Twisted affine Lie superalgebra of type Q and quantization of its enveloping superalgebra

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Abstract

We introduce a new quantum group which is a quantization of the enveloping superalgebra of a twisted affine Lie superalgebra of type Q. We study generators and relations for superalgebras in the finite and twisted affine cases, and also universal central extensions. Afterwards, we apply the FRT formalism to a certain solution of the quantum Yang-Baxter equation to define that new quantum group and we study some of its properties. We construct a functor of Schur-Weyl type which connects it to affine Hecke-Clifford algebras and prove that it provides an equivalence between two categories of modules.

1 Introduction

Soon after the emergence of quantum groups attached to Kac-Moody Lie algebras in the 1980's, it was realized that it was also possible to quantize the enveloping superalgebras of certain Lie superalgebras, in particular those of classical type [Ya]. Simple Lie superalgebras were classified by V. Kac in [Ka] and, among the classical ones, are the so-called strange Lie superalgebras $\mathfrak{psq}_n$ of type Q. Quantum groups $\mathfrak{U}_q\mathfrak{q}_n$ for the Lie superalgebras $\mathfrak{q}_n$ (closely related to $\mathfrak{psq}_n$) were first constructed by G. Olshanski in [Ol]; moreover, he was able to relate them to Hecke-Clifford algebras (also called Sergeev algebras) by extending the classical Schur-Weyl construction.

Among quantum Kac-Moody groups of non-finite type, the quantized enveloping algebras of affine Lie algebras and the Yangians are those which have been the most studied. In [Na2], M. Nazarov associated a Yangian to a twisted Lie superalgebra of type Q. (See also [St].) Our goal in this paper is to construct a q-deformation, which is a Hopf algebra, of the enveloping superalgebra of a twisted affinization of a Lie superalgebra of type Q. (Previous results are contained in [Na1]; this article also considers the strange Lie superalgebras of type P and their associated Yangians.) Other mathematicians have been interested recently in Lie superalgebras of type Q: for instance, [GJKK] provides several interesting results about highest weight modules over $\mathfrak{U}_q\mathfrak{q}_n$ and it is announced in [GJKKK] that the authors have been able to develop a theory of crystal bases for this quantum supergroup. Blocks of the category $\mathcal{O}$ over $\mathfrak{q}_n$ were studied in [FrMa] and characters of finite dimensional $\mathfrak{q}_n$-modules were computed in [PeSe1, PeSe2]. A conjecture regarding characters of infinite dimensional modules over $\mathfrak{q}_n$ is presented in [Br]. The representation theory of (twisted) affine Lie superalgebras of type Q has not been much studied so far, but one notable article is [GoSe] which, however, focuses on Verma modules over a twisted affine Lie superalgebra of type Q different from the one that we consider.

We start by defining $\mathfrak{q}_n$ and $\mathfrak{sq}_n$ and give presentations of $\mathfrak{sq}_n$ in terms of generators and relations. Afterwards, in subsection 3.2, we present some results about central extensions of a certain matrix Lie superalgebra (more general results are contained in [ChGu]) and use them to study $\widehat{\mathfrak{q}}_n^{tw}$, the twisted affine Lie superalgebra of type Q, which is a central extension of $\mathcal{L}_{tw}\mathfrak{q}_n$, the twisted loop superalgebra of $\mathfrak{q}_n$. $\mathcal{L}_{tw}\mathfrak{q}_n$ bears some ressemblance to the $\mathfrak{sl}_n$-Onsager algebra [IvUg]. It should be possible to extend the results of subsection 3.2 and those in [ChGu] to the double affine setting: this may be useful in combining the ideas in [Gu1, Gu2] with the double affine Hecke-Clifford algebras studied in [KhWa2, KhWa3, Kh] to produce new double affine quantum superalgebras, but we do not consider this question in this paper.

The second half of the paper concerns the quantum setup. We recall the definition of affine Hecke-Clifford algebras in section 4. For the symmetric group, these were first studied in [JoNa]; in [KhWa1], the authors succeeded in associating similar algebras to the other classical Weyl groups. In section 5, we recall Olshansk's construction of the quantum group of type Q. Afterwards, we introduce its affine version, $\mathfrak{U}_q\widehat{\mathfrak{q}}_n^{tw}$, by using a certain matrix S, which is a solution of the quantum Yang-Baxter equation, and the FRT-presentation first developed in [FRT]. In the following section, we prove that $\mathfrak{U}_q\widehat{\mathfrak{q}}_n^{tw}$ is a Hopf algebra deformation of the enveloping superalgebra of $\widehat{\mathfrak{q}}_n^{tw}$ and provides a quantization of a certain co-Poisson Hopf algebra structure coming from a Lie bisuperalgebra structure on $\widehat{\mathfrak{q}}_n^{tw}$.

In the last section, we introduce the Schur-Weyl functor, extending the work of Olshanski [Ol] to the affine setting, thus obtaining a q-analog of one of the main theorems in [Na2] and a super analog of the central construction in [ChPr]. We prove that it provides an equivalence between certain categories and we establish some of its properties - for instance, that it sends an irreducible module to zero or to an irreducible one.

It should be possible to realize $\mathcal{L}_{tw}\mathfrak{q}_n$ via a centralizer construction by combining the work of [NaSe] with [HoMo], but we do not treat this question here.

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3 The Lie superalgebra of type Q and its associated twisted affine Lie superalgebra

We start by recalling definitions relevant for our work. Unless specified, all the (Lie) algebras and superalgebras are defined over $\mathbb{C}$.

3.1 Finite type

The space $\mathbb{C}(n|n) = \mathbb{C}^n \oplus \mathbb{C}^n$ is $\mathbb{Z}/2\mathbb{Z}$-graded; we denote its standard basis by $e_{-n}, \dots, e_{-1}, e_1, \dots, e_n$. The parity of e_i equals $|i|$ where $|i| = 1$ if $i < 0$ and $|i| = 0$ if $i > 0$. $\mathrm{End}_{\mathbb{C}}(\mathbb{C}(n|n))$ is a superalgebra, i.e. a $\mathbb{Z}/2\mathbb{Z}$-graded algebra. It has a basis given by matrices E_{ab} with $-n \le a, b \le n$, $ab \ne 0$ and the parity of E_{ab} is $|E_{ab}| = |a| + |b|$.

Definition 3.1. *The general linear Lie superalgebra* $\mathfrak{gl}_{n|n}$ *is defined as the space* $\mathrm{End}_{\mathbb{C}}(\mathbb{C}(n|n))$ *endowed with the superbracket* $[m_1, m_2] = m_1m_2 - (-1)^{|m_1||m_2|}m_2m_1$, $m_1, m_2 \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}(n|n))$.

The supertrace on $\mathfrak{gl}_{n|n}$ is given by $Str(E_{ab}) = \delta_{ab}(-1)^{|a|}$ and is extended linearly to all of $\mathfrak{gl}_{n|n}$. The special linear Lie superalgebra is defined to be $\mathfrak{sl}_{n|n} = \{E \in \mathfrak{gl}_{n|n} | Str(E) = 0\}$. Let $J \in \mathfrak{gl}_{n|n}$ be the element $J = \sum_{a=1}^{n}(E_{a,-a} - E_{-a,a})$. The map $\iota : \mathfrak{gl}_{n|n} \longrightarrow \mathfrak{gl}_{n|n}$ given by $E_{ab} \mapsto E_{-a,-b}$ is an involutive automorphism which can be restricted to $\mathfrak{sl}_{n|n}$.

The two closely related Lie superalgebras $\mathfrak{q}_n$ and $\mathfrak{sq}_n$ that we now define will be among the main objects of interest in this article.

Definition 3.2. *The Lie superalgebra* $\mathfrak{q}_n$ *can be defined equivalently as either the centralizer of* J *in* $\mathfrak{gl}_{n|n}$ *or the fixed-point subalgebra of* $\mathfrak{gl}_{n|n}$ *with respect to* ι*. We set* $\mathfrak{sq}_n = [\mathfrak{q}_n, \mathfrak{q}_n]$.

$\mathfrak{sq}_n$ is the Lie subsuperalgebra of $\mathfrak{q}_n$ consisting of matrices whose upper-right and lower-left $n \times n$ blocks have trace zero (but not necessarily the two diagonal $n \times n$ blocks). Note that $\mathfrak{sq}_n$ contains the identity matrix. The simple Lie superalgebra of type Q_n, as defined in [Ka], is the quotient $\mathfrak{sq}_n/\mathbb{C}I_{2n}$, which is usually denoted $\mathfrak{psq}_n$.

A basis of $\mathfrak{q}_n$ is given by $\mathsf{E}^0_{ab} = E_{ab} + E_{-a,-b}$ and $\mathsf{E}^1_{ab} = E_{-a,b} + E_{a,-b}$ for $1 \le a, b \le n$. Its even part is isomorphic to the Lie algebra $\mathfrak{gl}_n$ and its odd part to $\mathfrak{gl}_n$, viewed as a $\mathfrak{gl}_n$-module via the adjoint action. As for $\mathfrak{sq}_n$, its even part is $\mathfrak{gl}_n$, but its odd part is $\mathfrak{sl}_n$, so a basis is given by E^0_{ab}, E^0_{aa}, E^1_{ab} with $1 \le a \ne b \le n$ and by $\mathsf{H}^1_a = \mathsf{E}^1_{aa} - \mathsf{E}^1_{a+1,a+1}$ for $1 \le a \le n-1$.

It may also be useful to use the following notation for a basis (which is what is done in [Ol]). For any $-n \le a, b \le n$ with $ab \ne 0$ and $a \le b$, we set $\mathsf{L}_{ab} = \mathsf{E}^1_{-a,b}$ if $a < 0 < b$, $\mathsf{L}_{ab} = \mathsf{E}^0_{ab}$ if $0 < a \le b$, $\mathsf{L}_{ab} = \mathsf{E}^0_{-a,-b}$ if $a \le b < 0$. (Note that $\mathsf{L}_{aa} = \mathsf{L}_{-a,-a}$.)

There is another way to view $\mathfrak{q}_n$ and $\mathfrak{sq}_n$ which is relevant considering the connection between $\mathfrak{q}_n$ and Hecke-Clifford algebras. Let Λ be the Clifford algebra on just one generator $\mathbf{c}$, so $\Lambda = \mathrm{span}_{\mathbb{C}}\{1, \mathbf{c}\}$ with $\mathbf{c}^2 = 1$. We view Λ as a superalgebra with $\deg(\mathbf{c}) = 1$. We can apply the following general result.

Proposition 3.1. *Let A be a unital superalgebra, so A is a $\mathbb{Z}/2\mathbb{Z}$-graded algebra, $A = A_0 \oplus A_1$ and $1 \in A_0$. The space $\mathfrak{gl}_n(A)$ of $n \times n$-matrices with entries in A can be turned into a Lie superalgebra by defining the superbracket $[\cdot,\cdot]$ in the following way:*

$$[m_1 \otimes a_1, m_2 \otimes a_2] = (m_1 m_2) \otimes (a_1 a_2) - (-1)^{|a_1||a_2|}(m_2 m_1) \otimes (a_2 a_1)$$

(Here, $|a_i|$ denotes the parity of a_i.)

We will apply this proposition again in subsection 3.2.

Definition 3.3. *With A as in the preceding proposition, we define the Lie superalgebra $\mathfrak{sl}_n(A)$ to be the derived Lie superalgebra $\mathfrak{sl}_n(A) = [\mathfrak{gl}_n(A), \mathfrak{gl}_n(A)]$.*

Remark 3.1. *We have a vector space decomposition $\mathfrak{sl}_n(A) = \mathfrak{sl}_n \otimes_{\mathbb{C}} A \oplus I \otimes [A, A]$.*

We can now give a second description of $\mathfrak{q}_n$ and $\mathfrak{sq}_n$. Note that, if $A = \Lambda$, then $[A, A] = \mathbb{C}$.

Lemma 3.1. *The Lie superalgebra $\mathfrak{sq}_n$ (resp. $\mathfrak{q}_n$) is isomorphic to $\mathfrak{sl}_n(\Lambda)$ (resp. $\mathfrak{gl}_n(\Lambda)$).*

Proof. It is enough to prove the statement for $\mathfrak{q}_n$ and $\mathfrak{gl}_n(\Lambda)$, and for this we give an explicit isomorphism $\mathfrak{gl}_n(\Lambda) \xrightarrow{\sim} \mathfrak{q}_n$: $E_{ab}(1) \mapsto \mathsf{E}^0_{ab}$ $E_{ab}(\mathbf{c}) \mapsto \mathsf{E}^1_{ab}$. □

The rest of the propositions of this subsection are analogs of the classical theorem of Serre for semisimple Lie algebras. The last one will be useful in the quantum affine setting. Similar presentations are known for basic classical Lie superalgebras (see [FSS], and references therein, where it is explained that one often has to introduce relations involving more than two generators, as in proposition 3.3 below).

A presentation of the Lie superalgebra $\mathfrak{q}_n$ is given in [LeSe] and the q-deformation of the relations can be found in [GJKK]. The reference [LeSe] is not available to the authors and, since it is probably not easily available to many readers, we have decided to include a proof for completeness. (Actually, our proof is for $\mathfrak{sq}_n$ since we consider this Lie superalgebra in the following proposition, but could be modified accordingly for $\mathfrak{q}_n$ by changing slightly $H_{i,1}, 1 \le i \le n-1$ and adding a new odd generator $H_{n,1}$.)

Proposition 3.2. *The Lie superalgebra $\mathfrak{sq}_n$ is isomorphic to the Lie superalgebra $\mathfrak{g}$ generated by $X^{\pm}_{i,j}, H_{i,1}$ with $1 \le i \le n-1, j = 0, 1$ and by $H_{i,0}$ with $1 \le i \le n$ (the second index indicates the parity) which satisfy the relations: (below, $\|\cdot\|$ denotes the absolute value, not the parity)*

$$[H_{i_1,1}, H_{i_2,1}] = 2\delta_{i_1 i_2}(H_{i_1,0} + H_{i_1+1,0}) - 2\delta_{i_1+1,i_2} H_{i_2,0} - 2\delta_{i_2+1,i_1} H_{i_1,0}, \tag{1}$$

$$[H_{i_1,j_1}, H_{i_2,j_2}] = 0 \ \textit{if } (j_1, j_2) \ne (1,1), \quad [H_{i_1,0}, X^{\pm}_{i_2,j_2}] = \pm(\delta_{i_1 i_2} - \delta_{i_1,i_2+1}) X^{\pm}_{i_2,j_2}, \tag{2}$$

$$[H_{i_1,1}, X^{\pm}_{i_2,j_2}] = (\pm 1)^{j_2+1}(2\delta_{i_1 i_2}\delta_{j_2 0} - (-1)^{j_2}\delta_{i_1,i_2+1} - \delta_{i_1+1,i_2}) X^{\pm}_{i_2,j_2+1}, \tag{3}$$

$$[X^+_{i_1,j_1}, X^-_{i_2,j_2}] = \delta_{i_1 i_2} H_{i_1,j_1+j_2} \ \textit{if } j_1 \ne j_2, \quad [X^+_{i_1,j_1}, X^-_{i_2,j_1}] = \delta_{i_1 i_2}(H_{i_1,0} - (-1)^{j_1} H_{i_1+1,0}), \tag{4}$$

$$[X^{\pm}_{i_1,j_1}, X^{\pm}_{i_2,j_2}] = (\pm 1)^{j_1+j_2+1}[X^{\pm}_{i_1,j_1+1}, X^{\pm}_{i_2,j_2+1}] \text{ if } i_1 \le i_2, \tag{5}$$

$$ad(X^{\pm}_{i_1,j_1})^2(X^{\pm}_{i_2,j_2}) = 0 \text{ for any } 1 \le i_1, i_2 \le n-1,\ j_1,j_2 = 0,1 \text{ if } \|i_1 - i_2\| = 1. \tag{6}$$

$$[X^{\pm}_{i_1,j_1}, X^{\pm}_{i_2,j_2}] = 0 \text{ for any } 1 \le i_1, i_2 \le n-1,\ j_1,j_2 = 0,1 \text{ if } \|i_1 - i_2\| \neq 1. \tag{7}$$

Remark 3.2. *The second relation in* (4) *when* $j_1 = 1$ *can be deduced from the other ones.*

Proof. We define a map $\varphi : \mathfrak{g} \twoheadrightarrow \mathfrak{sq}_n$ by $X^+_{i,j} \mapsto \mathsf{E}^j_{i,i+1},\ X^-_{i,j} \mapsto \mathsf{E}^j_{i+1,i},\ H_{i,0} \mapsto \mathsf{E}^0_{ii},\ H_{i,1} \mapsto \mathsf{H}^1_i$.

We check, for instance, that it respects relation (5) in the $+$ case when $j_1 = 1 = j_2$ and $i_2 = i_1 + 1$.

$$\begin{aligned}
[\varphi(X^+_{i_1,1}), \varphi(X^+_{i_2,1})] &= [E_{-i_1,i_1+1} + E_{i_1,-i_1-1}, E_{-i_1-1,i_1+2} + E_{i_1+1,-i_1-2}] \\
&= E_{-i_1,-i_1-2} + E_{i_1,i_1+2} = [E_{i_1,i_1+1} + E_{-i_1,-i_1-1}, E_{i_1+1,i_1+2} + E_{-i_1-1,-i_1-2}] \\
&= [\varphi(X^+_{i_1,0}), \varphi(X^+_{i_2,0})]
\end{aligned}$$

Serre's theorem for $\mathfrak{sl}_n$ (slightly adapted to $\mathfrak{gl}_n$) implies that we have a map $\mathfrak{gl}_n \to \mathfrak{g}$ which, composed with $\varphi : \mathfrak{g} \twoheadrightarrow \mathfrak{sq}_n$, is an isomorphism onto the even part of $\mathfrak{sq}_n$. We thus have an adjoint action of $\mathfrak{gl}_n$ on $\mathfrak{g}$. Equations (1)-(5) imply that $\mathfrak{g}$ can be decomposed as $\mathfrak{g} = \mathfrak{g}^- + \mathfrak{g}^0 + \mathfrak{g}^+$ where $\mathfrak{g}^{\pm}$ is the subalgebra generated by $X^{\pm}_{i,j}$ for $1 \le i \le n-1, j = 0,1$ and $\mathfrak{g}^0$ is the span of the $H_{i,j}$. Furthermore, each of these is a Lie subsuperalgebra and, considering the weights of the adjoint action of the standard Cartan subalgebra of $\mathfrak{gl}_n$, we deduce that that decomposition is a direct sum. Note that the map $X^+_{i,j} \mapsto X^-_{i,j}, H_{i,j} \mapsto H_{i,j}$ is an anti-involution of $\mathfrak{g}$. It is thus enough to prove the following claim: a vector space basis of $\mathfrak{g}^+$ is given by:

$$\left\{ \left[X^+_{i_1,0}, \left[X^+_{i_1+1,0}, \left[\cdots, [X^+_{i_2-1,0}, X^+_{i_2,j}]\cdots\right]\right]\right] | 1 \le i_1 \le i_2 \le n-1,\ j = 0,1 \right\}.$$

The proof of this claim is by induction using relations (5),(6),(7). (The order of the brackets does not matter, so we just picked a convenient one.) □

Let us give another presentation of $\mathfrak{sq}_n$. It is inspired by the presentation of $\mathfrak{sl}_{n|n}$ given in [Ya] and also by the fact that one of the extra generator corresponds, in the Chevalley-Kac presentation of the loop algebra $\mathfrak{sl}_n(\mathbb{C}[u,u^{-1}])$, to $E_{n1} \otimes u$.

