{ii_3}(\mathcal{X}^{\pm}_{i,k_1}, \mathcal{X}^{\pm}_{i,k_2}, \mathcal{X}^{\pm}_{i_3,k_3}) u_1^{-k_1} u_2^{-k_2} u_3^{-k_3} \; + \; \{u_1 \longleftrightarrow u_2\} = 0,$$

where $C_{ii_3}(x,y,z) = \big(zxy - (q^{d_{ii_3}} + q^{-d_{ii_3}})xzy + xyz\big)$.

Notice that $C_{ii_3}(x,y,z)$ is linear in each of x, y, z. So using again (25) and induction, we obtain

$$\mathrm{Sym}\left\{\sum_{k_1,k_2,k_3} P_{ii_3}(u_1,u_2) C_{ii_3}(\mathbf{X}^{\pm}_{i,r_1,k_1}, \mathbf{X}^{\pm}_{i,r_2,k_2}, \mathbf{X}^{\pm}_{i_3,r_3,k_3}) u_1^{-k_1} u_2^{-k_2} u_3^{-k_3}\right\} = 0.$$

Thus

$$\mathrm{Sym}\left\{P_{ii_3}(u_1,u_2) C_{ii_3}(\mathbf{X}^{\pm}_{i,r_1,k_1}(u_1,v_1), \mathbf{X}^{\pm}_{i,r_2,k_2}(u_2,v_2), \mathbf{X}^{\pm}_{i_3,r_3,k_3}(u_3,v_3))\right\} = 0. \tag{34}$$

Since we have $P_{ij}(u_1,u_2) = p_{ij}(u_1,u_2) + O(h)$ and $C_{ii_3} = (zxy - 2xzy + xyz) + O(h)$, we can see that, after passing to $\mathbf{K}^{r_1+r_2+r_3}/\mathbf{K}^{r_1+r_2+r_3+1}$, the equation (34) becomes equation (20). Similarly we can prove (21) by using (7) and (8).

5. Relation (14)*.* This applies only to the case $m = 1$. Since $\mathcal{X}^{\pm}_{i,r}\mathcal{X}^{\pm}_{j,s} = \mathcal{X}^{\pm}_{j,s}\mathcal{X}^{\pm}_{i,r}$ if $c_{ij} = 0$, it follows that $\pi(X^{\pm}_{i,r})\pi(X^{\pm}_{j,s}) = \pi(X^{\pm}_{j,s})\pi(X^{\pm}_{i,r})$ in this case, which is relation (14) when $c_{ij} = 0$. From (9), we deduce by induction that, for any $k_1, \ldots, k_N, k \in \mathbb{Z}$, $r_1, \ldots, r_N \in \mathbb{Z}_{\geq 0}$:

$$\sum_{\tau \in S_N} \sum_{\ell=0}^{N} (-1)^{\ell} \begin{bmatrix} N \\ \ell \end{bmatrix}_q \mathbf{X}^{\pm}_{i,r_{\tau(1)},k_{\tau(1)}} \cdots \mathbf{X}^{\pm}_{i,r_{\tau(\ell)},k_{\tau(\ell)}} \mathbf{X}^{\pm}_{j,s,k} \mathbf{X}^{\pm}_{i,r_{\tau(\ell+1)},k_{\tau(\ell+1)}} \cdots \mathbf{X}^{\pm}_{i,r_{\tau(N)},k_{\tau(N)}} = 0.$$

Passing to the quotient $\mathbf{K}^r/\mathbf{K}^{r+1}$ where $r = s + \sum_{i=1}^{N} r_i$, we obtain, when $k = 0$ and $k_i = 0$ for $1 \leq i \leq N$, relation (14) but with $X^{\pm}_{i,r_{\tau(a)}}$ replaced by $\pi(X^{\pm}_{i,r_{\tau(a)}})$.

From steps 1-5, we can conclude that π is an algebra homomorphism.

We now have to check that the images $\overline{\mathbf{X}}^{\pm}_{i,r,k}, \overline{\mathbf{H}}_{i,r,k}$ of $\mathbf{X}^{\pm}_{i,r,k}, \mathbf{H}_{i,r,k}$ for $r \in \mathbb{Z}_{\geq 0}, 0 \leq k \leq m-1$ generate $gr_{\mathbf{K}}(\mathfrak{U}_h(L(\mathfrak{g})^{\sigma}))$. This will prove that the homomorphism π is surjective. Let $\mathbf{Y}$ be the subalgebra of $gr_{\mathbf{K}}(\mathfrak{U}_h(L(\mathfrak{g})^{\sigma}))$ generated by $\overline{\mathbf{X}}^{\pm}_{i,r,k}, \overline{\mathbf{H}}_{i,r,k}$ $\forall\, r \in \mathbb{Z}_{\geq 0}$ and for $0 \leq k \leq m-1$. The kernel $\mathbf{K}$ is generated as an ideal by $h\mathfrak{U}_h(L(\mathfrak{g})^{\sigma})$ and $\mathcal{X}^{\pm}_{i,r+m} - \mathcal{X}^{\pm}_{i,r}, \mathcal{H}_{i,r+m} - \mathcal{H}_{i,r}$ for $i \in I, r \in \mathbb{Z}$, so we have to see that $\overline{\mathcal{X}^{\pm}_{i,r+m} - \mathcal{X}^{\pm}_{i,r}}$ and $\overline{\mathcal{H}_{i,r+m} - \mathcal{H}_{i,r}}$ are in $\mathbf{Y}$. This follows from lemma 2.3 which implies that, if $r \equiv k \bmod m$ with $0 \leq k \leq m-1$, then $\overline{\mathcal{X}^{\pm}_{i,r+m} - \mathcal{X}^{\pm}_{i,r}} = \overline{\mathbf{X}}^{\pm}_{i,1,r} = \overline{\mathbf{X}}^{\pm}_{i,1,k}$, so $\overline{\mathcal{X}^{\pm}_{i,r+m} - \mathcal{X}^{\pm}_{i,r}} \in \mathbf{Y}$ because $\overline{\mathbf{X}}^{\pm}_{i,1,k} \in \mathbf{Y}$. The same argument works for $\overline{\mathcal{H}_{i,r+m} - \mathcal{H}_{i,r}}$.