Proposition 3.3. *Suppose that* $n \ge 4$. $\mathfrak{sq}_n$ *is isomorphic to the Lie superalgebra* $\mathfrak{a}$ *generated by the even elements* e_i, f_i, h_i, h_n *with* $1 \le i \le n-1$ *and by the odd elements* $e_n, \widetilde{h}_n$ *which satisfy the following relations:*

The elements e_i, f_i, h_i, h_n *with* $1 \le i \le n-1$ *satisfy the standard relations (cf. Serre's theorem for* $\mathfrak{sl}_n$*) of the Lie algebra* $\mathfrak{gl}_n$*. Furthermore,*

$$[e_n, e_n] = 0, \quad [e_i, e_n] = 0 \text{ if } i \neq 1, n-1, \quad [e_n, [e_n, e_1]] = 0, \quad [e_n, [e_n, e_{n-1}]] = 0, \tag{8}$$

$$[f_i, e_n] = 0\ \forall\, i = 1, \ldots, n-1, \quad [e_1, [e_1, e_n]] = 0, \quad [e_{n-1}, [e_{n-1}, e_n]] = 0, \tag{9}$$

$$[h_i, e_n] = 0,\ i \neq 1, n, \quad [h_1, e_n] = -e_n = -[h_n, e_n], \tag{10}$$

$$\widetilde{h}_n = \left[e_n, \left[\cdots [[e_1, e_2], e_3], \cdots, e_{n-1}\right]\right], \quad [\widetilde{h}_n, \widetilde{h}_n] = 2(h_1 + h_n), \tag{11}$$

$$\left[e_n, \left[e_{n-1}, [e_1, e_n]\right]\right] = 0, \quad \left[\widetilde{h}_n, [e_n, e_1]\right] = \left[\cdots [[f_{n-1}, f_{n-2}], \cdots], f_2\right] \tag{12}$$

Proof. We will need the relation

$$\left[[\widetilde{h}_n, e_{n-1}], e_n\right] = -\left[\cdots [[f_{n-2}, f_{n-3}], f_{n-4}], \cdots, f_1\right], \tag{13}$$

which follows from (12). Let us prove it first. We will need to use $[\widetilde{h}_n, e_n] = 0$: this is proved later, independently of the proof of (13). Set $e_{2,n} = \Big[\big[\cdots[e_2, e_3],\big], \cdots e_{n-1}\Big]$. (12) implies that $[[\widetilde{h}_n, e_{n-1}], [e_n, e_1]] + \Big[\widetilde{h}_n, [[e_n, e_1], e_{n-1}]\Big] = -\Big[\cdots[[f_{n-2}, f_{n-3}], \cdots], f_2\Big]$, so, applying $ad(f_1)$ and using that $\big[f_1, [\widetilde{h}_n, e_{n-1}]\big] = 0$, we find that

$$\begin{aligned}
&\Big[\cdots[[f_{n-2}, f_{n-3}], \cdots], f_1\Big] = -\Big[[\widetilde{h}_n, f_1], [[e_n, e_1], e_{n-1}]\Big] - [[\widetilde{h}_n, e_{n-1}], [e_n, h_1]] - \Big[\widetilde{h}_n, [[e_n, h_1], e_{n-1}]\Big] \\
&= \Big[[e_n, e_{2,n}], [[e_n, e_1], e_{n-1}]\Big] - [[\widetilde{h}_n, e_{n-1}], e_n] - [\widetilde{h}_n, [e_n, e_{n-1}]] = \Big[e_n, [e_{2,n}, [[e_n, e_1], e_{n-1}]]\Big] \text{ by (12)} \\
&= \Big[e_n, [[e_{2,n}, [e_n, e_1]], e_{n-1}]\Big] = \Big[e_n, [[[e_{2,n}, e_n], e_1], e_{n-1}]\Big] + \Big[e_n, [[e_n, [e_{2,n}, e_1]], e_{n-1}]\Big] \\
&= \Big[e_n, [[[e_{2,n}, e_n], e_{n-1}], e_1]\Big] - [e_n, [\widetilde{h}_n, e_{n-1}]] = \Big[e_n, [[e_{2,n}, [e_n, e_{n-1}]], e_1]\Big] - [e_n, [\widetilde{h}_n, e_{n-1}]] \\
&= -\big[e_n, [\widetilde{h}_n, e_{n-1}]\big]
\end{aligned}$$

because $\big[e_{2,n}, [e_n, e_{n-1}]\big] = 0$. Indeed,

$$\big[e_{2,n}, [e_n, e_{n-1}]\big] = \Big[\big[[e_{2,n-2}, e_{n-2}], e_{n-1}\big], [e_n, e_{n-1}]\Big] = \Big[\big[e_{2,n-2}, [e_{n-2}, [e_n, e_{n-1}]]\big], e_{n-1}\Big]$$

because $\big[e_{n-1}, [e_n, e_{n-1}]\big] = 0$; moreover,

$$\big[e_{2,n}, [e_n, e_{n-1}]\big] = \big[[e_{2,n}, e_n], e_{n-1}\big] = \Big[\big[e_{2,n-2}, [e_{n-2}, [e_{n-1}, e_n]]\big], e_{n-1}\Big]$$

since $[e_{2,n}, e_{n-1}] = 0$, so $\big[e_{2,n}, [e_n, e_{n-1}]\big] = -\big[e_{2,n}, [e_n, e_{n-1}]\big]$.

We now define an epimorphism $\pi : \mathfrak{a} \twoheadrightarrow \mathfrak{sq}_n$ and we show that the dimension of $\mathfrak{a}$ is at most the dimension of $\mathfrak{sq}_n$, so that π must be an isomorphism.

Set $\pi(h_i) = \mathsf{E}^0_{ii}$ for $1 \le i \le n$, $\pi(e_i) = \mathsf{E}^0_{i,i+1}$ and $\pi(f_i) = \mathsf{E}^0_{i+1,i}$ for $1 \le i \le n-1$; set also $\pi(e_n) = \mathsf{E}^1_{n1}$ and $\pi(\widetilde{h}_n) = \mathsf{E}^1_{nn} - \mathsf{E}^1_{11}$. One can readily check that π respects the defining relations of $\mathfrak{a}$. $\pi(e_n)$ generates the odd part of $\mathfrak{sq}_n$ as a module over its even part (via the adjoint representation), so π is indeed onto.

Let us determine a spanning set of vectors for $\mathfrak{a}$. Let $\widetilde{\mathfrak{a}}_0$ be the Lie subalgebra of $\mathfrak{a}$ generated by $e_i, f_i, h_i, h_n, 1 \le i \le n-1$. Let $\widetilde{\mathfrak{a}}_1$ be the cyclic $\widetilde{\mathfrak{a}}_0$-submodule of $\mathfrak{a}$ generated by e_n via the adjoint representation of $\mathfrak{a}$ on itself, but restricted to $\widetilde{\mathfrak{a}}_0$. $\widetilde{\mathfrak{a}}_0$ is isomorphic to $\mathfrak{gl}_n$ and π maps $\widetilde{\mathfrak{a}}_1$ onto the odd part of $\mathfrak{sq}_n$.

Since $[f_i, e_n] = 0$, e_n is a lowest weight vector of $\mathfrak{gl}_n \cong \widetilde{\mathfrak{a}}_0$. Moreover, $ad(e_i)^2(e_n) = 0$ for $i = 1, n-1$ and $[e_i, e_n] = 0$ if $i \neq 1, n-1$. It follows that $\widetilde{\mathfrak{a}}_1$ is isomorphic to the adjoint representation of $\mathfrak{gl}_n$ on $\mathfrak{sl}_n$. Therefore, it is enough to prove that $\mathfrak{a} = \widetilde{\mathfrak{a}}_0 \oplus \widetilde{\mathfrak{a}}_1$. Let us show that, if $X_1, X_2 \in \widetilde{\mathfrak{a}}_1$, then $[X_1, X_2] \in \widetilde{\mathfrak{a}}_0$. Using the super Jacobi identity and the fact that $\widetilde{\mathfrak{a}}_1$ is generated by e_n under the adjoint action of $\widetilde{\mathfrak{a}}_0$, we can reduce the problem to proving that $[e_n, X] \in \widetilde{\mathfrak{a}}_0$ if $X \in \widetilde{\mathfrak{a}}_1$.

Set

$$\alpha_i = \Big[\cdots[[e_1, e_2], e_3], \cdots, e_i\Big] \text{ and } \beta_j = \Big[\cdots[[e_j, e_{j+1}], e_{j+2}], \cdots, e_{n-1}\Big] \text{ for } i, j = 1, \ldots, n-1.$$

Then, by (12) and the fact that $[\widetilde{h}_n, e_n] = 0$ (proved below), we can see that

$$\Big[[e_n, \beta_2], [e_n, e_1]\Big] = -\Big[\cdots[[f_{n-1}, f_{n-2}], f_{n-3}], \cdots, f_1\Big] \tag{14}$$

Since $\widetilde{\mathfrak{a}}_1$ is isomorphic to the adjoint representation of $\mathfrak{gl}_n$ on $\mathfrak{sl}_n$, $\widetilde{\mathfrak{a}}_1$ has a basis

$$\{e_n;\ [\alpha_i, e_n], 1 \le i \le n-1;\ [\beta_j, e_n], 2 \le j \le n-1;\ [\alpha_i, [\beta_j, e_n]], 1 \le i, j \le n-1\}.$$

Now we prove $[e_n, \widetilde{\mathfrak{a}}_1] \subseteq \widetilde{\mathfrak{a}}_0$ in seven steps.

1) $[e_n, e_n] = 0$. This is the first relation in (8).

2) $\big[e_n, [e_n, \alpha_i]\big] = 0$, for $1 \leq i \leq n-1$. If $i \neq n-1$, then $\big[e_n, [e_n, \alpha_i]\big] = 0$ since $\big[e_n, [e_n, e_1]\big] = 0$. If $i = n-1$, then $[e_n, \alpha_{n-1}] = \widetilde{h}_n$, so $[e_n, [e_n, \alpha_{n-1}]] = 0$. Indeed,

$$
\begin{aligned}
[e_n, \widetilde{h}_n] &= \Big[e_n, \big[e_n, [\alpha_{n-2}, e_{n-1}]\big]\Big] = \Big[e_n, \big[[e_n, \alpha_{n-2}], e_{n-1}\big]\Big] + \Big[e_n, \big[\alpha_{n-2}, [e_n, e_{n-1}]\big]\Big] \\
&= -\Big[\big[e_n, \alpha_{n-2}\big], \big[e_n, e_{n-1}\big]\Big] + \Big[\big[e_n, \alpha_{n-2}\big], \big[e_n, e_{n-1}\big]\Big] = 0
\end{aligned}
$$

(It is also true that $[\widetilde{h}_n, e_i] = 0$ for $2 \leq i \leq n-2$.)

3) $\big[e_n, [e_n, \beta_j]\big] = 0$, for $2 \leq j \leq n-1$, since $\big[e_n, [e_n, e_{n-1}]\big] = 0$.

4) $\Big[e_n, \big[\alpha_{n-1}, [\beta_j, e_n]\big]\Big] \in \widetilde{\mathfrak{a}}_0$ for $1 \leq j \leq n-1$. For $j = 1$ we have $[\beta_1, e_n] = -\widetilde{h}_n$, hence, since $[\widetilde{h}_n, e_n] = 0$,

$$
\Big[e_n, \big[\alpha_{n-1}, [\beta_1, e_n]\big]\Big] = \big[[e_n, \alpha_{n-1}], [\beta_1, e_n]\big] = -[\widetilde{h}_n, \widetilde{h}_n] = -2(h_1 + h_n) \in \widetilde{\mathfrak{a}}_0 \text{ by (11).}
$$

If $j \neq 1$ and we write $\beta_j = [\widetilde{\beta}_j, e_{n-1}]$, we have

$$
\begin{aligned}
\Big[e_n, \big[\alpha_{n-1}, [\beta_j, e_n]\big]\Big] &= \Big[\widetilde{h}_n, \big[[\widetilde{\beta}_j, e_{n-1}], e_n\big]\Big] = \Big[\big[\widetilde{h}_n, [\widetilde{\beta}_j, e_{n-1}]\big], e_n\Big] \\
&= \Big[\big[\widetilde{\beta}_j, [\widetilde{h}_n, e_{n-1}]\big], e_n\Big] = \Big[\widetilde{\beta}_j, \big[[\widetilde{h}_n, e_{n-1}], e_n\big]\Big]
\end{aligned}
$$

Hence by equation (13), we have $\Big[e_n, \big[\alpha_{n-1}, [\beta_j, e_n]\big]\Big] \in \widetilde{\mathfrak{a}}_0$.

5) Since $[\beta_1, e_n] = -\widetilde{h}_n$, so using equation (12) and $[\widetilde{h}_n, e_n] = 0$, we can show $\Big[e_n, \big[\alpha_i, [\beta_1, e_n]\big]\Big] \in \widetilde{\mathfrak{a}}_0$ for $1 \leq i \leq n-2$.

6) If $2 \leq j \neq i+1 \leq n-1$, then $\Big[e_n, \big[\alpha_i, [\beta_j, e_n]\big]\Big] = 0$.

(i) If $j \neq 2$, writing $\alpha_i = [e_1, \widetilde{\alpha}_i]$ and $\beta_j = [\widetilde{\beta}_j, e_{n-1}]$, then

$$
\Big[\beta_j, \big[e_n, \alpha_i\big]\Big] = \Big[\beta_j, \big[[e_n, e_1], \widetilde{\alpha}_i\big]\Big] = \Big[\big[[\widetilde{\beta}_j, e_{n-1}], [e_n, e_1]\big], \widetilde{\alpha}_i\Big] = \Big[\big[\widetilde{\beta}_j, [e_{n-1}, [e_n, e_1]]\big], \widetilde{\alpha}_i\Big].
$$

Hence, by equation (12), $\Big[e_n, \big[\beta_j, [e_n, \alpha_i]\big]\Big] = \Big[\big[\widetilde{\beta}_j, [e_n, [e_{n-1}, [e_n, e_1]]]\big], \widetilde{\alpha}_i\Big] = 0$.

(ii) We have $\Big[\beta_2, \big[e_n, \alpha_i\big]\Big] = \Big[\beta_2, \big[[e_n, e_1], \widetilde{\alpha}_i\big]\Big] = \Big[\big[\beta_2, [e_n, e_1]\big], \widetilde{\alpha}_i\Big]$. Thus by equation (14) and $\widetilde{\mathfrak{a}}_0 \cong \mathfrak{gl}_n$, $\Big[e_n, \big[\beta_2, [e_n, \alpha_i]\big]\Big] = \Big[\big[e_n, [\beta_2, [e_n, e_1]]\big], \widetilde{\alpha}_i\Big] = 0$.

7) $\Big[\big[e_n, \beta_{i+1}\big], \big[e_n, \alpha_i\big]\Big] \in \widetilde{\mathfrak{a}}_0$ for $1 \leq i \leq n-2$. The case $i = 1$ is equation (14), so let us assume that $i \neq 1$.

$$
\begin{aligned}
\Big[\big[e_n, \beta_{i+1}\big], \big[e_n, \alpha_i\big]\Big] &= \Big[\big[e_n, \beta_{i+1}\big], \big[e_n, [\alpha_{i-1}, e_i]\big]\Big] = \Big[\big[e_n, \beta_{i+1}\big], \big[[e_n, \alpha_{i-1}], e_i\big]\Big] \\
&= \Big[\big[[e_n, \beta_{i+1}], [e_n, \alpha_{i-1}]\big], e_i\Big] - \Big[\big[e_n, \alpha_{i-1}\big], \big[[e_n, \beta_{i+1}], e_i\big]\Big] \\
&= -\Big[\big[e_n, \alpha_{i-1}\big], \big[e_n, [\beta_{i+1}, e_i]\big]\Big] = \Big[\big[e_n, \beta_i\big], \big[e_n, \alpha_{i-1}\big]\Big]
\end{aligned}
$$

Hence by equation (14), we have $\Big[\big[e_n, \beta_{i+1}\big], \big[e_n, \alpha_i\big]\Big] = \Big[\big[e_n, \beta_2\big], \big[e_n, \alpha_1\big]\Big] \in \widetilde{\mathfrak{a}}_0$ for all $1 \leq i \leq n-2$. □

The last presentation that we give has the advantage of involving a smaller number of generators than the one given in proposition 3.2 with relations that are simpler than those in proposition 3.3. Olshanski was probably referring to a similar presentation for $\mathfrak{q}_n$ in his third remark in [Ol].