We are left to show that the map $\pi : Y(\mathfrak{g},\sigma) \to gr_{\mathbf{K}}(\mathfrak{U}_h(L(\mathfrak{g})^{\sigma}))$ is injective. This is where we need proposition 1.1. We will assume that this proposition holds also when σ is non-trivial. For each $\alpha \in \Delta^+$ and for each integer k such that $0 \leq k \leq |\mathcal{O}(\alpha)|-1$ where $|\mathcal{O}(\alpha)|$ is the size of the orbit $\mathcal{O}(\alpha)$, choose a sequence $i_1, \ldots, i_t \in I$ such that setting

$$E^{\pm}_{\alpha,k} = \left[\frac{1}{m}\sum_{s=0}^{m-1} X^{\pm}_{\sigma^s(i_1),0}, \left[\frac{1}{m}\sum_{s=0}^{m-1} X^{\pm}_{\sigma^s(i_2),0}, \cdots \left[\frac{1}{m}\sum_{s=0}^{m-1} X^{\pm}_{\sigma^s(i_{t-1}),0}, \frac{1}{m}\sum_{s=0}^{m-1} \omega^{-ks} X^{\pm}_{\sigma^s(i_t),0}\right] \cdots \right]\right]$$

gives a non-zero vector in $\left(\sum_{s=0}^{|\mathcal{O}(\alpha)|-1} \mathfrak{g}_{\sigma^s(\alpha)}\right) \cap \mathfrak{g}_k$ (here we view $E^{\pm}_{\alpha,k}$ as belonging to $\mathfrak{U}(\mathfrak{g}) \subset Y_h(\mathfrak{g})$). We require the same choice of indices $i_1, \ldots, i_t$ for $\alpha, \ldots, \sigma^{m-1}(\alpha)$, so that $E^{\pm}_{\sigma^s(\alpha),k} = \omega^{ks} E^{\pm}_{\alpha,k}$. Set $X^{\pm}_{\alpha,r,k} = \left[X^{\pm}_{i_1,0,0}, \left[X^{\pm}_{i_2,0,0}, \cdots \left[X^{\pm}_{i_{t-1},0,0}, X^{\pm}_{i_t,r,k}\right] \cdots\right]\right]$. The elements $\mathcal{X}^{\pm}_{\alpha,r,k} \in \mathfrak{U}_h(L(\mathfrak{g})^\sigma)$ are defined similarly: $\mathcal{X}^{\pm}_{\alpha,r,k} = \left[\mathcal{X}^{\pm}_{i_1,0}, \left[\mathcal{X}^{\pm}_{i_2,0}, \cdots \left[\mathcal{X}^{\pm}_{i_{t-1},0}, \mathcal{X}^{\pm}_{i_t,k+rm}\right] \cdots\right]\right]$ and we have $X^{\pm}_{\sigma(\alpha),r,k} = \omega^k X^{\pm}_{\alpha,r,k}$, $\mathcal{X}^{\pm}_{\sigma(\alpha),r,k} = \omega^k \mathcal{X}^{\pm}_{\alpha,r,k}$. Under $\mathfrak{U}_h(L(\mathfrak{g})^\sigma)/h\mathfrak{U}_h(L(\mathfrak{g})^\sigma) \cong \mathfrak{U}(L(\mathfrak{g})^\sigma)$, $\overline{\mathcal{X}^{\pm}_{\alpha,r,k}}$ is equal to $E^{\pm}_{\alpha,k} \otimes u^{k+rm}$ (see [Dr2]), and in $Y_h(\mathfrak{g},\sigma)/hY_h(\mathfrak{g},\sigma) \cong \mathfrak{U}(L(\mathfrak{g}))$, $\overline{X^{\pm}_{\alpha,r,k}} = m^{r+t} E^{\pm}_{\alpha,k} \otimes u^r$ (see [Le]).

$Y_h(\mathfrak{g})$ (so $Y_h(\mathfrak{g},\sigma)$ with $\sigma = \mathrm{id}, m = 1$) is a free $\mathbb{C}[h]$-module spanned by the set of ordered monomials in the elements $X^{\pm}_{\alpha,r}, H_{i,r}$(for any choice of total ordering on these elements) - see [Le]. Because of theorem 1.1, for any σ, a basis for $Y_h(\mathfrak{g},\sigma)$ as a module over $\mathbb{C}[h]$ is provided by the set B of ordered monomials in the elements $X^{\pm}_{\alpha,r,k}, H_{i,r,k}$ for all $\alpha \in \Delta^+, i \in I, r \geq 0$ and for $0 \leq k \leq |\mathcal{O}(\alpha)|-1$ (respectively, $0 \leq k \leq |\mathcal{O}(\alpha_i)|-1$). If α and β are in the same orbit under σ, then $X^{\pm}_{\alpha,r,k}$ and $X^{\pm}_{\beta,r,k}$ are scalar multiples of each other, so it is enough, for each r,k, to choose only one generator for each root orbit. Let B_s be the basis over $\mathbb{C}$ of the piece of $Y_h(\mathfrak{g},\sigma)$ of degree s provided by elements of the form $h^t M$ where $M \in B$ has degree $s-t$.

Because we are assuming that proposition 1.1 holds true for $\mathfrak{U}_h(L(\mathfrak{g})^\sigma)$, ordered monomials in the elements $\mathcal{X}^{\pm}_{\alpha,r}, \mathcal{H}_{i,r}$ provide a (topological) basis of $\mathfrak{U}_h(L(\mathfrak{g})^\sigma)$ over $\mathbb{C}[[h]]$. (As in the previous paragraph, some restrictions should be imposed on r, α, i to avoid zero elements and repetitions.) However, we need to compute the associated graded ring for the descending filtration given by the powers of $\mathbf{K}$ and this necessitates a different basis. For $r \in \mathbb{Z}_{\geq 0}, 0 \leq k \leq m-1$ and $* = +$ or $* = -$, set $\mathbf{X}^*_{\alpha,\pm r,\pm k} = \sum_{s=0}^{r} (-1)^{r-s}\binom{r}{s}\mathcal{X}^*_{\alpha,\pm s,\pm k}$ and $\mathbf{H}_{i,\pm r,\pm k} = \sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}\frac{\Psi^+_{i,\pm k \pm ms} - \Psi^-_{i,\pm k \pm ms}}{h}$.

From proposition 1.1, a free topological basis of $\mathfrak{U}_h(L(\mathfrak{g})^\sigma)$ is provided by the set $\mathbf{B}$ of ordered monomials (for some choice of total ordering) in the set of non-zero elements among $\mathbf{X}^+_{\alpha,\pm r,\pm k}, \mathbf{X}^-_{\alpha,\pm r,\pm k}, \mathbf{H}_{i,\pm r,\pm k}$ for $r \in \mathbb{Z}_{\geq 0}$ and for $0 \leq k \leq |\mathcal{O}(\alpha)|-1$ (respectively, $0 \leq k \leq |\mathcal{O}(\alpha_i)|-1$). Again, one should note here that if α and β are in the same orbit under σ, then $\mathbf{X}^*_{\alpha,\pm r,\pm k}$ is a multiple of $\mathbf{X}^*_{\beta,\pm r,\pm k}$, hence only one of the two should be included; the same holds for $\mathbf{H}_{i,\pm r,\pm k}$ and $\mathbf{H}_{j,\pm r,\pm k}$ if i and j are in the same orbit under σ.

We can now determine a basis of $gr_{\mathbf{K}}(\mathfrak{U}_h(L(\mathfrak{g})^\sigma))$. What we need is a basis of $\mathbf{K}^s/\mathbf{K}^{s+1}$ for each $s \in \mathbb{Z}_{\geq 0}$. Note that $\mathbf{X}^*_{\alpha,\pm r,\pm k}, \mathbf{H}_{i,\pm r,\pm k} \in \mathbf{K}^r$. Furthermore, it follows from the proof of lemma 2.2 that $\mathbf{X}^*_{\alpha,r,m-k} - (-1)^r \mathbf{X}^*_{\alpha,-r,-k} \in \mathbf{K}^{r+1}$ for any $r, k \geq 0, * = \pm$ and the same is true for $\mathbf{H}_{i,r,k}$. Therefore, to obtain a spanning set of $\mathbf{K}^s/\mathbf{K}^{s+1}$, it is enough to consider (ordered) monomials in the elements $\mathbf{X}^*_{\alpha,r,k}$ and $\mathbf{H}_{i,r,k}$ with $r \geq 0, 0 \leq k \leq m-1, * = \pm$. If $\mathcal{M}$ is such a monomial and $h^t\mathcal{M} \in \mathbf{K}^s$, then $h^t\mathcal{M} \in \mathbf{K}^s \cap h\mathfrak{U}_h(L(\mathfrak{g})^\sigma)$, hence $h^{t-1}\mathcal{M} \in \mathbf{K}^{s-1}$ (see [GaTL] section 6.3; proposition 1.1 is necessary here), thus $\mathcal{M} \in \mathbf{K}^{s-t}$. If the sum of all the indices r appearing in such a monomial $\mathcal{M}$ is $\geq s$, then $\mathcal{M} \in \mathbf{K}^s$. The converse is also true: if $\mathcal{M} \in \mathbf{K}^s$, the image of $\mathcal{M}$ under the composite $\mathfrak{U}_h(L(\mathfrak{g})^\sigma) \twoheadrightarrow \mathfrak{U}(L(\mathfrak{g})^\sigma) \hookrightarrow \mathfrak{U}(L(\mathfrak{g})) \twoheadrightarrow \mathfrak{U}(\mathfrak{g} \otimes_{\mathbb{C}} (\mathbb{C}[t,t^{-1}]/(t-1)^s))$ is 0 since $\mathbf{K}^s$ belongs to the kernel of this composite, and we deduce from this that the sum of all the indices r appearing in $\mathcal{M}$ is $\geq s$.