Proposition 3.4. *$\mathfrak{sq}_n$ is isomorphic to the Lie superalgebra $\mathfrak{k}$ generated by the even elements e_i, f_i, h_i, h_n with $1 \leq i \leq n-1$ and by the odd elements $\widetilde{e}_1, \widetilde{f}_1, \widetilde{h}_1$ which satisfy the following relations: the elements e_i, f_i, h_i, h_n with $1 \leq i \leq n-1$ satisfy the standard relations of Serre's theorem for $\mathfrak{sl}_n$ extended to $\mathfrak{gl}_n$ and, furthermore,*

$$[e_i, \widetilde{e}_1] = 0 \text{ if } i \neq 1,2, \quad [e_1, \widetilde{e}_1] = 0 = \big[e_2, [e_2, \widetilde{e}_1]\big], \;\; \big[\widetilde{e}_1, [\widetilde{e}_1, e_2]\big] = 0 = \big[\widetilde{e}_1, [e_1, e_2]\big], \tag{15}$$

$$[f_i, \widetilde{f}_1] = 0 \text{ if } i \neq 1,2, \quad [f_1, \widetilde{f}_1] = 0 = \big[f_2, [f_2, \widetilde{f}_1]\big], \;\; \big[\widetilde{f}_1, [\widetilde{f}_1, f_2]\big] = 0 = \big[\widetilde{f}_1, [f_1, f_2]\big], \tag{16}$$

$$[f_i, \widetilde{e}_1] = 0 = [e_i, \widetilde{f}_1] \text{ if } i \neq 1, \; [\widetilde{e}_1, f_1] = \widetilde{h}_1 = [e_1, \widetilde{f}_1], \; [\widetilde{e}_1, \widetilde{f}_1] = h_1 + h_2, \; [\widetilde{e}_1, \widetilde{e}_1] = 0 = [\widetilde{f}_1, \widetilde{f}_1], \tag{17}$$

$$[h_i, \widetilde{e}_1] = 0 = [h_i, \widetilde{f}_1] \text{ if } i \neq 1,2, \quad [h_1, \widetilde{e}_1] = \widetilde{e}_1 = -[h_2, \widetilde{e}_1], \quad [h_1, \widetilde{f}_1] = -\widetilde{f}_1 = -[h_2, \widetilde{f}_1], \tag{18}$$

Remark 3.3. *The following commutator relations between the generators can be deduced from these relations:*

$$[\widetilde{h}_1, h_j] = 0 \;\; \forall\, 1 \leq j \leq n, \quad [\widetilde{h}_1, \widetilde{h}_1] = 2h_1 + 2h_2, \tag{19}$$

$$[\widetilde{h}_1, \widetilde{e}_1] = 0, \quad [\widetilde{h}_1, \widetilde{f}_1] = 0, \quad [\widetilde{h}_1, e_1] = 2\widetilde{e}_1, \; [\widetilde{h}_1, f_1] = -2\widetilde{f}_1, \quad [\widetilde{h}_1, e_i] = 0 = [\widetilde{h}_1, f_i] \text{ if } i > 2 \tag{20}$$

$$[\widetilde{h}_1, e_2] = -\big[f_1, [\widetilde{e}_1, e_2]\big], \quad [\widetilde{h}_1, f_2] = \big[[f_2, \widetilde{f}_1], e_1\big] \tag{21}$$

Proof. We define a Lie superalgebra homomorphism $\pi : \mathfrak{k} \twoheadrightarrow \mathfrak{sq}_n$ which is the same on $e_i, f_i, h_i, h_n, 1 \leq i \leq n-1$ as the one in the proof of proposition 3.3 and for which $\pi(\widetilde{e}_1) = \mathsf{E}^1_{12}, \pi(\widetilde{f}_1) = \mathsf{E}^1_{21}$ and $\pi(\widetilde{h}_1) = \mathsf{E}^1_{11} - \mathsf{E}^1_{22}$. To prove that π is an isomorphism, we show that the dimension of $\mathfrak{k}$ is at most $2n^2 - 1$.

The elements $e_i, f_i, h_i, h_n, 1 \leq i \leq n-1$ generate an even subalgebra $\mathfrak{k}_0$ isomorphic to $\mathfrak{gl}_n$. Let $\mathfrak{k}_1$ be the $\mathfrak{k}_0$-submodule of $\mathfrak{k}$ generated by $\widetilde{e}_1$. $\mathfrak{k}_1$ is also generated by $\bar{e} = \Big[\big[\cdots[\widetilde{e}_1, e_2], \cdots\big], e_{n-1}\Big]$. $ad(e_i)(\bar{e}) = 0$ if $1 \leq i \leq n-1$, so $\bar{e}$ is a highest weight vector; since $ad(f_i)(\bar{e}) = 0$ if $2 \leq i \leq n-2$ and $ad(f_i)^2(\bar{e}) = 0$ if $i = 1, n-1$, it follows that $\mathfrak{k}_1$ is isomorphic to the adjoint representation of $\mathfrak{gl}_n$ on $\mathfrak{sl}_n$.

As in the proof of proposition 3.3, it is enough to explain why $[\mathfrak{k}_1, \mathfrak{k}_1] \subset \mathfrak{k}_0$. Since $\mathfrak{k}_1$ is isomorphic to the adjoint representation of $\mathfrak{gl}_n$ on $\mathfrak{sl}_n$, it is spanned by $\widetilde{e}_{1j}, \widetilde{f}_{i1}, \widetilde{e}_{ij} = [f_{i1}, \widetilde{e}_{1j}]$ with $\widetilde{e}_{1j} = \Big[\big[\cdots[\widetilde{e}_1, e_2], \cdots\big], e_{j-1}\Big]$, $\widetilde{f}_{i1} = \Big[f_{i-1}, \big[\cdots, [f_2, \widetilde{f}_1]\cdots\big]\Big]$ for all $2 \leq i, j \leq n$. (We define f_{i1} (resp. e_{1j}) as $\widetilde{f}_{i1}$ (resp. $\widetilde{e}_{1j}$) but with $\widetilde{f}_1$ replaced by f_1 (resp. $\widetilde{e}_1$ replaced by e_1)). We can also write $\widetilde{e}_{ij} = [\widetilde{f}_{i1}, e_{1j}]$.

We must prove that $[\widetilde{e}_{i_1 j_1}, \widetilde{e}_{i_2 j_2}] \in \mathfrak{k}_0$ for any $1 \leq i_1, i_2, j_1, j_2 \leq n$ (with $\widetilde{e}_{i1} = \widetilde{f}_{i1}$). It is actually enough to show that $[\widetilde{e}_1, \mathfrak{k}_1] \subset \mathfrak{k}_0$. (This implies that $[\widetilde{f}_1, \mathfrak{k}_1] \subset \mathfrak{k}_0$.) This is indeed the case since we have:

$$[\widetilde{e}_1, \widetilde{e}_{1j}] = \big[\widetilde{e}_1, [\widetilde{e}_1, e_{2j}]\big] = 0,$$

$$[\widetilde{e}_1, \widetilde{f}_{i1}] = \big[\widetilde{e}_1, [f_{i2}, \widetilde{f}_1]\big] = \big[[\widetilde{e}_1, f_{i2}], \widetilde{f}_1\big] + \big[f_{i2}, [\widetilde{e}_1, \widetilde{f}_1]\big] = [f_{i2}, h_1 + h_2] \in \mathfrak{k}_0,$$

$$[\widetilde{e}_1, \widetilde{e}_{ij}] = \big[\widetilde{e}_1, [\widetilde{f}_{i1}, e_{1j}]\big] = \big[[\widetilde{e}_1, \widetilde{f}_{i1}], e_{1j}\big] - \big[\widetilde{f}_{i1}, [\widetilde{e}_1, e_{1j}]\big] \subseteq [\mathfrak{k}_0, e_{1j}] \subseteq \mathfrak{k}_0,$$

where $e_{2j}, f_{i2} \in \mathfrak{k}_0$ are such that $\widetilde{e}_{1j} = [\widetilde{e}_1, e_{2j}]$ and $\widetilde{f}_{i1} = [f_{i2}, \widetilde{f}_1]$. □

3.2 Affine type

In [Na2], the author considers a twisted polynomial current Lie superalgebra $\overline{\mathcal{L}}_{tw}\mathfrak{q}_n$. We do the same, but include Laurent polynomials. We also present a few results about $\overline{\mathcal{L}}_{tw}\mathfrak{q}_n$.

The involution ι can be extended to $\mathfrak{gl}_{n|n}\otimes_{\mathbb{C}}\mathbb{C}[u^{\pm 1}]$ by letting it act only on $\mathfrak{gl}_{n|n}$. Set $\mathfrak{gl}_{n|n}[u^{\pm 1}]=\mathfrak{gl}_{n|n}\otimes_{\mathbb{C}}\mathbb{C}[u^{\pm 1}]$.

Definition 3.4. *We define the twisted Lie superalgebras $\mathcal{L}_{tw}\mathfrak{q}_n$ and $\mathcal{L}_{tw}{}'\mathfrak{q}_n$ to be*

$$\mathcal{L}_{tw}\mathfrak{q}_n=\{X(u)\in\mathfrak{gl}_{n|n}[u^{\pm 1}]\mid \iota(X(u))=X(u^{-1})\},\ \ \mathcal{L}_{tw}{}'\mathfrak{q}_n=[\mathcal{L}_{tw}\mathfrak{q}_n,\mathcal{L}_{tw}\mathfrak{q}_n]$$

We let $\overline{\mathcal{L}}_{tw}\mathfrak{q}_n$ be defined as (see [Na2], where the author considers only polynomials in u):

$$\overline{\mathcal{L}}_{tw}\mathfrak{q}_n=\{X(u)\in\mathfrak{gl}_{n|n}[u^{\pm 1}]\mid \iota(X(u))=X(-u)\}\ \text{and set}\ \overline{\mathcal{L}}_{tw}{}'\mathfrak{q}_n=[\overline{\mathcal{L}}_{tw}\mathfrak{q}_n,\overline{\mathcal{L}}_{tw}\mathfrak{q}_n].$$

A vector space basis for $\mathcal{L}_{tw}\mathfrak{q}_n$ is provided by:

$$\left\{\mathbf{E}^0_{ab}(k)=E_{ab}(u^k)+E_{-a,-b}(u^{-k}),\mathbf{E}^1_{ab}(k)=E_{-a,b}(u^{-k})+E_{a,-b}(u^k)|1\le a,b\le n,k\in\mathbb{Z}\right\}$$

As for $\mathcal{L}_{tw}{}'\mathfrak{q}_n$, a basis is provided by $\{\mathbf{E}^0_{ab}(k),\mathbf{E}^1_{ab}(k)|1\le a\ne b\le n,k\in\mathbb{Z}\}\cup\{\mathbf{H}^i_a(k)=\mathbf{E}^i_{aa}(k)-\mathbf{E}^i_{a+1,a+1}(k),\,i=0,1,\,k\in\mathbb{Z}\}\cup\{\mathbf{I}^0(k)=\sum_{a=1}^n(\mathbf{E}^0_{aa}(k)+\mathbf{E}^0_{aa}(-k))|k\in\mathbb{Z}_{\ge 0}\}\cup\{\mathbf{I}^1(k)=\sum_{a=1}^n(\mathbf{E}^1_{aa}(k)-\mathbf{E}^1_{aa}(k+2))|k\in\mathbb{Z}\}$. We have an exact sequence of Lie superalgebras:

$$0\longrightarrow\mathcal{L}_{tw}{}'\mathfrak{q}_n\longrightarrow\mathcal{L}_{tw}\mathfrak{q}_n\longrightarrow\mathfrak{L}\longrightarrow 0$$

where $\mathfrak{L}$ is the abelian Lie superalgebra spanned by the even elements $I^0(k)$ for $k\in\mathbb{Z}_{>0}$ and the odd elements $I^1(k)$ for $k\in\mathbb{Z}_{\ge 0}$ which correspond to the images of $\sum_{i=1}^n(\mathbf{E}^0_{ii}(k)-\mathbf{E}^0_{ii}(-k))$ and of $\sum_{i=1}^n(\mathbf{E}^1_{ii}(k)+\mathbf{E}^1_{ii}(-k))$ in the quotient $\mathcal{L}_{tw}\mathfrak{q}_n/\mathcal{L}_{tw}{}'\mathfrak{q}_n$.

Some of the propositions of subsection 3.1 have analogs in this affine setup, but first we have to introduce the following smash product.

Definition 3.5. *Let $\mathbf{\Lambda}$ be the algebra $\mathbb{C}\langle u,u^{-1},\mathbf{c}\rangle/(\mathbf{c}^2-1,\mathbf{c}u-u^{-1}\mathbf{c})$.*

The algebra $\mathbf{\Lambda}$ is $\mathbb{Z}/2\mathbb{Z}$-graded with even part equal to $\mathbb{C}[u,u^{-1}]$ and odd part equal to $\mathbb{C}[u,u^{-1}]\mathbf{c}$. The even part of $[\mathbf{\Lambda},\mathbf{\Lambda}]$ is equal to $[\mathbf{\Lambda},\mathbf{\Lambda}]^0=\{p\in\mathbb{C}[u,u^{-1}]|p(u)=p(u^{-1})\}$ and its odd part is $[\mathbf{\Lambda},\mathbf{\Lambda}]^1=\{p\mathbf{c}\in\mathbb{C}[u,u^{-1}]\mathbf{c}|p(1)=0=p(-1)\}$. Here, $[\mathbf{\Lambda},\mathbf{\Lambda}]$ is computed using the super bracket $[\ell_1,\ell_2]=\ell_1\ell_2-(-1)^{|\ell_1||\ell_2|}\ell_2\ell_1$ with $\ell_1,\ell_2\in\mathbf{\Lambda}$ and $|\ell_i|$ is the parity of ℓ_i; in particular, $[\mathbf{c},\mathbf{c}]=2$. Thus, the even part of the Lie superalgebra $\mathfrak{sl}_n(\mathbf{\Lambda})$ is $\mathfrak{gl}_n([\mathbf{\Lambda},\mathbf{\Lambda}]^0)+\mathfrak{sl}_n(\mathbb{C}[u,u^{-1}])$ and its odd part is $\mathfrak{gl}_n\otimes_{\mathbb{C}}[\mathbf{\Lambda},\mathbf{\Lambda}]^1+\mathfrak{sl}_n\otimes_{\mathbb{C}}\mathbb{C}[u,u^{-1}]\mathbf{c}$.

Lemma 3.2. *The Lie superalgebra $\mathcal{L}_{tw}\mathfrak{q}_n$ ($\mathcal{L}_{tw}{}'\mathfrak{q}_n$) is isomorphic to $\mathfrak{gl}_n(\mathbf{\Lambda})$ (resp. to $\mathfrak{sl}_n(\mathbf{\Lambda})$).*

Proof. It is enough to check that the following formulas define an isomorphism $\mathfrak{gl}_n(\mathbf{\Lambda})\xrightarrow{\sim}\mathcal{L}_{tw}\mathfrak{q}_n$: $E_{ab}(u^k)\mapsto\mathbf{E}^0_{ab}(k)$, $E_{ab}(u^k\mathbf{c})\mapsto\mathbf{E}^1_{ab}(k)$. □

Remark 3.4. *The Lie superalgebras $\mathcal{L}_{tw}\mathfrak{q}_n$ and $\mathcal{L}_{tw}{}'\mathfrak{q}_n$ differ from the Lie superalgebra $\mathfrak{q}(n)^{(2)}$ studied in [GoSe]. The loop version of this Kac-Moody superalgebra of type $\mathfrak{q}(n)^{(2)}$ is almost equal to $\mathfrak{sl}_n(\widetilde{\mathbf{\Lambda}})$ where $\widetilde{\mathbf{\Lambda}}\cong\mathbb{C}\langle u^{\pm 1},\mathbf{c}^{\pm 1}\rangle/(\mathbf{c}^2-u,\mathbf{c}u-u\mathbf{c})$: the difference is that the even part of the former is $\mathfrak{sl}_n(\mathbb{C}[t^{\pm 2}])$, not $\mathfrak{gl}_n(\mathbb{C}[t^{\pm 2}])$.*

As for the twisted polynomial Lie superalgebra considered in [Na2], it is a subsuperalgebra of $\overline{\mathcal{L}}_{tw}\mathfrak{q}_n$, which is isomorphic to $\mathfrak{gl}_n(\overline{\mathbf{\Lambda}})$ where $\overline{\mathbf{\Lambda}}=\mathbb{C}\langle v,v^{-1},\mathbf{c}\rangle/(\mathbf{c}^2-1,\mathbf{c}v+v\mathbf{c})$.

The Lie superalgebra $\widehat{\mathfrak{gl}}_{n|n}[u^{\pm 1}]$ is defined to be equal to $\mathfrak{gl}_{n|n}[u^{\pm 1}]\oplus\mathbb{C}\cdot\mathfrak{c}$ as a vector space with $\mathfrak{c}$ a central even element and the super bracket is given by:

$$[E_{a_1b_1}\otimes u^{r_1},E_{a_2b_2}\otimes u^{r_2}]=[E_{a_1b_1},E_{a_2b_2}]\otimes u^{r_1+r_2}+\delta_{r_1,-r_2}Str(E_{a_1b_1}E_{a_2b_2})r_1\mathfrak{c}$$

where Str denotes the supertrace. (If we replace $\mathfrak{gl}_{n|n}$ by $\mathfrak{sl}_{n|n}$, we get the universal central extension of $\mathfrak{sl}_{n|n} \otimes_{\mathbb{C}} \mathbb{C}[u^{\pm 1}]$: see [IoKo], theorem 4.7.) The involution ι can be extended to $\widehat{\mathfrak{gl}}_{n|n}[u^{\pm 1}]$ by fixing the extra central element $\mathfrak{c}$.

Definition 3.6. *The central extension $\widehat{\mathfrak{q}}_n^{tw}$ of $\mathcal{L}_{tw}\mathfrak{q}_n$ is defined as $\mathcal{L}_{tw}\mathfrak{q}_n$ in definition 3.4, except that $\mathfrak{gl}_{n|n}[u^{\pm 1}]$ has to be replaced by $\widehat{\mathfrak{gl}}_{n|n}[u^{\pm 1}]$. We let $\widehat{\mathfrak{q}}_n^{tw\prime}$ be its derived Lie subsuperalgebra.*

We would like to see that $\widehat{\mathfrak{q}}_n^{tw\prime}$ is actually the universal central extension of $\mathcal{L}_{tw}{}'\mathfrak{q}_n$. We need to compute $HC_1(\mathbf{\Lambda})$ to see that it is one dimensional and use theorem 3.1 from [ChGu].

Definition 3.7. *For $n \geq 3$, the Steinberg Lie superalgebra $\mathfrak{st}_n(\mathbf{\Lambda})$ is defined to be the Lie superalgebra over $\mathbb{C}$ generated by the homogeneous elements $F_{ij}(a)$, $a \in \mathbf{\Lambda}$ homogeneous, $1 \leq i \neq j \leq n$ and $\deg F_{ij}(a) = \deg a = |a|$, subject to the following relations for $a, b \in \mathbf{\Lambda}$:*

$$a \mapsto F_{ij}(a) \text{ is a } \mathbb{C}\text{-linear map,} \tag{22}$$

$$[F_{ij}(a), F_{jk}(b)] = F_{ik}(ab), \text{ for distinct } i, j, k, \tag{23}$$

$$[F_{ij}(a), F_{kl}(b)] = 0, \text{ for } i \neq j \neq k \neq l \neq i, \tag{24}$$

Theorem 3.1. *[ChGu] Suppose that $n \geq 3$. The Steinberg Lie superalgebra $\mathfrak{st}_n(\mathbf{\Lambda})$ is the universal central extension of $\mathfrak{sl}_n(\mathbf{\Lambda})$ and the kernel of the epimorphism $\mathfrak{st}_n(\mathbf{\Lambda}) \twoheadrightarrow \mathfrak{sl}_n(\mathbf{\Lambda})$ is isomorphic to $HC_1(\mathbf{\Lambda})$, a $\mathbb{Z}/2\mathbb{Z}$-graded version of the first cyclic homology group of $\mathbf{\Lambda}$.*

Let us now compute $HC_1(\mathbf{\Lambda})$. Let $\langle \mathbf{\Lambda}, \mathbf{\Lambda} \rangle$ be the quotient of $\mathbf{\Lambda} \otimes \mathbf{\Lambda}$ by the subspace I spanned by $a_1 \otimes a_2 + (-1)^{|a_1||a_2|} a_2 \otimes a_1$ and $(-1)^{|a_2||a_1|} a_2 a_3 \otimes a_1 + (-1)^{|a_3||a_2|} a_3 a_1 \otimes a_2 + (-1)^{|a_3||a_1|} a_1 a_2 \otimes a_3$ for $a_1, a_2, a_3 \in \mathbf{\Lambda}$. The first cyclic homology group $HC_1(\mathbf{\Lambda})$ is, by definition, the kernel of the map $\langle \mathbf{\Lambda}, \mathbf{\Lambda} \rangle \twoheadrightarrow [\mathbf{\Lambda}, \mathbf{\Lambda}]$ given by $a_1 \otimes a_2 \mapsto [a_1, a_2] = a_1 a_2 - (-1)^{|a_1||a_2|} a_2 a_1$.