Therefore, a basis for $\mathbf{K}^s/\mathbf{K}^{s+1}$ is provided by the set $\widetilde{\mathbf{B}}_s$ of elements of the form $\overline{h^t\mathcal{M}}$ where $\mathcal{M}$ is an ordered monomial in the elements $\mathbf{X}^+_{\alpha,r,k}, \mathbf{X}^-_{\alpha,r,k}$ and $\mathbf{H}_{i,r,k}$ which are nonzero and the sum of the indices r in $\mathcal{M}$ is equal to $s-t$. (Again, it should be taken into account that some of the elements are scalar multiples of each other if they correspond to roots in the same σ-orbit, as for $Y_h(\mathfrak{g},\sigma)$.) It follows that π maps the basis B_s of each graded piece of $Y_h(\mathfrak{g},\sigma)$ of degree s to the basis $\widetilde{\mathbf{B}}_s$ of $\mathbf{K}^s/\mathbf{K}^{s+1}$: this ends the proof that π is an isomorphism. □

3. From quantum toroidal algebras to affine Yangians and deformed double current algebras

The ideas of the previous section can be extended to the quantum toroidal case: this was used in [Gu1, Gu2] and we give here just a few explanations.

Definition 3.1. *The quantum toroidal algebra $\mathfrak{U}_h^{tor}(\mathfrak{g})$ (respectively, the affine Yangian $\widehat{Y}(\mathfrak{g})$) is the algebra which is defined exactly as $\mathfrak{U}_h(L(\mathfrak{g}))$ (respectively, as $Y(\mathfrak{g})$), except that the Cartan matrix of finite type must be replaced by the corresponding affine Cartan matrix.*

Remark 3.1. *To obtain the "full" quantum toroidal algebra, a central element should be added. This explains why it is necessary to take the quotient by the one-dimensional subspace spanned by $t^{-1}dt$ in the next lemma.*

Lemma 3.1. *$\mathfrak{U}_h^{tor}(\mathfrak{g})/h\mathfrak{U}_h^{tor}(\mathfrak{g})$ is isomorphic to the enveloping algebra of the quotient of the universal central extension $\widehat{\mathfrak{g}}(\mathbb{C}[s^{\pm 1},t^{\pm 1}])$ of $\mathfrak{g}(\mathbb{C}[s^{\pm 1},t^{\pm 1}])$ by the one-dimensional subspace spanned by the central element $t^{-1}dt$ (under the identification of the center with the quotient $\frac{\Omega^1(\mathbb{C}[s^{\pm 1},t^{\pm 1}])}{d(\mathbb{C}[s^{\pm 1},t^{\pm 1}])}$ of the space of all 1-forms on $\mathbb{C}^\times \times \mathbb{C}^\times$ by the subspace of all exact forms - see* [MRY]*).*

Proof. This follows, for instance, by comparing the relations in definition 3.1 and those in section 3 in [MRY]. □

It is not known in general if the quantum toroidal algebras are flat deformations of the enveloping algebra $\mathfrak{U}\left(\widehat{\mathfrak{g}}(\mathbb{C}[s^{\pm 1},t^{\pm 1}])/\mathbb{C}t^{-1}dt\right)$. At least, it was proved in [He2] that they possess a multiplicative triangular decomposition.

$\widehat{Y}(\mathfrak{g})$ admits a filtration by giving generators $X_{i,r}^{\pm}, H_{i,r}$ degree r and its associated graded ring is a quotient of the enveloping algebra of the universal central extension $\widehat{\mathfrak{g}}(\mathbb{C}[s^{\pm 1},t])$ of $\mathfrak{g}(\mathbb{C}[s^{\pm 1},t])$. (Conjecturally, they are isomorphic.) For any positive real root α of the affine Lie algebra $\widehat{\mathfrak{g}}(\mathbb{C}[s^{\pm 1}])$ and for each $r \in \mathbb{Z}_{\geq 0}$, one can define elements $X_{\alpha,r}^{\pm} \in \widehat{Y}(\mathfrak{g})$ with principal symbol in $gr(\widehat{Y}(\mathfrak{g}))$ corresponding to $X_\alpha^{\pm} t^r$; it is also possible to define elements $H_{i,p,r} \in \widehat{Y}(\mathfrak{g})$ for $i \in I, p \in \mathbb{Z}, r \in \mathbb{Z}_{\geq 0}$ corresponding to $H_i s^p t^r$ in $gr(\widehat{Y}(\mathfrak{g}))$. However, ordered monomials in the elements $X_{\alpha,r}^{\pm}, H_{i,p,r}$ are not enough to form a spanning set of $\widehat{Y}(\mathfrak{g})$: one also has to lift to $\widehat{Y}(\mathfrak{g})$ the central elements of $\widehat{\mathfrak{g}}(\mathbb{C}[s^{\pm 1},t])$.

We have a sequence of algebra homomorphisms

$$\mathfrak{U}_h^{tor}(\mathfrak{g}) \twoheadrightarrow \mathfrak{U}_h^{tor}(\mathfrak{g})/h\mathfrak{U}_h^{tor}(\mathfrak{g}) \xrightarrow{\sim} \mathfrak{U}\big(\widehat{\mathfrak{g}}(\mathbb{C}[s^{\pm 1},t^{\pm 1}])/t^{-1}dt\big) \overset{t\mapsto 1}{\twoheadrightarrow} \widehat{\mathfrak{g}}(\mathbb{C}[s^{\pm 1}]);$$

let $\mathbb{K}$ be the kernel of this composite and let $\widetilde{\widehat{Y}}(\mathfrak{g})$ be the $\mathbb{C}[[h]]$-subalgebra of $\mathfrak{U}_h^{tor}(\mathfrak{g}) \otimes_{\mathbb{C}[[h]]} \mathbb{C}((h))$ generated by $\mathfrak{U}_h^{tor}(\mathfrak{g})$ and by $h^{-1}\mathbb{K}$. One can define a homomorphism $\psi : \widehat{Y}(\mathfrak{g}) \longrightarrow \widetilde{\widehat{Y}}(\mathfrak{g})/h\widetilde{\widehat{Y}}(\mathfrak{g})$ (the formulas (26) are also valid when i is the extending vertex (usually labelled 0) of the affine Dynkin diagram of $\widehat{\mathfrak{g}}$) and show that it is onto as in the case of $Y(\mathfrak{g})$. Using the same ideas as in the proof of the main theorem, one could try to prove the following conjecture.

Conjecture 3.1. *$\widehat{Y}(\mathfrak{g})$ is isomorphic to $\widetilde{\widehat{Y}}(\mathfrak{g})/h\widetilde{\widehat{Y}}(\mathfrak{g})$.*

To prove this, one should first establish that $\mathfrak{U}_h^{tor}(\mathfrak{g})$ is a flat deformation of $\mathfrak{U}\big(\widehat{\mathfrak{g}}(\mathbb{C}[s^{\pm 1},t^{\pm 1}])/\mathbb{C}t^{-1}dt\big)$. In type A, the quantum toroidal algebras $\mathfrak{U}_{q,d}^{tor}(\mathfrak{sl}_n)$ studied in [VaVa] depend on two parameters $d, q \in \mathbb{C}^\times$. One can define a similar algebra $\mathfrak{U}_{q,d,h}^{tor}(\mathfrak{sl}_n)$ over $\mathbb{C}[[h]]$ using the same relations and setting $q = e^{\frac{\lambda}{2}h}, d = e^{\beta h}$ for some $\lambda, \beta \in \mathbb{C}$. Moreover, the affine Yangian of type A depends also on two parameters λ, β - see [Gu1]. Conjecture 3.1 is true when $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ in this two-parameter setting: in [Gu1], it was possible to avoid the question of the flatness of $\mathfrak{U}_{q,d,h}^{tor}(\mathfrak{g})$ by using a family of representations coming from the Schur-Weyl functor.

Let $\widetilde{S}$ be the subalgebra of $\widehat{Y}(\mathfrak{g}) \otimes_{\mathbb{C}} \mathbb{C}[h]$ generated by $h^r X_{i,r}^{\pm}, h^r H_{i,r}, i \in \widehat{I}, r \geq 0$. Set $\widehat{S} = \widetilde{S}/h\widetilde{S}$. Since $\widetilde{S}$ is the Rees ring of $\widehat{Y}(\mathfrak{g})$ (for the same filtration as above), $\widehat{S} \cong \mathrm{gr}(\widehat{Y}(\mathfrak{g}))$. There is a canonical map

$\mathfrak{U}\big(\widehat{\mathfrak{g}}(\mathbb{C}[s^{\pm 1},t])\big) \twoheadrightarrow \mathrm{gr}(\widehat{Y}(\mathfrak{g}))$: it was proven in [Gu1] that this map is an isomorphism when $\mathfrak{g} = \mathfrak{sl}_n$. In general, this is not known although believed to be true, so we will proceed by assuming that we have an isomorphism $\mathfrak{U}\big(\widehat{\mathfrak{g}}(\mathbb{C}[s^{\pm 1},t])\big) \xrightarrow{\sim} \mathrm{gr}(\widehat{Y}(\mathfrak{g}))$. Under this assumption, we have a map $\widehat{S} \twoheadrightarrow \mathfrak{U}\big(\mathfrak{g}(\mathbb{C}[s^{\pm 1},t])\big)$.

Consider the composite $\widetilde{S} \twoheadrightarrow \widehat{S} \twoheadrightarrow \mathfrak{U}(\mathfrak{sl}_n[s^{\pm 1},t]) \twoheadrightarrow \mathfrak{U}(\mathfrak{sl}_n[t])$, where the last map is obtained by setting $s = 1$. Let K be the kernel of this composite map. Let S be the $\mathbb{C}[h]$-subalgebra of $\widehat{Y}(\mathfrak{g}) \otimes_{\mathbb{C}} \mathbb{C}[h,h^{-1}]$ generated by $\widetilde{S}$ and $h^{-1}\mathsf{K}$. It is argued in [Gu2] that S/hS should be an interesting deformation of the enveloping algebra of the universal central extension of $\mathfrak{g}(\mathbb{C}[s,t])$; when $\mathfrak{g} = \mathfrak{sl}_n$, S/hS is the deformed double current algebra of type A studied in [Gu1]. It is also possible to define the affine Yangian $\widehat{Y}_h(\mathfrak{g})$ over $\mathbb{C}[h]$ by adding a central variable h to the right-hand side of the relations (11),(13) as in [GaTL] and define a subalgebra of $\widehat{Y}_h(\mathfrak{g}) \otimes_{\mathbb{C}[h]} \mathbb{C}[h,h^{-1}]$ similar to S. Note that there is a homomorphism $\widehat{Y}_h(\mathfrak{g}) \twoheadrightarrow \widetilde{S}$ which induces a map $\widehat{Y}_h(\mathfrak{g})/h\widehat{Y}_h(\mathfrak{g}) \longrightarrow \widehat{S}$ which is conjecturally an isomorphism.

Appendix A

Let us show that φ preserves the relation (20). Assume $c_{ii_3} = -1$ and $\sigma(i) \neq i_3$.

Assume first that $m = 2$. Then $\sigma^2 = 1$ and $\varphi(X^{\pm}_{i,r,k}) = 2^r(X^{\pm}_{i,r} + (-1)^k X^{\pm}_{\sigma(i),r})$. We consider the following cases.