In $\langle \mathbf{\Lambda}, \mathbf{\Lambda} \rangle$, $u^i \mathbf{c}^j \otimes u^k \mathbf{c}^l$ can be expressed as a linear combination of $u^i \mathbf{c} \otimes u$, $u^i \mathbf{c} \otimes u^{-1}$, $u^{-1} \otimes u$ and $u^i \mathbf{c}^j \otimes \mathbf{c}$ with $i \in \mathbb{Z}, j = 0, 1$. Observe that, in $\langle \mathbf{\Lambda}, \mathbf{\Lambda} \rangle$,

$$u^i \mathbf{c} \otimes u^{-1} = u^i \otimes \mathbf{c} u^{-1} + \mathbf{c} \otimes u^{i-1} = u^i \otimes u \mathbf{c} + \mathbf{c} \otimes u^{i-1} = u^{i+1} \otimes \mathbf{c} + \mathbf{c} u^i \otimes u + \mathbf{c} \otimes u^{i-1}.$$

This implies that $\langle \mathbf{\Lambda}, \mathbf{\Lambda} \rangle$ is spanned by $u^i \mathbf{c} \otimes u$, $u^{-1} \otimes u$ and $u^i \mathbf{c}^j \otimes \mathbf{c}$ with $i \in \mathbb{Z}, j = 0, 1$. Moreover, if $i > 0$, then

$$\begin{aligned} u^i \otimes \mathbf{c} &= u^{i-1} \otimes u \mathbf{c} + u \otimes \mathbf{c} u^{i-1} = u^{i-2} \otimes u^2 \mathbf{c} + u \otimes u^{-i+3} \mathbf{c} + u \otimes u^{-i+1} \mathbf{c} \\ &= u^{i-3} \otimes u^3 \mathbf{c} + u \otimes u^{-i+5} \mathbf{c} + u \otimes u^{-i+3} \mathbf{c} + u \otimes u^{-i+1} \mathbf{c} = \cdots \\ &= u^2 \otimes u^{i-2} \mathbf{c} + u \otimes u^{i-5} \mathbf{c} + \cdots + u \otimes u^{-i+3} \mathbf{c} + u \otimes u^{-i+1} \mathbf{c} \\ &= u \otimes u^{i-1} \mathbf{c} + u \otimes u^{i-3} \mathbf{c} + u \otimes u^{i-5} \mathbf{c} + \cdots + u \otimes u^{-i+3} \mathbf{c} + u \otimes u^{-i+1} \mathbf{c} \end{aligned}$$

and a similar identity holds when $i < 0$.

This shows that $\langle \mathbf{\Lambda}, \mathbf{\Lambda} \rangle$ is spanned by $u^i \mathbf{c} \otimes u$, $u^{-1} \otimes u$ and $u^i \mathbf{c} \otimes \mathbf{c}$ with $i \in \mathbb{Z}$. It is even enough to restrict to $u^i \mathbf{c} \otimes \mathbf{c}$ with $i \geq 0$ because $u^i \mathbf{c} \otimes \mathbf{c} - u^{-i} \mathbf{c} \otimes \mathbf{c} = 0$. Using the following computations, we can conclude that $HC_1(\mathbf{\Lambda})$ is spanned by $u^{-1} \otimes u$: $[u^i \mathbf{c}, u] = (u^{i-1} - u^{i+1}) \mathbf{c}$, $[u^i \mathbf{c}, \mathbf{c}] = u^i + u^{-i}$.

We can also perform similar computations by replacing $\mathbf{\Lambda}$ with $\overline{\mathbf{\Lambda}}$. The results in [ChGu] apply also to $\mathfrak{sl}_n(\overline{\mathbf{\Lambda}})$ and it is natural to wonder whether the universal central extension of $\mathfrak{sl}_n(\overline{\mathbf{\Lambda}})$ is trivial or not. It turns out to be trivial because $HC_1(\overline{\mathbf{\Lambda}}) = 0$. Let us see why this is the case.

In $\langle \overline{\mathbf{\Lambda}}, \overline{\mathbf{\Lambda}} \rangle$, we have that $v^i \mathbf{c}^j \otimes v^k \mathbf{c}^l$ can be expressed as a linear combination of $v^i \mathbf{c}^j \otimes v$ with $i \geq -1$, $v^i \mathbf{c}^j \otimes v^{-1}$ with $i \leq 0$, and $v^i \mathbf{c}^j \otimes \mathbf{c}$ with $i \in \mathbb{Z}$, for $j = 0, 1$.

Suppose that i is odd. In $\langle \overline{\mathbf{\Lambda}}, \overline{\mathbf{\Lambda}} \rangle$, $v^i \mathbf{c} \otimes \mathbf{c} - \mathbf{c} v^i \otimes \mathbf{c} + \mathbf{c}\mathbf{c} \otimes v^i = 0$, i.e., $(1 + (-1)^{i+1}) v^i \mathbf{c} \otimes \mathbf{c} = 0$, so $v^i \mathbf{c} \otimes \mathbf{c} = 0$. Suppose i is even. Then $v^i \mathbf{c}\mathbf{c} \otimes \mathbf{c} + \mathbf{c} v^i \mathbf{c} \otimes \mathbf{c} + \mathbf{c}\mathbf{c} \otimes v^i \mathbf{c} = 0$, i.e., $(1 + (-1)^i) v^i \otimes \mathbf{c} = 0$, so $v^i \otimes \mathbf{c} = 0$.

It is also true that, in $\langle\overline{\mathbf{\Lambda}},\overline{\mathbf{\Lambda}}\rangle$, $v^{2k+1}\otimes\mathbf{c}+v^{2k}\mathbf{c}\otimes v+\mathbf{c}v\otimes v^{2k}=0$ and $v^{2k+1}\otimes\mathbf{c}+v\mathbf{c}\otimes v^{2k}+\mathbf{c}v^{2k}\otimes v=0$. Substracting these two equations yields $v\mathbf{c}\otimes v^{2k}=0$ and, therefore, $v^{2k+1}\otimes\mathbf{c}=-v^{2k}\mathbf{c}\otimes v$.

A bit surprisingly, we can show also that $v^{-1}\otimes v=0$ in $\langle\overline{\mathbf{\Lambda}},\overline{\mathbf{\Lambda}}\rangle$:

$$v^{-1}\otimes v=v^{-1}\mathbf{c}^2\otimes v=\mathbf{c}v\otimes v^{-1}\mathbf{c}-\mathbf{c}\otimes\mathbf{c}=-v\mathbf{c}\otimes v^{-1}\mathbf{c}-\mathbf{c}\otimes\mathbf{c}=\mathbf{c}v^{-1}\mathbf{c}\otimes v-v^{-1}\mathbf{c}v\otimes\mathbf{c}-\mathbf{c}\otimes\mathbf{c}=-v^{-1}\otimes v$$

Therefore, the space $\langle\overline{\mathbf{\Lambda}},\overline{\mathbf{\Lambda}}\rangle$ is spanned by $v^i\mathbf{c}\otimes v$ with $i\geq -1$, $v^i\mathbf{c}\otimes v^{-1}$ with $i\leq 0$, $v^i\mathbf{c}\otimes\mathbf{c}$ with i even. Using the following computations in $[\overline{\mathbf{\Lambda}},\overline{\mathbf{\Lambda}}]$, we can conclude that $HC_1(\overline{\mathbf{\Lambda}})$ is $\{0\}$: $[v^i\mathbf{c}^j,v]=((-1)^j-1)v^{i+1}\mathbf{c}^j$ ($=0$ if and only if $j=0$), $[v^i\mathbf{c}^j,v^{-1}]=((-1)^j-1)v^{i-1}\mathbf{c}^j$ ($=0$ if and only if $j=0$), and $[v^i\mathbf{c},\mathbf{c}]=(1+(-1)^i)v^i$ ($=0$ if and only if i is odd).

We will now give a couple of presentations by generators and relations of the twisted affine Lie superalgebras, as we did in the previous subsection for $\mathfrak{q}_n$ and $\mathfrak{sq}_n$. Recall that $\mathcal{L}_{tw}{}'\mathfrak{q}_n\cong\mathfrak{sl}_n(\mathbf{\Lambda})$ by lemma 3.2, so $\widehat{\mathfrak{q}}_n^{tw\prime}$ is isomorphic to $\widehat{\mathfrak{sl}}_n(\mathbf{\Lambda})$, the universal central extension of $\mathfrak{sl}_n(\mathbf{\Lambda})$.

Proposition 3.5. *Suppose that $n\geq 5$. $\widehat{\mathfrak{sl}}_n(\mathbf{\Lambda})$ is isomorphic to the Lie superalgebra $\mathfrak{e}$ which is generated by the even elements $F_{ab}(1),F_{ab}(u),F_{ab}(u^{-1})$ and the odd elements $F_{ab}(\mathbf{c})$ for $1\leq a\neq b\leq n$ which satisfy the following identities:*
If a,b,c are all distinct and a,c,d also,

$$[F_{ab}(u^i),F_{bc}(\mathbf{c})]=[F_{ad}(\mathbf{c}),F_{dc}(u^{-i})] \text{ and } [F_{ab}(u^i),F_{bc}(u^i)]=[F_{ad}(u^i),F_{dc}(u^i)] \text{ for } i=-1,1, \quad (25)$$

$$[F_{ab}(\mathbf{c}^i),F_{bc}(\mathbf{c}^j)]=F_{ac}(\mathbf{c}^{i+j}) \text{ for } i,j,k=0,1 \text{ and } k\equiv i+j \bmod 2 \quad (26)$$

$$[F_{ab}(u^i),F_{bc}(u^j)]=F_{ac}(u^{i+j}) \text{ for } i,j,i+j=-1,0,1. \quad (27)$$

If $a\neq b\neq c\neq d\neq a$,

$$[F_{ab}(u^i),F_{cd}(\mathbf{c}^j)]=0=[F_{ab}(u^i),F_{cd}(u^j)],\;\; [F_{ab}(\mathbf{c}^i),F_{cd}(\mathbf{c}^j)]=0, \text{ for } i,j=-1,0,1 \quad (28)$$

Proof. We need theorem 3.1. We have to explain how to construct in $\mathfrak{e}$ elements $F_{ab}(u^k\mathbf{c}^i)$ $\forall\, k\in\mathbb{Z},i=0,1,a\neq b$ which satisfy the relations (23),(24), i.e.,

$$[F_{ab}(u^k\mathbf{c}^i),F_{cd}(u^l\mathbf{c}^j)]=\delta_{bc}F_{ad}(u^{k+(-1)^i l}\mathbf{c}^{i+j}) \text{ for } a\neq b,c\neq d\neq a. \quad (29)$$

Since $\{u^k\mathbf{c}^i|k\in\mathbb{Z},i=0,1\}$ is a vector space basis of $\mathbf{\Lambda}$, we could then define $F_{ab}(\ell)$ for any $\ell\in\mathbf{\Lambda}$ by linearity and the equality $[F_{ab}(\ell_1),F_{cd}(\ell_2)]=\delta_{bc}F_{ad}(\ell_1\ell_2)$ for $a\neq b,c\neq d\neq a$ would then follow from (29).

First we show (29) for $i=0,j=0,1$ and $k,l\in\mathbb{Z}_{\geq 0}$ (or $\mathbb{Z}_{\leq 0}$). We just show the positive case - the negative case is similar. Because of relations (25)-(28), for $1\leq a\neq c\leq n$, we have well defined elements $F_{ac}(u^m\mathbf{c}^j)$ given by $F_{ac}(u^m\mathbf{c}^j)=[F_{ab}(u),F_{bc}(u^{m-1}\mathbf{c}^j)]$ for $1\leq m\leq 2,j=0,1,1\leq b\leq n,b\neq a,c$. (They are well defined since they do not depend on the choice of b.) Equation (29) is then true when $0\leq k+l\leq 2,i=0,j=0,1$.

Let us assume now that we have constructed elements $F_{ac}(u^k\mathbf{c}^j)$ for $0\leq k\leq m,m\geq 2,j=0,1$ which satisfy relation (29) when $0\leq k+l\leq m,i=0,j=0,1$. Suppose that $a\neq d\neq b\neq c\neq a$ and pick $e\neq a,b,c,d$ (this is where our assumption $n\geq 5$ is useful); then $[F_{ad}(u),F_{bc}(u^m\mathbf{c}^j)]=\big[F_{ad}(u),[F_{be}(u),F_{ec}(u^{m-1}\mathbf{c}^j)]\big]=0$ by induction. For $a\neq c$, pick $b\neq a,c$ and set $F_{ac}(u^{m+1}\mathbf{c}^j)=[F_{ab}(u),F_{bc}(u^m\mathbf{c}^j)]$. This does not depend on the choice of b, for if $d\neq a,b,c$, then

$$[F_{ab}(u),F_{bc}(u^m\mathbf{c}^j)]=\big[[F_{ad}(u),F_{db}(1)],F_{bc}(u^m\mathbf{c}^j)\big]=[F_{ad}(u),F_{dc}(u^m\mathbf{c}^j)]$$

since we proved just before that $[F_{ad}(u),F_{bc}(u^m\mathbf{c}^j)]=0$.

We thus have well defined elements $F_{ac}(u^{m+1}\mathbf{c}^j),j=0,1$. If $a\neq b$, $c\neq d\neq a$, pick $e\neq a,b,c,d$; then we have

$$[F_{ab}(1),F_{cd}(u^{m+1}\mathbf{c}^j)]=\big[F_{ab}(1),[F_{ce}(u),F_{ed}(u^m\mathbf{c}^j)]\big]=\delta_{bc}[F_{ae}(u),F_{ed}(u^m\mathbf{c}^j)]=\delta_{bc}F_{ad}(u^{m+1}\mathbf{c}^j).$$

If $k \geq 2$ and $k+l=m+1$, then, by induction,

$$[F_{ab}(u^k), F_{cd}(u^l\mathbf{c}^j)] = \big[[F_{ae}(u), F_{eb}(u^{k-1})], F_{cd}(u^l\mathbf{c}^j)\big] = \delta_{bc}[F_{ae}(u), F_{ed}(u^{k-1+l}\mathbf{c}^j)] = \delta_{bc}F_{ad}(u^{m+1}\mathbf{c}^j).$$

Now we show that (29) holds for $i=0, j=0,1$, and $kl<0$. We consider only the case $k>0$ and $l<0$. If $a \neq b$, $c \neq d \neq a$, pick $e \neq a,b,c,d$; by induction on $\|l\|$, we have

$$[F_{ab}(u), F_{cd}(u^l\mathbf{c}^j)] = \big[F_{ab}(u), [F_{ce}(u^{-1}), F_{ed}(u^{l+1}\mathbf{c}^j)]\big] = \delta_{bc}[F_{ae}(1), F_{ed}(u^{l+1}\mathbf{c}^j)] = \delta_{bc}F_{ad}(u^{l+1}\mathbf{c}^j).$$

By induction on $k+\|l\|$, we deduce that, for $k \geq 2$,

$$[F_{ab}(u^k), F_{cd}(u^l\mathbf{c}^j)] = \big[[F_{ae}(u^{k-1}), F_{eb}(u)], F_{cd}(u^l\mathbf{c}^j)\big] = \delta_{bc}[F_{ae}(u^{k-1}), F_{ed}(u^{l+1}\mathbf{c}^j)] = \delta_{bc}F_{ad}(u^{k+l}\mathbf{c}^j).$$

We are left to show (29) for $i=1, j=0,1$ and $k,l \in \mathbb{Z}$. If $a \neq b$, $c \neq d \neq a$, pick $e \neq a,b,c,d$; by induction on $l>0$, we obtain

$$\begin{aligned}
[F_{ab}(\mathbf{c}), F_{cd}(u^{\pm l}\mathbf{c}^j)] &= \big[F_{ab}(\mathbf{c}), [F_{ce}(u^{\pm 1}), F_{ed}(u^{\pm l\mp 1}\mathbf{c}^j)]\big] = \big[[F_{ab}(\mathbf{c}), F_{ce}(u^{\pm 1})], F_{ed}(u^{\pm l\mp 1}\mathbf{c}^j)\big] \\
&= \big[[F_{ab}(u^{\mp 1}), F_{ce}(\mathbf{c})], F_{ed}(u^{\pm l\mp 1}\mathbf{c}^j)\big] = \big[F_{ab}(u^{\mp 1}), [F_{ce}(\mathbf{c}), F_{ed}(u^{\pm l\mp 1}\mathbf{c}^j)]\big] \\
&= [F_{ab}(u^{\mp 1}), F_{cd}(u^{\mp l\pm 1}\mathbf{c}^{j+1})] = \delta_{bc}F_{ad}(u^{\mp l}\mathbf{c}^{j+1}).
\end{aligned}$$

Finally, for arbitrary $k,l \in \mathbb{Z}$,

$$\begin{aligned}
[F_{ab}(u^k\mathbf{c}), F_{cd}(u^l\mathbf{c}^j)] &= \big[[F_{ae}(u^k), F_{eb}(\mathbf{c})], F_{cd}(u^l\mathbf{c}^j)\big] = \big[F_{ae}(u^k), [F_{eb}(\mathbf{c}), F_{cd}(u^l\mathbf{c}^j)]\big] \\
&= \delta_{bc}[F_{ae}(u^k), F_{ed}(u^{-l}\mathbf{c}^{j+1})] = \delta_{bc}F_{ad}(u^{k-l}\mathbf{c}^{j+1})
\end{aligned}$$

□

We end this subsection by giving an affine version of proposition 3.4 and we will need the previous proposition to prove it.