Case 1: $\sigma(i) = i$. We have $p_{ii_3}(v_1,v_2) = 1$ and $c_{i\sigma(i_3)} = -1$. (20) is equivalent to the following identities in $Y(\mathfrak{g})$ (we use (14) here):

$$\begin{aligned}
\mathrm{Sym}\Big(&(-1)^{k_2}\big[[X^{\pm}_{i_3,r_3}, X^{\pm}_{i,r_1}], X^{\pm}_{\sigma(i),r_2}\big] + (-1)^{k_1}\big[[X^{\pm}_{i_3,r_3}, X^{\pm}_{\sigma(i),r_1}], X^{\pm}_{i,r_2}\big] \\
&+(-1)^{k_1+k_2}\big[[X^{\pm}_{i_3,r_3}, X^{\pm}_{\sigma(i),r_1}], X^{\pm}_{\sigma(i),r_2}\big] + (-1)^{k_3}\big[[X^{\pm}_{\sigma(i_3),r_3}, X^{\pm}_{i,r_1}], X^{\pm}_{i,r_2}\big] \\
&+(-1)^{k_2+k_3}\big[[X^{\pm}_{\sigma(i_3),r_3}, X^{\pm}_{i,r_1}], X^{\pm}_{\sigma(i),r_2}\big] + (-1)^{k_1+k_3}\big[[X^{\pm}_{\sigma(i_3),r_3}, X^{\pm}_{\sigma(i),r_1}], X^{\pm}_{i,r_2}\big]\Big) = 0
\end{aligned} \tag{35}$$

Here, Sym means symmetrization with respect to (r_1,k_1) and (r_2,k_2). Using $\sigma(i) = i$ and (14) again, we can see that this equality holds.

Case 2: $\sigma(i_3) \neq i_3$, $c_{i\sigma(i)} = 0$. In this case, we have $p_{ii_3}(v_1,v_2) = 1$ and $c_{i\sigma(i_3)} = 0$. (20) is also equivalent to (35). Using (14), we have $[X^{\pm}_{\sigma(i),s}, X^{\pm}_{i_3,r}] = [X^{\pm}_{\sigma(i_3),s}, X^{\pm}_{i,r}] = [X^{\pm}_{\sigma(i),s}, X^{\pm}_{i,r}] = 0$ which implies (20).

Case 3: $\sigma(i_3) = i_3$, $c_{i\sigma(i)} = 0$. We have $p_{ii_3}(v_1,v_2) = v_1 + v_2$ and $c_{\sigma(i)i_3} = -1$. If we replace $p_{ii_3}(v_1,v_2)$ by just v_1 on the left hand side of (20), compute the coefficient of $u_1^{r_1}u_2^{r_2}u_3^{r_3}v_1^{k_1}v_2^{k_2}v_3^{k_3}$ and apply φ to it, we find that it equals (after some minor simplifications)

$$\begin{aligned}
\mathrm{Sym}\Big(&(-1)^{k_2}\big[[X^{\pm}_{i_3,r_3}, X^{\pm}_{i,r_1}], X^{\pm}_{\sigma(i),r_2}\big] - (-1)^{k_1}\big[[X^{\pm}_{i_3,r_3}, X^{\pm}_{\sigma(i),r_1}], X^{\pm}_{i,r_2}\big] \\
&+(-1)^{k_2+k_3}\big[[X^{\pm}_{\sigma(i_3),r_3}, X^{\pm}_{i,r_1}], X^{\pm}_{\sigma(i),r_2}\big] - (-1)^{k_1+k_3}\big[[X^{\pm}_{\sigma(i_3),r_3}, X^{\pm}_{\sigma(i),r_1}], X^{\pm}_{i,r_2}\big] \\
&-(-1)^{k_1+k_2}\big[[X^{\pm}_{i_3,r_3}, X^{\pm}_{\sigma(i),r_1}], X^{\pm}_{\sigma(i),r_2}\big] + (-1)^{k_3}\big[[X^{\pm}_{\sigma(i_3),r_3}, X^{\pm}_{i,r_1}], X^{\pm}_{i,r_2}\big]\Big)
\end{aligned}$$

If now we replace $p_{ii_3}(v_1,v_2)$ by just v_2 on the left hand side of (20), compute again the coefficient of $u_1^{r_1}u_2^{r_2}u_3^{r_3}v_1^{k_1}v_2^{k_2}v_3^{k_3}$ and apply φ to it, we find that it equals (after some minor simplifications)

$$\begin{aligned}
\mathrm{Sym}\Big(&-(-1)^{k_2}\big[[X^{\pm}_{i_3,r_3}, X^{\pm}_{i,r_1}], X^{\pm}_{\sigma(i),r_2}\big] + (-1)^{k_1}\big[[X^{\pm}_{i_3,r_3}, X^{\pm}_{\sigma(i),r_1}], X^{\pm}_{i,r_2}\big] \\
&-(-1)^{k_2+k_3}\big[[X^{\pm}_{\sigma(i_3),r_3}, X^{\pm}_{i,r_1}], X^{\pm}_{\sigma(i),r_2}\big] + (-1)^{k_1+k_3}\big[[X^{\pm}_{\sigma(i_3),r_3}, X^{\pm}_{\sigma(i),r_1}], X^{\pm}_{i,r_2}\big] \\
&-(-1)^{k_1+k_2}\big[[X^{\pm}_{i_3,r_3}, X^{\pm}_{\sigma(i),r_1}], X^{\pm}_{\sigma(i),r_2}\big] + (-1)^{k_3}\big[[X^{\pm}_{\sigma(i_3),r_3}, X^{\pm}_{i,r_1}], X^{\pm}_{i,r_2}\big]\Big)
\end{aligned}$$

The sum of these two expressions vanishes in $Y(\mathfrak{g})$.