Proposition 3.6. *Suppose that $n \geq 5$. The Lie superalgebra $\widehat{\mathfrak{q}}_n^{tw\prime}$ is isomorphic to the Lie superalgebra $\mathfrak{f}$ generated by the even elements $e_{i,r}, f_{i,r}, h_{i,r}, h_{n,0}$ with $1 \leq i \leq n-1, r=-1,0,1$, by the odd elements $\widetilde{h}_{1,0}, \widetilde{e}_{1,0}, \widetilde{f}_{1,0}$ and by a central element c which satisfy the following relations:*
The elements with the second index equal to 0 satisfy the relations in proposition 3.4;
For any $1 \leq i_1, i_2 \leq n, r_1, r_2 = -1,0,1$ (respecting some restrictions below):

$$[h_{i_1,\pm 1}, e_{i_2,0}] = [h_{i_1,0} - h_{i_1+1,0}, e_{i_2,\pm 1}], \;\; [h_{i_1,0}, e_{i_2,r_2}] = (\delta_{i_1,i_2} - \delta_{i_1,i_2+1})e_{i_2,r_2}; \tag{30}$$

$$[h_{i_1,\pm 1}, f_{i_2,0}] = [h_{i_1,0} - h_{i_1+1,0}, f_{i_2,\pm 1}], \;\; [h_{i_1,0}, f_{i_2,r_2}] = (\delta_{i_1,i_2+1} - \delta_{i_1,i_2})f_{i_2,r_2}; \tag{31}$$

$$[e_{i_1,r_1}, f_{i_2,r_2}] = \delta_{i_1 i_2} h_{i_1,r_1+r_2} \text{ for } r_1+r_2=-1,1, \;\; [h_{i_1,r_1}, h_{i_2,r_2}] = \delta_{r_1,-r_2}\mathsf{c}; \tag{32}$$

$$[e_{i_1,r_1}, e_{i_2,r_2}] = 0 = [f_{i_1,r_1}, f_{i_2,r_2}] \text{ if } i_1=i_2 \text{ or } \|i_1-i_2\|>1; \tag{33}$$

$$[e_{i_1,\pm 1}, e_{i_2,0}] = [e_{i_1,0}, e_{i_2,\pm 1}], \; [f_{i_1,\pm 1}, f_{i_2,0}] = [f_{i_1,0}, f_{i_2,\pm 1}], \; [e_{i_1,\pm 1}, f_{i_2,\mp 1}] = \delta_{i_1 i_2}(h_{i_1,0} - h_{i_1+1,0} \pm 2\mathsf{c}); \tag{34}$$

$$\big[e_{i_1,r_1}, [e_{i_1,r_2}, e_{i_3,r_3}]\big] = 0 = \big[f_{i_1,r_1}, [f_{i_1,r_2}, f_{i_3,r_3}]\big] \;\; \text{if } \|i_1-i_3\|=1, \; r_1+r_2, r_3=-1,0,1; \tag{35}$$

$$[\widetilde{e}_{1,0}, e_{i_2,r_2}] = 0 = [\widetilde{f}_{1,0}, f_{i_2,r_2}] \text{ if } i_2>2, \; [\widetilde{e}_{1,0}, f_{i_2,r_2}] = 0 = [\widetilde{f}_{1,0}, e_{i_2,r_2}] \text{ if } i_2 \geq 2; \tag{36}$$

$$[\widetilde{e}_{1,0}, e_{2,r_2}] = \Big[e_{1,-r_2}, \big[f_{1,0}, [\widetilde{e}_{1,0}, e_{2,0}]\big]\Big], \; [f_{2,r_2}, \widetilde{f}_{1,0}] = \Big[\big[[f_{2,0}, \widetilde{f}_{1,0}], e_{1,0}\big], f_{1,-r_2}\Big]. \tag{37}$$

Proof. We use proposition 3.5: we introduce elements $F_{ab}(u^{\pm 1}), F_{ab}(\mathbf{c}^i), i = 0,1$ in $\mathfrak{f}$ for $1 \le a \neq b \le n$ and we show that they satisfy the relations (25) - (28). The relations defining $\mathfrak{f}$ show that the Lie subalgebra of $\mathfrak{f}$ generated by the even generators contains the affine Lie algebra $\widehat{\mathfrak{sl}}_n(\mathbb{C}[u,u^{-1}])$. This gives us the elements $F_{ab}(u^{\pm 1})$ that we need. (For instance, $h_{i,\pm 1}$ corresponds to $\mathbf{E}^0_{ii}(\pm 1) - \mathbf{E}^0_{i+1,i+1}(\pm 1)$.) Moreover, we can define elements $F_{ab}(\mathbf{c}^j), j = 0,1$ as in proposition 3.4 and this proposition actually shows that the Lie subsuperalgebra generated by the elements with $r = 0$ is equal to $\mathfrak{sq}_n$. It is thus enough to check the first relations in (25) and (28).

We must prove that $[F_{ab}(\mathbf{c}), F_{cd}(u)] = 0$ if $a \neq b \neq c \neq d \neq a$. It follows from (36) that $[F_{12}(\mathbf{c}), F_{cd}(u)] = 0$ for any $3 \le c \le n, 2 \le d \le n, c \neq d$, from which we can deduce that $[F_{1b}(\mathbf{c}), F_{cd}(u)] = 0$ for any $2 \le b,c,d \le n, b \neq c \neq d$ and then also when $c = 1$. Using the adjoint action of $\mathfrak{gl}_n$, we then see that $[F_{ab}(\mathbf{c}), F_{cd}(u)] = 0$ if $a \neq b \neq c \neq d \neq a$ and $2 \le a,b,c,d \le n$; the general case follows from all this. The proof that $[F_{ab}(\mathbf{c}), F_{cd}(u^{-1})] = 0$ is similar.

We must also show that $[F_{ab}(\mathbf{c}), F_{bc}(u^r)] = [F_{ad}(u^{-r}), F_{dc}(\mathbf{c})], r = -1, 1$ when $a \neq b \neq c \neq a \neq d \neq c$. Let us prove this first when $b = d$. Relation (37) says that $[F_{12}(\mathbf{c}), F_{23}(u^r)] = [F_{12}(u^{-r}), F_{23}(\mathbf{c})]$. From this, we deduce that $[F_{a2}(\mathbf{c}), F_{2c}(u^r)] = [F_{a2}(u^{-r}), F_{2c}(\mathbf{c})]$ for any $a \neq 2 \neq c \neq a$. If $b \neq 2, a, c$, then $[F_{a2}(\mathbf{c}), F_{bc}(u^r)] = 0$ by the previous paragraph, so using $ad(F_{2b}(1))$ we see that $[F_{ab}(\mathbf{c}), F_{bc}(u^r)] = [F_{a2}(\mathbf{c}), F_{2c}(u^r)]$; similarly $[F_{ab}(u^{-r}), F_{bc}(\mathbf{c})] = [F_{a2}(u^{-r}), F_{2c}(\mathbf{c})]$. Using all this, we can deduce the desired conclusion. □

4 Hecke-Clifford algebras

In order to extend some of the results of M. Nazarov in [Na2] to the quantum affine setup, we will need the affine Hecke-Clifford algebra.

Definition 4.1. *[JoNa] Let $q \in \mathbb{C}^\times, l \in \mathbb{Z}_{\ge 1}$ and set $\epsilon = q - q^{-1}$. The affine Hecke-Clifford algebra $\mathbf{H}_{q,l}$ (which is called the affine Sergeev algebra in [JoNa]) is the algebra generated by elements $T_1, \ldots, T_{l-1}$, pairwise commuting elements $X_1^{\pm 1}, \ldots, X_l^{\pm 1}$, and anti-commuting elements $\mathsf{c}_1, \ldots, \mathsf{c}_l$, which satisfy the relations:*

$$(T_i - q)(T_i + q^{-1}) = 0,\ i = 1, \ldots, l-1, \ \ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},\ i = 1, \ldots, l-2$$

$$T_i T_j = T_j T_i \text{ if } \|i-j\| \ge 2, \ \ \mathsf{c}_i^2 = -1, \ \ \mathsf{c}_i \mathsf{c}_j = -\mathsf{c}_j \mathsf{c}_i \text{ if } 1 \le i \neq j \le l, \ \ T_i \mathsf{c}_i = \mathsf{c}_{i+1} T_i$$

$$\mathsf{c}_i X_i = X_i^{-1} \mathsf{c}_i, \ \ \mathsf{c}_i X_j = X_j \mathsf{c}_i \text{ if } j \neq i, \ \ T_i \mathsf{c}_j = \mathsf{c}_j T_i \text{ and } T_i X_j = X_j T_i \text{ if } j \neq i, i+1$$

$$T_i X_i = X_{i+1} T_i - \epsilon (X_{i+1} - \mathsf{c}_i \mathsf{c}_{i+1} X_i), \ \ T_i X_{i+1} = X_i T_i + \epsilon (1 + \mathsf{c}_i \mathsf{c}_{i+1}) X_{i+1},$$

Remark 4.1. *The second relation on the last line can be deduced from the others: we have included it only for convenience.*

The subalgebra $\mathcal{H}_{q,l}$ generated by the elements $T_i, 1 \le i \le l-1$ and $\mathsf{c}_i, 1 \le i \le l$ is isomorphic to the finite Hecke-Clifford algebra which plays a role in [Ol]. $\mathbf{H}_{q,l}$ becomes a $\mathbb{Z}/2\mathbb{Z}$-graded algebra if we declare T_i, X_j to be even and c_j to be odd for all $1 \le i \le l-1, 1 \le j \le l$.

In the computations below, the following relations will also be useful: $T_i \mathsf{c}_{i+1} = \mathsf{c}_i T_i - \epsilon(\mathsf{c}_i - \mathsf{c}_{i+1})$ and $(T_i - \epsilon \mathsf{c}_i \mathsf{c}_{i+1}) X_i T_i = X_{i+1}$. Moreover, the inverse of $T_i - \epsilon \mathsf{c}_i \mathsf{c}_{i+1}$ is $T_i - \epsilon(1 + \mathsf{c}_i \mathsf{c}_{i+1})$.

5 Quantum groups associated to $\mathfrak{q}_n$ and $\widehat{\mathfrak{q}}_n^{tw}$

After recalling the quantum supergroup introduced in [Ol], we define in the second subsection a new twisted affine quantum group associated to $\mathfrak{q}_n$. For the rest of this paper, we will view q as a variable and will consider algebras over $\mathbb{C}(q)$ and over $\mathcal{A}$, the localization of $\mathbb{C}[q,q^{-1}]$ at the ideal $(q-1)$. We will denote $\mathbb{C}(q) \otimes_{\mathbb{C}} \mathbb{C}(n|n)$ by $\mathbb{C}_q(n|n)$.

5.1 Finite case

G. Olshanski found a quantization of $\mathfrak{U}\mathfrak{q}_n$ in [Ol] and gave a presentation in terms of a matrix $S \in \mathrm{End}_{\mathbb{C}(q)}(\mathbb{C}_q(n|n))^{\otimes 2}$ which satisfies the quantum Yang-Baxter equation. However, the classical limit of S differs from the unique even R-matrix in $\mathfrak{q}_n^{\otimes 2}$ which determines the cobracket. (Actually, the latter does not satisfy the classical Yang-Baxter equation.) We recall a couple of definitions from [Ol].

Definition 5.1. *We denote by S the element of* $\mathrm{End}_{\mathbb{C}_q}(\mathbb{C}_q(n|n))^{\otimes 2}$ *given by the formula:*

$$S = \sum_{a=1}^{n}(1+(q-1)\mathsf{E}^0_{aa})\otimes E_{aa} + \sum_{a=1}^{n}(1+(q^{-1}-1)\mathsf{E}^0_{aa})\otimes E_{-a,-a} + \epsilon\left(\sum_{\substack{a,b=1\\a>b}}^{n}\mathsf{E}^0_{ab}\otimes E_{ba} - \sum_{\substack{a,b=1\\a<b}}^{n}\mathsf{E}^0_{ab}\otimes E_{-b,-a} - \sum_{a,b=1}^{n}\mathsf{E}^1_{ab}\otimes E_{-b,a}\right) \tag{38}$$

S can be viewed as an element of $\mathfrak{q}_n \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}_q}(\mathbb{C}_q(n|n))$ which is upper triangular. It will also be useful in computations below to rewrite it as

$$S = \sum_{i,j\in\{\pm1,\ldots,\pm n\}} q^{(\delta_{ij}+\delta_{-i,j})(1-2|j|)}E_{ii}\otimes E_{jj} + \epsilon\left(\sum_{\substack{i,j\in\{\pm1,\ldots,\pm n\}\\i>j}}(-1)^{|j|}E_{ij}\otimes E_{ji} + \sum_{\substack{i,j\in\{\pm1,\ldots,\pm n\}\\i>j}}(-1)^{|j|}E_{-i,-j}\otimes E_{ji}\right) \tag{39}$$

Definition 5.2. *[Ol] The quantized enveloping superalgebra of* $\mathfrak{q}_n$ *is the* $\mathbb{Z}/2\mathbb{Z}$*-graded algebra* $\mathfrak{U}_q\mathfrak{q}_n$ *generated by elements* t_{ij} *with* $i \le j$ *and* $i,j \in \{\pm1,\ldots,\pm n\}$ *which satisfy the following relations:*

$$t_{ii}t_{-i,-i} = 1 = t_{-i,-i}t_{ii}, \quad T_{12}T_{13}S_{23} = S_{23}T_{13}T_{12} \tag{40}$$

where $T = \sum_{i\le j} t_{ij}\otimes_{\mathbb{C}} E_{ij}$ *and the last equality holds in* $\mathfrak{U}_q\mathfrak{q}_n \otimes_{\mathbb{C}(q)} \mathrm{End}_{\mathbb{C}(q)}(\mathbb{C}_q(n|n))^{\otimes 2}$*. The* $\mathbb{Z}/2\mathbb{Z}$*-degree of* t_{ij} *is* $|i|+|j|$*.*

The elements t_{ij} with $i \le j$ and $|i| = |j|$ generate a subalgebra isomorphic to $\mathfrak{U}_q\mathfrak{gl}_n(\mathbb{C})$: see the first remark in [Ol]. $\mathfrak{U}_q\mathfrak{q}_n$ is a quantization of $\mathfrak{U}\mathfrak{q}_n$ and the Lie bialgebra structure on $\mathfrak{q}_n$ is the one associated with the following Manin triple [Ol]: $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\mathfrak{g} = \mathfrak{gl}_{n|n}$, $\mathfrak{g}_1 = \mathfrak{q}_n$, $\mathfrak{g}_2 = \sum_{a=1}^{n}\mathbb{C}(E_{aa}-E_{-a,-a}) + \sum_{\substack{-n\le i<j\le n\\ ij\neq 0}}\mathbb{C}E_{ij}$ and these two Lie subsuperalgebras are isotropic with respect to the invariant bilinear form ω given by $\omega(X,Y) = \frac{1}{2}Str(XY)$, $X,Y \in \mathfrak{gl}_{n|n}$. In section 6, we will give a twisted affine extension of this bialgebra structure.

5.2 Affine case

Set $\mathbb{C}_q(x,y) = \mathbb{C}(q)\otimes_{\mathbb{C}}\mathbb{C}(x,y)$. We now introduce the matrix (with coefficients in $\mathbb{C}_q(x,y)$) which we will need to define the twisted quantum affine superalgebra of $\mathfrak{q}_n$.

Definition 5.3. *Let* $S(x,y)$ *be the element of* $\mathrm{End}_{\mathbb{C}(q)}(\mathbb{C}_q(n|n))^{\otimes 2}\otimes_{\mathbb{C}(q)}\mathbb{C}_q(x,y)$ *given by*

$$S(x,y) = S + \frac{\epsilon P}{x^{-1}y-1} + \frac{\epsilon P J_1 J_2}{xy-1}$$

where $P = \sum_{\substack{i,j=-n\\ ij\neq 0}}^{n}(-1)^{|j|}E_{ij}\otimes E_{ji}$ *is the superpermutation operator and* $J_1 = J\otimes 1, J_2 = 1\otimes J$.

The proof of the following lemma explains where the idea of defining $S(x,y)$ in this way comes from.

Lemma 5.1. *$S(x,y)$ satisfies the quantum Yang-Baxter equation with parameters:*

$$S_{12}(x,y)S_{13}(x,z)S_{23}(y,z) = S_{23}(y,z)S_{13}(x,z)S_{12}(x,y).$$

Proof. Set $\widetilde{S}(x,y) = PS(x,y)$. $S(x,y)$ satisfies the quantum Yang-Baxter equation with parameters if and only if $\widetilde{S}(x,y)$ satisfies the braid relation $\widetilde{S}_{12}(y,z)\widetilde{S}_{23}(x,z)\widetilde{S}_{12}(x,y) = \widetilde{S}_{23}(x,y)\widetilde{S}_{12}(x,z)\widetilde{S}_{23}(y,z)$. Indeed, starting from this braid relation and multiplying on the left on both sides by P_{12} gives the equality $S_{12}(y,z)\widetilde{S}_{23}(x,z)\widetilde{S}_{12}(x,y) = \widetilde{S}_{13}(x,y)S_{12}(x,z)\widetilde{S}_{23}(y,z)$. Multiplying again on the left by P_{23} and P_{12} yields $S_{23}(y,z)S_{13}(x,z)S_{12}(x,y) = S_{12}(x,y)S_{13}(x,z)S_{23}(y,z)$.

To complete the proof, we have to observe that $\widetilde{S}(x,y)$ is the image in $\mathrm{End}_{\mathbb{C}(q)}(\mathbb{C}_q(n|n)^{\otimes 2}) \otimes_{\mathbb{C}(q)} \mathbb{C}_q(x,y)$ of the element $\psi(x,y)$ defined in equation 3.13 in [JoNa], where $\mathbb{C}_q(n|n)^{\otimes 2}$ is viewed as a representation of $\mathcal{H}_{q,2}$ as in theorem 5.2 in [Ol]. Finally, lemma 4.1 in [JoNa] states that $\psi(x,y)$ satisfies the braid relation. $\square$

Using $S(x,y)$ and the formalism developed in [FRT], we can now introduce a new quantum affine superalgebra.

Definition 5.4. *The twisted affine quantum superalgebra $\mathfrak{U}_q\widehat{\mathfrak{q}}_n^{tw}$ is the algebra over $\mathbb{C}(q)$ generated by elements $t_{ij}^{(r)}, r \in \mathbb{Z}_{\geq 0}, -n \leq i,j \leq n, ij \neq 0$ and by an invertible, central even element $\mathcal{C}$ which satisfy the following relations:*

$$t_{ij}^{(0)} = 0 \text{ if } i > j, \;\; t_{ii}^{(0)}t_{-i,-i}^{(0)} = 1 = t_{-i,-i}^{(0)}t_{ii}^{(0)} \tag{41}$$

$$T_{12}(w)T_{13}(z)S_{23}(\mathcal{C}^{-1}w, \mathcal{C}^{-1}z) = S_{23}(\mathcal{C}w, \mathcal{C}z)T_{13}(z)T_{12}(w) \tag{42}$$

where $T(z) = \sum_{\substack{i,j=-n \\ ij\neq 0}}^{n} t_{ij}(z) \otimes E_{ij} \in \mathfrak{U}_q\widehat{\mathfrak{q}}_n^{tw}[[z^{-1}]] \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(\mathbb{C}(n|n))$ and $t_{ij}(z) = \sum_{r\geq 0} t_{ij}^{(r)} z^{-r}$. The twisted quantum loop superalgebra $\mathfrak{U}_q\mathcal{L}_{tw}\mathfrak{q}_n$ is the quotient $\mathfrak{U}_q\widehat{\mathfrak{q}}_n^{tw}/(\mathcal{C}-1)$.

$\mathfrak{U}_q\widehat{\mathfrak{q}}_n^{tw}$ is a Hopf algebra with coproduct $\Delta(T(z)) = T_{13}(z(1\otimes\mathcal{C}))T_{23}(z(\mathcal{C}^{-1}\otimes 1)) \in (\mathfrak{U}_q\widehat{\mathfrak{q}}_n^{tw})^{\otimes 2}[[z^{-1}]] \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(\mathbb{C}(n|n))$, $\Delta(\mathcal{C}) = \mathcal{C} \otimes_{\mathbb{C}} \mathcal{C}$, or, more explicitly,

$$\Delta(t_{ij}^{(r)}) = \sum_{s=0}^{r} \sum_{k=-n,k\neq 0}^{n} (-1)^{(|i|+|k|)(|k|+|j|)} t_{ik}^{(s)}\mathcal{C}^{s-r} \otimes t_{kj}^{(r-s)}\mathcal{C}^{s}. \tag{43}$$

The antipode is given by $T(z) \mapsto T(z)^{-1}, \mathcal{C} \mapsto \mathcal{C}^{-1}$. $\mathfrak{U}_q\widehat{\mathfrak{q}}_n^{tw}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded algebra: the generator $t_{ij}^{(r)}$ has degree $|i|+|j|$.

6 Quantization

We explain in this section how $\mathfrak{U}_q\widehat{\mathfrak{q}}_n^{tw}$ can be viewed as a quantization of a Lie bialgebra structure on $\widehat{\mathfrak{q}}_n^{tw}$. We prove that it is a deformation of $\mathfrak{U}\widehat{\mathfrak{q}}_n^{tw}$ (theorem 6.1) and that the classical limit of the coproduct Δ recovers the cobracket δ defined in (44) (proposition 6.1).

Set $\mathsf{s}(x,y) = \mathsf{s} + \frac{2P}{x^{-1}y-1} + \frac{2PJ_1J_2}{xy-1}$ where

$$\mathsf{s} = \sum_{a=1}^{n} \mathsf{E}_{aa}^{0} \otimes (E_{aa} - E_{-a,-a}) + 2\sum_{1\leq b<a\leq n} \mathsf{E}_{ab}^{0} \otimes E_{ba} - 2\sum_{1\leq a<b\leq n} \mathsf{E}_{ab}^{0} \otimes E_{-b,-a} - 2\sum_{a,b=1}^{n} \mathsf{E}_{a,b}^{1} \otimes E_{-b,a}$$

is the element defined in (3.3) in [Ol]. $\mathsf{s}(x,y)$ satisfies $\mathsf{s}_{12}(x,y) + \mathsf{s}_{21}(y,x) = 0$ since $\mathsf{s}_{12} + \mathsf{s}_{21} = 2P$ (where $\mathsf{s}_{21}(y,x) = P\mathsf{s}_{12}(y,x)P$) and the classical Yang-Baxter equation $[\mathsf{s}_{12}(x,y), \mathsf{s}_{13}(x,z)] + [\mathsf{s}_{12}(x,y), \mathsf{s}_{23}(y,z)] + [\mathsf{s}_{13}(x,z), \mathsf{s}_{23}(y,z)] = 0$ (since s does [Ol]).

Set $\mathbf{E}_{ij}(r) = E_{ij}u^r + E_{-i,-j}u^{-r}$. $\widehat{\mathfrak{q}}_n^{tw}$ becomes a Lie superbialgebra if we define a cobracket δ by $\delta(\mathfrak{c}) = 0$ and (for $r > 0$):

$$\begin{aligned}\delta(\mathbf{E}_{ij}(r)) = 2\sum_{s=0}^{r}\sum_{\substack{k=-n\\k\neq 0}}^{n}(-1)^{|k|}\big((-1)^{(|i|+|k|)(|j|+|k|)}\mathbf{E}_{kj}(r-s)\otimes\mathbf{E}_{ik}(s) - \mathbf{E}_{ik}(s)\otimes\mathbf{E}_{kj}(r-s)\big)\\ + (-1)^{|i|}\Big(\big(\mathbf{E}_{ii}(0) - (-1)^{|i|+|j|}\mathbf{E}_{jj}(0)\big)\otimes\mathbf{E}_{ij}(r) - \mathbf{E}_{ij}(r)\otimes\big(\mathbf{E}_{ii}(0) - (-1)^{|i|+|j|}\mathbf{E}_{jj}(0)\big)\Big)\\ + 2r\big(\mathbf{E}_{ij}(r)\otimes\mathfrak{c} - \mathfrak{c}\otimes\mathbf{E}_{ij}(r)\big)\end{aligned} \tag{44}$$

When $s = 0$, the first sum should be over k such that $k < i$, and when $s = r$ it should be over k such that $k > j$. This superbialgebra structure comes from the following Manin triple, which is an affine extension of the one given in definition 3.1 in [Ol]. Set $\mathfrak{p} = \widehat{\mathfrak{gl}}_{n|n}[u^{\pm 1}] \oplus \mathbb{C}\mathfrak{d}$ where $\mathfrak{d}$ is the usual derivation, so $[\mathfrak{d}, E_{ij}u^r] = rE_{ij}u^r$. $\mathfrak{d}$ is an even element and $[\mathfrak{c}, \mathfrak{d}] = 0$. Consider the following pair of Lie subsuperalgebras of $\mathfrak{p}$: $\mathfrak{p}_1 = \widehat{\mathfrak{q}}_n^{tw}$ and

$$\mathfrak{p}_2 = \sum_{a=1}^{n}\mathbb{C}(E_{aa} - E_{-a,-a}) + \sum_{\substack{-n\leq i<j\leq n\\ ij\neq 0}}\mathbb{C}E_{ij} + \sum_{r\in\mathbb{Z}_{>0}}\sum_{\substack{i,j=-n\\ ij\neq 0}}^{n}\mathbb{C}E_{ij}u^{-r} + \mathbb{C}\mathfrak{d}.$$

Note that $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$. There is an even, invariant, supersymmetric, nondegenerate bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{p}$ given by $\langle E_{ij}u^{r_1}, E_{kl}u^{r_2}\rangle = \frac{1}{2}Str(E_{ij}E_{kl})\delta_{r_1,-r_2}$, $\langle\mathfrak{c},\mathfrak{d}\rangle = \frac{1}{2}$ and $\langle\mathfrak{c}, E_{ij}u^r\rangle = \langle\mathfrak{c},\mathfrak{c}\rangle = 0 = \langle\mathfrak{d}, E_{ij}u^r\rangle = \langle\mathfrak{d},\mathfrak{d}\rangle$. Since $\mathfrak{p}_2$ is a Lie superalgebra, its bracket induces a dual map $\widetilde{\delta} : \mathfrak{p}_2^* \longrightarrow (\Lambda^2\mathfrak{p}_2)^* \subset (\mathfrak{p}_2 \otimes_{\mathbb{C}} \mathfrak{p}_2)^*$. Using $\langle\cdot,\cdot\rangle$, we can view $\mathfrak{p}_1$ as a subspace of $\mathfrak{p}_2^*$ and $\Lambda^2\mathfrak{p}_1 \subset \Lambda^2\mathfrak{p}_2^* \subset (\Lambda^2\mathfrak{p}_2)^*$. It turns out that $\widetilde{\delta}(\mathfrak{p}_1) \subset \Lambda^2\mathfrak{p}_1$, so $\widetilde{\delta}|_{\mathfrak{p}_1}$ can be used to define a cobracket δ on $\mathfrak{p}_1$, which is the one given above.

The cobracket δ induces a co-Poisson Hopf algebra structure on $\mathfrak{U}\widehat{\mathfrak{q}}_n^{tw}$. We will explain how $\mathfrak{U}_q\widehat{\mathfrak{q}}_n^{tw}$ provides a Hopf algebra deformation of $\mathfrak{U}\widehat{\mathfrak{q}}_n^{tw}$ and we will consider afterwards how the Poisson cobracket on $\mathfrak{U}\widehat{\mathfrak{q}}_n^{tw}$ can be retrieved from the coproduct on $\mathfrak{U}_q\widehat{\mathfrak{q}}_n^{tw}$.

Let us write down explicitly relation (42); this will be useful below. Set $p(i,j) = |i| + |j|, \varphi(i,j) = (\delta_{i,j} + \delta_{i,-j})\mathrm{sign}(j)$ and $\theta(i,j,k) = \mathrm{sign}(\mathrm{sign}(i) + \mathrm{sign}(j) + \mathrm{sign}(k)) = (-1)^{|i||j|+|j||k|+|k||i|}$. Expanding $\frac{P}{w^{-1}z-1}$ and $\frac{PJ_1J_2}{wz-1}$ as $\frac{P}{w^{-1}z-1} = -\left(\sum_{t=0}^{\infty}(w^{-1}z)^t\right)P$ and

$$\frac{PJ_1J_2}{wz-1} = -(wz)^{-1}\left(\sum_{t=0}^{\infty}(wz)^{-t}\right)\sum_{\substack{i,j=-n\\ ij\neq 0}}^{n}(-1)^{|j|}E_{ij}\otimes E_{-j,-i},$$

we find that

$$\begin{aligned}&(-1)^{p(i,j)p(k,l)}q^{\varphi(j,l)}t_{ij}^{(r)}t_{kl}^{(s)} + \{j<l\}\theta(i,j,k)\epsilon t_{il}^{(r)}t_{kj}^{(s)} + \{-l<j\}\theta(-i,-j,k)\epsilon t_{i,-l}^{(r)}t_{k,-j}^{(s)}\\ &- \epsilon\theta(i,j,k)\sum_{t=0}^{r}t_{il}^{(r-t)}t_{kj}^{(s+t)} - \epsilon\theta(i,j,-k)\sum_{t=0}^{\min\{r-1,s-1\}}\mathcal{C}^{2(t+1)}t_{i,-l}^{(r-1-t)}t_{k,-j}^{(s-1-t)}\\ &= q^{\varphi(i,k)}t_{kl}^{(s)}t_{ij}^{(r)} + \{k<i\}\theta(i,j,k)\epsilon t_{il}^{(s)}t_{kj}^{(r)} + \{i<-k\}\theta(-i,-j,k)\epsilon t_{-i,l}^{(s)}t_{-k,j}^{(r)}\\ &- \epsilon\theta(i,j,k)\sum_{t=0}^{r}t_{il}^{(s+t)}t_{kj}^{(r-t)} - \epsilon\theta(i,j,-k)\sum_{t=0}^{\min\{r-1,s-1\}}\mathcal{C}^{-2(t+1)}t_{-i,l}^{(s-1-t)}t_{-k,j}^{(r-1-t)}\end{aligned} \tag{45}$$

where $\{\bullet < \bullet\}$ equals 1 if the inequality is satisfied and 0 otherwise.

Let $\mathfrak{U}_{\mathcal{A}}\widehat{\mathfrak{q}}_n^{tw}$ be the $\mathcal{A}$-subalgebra of $\mathfrak{U}_q\widehat{\mathfrak{q}}_n^{tw}$ generated by $\tau_{ij}^{(r)}, -n \leq i,j \leq n, ij \neq 0, r \in \mathbb{Z}_{\geq 0}$ and by γ where

$$\tau_{ij}^{(r)} = \frac{t_{ij}^{(r)}}{q-q^{-1}} \text{ if } i\neq j \text{ or } r\neq 0, \quad \tau_{ii}^{(0)} = \frac{t_{ii}^{(0)}-1}{q-1}, \quad \gamma = \frac{\mathcal{C}-1}{q-1}.$$

Theorem 6.1. *The quotient* $\mathfrak{U}_{\mathcal{A}}\widehat{\mathfrak{q}}_n^{tw}/(q-1)\mathfrak{U}_{\mathcal{A}}\widehat{\mathfrak{q}}_n^{tw}$ *is isomorphic to* $\mathfrak{U}\widehat{\mathfrak{q}}_n^{tw}$ *as an algebra over* $\mathbb{C}$.

Proof. We first construct an epimorphism $\Phi : \mathfrak{U}\widehat{\mathfrak{q}}_n^{tw} \twoheadrightarrow \mathfrak{U}_{\mathcal{A}}\widehat{\mathfrak{q}}_n^{tw}/(q-1)\mathfrak{U}_{\mathcal{A}}\widehat{\mathfrak{q}}_n^{tw}$.

For $r \in \mathbb{Z}_{>0}$, $-n \le i,j \le n, ij \neq 0$, set $\Phi(\mathbf{E}_{ij}(r)) = (-1)^{|j|}\overline{\tau}_{ji}^{(r)}$ where $\overline{\tau}_{ji}^{(r)}$ is the image of $\tau_{ji}^{(r)}$ in the quotient $\mathfrak{U}_{\mathcal{A}}\widehat{\mathfrak{q}}_n^{tw}/(q-1)\mathfrak{U}_{\mathcal{A}}\widehat{\mathfrak{q}}_n^{tw}$. If $r = 0$, we can assume that $j \le i$ (since $\mathbf{E}_{ij}(0) = \mathbf{E}_{-i,-j}(0)$) and set $\Phi(\mathbf{E}_{ij}(0)) = (-1)^{|j|}\overline{\tau}_{ji}^{(0)}$ if $j < i$ and $\Phi(\mathbf{E}_{ii}(0)) = (-1)^{|i|}\overline{\tau}_{ii}^{(0)}$. Set $\Phi(\mathfrak{c}) = -\gamma$. We have to show that Φ satisfies $[\Phi(X_1), \Phi(X_2)] = \Phi([X_1, X_2]) \;\forall\, X_1, X_2 \in \widehat{\mathfrak{q}}_n^{tw}$. We will not check all the cases, but some of them to show how one should proceed.

A straightforward computation shows that

$$\begin{aligned}&[(-1)^{|i|}\mathbf{E}_{ji}(r), (-1)^{|k|}\mathbf{E}_{lk}(s)] = (-1)^{|i|+|k|}(\delta_{il}\mathbf{E}_{jk}(r+s) + \delta_{i,-l}\mathbf{E}_{j,-k}(r-s))\\ &- (-1)^{|i||j|+|i||l|+|j||l|+|k|}((-1)^{|i|+|j|}\delta_{kj}\mathbf{E}_{li}(r+s) + \delta_{k,-j}\mathbf{E}_{l,-i}(s-r)) + 2\delta_{i,-l}(-1)^{|i|+|k|}\delta_{rs}Str(E_{j,-k})r\mathfrak{c}\end{aligned} \tag{46}$$

¿From (45), we deduce the following relation in $\mathfrak{U}_{\mathcal{A}}\widehat{\mathfrak{q}}_n^{tw}/(q-1)\mathfrak{U}_{\mathcal{A}}\widehat{\mathfrak{q}}_n^{tw}$ if $s > r > 0$:

$$\begin{aligned}[\overline{\tau}_{ij}^{(r)}, \overline{\tau}_{kl}^{(s)}] &= \delta_{il}(-1)^{|i|}\overline{\tau}_{kj}^{(s+r)} + \delta_{i,-l}(-1)^{|i|}\overline{\tau}_{k,-j}^{(s-r)}\\ &\quad - \delta_{jk}(-1)^{|i||j|+|i||l|+|j||l|}\overline{\tau}_{il}^{(r+s)} - \delta_{-k,j}(-1)^{|i||j|+|i||l|+|j||l|+|i|+|j|}\overline{\tau}_{-i,l}^{(s-r)}\end{aligned} \tag{47}$$

If $s = r > 0$, we find that (using $\overline{\tau}_{jj}^{(0)} = -\overline{\tau}_{-j,-j}^{(0)}$):

$$\begin{aligned}[\overline{\tau}_{ij}^{(r)}, \overline{\tau}_{kl}^{(r)}] &= \delta_{il}(-1)^{|i|}\overline{\tau}_{kj}^{(2r)} + \delta_{i,-l}(-1)^{|i|}\overline{\tau}_{k,-j}^{(0)} + \delta_{j,-k}(-1)^{|i||j|+|i||l|+|j||l|+|i|+|j|}\overline{\tau}_{i,-l}^{(0)}\\ &\quad + 2\delta_{i,-l}\delta_{j,-k}(-1)^{|i|}r\gamma - \delta_{jk}(-1)^{|i||j|+|i||l|+|j||l|}\overline{\tau}_{il}^{(2r)}\\ &\quad - \delta_{-k,j}(1-\delta_{-i,l})(-1)^{|i||j|+|i||l|+|j||l|+|i|+|j|}\overline{\tau}_{-i,l}^{(0)} - \delta_{-i,l}(1-\delta_{-k,j})(-1)^{|i|}\overline{\tau}_{-k,j}^{(0)}\end{aligned} \tag{48}$$

We can now verify that the equality $[\Phi(X_1), \Phi(X_2)] = \Phi([X_1, X_2])$ holds when $X_1 = (-1)^{|i|}\mathbf{E}_{ji}(r)$ and $X_2 = (-1)^{|k|}\mathbf{E}_{lk}(s)$ in the two cases $s > r > 0$ and $s = r > 0$ by comparing (46) with (47) and (48). Note that, in (47), when $r - s < 0$, $\Phi(\mathbf{E}_{j,-k}(r-s))$ is not defined: this is not a problem since $\mathbf{E}_{j,-k}(r-s) = \mathbf{E}_{-j,k}(s-r)$. We can also verify that equality when $s > r = 0$ if we use the following relation (with $i \le j$) which can be deduced from (45):

$$\begin{aligned}[\overline{\tau}_{ij}^{(0)}, \overline{\tau}_{kl}^{(s)}] &= (-1)^{|i|}\delta_{il}\overline{\tau}_{kj}^{(s)} + (-1)^{|i|}\delta_{i,-l}(1-\delta_{ij})\overline{\tau}_{k,-j}^{(s)}\\ &\quad - \delta_{kj}(-1)^{|i||j|+|i||l|+|j||l|}\overline{\tau}_{il}^{(s)} - (-1)^{|i||j|+|i||l|+|j||l|+|j|+|i|}\delta_{j,-k}(1-\delta_{ij})\overline{\tau}_{-i,l}^{(s)}\end{aligned} \tag{49}$$

The case $s = r = 0$ is treated in [Ol] and the case $r > s$ can be checked similarly.