Case 4: $c_{i\sigma(i)} = -1$. Then $c_{\sigma(i)i_3} = 0$ and $p_{ii_3}(v_1, v_2) = v_1 + v_2$. This is analogous to the previous case, but both sums simplify even more since $c_{\sigma(i)i_3} = 0$.

Assume now that $m = 3$. Then $\sigma^3 = 1$ and $\varphi(X^{\pm}_{i,r,k}) = 3^r(X^{\pm}_{i,r} + \omega^{-k}X_{\sigma(i),r} + \omega^{-2k}X_{\sigma^2(i),r})$.

We have the following cases.

Case 1: $\sigma(i) = i$. We have $p_{ii_3}(v_1, v_2) = 1$ and $c_{i\sigma^s(i_3)} = -1$ for $s = 0, 1, 2$. The relation (20) is equivalent to the following identity:

$$\mathrm{Sym}\left(\sum_{s_1,s_2,s_3=0}^{2} \omega^{-(s_1k_1+s_2k_2+s_3k_3)}\big[[X^{\pm}_{\sigma^{s_3}(i_3),r_3}, X^{\pm}_{i,r_1}], X^{\pm}_{i,r_2}\big]\right) = 0$$

where Sym denotes symmetrization over $(r_1, k_1), (r_2, k_2)$. The previous equality holds since, from (14), it follows that $\big[[X^{\pm}_{\sigma^{s_3}(i_3),r_3}, X^{\pm}_{i,r_2}], X^{\pm}_{i,r_1}\big] + \big[[X^{\pm}_{\sigma^{s_3}(i_3),r_3}, X^{\pm}_{i,r_1}], X^{\pm}_{i,r_2}\big] = 0$.

Case 2: $\sigma(i_3) = i_3$, $c_{i\sigma(i)} = 0$. We have $p_{ii_3}(v_1, v_2) = v_1^2 + v_1v_2 + v_2^2$ and $c_{\sigma^s(i)i_3} = -1$ for $s = 0, 1, 2$. The relation (20) can be written as

$$\begin{aligned}\mathrm{Sym}\Bigg(\sum_{s_1,s_2,s_3=0}^{2} &\omega^{s_3k_3}\big(\omega^{-(s_1(k_1+2)+s_2k_2)} + \omega^{-(s_1(k_1+1)+s_2(k_2+1))} \\ &+ \omega^{-(s_1k_1+s_2(k_2+2))}\big)\big[[X^{\pm}_{i_3,r_3}, X^{\pm}_{\sigma^{s_2}(i),r_2}], X^{\pm}_{\sigma^{s_1}(i),r_1}\big]\Bigg) = 0\end{aligned}$$

Since $\big[[X^{\pm}_{i_3,r_3}, X^{\pm}_{\sigma^{s_2}(i),r_2}], X^{\pm}_{\sigma^{s_1}(i),r_1}\big] + \big[[X^{\pm}_{i_3,r_3}, X^{\pm}_{\sigma^{s_2}(i),r_1}], X^{\pm}_{\sigma^{s_1}(i),r_2}\big] = 0$ if $s_1 = s_2$ by relation (14), the left-hand side can be written as

$$\begin{aligned}\mathrm{Sym}\Big(\sum_{s,s_3=0}^{2}\sum_{t=1}^{2} &\omega^{s_3k_3}\big(\omega^{-(s(k_1+2)+(s+t)k_2)} + \omega^{-(s(k_1+1)+(s+t)(k_2+1))} + \\ &\omega^{-(sk_1+(s+t)(k_2+2))}\big)\big[[X^{\pm}_{i_3,r_3}, X^{\pm}_{\sigma^{s+t}(i),r_2}], X^{\pm}_{\sigma^{s}(i),r_1}\big]\Big) \\ = 3\delta_{k_3,0}\sum_{t=1}^{2}(1 + w^{-t} + w^{-2t})&\sum_{s=0}^{2}\omega^{-s(k_1+k_2)-2s-tk_2}\big[[X^{\pm}_{i_3,r_3}, X^{\pm}_{\sigma^{s+t}(i),r_2}], X^{\pm}_{\sigma^{s}(i),r_1}\big]\end{aligned}$$

which is zero.

Now let us prove (21). We have $c_{i\sigma(i)} = -1$ and $m = 2$. Applying φ to (21) reduces to the following identity:

$$\mathrm{Sym}\Big(2(-1)^{k_1+k_2}\big[[X^{\pm}_{i,r_3}, X^{\pm}_{\sigma(i),r_1}], X^{\pm}_{\sigma(i),r_2}\big] + 2(-1)^{k_2}\big[[X^{\pm}_{i,r_1}, X^{\pm}_{\sigma(i),r_2}], X^{\pm}_{i,r_3}\big]\Big) = 0$$

where Sym denotes symmetrization over $(r_1, k_1), (r_2, k_2), (r_3, k_3)$. This equality holds by (14).

Appendix B

We give a proof of lemma 2.1.

Let $\mathcal{H}_i^+(z)=\sum_{k\geq 1}k\mathcal{H}_{i,k}z^{-k-1}$ and $\mathcal{X}_j^{\pm}(w)=\sum_{l\in\mathbb{Z}}\mathcal{X}_{j,l}^{\pm}w^{-l}$. We have

$$\begin{aligned}
[\mathcal{H}_i^+(z),\mathcal{X}_j^{\pm}(w)] &= \sum_{k\geq 1,l\in\mathbb{Z}} k[\mathcal{H}_{i,k},\mathcal{X}_{j,l}^{\pm}]z^{-k-1}w^{-l}\\
&= \sum_{k\geq 1,l\in\mathbb{Z}}\left(\pm\sum_{d\in\mathbb{Z}/m\mathbb{Z}}\frac{q^{kd_ic_{i,\sigma^d(j)}}-q^{-kd_ic_{i,\sigma^d(j)}}}{h}\omega^{dk}\right)\mathcal{X}_{j,k+l}^{\pm}z^{-k-1}w^{-l}\\
&= \pm\sum_{d\in\mathbb{Z}/m\mathbb{Z}}\sum_{k\geq 1}\frac{q^{kd_ic_{i,\sigma^d(j)}}-q^{-kd_ic_{i,\sigma^d(j)}}}{h}\omega^{kd}z^{-1}\left(\frac{w}{z}\right)^k\mathcal{X}_j^{\pm}(w)\\
&= \pm\frac{1}{h}\sum_{d\in\mathbb{Z}/m\mathbb{Z}}\Big(\frac{1}{z-q^{d_ic_{i,\sigma^d(j)}}\omega^dw}-\frac{1}{z-q^{-d_ic_{i,\sigma^d(j)}}\omega^dw}\Big)\mathcal{X}_j^{\pm}(w)\\
&= \pm\sum_{d\in\mathbb{Z}/m\mathbb{Z}}\frac{(q^{d_ic_{i,\sigma^d(j)}}-q^{-d_ic_{i,\sigma^d(j)}})\omega^d}{(z-q^{d_ic_{i,\sigma^d(j)}}\omega^dw)(z-q^{-d_ic_{i,\sigma^d(j)}}\omega^dw)}w\mathcal{X}_j^{\pm}(w).
\end{aligned}$$

Define

$$\Psi_i^+(z)=\sum_{k\geq 0}\Psi_{i,k}^+z^{-k}=\exp\left(\frac{h}{2}\mathcal{H}_{i,0}\right)\exp\left(h\sum_{k\geq 1}\mathcal{H}_{i,k}z^{-k}\right). \tag{36}$$

Then the above cross relation between $\mathcal{H}_i^+(z)$ and $\mathcal{X}_j^{\pm}(w)$ is equivalent to the data of a $\alpha_{\pm}(z,w)\in\mathbb{C}[w,w^{-1}][[z]]$ such that

$$\Psi_i^+(z)\mathcal{X}_j^{\pm}(w)=\alpha_{\pm}(z,w)\mathcal{X}_j^{\pm}(w)\Psi_i^+(z). \tag{37}$$

Now we calculate $\alpha_{\pm}(z,w)$. Applying ∂_z to (37), we get

$$\partial_z\Psi_i^+(z)\mathcal{X}_j^{\pm}(w)=\partial_z\alpha_{\pm}(z,w)\mathcal{X}_j^{\pm}(w)\Psi_i^+(z)+\alpha_{\pm}(z,w)\mathcal{X}_j^{\pm}(w)\partial_z\Psi_i^+(z).$$

From (36), we have $\partial_z\Psi_i^+(z)=-h\mathcal{H}_i^+(z)\Psi_i^+(z)$. Thus we have

$$-h\alpha_{\pm}(z,w)[\mathcal{H}_i^+,\mathcal{X}_j^{\pm}(w)]\Psi_i^+(z)=\partial_z\alpha_{\pm}(z,w)\mathcal{X}_j^{\pm}(w)\Psi_i^+(z).$$

Therefore, $\alpha_{\pm}(z,w)$ satisfies the following differential equation:

$$\frac{\partial_z\alpha_{\pm}(z,w)}{\alpha_{\pm}(z,w)}=\mp\sum_{d\in\mathbb{Z}/m\mathbb{Z}}\frac{(q^{c_{i,\sigma^d(j)}}-q^{-d_ic_{i,\sigma^d(j)}})\omega^d}{(z-q^{d_ic_{i,\sigma^d(j)}}\omega^dw)(z-q^{-d_ic_{i,\sigma^d(j)}}\omega^dw)}w.$$

The solution is

$$\alpha_{\pm}(z,w)=C\cdot\prod_{d\in\mathbb{Z}/m\mathbb{Z}}\frac{z-q^{\mp d_ic_{i,\sigma^d(j)}}\omega^dw}{z-q^{\pm d_ic_{i,\sigma^d(j)}}\omega^dw}.$$

The constant term of $\Psi_i^+(z)$ is $\exp\left(\frac{h}{2}\mathcal{H}_{i,0}\right)$ and

$$\exp\left(\frac{h}{2}\mathcal{H}_{i,0}\right)\mathcal{X}_j^{\pm}(w)=q^{\pm\sum_{d\in\mathbb{Z}/m\mathbb{Z}}d_ic_{i,\sigma^d(j)}}\mathcal{X}_j^{\pm}(w)\exp\left(\frac{h}{2}\mathcal{H}_{i,0}\right),$$

which implies $C=q^{\pm\sum_{d\in\mathbb{Z}/m\mathbb{Z}}d_ic_{i,\sigma^d(j)}}$. So we have

$$\alpha_{\pm}(z,w)=\prod_{d\in\mathbb{Z}/m\mathbb{Z}}\frac{q^{\pm d_ic_{i,\sigma^d(j)}}z-\omega^dw}{z-\omega^dq^{\pm d_ic_{i,\sigma^d(j)}}w}=\frac{G_{ij}^{\pm}(z,w)}{F_{ij}^{\pm}(z,w)}.$$

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