We thus have an epimorphism $\Phi : \mathfrak{U}\widehat{\mathfrak{q}}_n^{tw} \twoheadrightarrow \mathfrak{U}_{\mathcal{A}}\widehat{\mathfrak{q}}_n^{tw}/(q-1)\mathfrak{U}_{\mathcal{A}}\widehat{\mathfrak{q}}_n^{tw}$. We will also need to consider $\widetilde{\Phi}$, which is defined as Φ, but with $\widehat{\mathfrak{q}}_n^{tw}$ replaced by $\mathcal{L}_{tw}\mathfrak{q}_n \cong \widehat{\mathfrak{q}}_n^{tw}/(\mathfrak{c})$ and $\mathfrak{U}_{\mathcal{A}}\widehat{\mathfrak{q}}_n^{tw}$ replaced by $\mathfrak{U}_{\mathcal{A}}\widehat{\mathfrak{q}}_n^{tw}/(\mathcal{C}-1)$. Actually, we will first prove that $\widetilde{\Phi}$ is an isomorphism, and for this we need evaluation representations. Let $x_1, \ldots, x_l$ be commuting variables. We can construct a representation of $\mathfrak{U}_{\mathcal{A}}\widehat{\mathfrak{q}}_n^{tw}$ on $\mathcal{A}[x_1^{\pm 1}, \ldots, x_l^{\pm 1}] \otimes_{\mathbb{C}} \mathbb{C}(n|n)^{\otimes l}$ via the map

$$\rho_l : \mathfrak{U}_{\mathcal{A}}\widehat{\mathfrak{q}}_n^{tw} \longrightarrow \mathrm{End}_{\mathcal{A}}(\mathcal{A}[x_1^{\pm 1}, \ldots, x_l^{\pm 1}]) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(\mathbb{C}(n|n))^{\otimes (l)}[[z^{-1}]]$$

determined by

$$\rho_l(T(z)) = S_{1,l+1}(x_1, z)S_{2,l+1}(x_2, z)\cdots S_{l,l+1}(x_l, z)$$

where both sides belong to $\mathrm{End}_{\mathcal{A}}(\mathcal{A}[x_1^{\pm 1}, \ldots, x_l^{\pm 1}]) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(\mathbb{C}(n|n))^{\otimes (l)}[[z^{-1}]] \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(\mathbb{C}(n|n))$, and by letting $\mathcal{C}$ act by multiplication by 1. That ρ_1 is a well-defined representation of $\mathfrak{U}_q\widehat{\mathfrak{q}}_n^{tw}$ follows from the fact that $S(x, y)$ satisfies the quantum Yang-Baxter equation (see lemma 5.1); the general case follows by applying the coproduct Δ to obtain a representation on $(\mathcal{A}[x,x^{-1}]\otimes_{\mathbb{C}}\mathbb{C}(n|n))^{\otimes_{\mathcal{A}}l}\cong\mathcal{A}[x_1^{\pm1},\ldots,x_l^{\pm1}]\otimes_{\mathbb{C}}\mathbb{C}(n|n)^{\otimes l}$. We can also view ρ_l as a representation of $\mathfrak{U}_{\mathcal{A}}\mathcal{L}_{tw}\mathfrak{q}_n$.

¿From the definition of $S(x,z)$, we can see that ρ_l descends to a representation $\overline{\rho_l}$ of the quotient $\mathfrak{U}_{\mathcal{A}}\mathcal{L}_{tw}\mathfrak{q}_n/(q-1)\mathfrak{U}_{\mathcal{A}}\mathcal{L}_{tw}\mathfrak{q}_n$ on the space $\mathbb{C}[x_1^{\pm1},\ldots,x_l^{\pm1}]\otimes_{\mathbb{C}}\mathbb{C}(n|n)^{\otimes l}$ and $\overline{\rho_l}\circ\widetilde{\Phi}$ yields the representation of $\mathcal{L}_{tw}\mathfrak{q}_n$ on the same space given by

$$\mathbf{E}_{ij}(s)(p\otimes\mathbf{v})=\sum_{k=1}^{l}(-1)^{|j|}\big(px_k^s\otimes E_{ij}^{(k)}(\mathbf{v})+px_k^{-s}\otimes E_{-i,-j}^{(k)}(\mathbf{v})\big)\text{ where }p=p(x_1^{\pm1},\ldots,x_l^{\pm1}).$$

Here, $E_{ij}^{(k)}\in\mathrm{End}_{\mathbb{C}}(\mathbb{C}(n|n)^{\otimes l})$ acts by the matrix E_{ij} only on the k^{th} factor in the tensor product $\mathbb{C}(n|n)^{\otimes l}$, but signs have to be taken into account: if $\mathbf{v}=v_1\otimes\cdots\otimes v_l$ which each v_i homogeneous, then $E_{ij}^{(k)}(\mathbf{v})=(-1)^{(|i|+|j|)\sum_{g=1}^{k-1}|v_g|}v_1\otimes\cdots\otimes v_{k-1}\otimes E_{ij}(v_k)\otimes v_{k+1}\otimes\cdots\otimes v_l$ where $|v_g|$ is the parity of v_g.

Let $K_l=\mathrm{Ker}(\overline{\rho_l}\circ\widetilde{\Phi})$. We want to show that $\cap_{l=1}^{\infty}K_l=\{0\}$ since this would imply that $\mathrm{Ker}(\widetilde{\Phi})=\{0\}$, hence completing the proof that $\widetilde{\Phi}$ is an isomorphism. Since $\mathcal{L}_{tw}\mathfrak{q}_n\subset\mathfrak{gl}_{n|n}[u^{\pm1}]$, it is enough to prove that the intersection of all the kernels of the representations of $\mathfrak{gl}_{n|n}[u^{\pm1}]$ on $\mathbb{C}[x_1^{\pm1},\ldots,x_l^{\pm1}]\otimes_{\mathbb{C}}\mathbb{C}(n|n)^{\otimes l}$ (given by $(E_{ij}u^r)(p\otimes\mathbf{v})=\sum_{k=1}^{l}px_k^r\otimes E_{ij}^{(k)}(\mathbf{v})$) is equal to zero. This is proved in [Na2]: see the proof of proposition 2.2, which ultimately boils down to applying the Poincaré-Birkhoff-Witt theorem for Lie superalgebras. (In [Na2], the author considers only polynomials in non-negative powers of $x_1,\ldots,x_l$, but the proofs works also for Laurent polynomials.)

To show that Φ is an isomorphism, we need to consider tensor products $\rho_l\otimes_{\mathcal{A}}\tau_k$ where τ_k is a representation of $\mathfrak{U}_{\mathcal{A}}\widehat{\mathfrak{q}}_n^{tw}$ on a $\mathcal{A}$-module on which $\mathcal{C}$ acts by multiplication by $q^k,k\in\mathbb{Z}$. Set $K_{l,k}=\mathrm{Ker}\big((\overline{\rho_l\otimes_{\mathcal{A}}\tau_k})\circ\Phi\big)$. $(\overline{\rho_l\otimes_{\mathcal{A}}\tau_k})\circ\Phi$ is a representation of $\mathfrak{U}\widehat{\mathfrak{q}}_n^{tw}$ on which $\mathfrak{c}$ acts by multiplication by k. One can then show that $\cap_{k\in\mathbb{Z}}\cap_{l=1}^{\infty}K_{l,k}=\{0\}$. □

It is possible to obtain an analogous proposition by replacing $\mathbb{C}(q),\mathcal{A}$ by $\mathbb{C}((h)),\mathbb{C}[[h]]$: one can define the Hopf algebra $\mathfrak{U}_{\mathbb{C}((h))}\widehat{\mathfrak{q}}_n^{tw}$ and its subalgebra $\mathfrak{U}_{\mathbb{C}[[h]]}\widehat{\mathfrak{q}}_n^{tw}$ by replacing q by $q=e^h\in\mathbb{C}[[h]]$, and show that $\mathfrak{U}\widehat{\mathfrak{q}}_n^{tw}$ is isomorphic to $\mathfrak{U}_{\mathbb{C}[[h]]}\widehat{\mathfrak{q}}_n^{tw}/h\mathfrak{U}_{\mathbb{C}[[h]]}\widehat{\mathfrak{q}}_n^{tw}$. $\tau_{ij}^{(r)}$ should be replaced by $(2h)^{-1}t_{ij}^{(r)}$ if $i\neq j$ or $r>0$ and by $h^{-1}(t_{ii}^{(0)}-1)$ if $i=j,r=0$; γ should be set equal to $h^{-1}(\mathcal{C}-1)$.

Let us now consider the classical limit of the coproduct on $\mathfrak{U}_{\mathbb{C}[[h]]}\widehat{\mathfrak{q}}_n^{tw}$. Let $(\cdot)^{\circ}$ be the involution on $(\mathfrak{U}_{\mathbb{C}[[h]]}\widehat{\mathfrak{q}}_n^{tw})^{\otimes2}$ given by $A_1\otimes A_2\mapsto(-1)^{|A_1||A_2|}A_2\otimes A_1$ where $|A_i|$ is the $\mathbb{Z}/2\mathbb{Z}$-degree of A_i for $i=1,2$.

Proposition 6.1. *For any $A\in\mathfrak{U}_{\mathbb{C}[[h]]}\widehat{\mathfrak{q}}_n^{tw}$, $\overline{h^{-1}(\Delta(A)-\Delta(A)^{\circ})}=\delta(\overline{A})\in(\mathfrak{U}\widehat{\mathfrak{q}}_n^{tw})^{\otimes2}$ where $\overline{\Delta(A)}$ denotes reduction modulo h and $\overline{A}$ is the image of A in $\mathfrak{U}_{\mathbb{C}[[h]]}\widehat{\mathfrak{q}}_n^{tw}/h\mathfrak{U}_{\mathbb{C}[[h]]}\widehat{\mathfrak{q}}_n^{tw}\cong\mathfrak{U}\widehat{\mathfrak{q}}_n^{tw}$.*

Proof. Only for this proof, we use the definition of $\tau_{ij}^{(r)},\gamma$ given just before the statement of the proposition with h in the denominator. Using the formula (43) for the coproduct Δ, we find that, for $-n\leq i\neq j\leq n$ and $r>0$, $\overline{h^{-1}(\Delta(\tau_{ij}^{(r)})-\Delta(\tau_{ij}^{(r)})^{\circ})}$ equals

$$\begin{aligned}&2\sum_{s=1}^{r-1}\sum_{\substack{k=-n\\k\neq0}}^{n}\big((-1)^{(|i|+|k|)(|k|+|j|)}\overline{\tau}_{ik}^{(s)}\otimes\overline{\tau}_{kj}^{(r-s)}-\overline{\tau}_{kj}^{(s)}\otimes\overline{\tau}_{ik}^{(r-s)}\big)\\&+2\sum_{\substack{k=-n\\k\neq0,i,j}}^{n}\big((-1)^{(|i|+|k|)(|k|+|j|)}(\overline{\tau}_{ik}^{(0)}\otimes\overline{\tau}_{kj}^{(r)}+\overline{\tau}_{ik}^{(r)}\otimes\overline{\tau}_{kj}^{(0)})-(\overline{\tau}_{kj}^{(r)}\otimes\overline{\tau}_{ik}^{(0)}+\overline{\tau}_{kj}^{(0)}\otimes\overline{\tau}_{ik}^{(r)})\big)\\&\quad+2\big(\overline{\tau}_{ii}^{(r)}\otimes\overline{\tau}_{ij}^{(0)}-\overline{\tau}_{ij}^{(0)}\otimes\overline{\tau}_{ii}^{(r)}+\overline{\tau}_{ij}^{(0)}\otimes\overline{\tau}_{jj}^{(r)}-\overline{\tau}_{jj}^{(r)}\otimes\overline{\tau}_{ij}^{(0)}\big)\\&\quad+\overline{\tau}_{ii}^{(0)}\otimes\overline{\tau}_{ij}^{(r)}-\overline{\tau}_{ij}^{(r)}\otimes\overline{\tau}_{ii}^{(0)}+\overline{\tau}_{ij}^{(r)}\otimes\overline{\tau}_{jj}^{(0)}-\overline{\tau}_{jj}^{(0)}\otimes\overline{\tau}_{ij}^{(r)}+2r(\overline{\tau}_{ij}^{(r)}\otimes\overline{\gamma}-\overline{\gamma}\otimes\overline{\tau}_{ij}^{(r)})\end{aligned}$$

Using formula (44) and the fact that $\tau_{ij}^{(0)} = 0$ if $i > j$, one can see that the right-hand side is equal to $\delta((-1)^{|i|}\mathbf{E}_{ji}(r))$. □

7 Schur-Weyl functor

We construct a functor, that we call the Schur-Weyl functor, which connects affine Hecke-Clifford algebras with twisted quantum loop algebras of type Q. Theorem 7.1 below extends theorem 5.2 in [Ol] to $\mathfrak{U}_q\widehat{\mathfrak{q}}_n^{tw}$. It can also be viewed as a super analog of the main theorem in [ChPr] and a q-version of proposition 5.2 in [Na2].

Theorem 5.2 in [Ol] states that $\mathbb{C}_q(n|n)$ is a representation of both $\mathcal{H}_{q,l}$ and $\mathfrak{U}_q\mathfrak{q}_n$ and, moreover, the actions of these two superalgebras supercommute. The element $T_a, 1 \le a \le l-1$ acts by $P_{a,a+1}S_{a,a+1}$ on $\mathbb{C}_q(n|n)^{\otimes l}$, the indices $a, a+1$ denoting on which two consecutive factors P and S act, whereas c_a acts via $J^{(a)} = 1 \otimes \cdots \otimes 1 \otimes J \otimes 1 \otimes \cdots \otimes 1 \in \mathrm{End}_{\mathbb{C}(q)}(\mathbb{C}_q(n|n))^{\otimes l}$. To see why $J^{(a)}J^{(b)} = -J^{(b)}J^{(a)}$ if $1 \le a \ne b \le l$, we have to keep in mind that $\mathrm{End}_{\mathbb{C}(q)}(\mathbb{C}_q(n|n))^{\otimes l}$ is identified with $\mathrm{End}_{\mathbb{C}(q)}(\mathbb{C}_q(n|n)^{\otimes l})$ via the following isomorphism (which can be extended from U, V to $V_1, \ldots V_l$): if U, V are two $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}(q)$-vector spaces, $f \in \mathrm{End}_{\mathbb{C}(q)}(U), g \in \mathrm{End}_{\mathbb{C}(q)}(V)$ are $\mathbb{Z}/2\mathbb{Z}$-graded linear maps of homogeneous degree and $u \in U, v \in V$, then $(f \otimes g)(u \otimes v) = (-1)^{p(g)p(u)} f(u) \otimes g(v)$ where $p(u), p(g) \in \{0, 1\}$ denote the $\mathbb{Z}/2\mathbb{Z}$-degrees of u and g.

In order to extend theorem 5.2 in [Ol] to $\mathfrak{U}_q\widehat{\mathfrak{q}}_n^{tw}$, we first reinterpret Olshanski's construction in a slightly different way. Let M be a $\mathbb{Z}/2\mathbb{Z}$-graded right module over $\mathcal{H}_{q,l}$. We can form the tensor product $M \otimes_{\mathcal{H}_{q,l}} \mathbb{C}_q(n|n)^{\otimes l}$. Using that the actions of $\mathfrak{U}_q\mathfrak{q}_n$ and $\mathcal{H}_{q,l}$ supercommute, we can see that $M \otimes_{\mathcal{H}_{q,l}} \mathbb{C}_q(n|n)^{\otimes l}$ becomes a left module over $\mathfrak{U}_q\mathfrak{q}_n$ if we define the action of $t_{ij}^{(0)}$ in the following way on an elementary tensor: $t_{ij}^{(0)}(m \otimes \mathbf{v}) = (-1)^{p(m)(|i|+|j|)} m \otimes t_{ij}^{(0)}(\mathbf{v})$. In other words, we identify $t_{ij}^{(0)}$ with the linear map $1 \otimes t_{ij}^{(0)} \in \mathrm{End}_{\mathbb{C}(q)}(M) \otimes_{\mathbb{C}(q)} \mathrm{End}_{\mathbb{C}(q)}(\mathbb{C}_q(n|n)^{\otimes l}) \cong \mathrm{End}_{\mathbb{C}(q)}(M \otimes_{\mathbb{C}(q)} \mathbb{C}_q(n|n)^{\otimes l})$ where the last isomorphism is the one from the previous paragraph.

Let $\overline{\mathcal{H}}_l$ be the $\mathbb{C}$-algebra defined by the same relations as those in definition 4.1 with $q = 1$. The irreducible $\mathbb{Z}/2\mathbb{Z}$-graded representations of $\overline{\mathcal{H}}_l$ are parametrized by the strict partitions of l [Se]. Let $\mathbb{F}$ be a splitting field for the semisimple $\mathbb{C}(q)$-algebra $\mathcal{H}_{q,l}$, so that $\mathcal{H}_{q,l} \otimes_{\mathbb{C}(q)} \mathbb{F}$ is a split semisimple algebra - see [JoNa]. It is proved in [JoNa] that the irreducible $\mathbb{Z}/2\mathbb{Z}$-graded representations of $\mathcal{H}_{q,l} \otimes_{\mathbb{C}(q)} \mathbb{F}$ are also parametrized by the strict partitions λ of l. (Compare with Tits' deformation theorem, which is theorem 7.4.6 in [GePf].) One of the main results of [JoNa] provides an explicit splitting field $\mathbb{F}$ for $\mathcal{H}_{q,l}$ which is a composite of quadratic extensions of $\mathbb{C}(q)$, hence is a Galois extension of $\mathbb{C}(q)$. The irreducible $\mathbb{Z}/2\mathbb{Z}$-graded representations of $\mathcal{H}_{q,l}$ can be recovered from those of $\mathcal{H}_{q,l} \otimes_{\mathbb{C}(q)} \mathbb{F}$ by taking direct sums of irreducible modules over an orbit for the action of the Galois group of F over $\mathbb{C}(q)$: see [CuRe] section 7B. It is proved in [Se] that the tensor space $\mathbb{C}(n|n)^{\otimes l}$ decomposes as a direct sum of subspaces of the form $W^\lambda \otimes_{\mathbb{C}} V^\lambda \cdot 2^{-\delta(|\lambda|)}$ where λ is a strict partition of l, V^λ is a representation of $\mathfrak{q}_n$ and $W^\lambda \otimes_{\mathbb{C}} V^\lambda \cdot 2^{-\delta(|\lambda|)}$ is either equal to $W^\lambda \otimes_{\mathbb{C}} V^\lambda$ if the number of parts in λ is even or to a proper submodule of $W^\lambda \otimes_{\mathbb{C}} V^\lambda$ if the number of parts in λ is odd. Moreover, V^λ is non-zero exactly when λ has at most n parts (theorem 4 in [Se]). Therefore, if $n > l$, this condition is always satisfied (and this is not the best lower bound since λ must be a strict partition of l, hence has always less then l parts if $l > 1$). This means that, when $n > l$, every irreducible $\mathbb{Z}/2\mathbb{Z}$-graded representation of $\overline{\mathcal{H}}_l$ occurs in $\mathbb{C}(n|n)^{\otimes l}$. The same is true for $\mathcal{H}_{q,l} \otimes_{\mathbb{C}(q)} \mathbb{F}$ and $\mathbb{F}(n|n)^{\otimes l}$; from this, it follows that every irreducible $\mathbb{Z}/2\mathbb{Z}$-graded representation of $\mathcal{H}_{q,l}$ must also occur in $\mathbb{C}_q(n|n)^{\otimes l}$ if $n > l$.

Let $W_{\mathbb{F}}^\lambda$ be the irreducible $\mathbb{Z}/2\mathbb{Z}$-graded representation of $\mathcal{H}_{q,l} \otimes_{\mathbb{C}(q)} \mathbb{F}$ corresponding to λ. Set $\mathcal{H}_{\mathbb{F},l} = \mathcal{H}_{q,l} \otimes_{\mathbb{C}(q)} \mathbb{F}$ and $\mathfrak{U}_{\mathbb{F}}\mathfrak{q}_n = \mathfrak{U}_q\mathfrak{q}_n \otimes_{\mathbb{C}(q)} \mathbb{F}$. $\mathbb{F}(n|n)$ decomposes into a direct sum of irreducible $\mathcal{H}_{\mathbb{F},l} \otimes_{\mathbb{F}} \mathfrak{U}_{\mathbb{F}}\mathfrak{q}_n$-modules of the form $W_{\mathbb{F}}^\lambda \otimes_{\mathbb{F}} V_{\mathbb{F}}^\lambda \cdot 2^{-\delta(|\lambda|)}$ as above. Let V be an irreducible right module over $\mathcal{H}_{\mathbb{F},l}$. Since $\mathrm{Hom}_{\mathbb{F}}(V \otimes_{\mathcal{H}_{\mathbb{F},l}} W_{\mathbb{F}}^\lambda, \mathbb{F}) \cong \mathrm{Hom}_{\mathcal{H}_{\mathbb{F},l}}(W_{\mathbb{F}}^\lambda, \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F}))$, it follows that $V \otimes_{\mathcal{H}_{\mathbb{F},l}} W_{\mathbb{F}}^\lambda$ is non-zero if and only if $V \cong W_{\mathbb{F}}^{\lambda,*}$ (the dual of $W_{\mathbb{F}}^\lambda$). Moreover, if λ has an even number of parts, then $W_{\mathbb{F}}^{\lambda,*} \otimes_{\mathcal{H}_{\mathbb{F},l}} W_{\mathbb{F}}^\lambda$ is one-dimensional and if λ has an odd number of parts, then $W_{\mathbb{F}}^{\lambda,*} \otimes_{\mathcal{H}_{\mathbb{F},l}} W_{\mathbb{F}}^{\lambda}$ is two-dimensional. (This is where we need the fact that $\mathcal{H}_{\mathbb{F},l}$ is split semisimple, so that the even part of the commutant of a simple $\mathcal{H}_{\mathbb{F},l}$-module reduces to scalars.) In this latter case, $W_{\mathbb{F}}^{\lambda} \otimes_{\mathbb{F}} V_{\mathbb{F}}^{\lambda} \cong W_{\mathbb{F}}^{\lambda} \otimes_{\mathbb{F}} V_{\mathbb{F}}^{\lambda} \cdot 2^{-\delta(|\lambda|)} \oplus W$ where W is a second irreducible module over $\mathcal{H}_{\mathbb{F},l} \otimes_{\mathbb{F}} \mathfrak{U}_{\mathbb{F}}\mathfrak{q}_n$ (see [Se] for the case of $\overline{\mathcal{H}}_l$); it follows that $W_{\mathbb{F}}^{\lambda,*} \otimes_{\mathcal{H}_{\mathbb{F},l}} W_{\mathbb{F}}^{\lambda} \otimes_{\mathbb{F}} V_{\mathbb{F}}^{\lambda} \cdot 2^{-\delta(|\lambda|)} \cong V_{\mathbb{F}}^{\lambda}$.

In the previous section, we explained how to obtain evaluation representations of $\mathfrak{U}_q\widehat{\mathfrak{q}}_n^{tw}$ on the space $\mathbb{C}(q)[x_1^{\pm 1}, \ldots, x_l^{\pm 1}] \otimes_{\mathbb{C}(q)} \mathbb{C}_q(n|n)^{\otimes l}$. (Strictly speaking, to obtain what one would normally call an evaluation representation, the variables $x_1, \ldots, x_l$ should be evaluated at some specific values in $\mathbb{C}$.) This construction can be generalized in the following way. Suppose that the variables $X_1^{\pm 1}, \ldots, X_l^{\pm 1}$ pairwise commute and are even elements. As in the previous section, it is possible to define an algebra homomorphism $\mathfrak{U}_q\widehat{\mathfrak{q}}_n^{tw} \longrightarrow \mathbb{C}(q)[X_1^{\pm 1}, \ldots, X_l^{\pm 1}] \otimes_{\mathbb{C}(q)} \mathrm{End}_{\mathbb{C}(q)}(\mathbb{C}_q(n|n))^{\otimes l}$ via the formula

$$T(z) \mapsto \prod_{1 \le p \le l}^{\rightarrow} \left(S_{p,l+1} + \frac{\epsilon}{X_p^{-1} z - 1} \otimes P_{p,l+1} + \frac{\epsilon}{X_p z - 1} \otimes P_{p,l+1} J_p J_{l+1} \right), \ \mathcal{C} \mapsto 1. \tag{50}$$

Let us view the affine Hecke-Clifford algebra $\mathbf{H}_{q,l}$ as an algebra over $\mathbb{C}(q)$ and defined by the same relations as in definition 4.1. If M is a $\mathbb{Z}/2\mathbb{Z}$-graded right module over $\mathbf{H}_{q,l}$, then, since $\mathbb{C}(q)[X_1^{\pm 1}, \ldots, X_l^{\pm 1}]$ can be viewed as a commutative subalgebra of $\mathbf{H}_{q,l}$, M can also be viewed as a module over $\mathbb{C}(q)[X_1^{\pm 1}, \ldots, X_l^{\pm 1}]$ corresponding to an algebra homomorphism $\mathbb{C}(q)[X_1^{\pm 1}, \ldots, X_l^{\pm 1}] \longrightarrow \mathrm{End}_{\mathbb{C}(q)}(M)$; by composition, we obtain an algebra homomorphism $\mathfrak{U}_q\widehat{\mathfrak{q}}_n^{tw} \longrightarrow \mathrm{End}_{\mathbb{C}(q)}(M) \otimes_{\mathbb{C}(q)} \mathrm{End}_{\mathbb{C}(q)}(\mathbb{C}_q(n|n)^{\otimes l}) \cong \mathrm{End}_{\mathbb{C}(q)}(M \otimes_{\mathbb{C}(q)} \mathbb{C}_q(n|n)^{\otimes l})$.

To state theorem 7.1, we will need to consider the following subalgebra: let $\widetilde{\mathfrak{U}}_{\mathbb{F}}$ be the $\mathbb{F}$-subalgebra of $\mathfrak{U}_q\mathcal{L}_{tw}\mathfrak{q}_n \otimes_{\mathbb{C}(q)} \mathbb{F}$ generated by $t_{ij}^{(0)}, 1 \le i \le j \le n$, and by $t_{n,-n}^{(1)}$; $\widetilde{\mathfrak{U}}_q$ is defined similarly over $\mathbb{C}(q)$. (The subsuperalgebra generated by $t_{ij}^{(0)}, 1 \le i \le j \le n$, is isomorphic to $\mathfrak{U}_{\mathbb{F}}\mathfrak{q}_n$.) Note that the Lie subsuperalgebra of $\mathcal{L}_{tw}\mathfrak{q}_n$ generated by $\mathfrak{q}_n$ and $\mathbf{E}_{n,n}^1(1)$ contains $\mathcal{L}_{tw}{}'\mathfrak{q}_n$, so $\widetilde{\mathfrak{U}}_q$ is quite a large subalgebra.

Definition 7.1. *We define* $\mathrm{mod}_R^{gr} - \mathbf{H}_{\mathbb{F},l}$ *to be the category of finite dimensional* $\mathbb{Z}/2\mathbb{Z}$*-graded right modules over* $\mathbf{H}_{\mathbb{F},l}$ *and* $\mathrm{mod}_{L,l}^{gr,0} - \widetilde{\mathfrak{U}}_{\mathbb{F}}$ *to be the category of finite dimensional* $\mathbb{Z}/2\mathbb{Z}$*-graded left modules over* $\widetilde{\mathfrak{U}}_{\mathbb{F}}$ *which, as modules over* $\mathfrak{U}_{\mathbb{F}}\mathfrak{q}_n$*, are direct sums of submodules of* $\mathbb{F}(n|n)^{\otimes l}$ *and on which the action of* $t_{n,-n}^{(1)}$ *satisfies the following technical condition: if* $(t_{n,-n}^{(1)})^k(x) = 0$ *for some* $x \in N$, $N \in \mathrm{mod}_{L,l}^{gr,0} - \widetilde{\mathfrak{U}}_{\mathbb{F}}, k \in \mathbb{Z}_{\ge 0}$*, then* $t_{n,-n}^{(1)}(x) = 0$.

Theorem 7.1. *Let* M *be a right* $\mathbb{Z}/2\mathbb{Z}$*-graded module over* $\mathbf{H}_{q,l}$*. Then* $M \otimes_{\mathcal{H}_{q,l}} \mathbb{C}_q(n|n)^{\otimes l}$ *is a* $\mathbb{Z}/2\mathbb{Z}$*-graded left module over* $\mathfrak{U}_q\widehat{\mathfrak{q}}_n^{tw}$ *on which* $\mathcal{C}$ *acts trivially and this defines a functor, denoted* SW *and called the Schur-Weyl functor, between the categories of* $\mathbb{Z}/2\mathbb{Z}$*-graded right (respectively, left) modules over* $\mathbf{H}_{q,l}$ *and* $\mathfrak{U}_q\widehat{\mathfrak{q}}_n^{tw}$*. Moreover, if* $l+1 \le n$*, it provides an equivalence of categories between* $\mathrm{mod}_R^{gr} - \mathbf{H}_{\mathbb{F},l}$ *and* $\mathrm{mod}_{L,l}^{gr,0} - \widetilde{\mathfrak{U}}_{\mathbb{F}}$*.*

Remark 7.1. *It is known [GJKK] that the category of tensor modules over* $\mathfrak{U}_q\mathfrak{q}_n$*, that is, the category of submodules of* $\mathbb{C}_q(n|n)^{\otimes l}$ $\forall\, l \in \mathbb{Z}_{>0}$*, is completely reducible. Moreover, in loc. cit., it is identified with a category of finite dimensional modules with non-negative highest weights (in a precise sense).*

If N *is a right module in* $\mathrm{mod}_{L,l}^{gr,0} - \widetilde{\mathfrak{U}}_{\mathbb{F}}$*, the second part of the theorem tells us that it is isomorphic, as* $\widetilde{\mathfrak{U}}_{\mathbb{F}}$*-module, to* $\mathsf{SW}(M)$ *for a right* $\mathbf{H}_{\mathbb{F},l}$*-module* M*. However, at this point, it is not possible to tell if the action of all* $t_{ij}^{(r)}$ *on* N *is the same as its action on* $\mathsf{SW}(M)$*, but we can prove this for* $t_{n,-n}^{(1)}$*. This explains why we consider* $\widetilde{\mathfrak{U}}_{\mathbb{F}}$*.*

Given a right $\mathbb{Z}/2\mathbb{Z}$-graded module M_0 over $\mathcal{H}_{q,l}$, we will also denote by $\mathsf{SW}(M_0)$ the left $\mathfrak{U}_q\mathfrak{q}_n$-module $M_0 \otimes_{\mathcal{H}_{q,l}} \mathbb{C}_q(n|n)^{\otimes l}$, and similarly over $\mathbb{F}$.

Proof. Since $\mathcal{C}$ acts trivially on $\mathsf{SW}(M)$, we will work with $\mathfrak{U}_q\mathcal{L}_{tw}\mathfrak{q}_n$ instead of $\mathfrak{U}_q\widehat{\mathfrak{q}}_n^{tw}$. We already know that $M \otimes_{\mathbb{C}(q)} \mathbb{C}_q(n|n)^{\otimes l}$ is a left module over $\mathfrak{U}_q\mathcal{L}_{tw}\mathfrak{q}_n$. We have to see that $t_{ij}^{(r)}(mT_a \otimes \mathbf{v} - m \otimes P_{a,a+1}S_{a,a+1}\mathbf{v})$

and $t_{ij}^{(r)}(m\mathsf{c}_a \otimes \mathbf{v} - m \otimes J^{(a)}\mathbf{v})$ belong to the subspace $\widetilde{M}$ defined by

$$\begin{aligned}\widetilde{M} =&\mathrm{span}_{\mathbb{C}(q)}\{\widetilde{m}T_b \otimes \widetilde{\mathbf{v}} - \widetilde{m} \otimes P_{b,b+1}S_{b,b+1}\widetilde{\mathbf{v}} | 1 \le b \le l-1, \widetilde{m} \in M, \widetilde{\mathbf{v}} \in \mathbb{C}_q(n|n)^{\otimes l}\} \\ &\oplus \mathrm{span}_{\mathbb{C}(q)}\{\widetilde{m}\mathsf{c}_i \otimes \widetilde{\mathbf{v}} - \widetilde{m} \otimes J^{(i)}\widetilde{\mathbf{v}} | \widetilde{m} \in M, 1 \le i \le l, \widetilde{\mathbf{v}} \in \mathbb{C}_q(n|n)^{\otimes l}\}\end{aligned}$$

in order to be able to conclude that $t_{ij}^{(r)}$ descends to the quotient $M/\widetilde{M} \cong M \otimes_{\mathcal{H}_{q,l}} \mathbb{C}_q(n|n)^{\otimes l}$ of $M \otimes_{\mathbb{C}(q)} \mathbb{C}_q(n|n)^{\otimes l}$. When $r = 0$, this follows from theorem 5.2 in [Ol]. The proof for any $r \in \mathbb{Z}_{\ge 0}$ can be reduced to the case when $l = 2, a = 1$ because $T_a \otimes 1 - 1 \otimes (PS)_{a,a+1}$ commutes with $S_{p,l+1}(X_p, z)$ if $p \neq a, a+1$ and $\mathsf{c}_a \otimes 1 - 1 \otimes J^{(a)}$ commutes with $S_{p,l+1}(X_p, z)$ if $p \neq a$. It is thus enough to prove that

$$\begin{aligned}&\left(1 \otimes S_{13} + \frac{\epsilon}{(X_1^{-1}z - 1)} \otimes P_{13} + \frac{\epsilon}{(X_1 z - 1)} \otimes P_{13}J_1J_3\right) \\ &\quad \cdot \left(1 \otimes S_{23} + \frac{\epsilon}{(X_2^{-1}z - 1)} \otimes P_{23} + \frac{\epsilon}{(X_2 z - 1)} \otimes P_{23}J_2J_3\right)(T_1 \otimes 1 \otimes 1 \otimes 1 - 1 \otimes P_{12}S_{12} \otimes 1)\end{aligned}$$

belongs to the right ideal in $\mathrm{End}^{\mathrm{opp}}_{\mathbb{C}(q)}(M) \otimes_{\mathbb{C}(q)} \mathrm{End}_{\mathbb{C}(q)}(\mathbb{C}_q(n|n))^{\otimes 3}$ generated by $T_1 \otimes 1 \otimes 1 \otimes 1 - 1 \otimes P_{12}S_{12} \otimes 1$ and $\mathsf{c}_i \otimes 1 \otimes 1 \otimes 1 - 1 \otimes J^{(i)} \otimes 1$ for $i = 1, 2$. (The analogous claim with $\mathsf{c}_1, J^{(1)}$ or $\mathsf{c}_2, J^{(2)}$ instead of $T_1, P_{12}S_{12}$ is easier to establish.) The proof of this claim involves rather long computations which we will not reproduce here.[1]

Since we will not provide the full details, we want at least to prove that $M \otimes_{\mathcal{H}_{q,l}} \mathbb{C}_q(n|n)^{\otimes l}$ is a module over a large subalgebra of $\mathfrak{U}_q\mathcal{L}_{tw}\mathfrak{q}_n$, namely over $\widetilde{\mathfrak{U}}_q$. Let us consider the action of $t_{n,-n}^{(1)}$ on $M \otimes_{\mathbb{C}(q)} \mathbb{C}_q(n|n)^{\otimes l}$. Using the fact that $t_{ij}^{(0)} = 0$ if $i > j$ and the formula (43) for $\Delta(t_{n,-n}^{(1)})$, we deduce that (if m is homogeneous)

$$t_{n,-n}^{(1)}(m \otimes \mathbf{v}) = (-1)^{p(m)}\epsilon \sum_{k=1}^{l} q^{\alpha_k(\mathbf{v}) - \beta_k(\mathbf{v})}\big(mX_k \otimes E_{-n,n}^{(k)}(\mathbf{v}) + mX_k^{-1} \otimes E_{n,-n}^{(k)}(\mathbf{v})\big)$$

where $\alpha_k(\mathbf{v}) = \#\{1 \le j < k | i_j = -n \text{ or } i_j = n\}$ and $\beta_k(\mathbf{v}) = \#\{k < j \le l | i_j = -n \text{ or } i_j = n\}$ if $\mathbf{v} = v_{i_1} \otimes \cdots \otimes v_{i_l}$ and $\{v_{-n}, v_{-n+1}, \ldots, v_n\}$ is the standard basis of $\mathbb{C}_q(n|n)$.

Set $h_{\pm n} = \sum_{b=1}^{n} q^{\pm\delta_{bn} \pm \delta_{b,-n}} \mathsf{E}_{bb}^0 \in \mathrm{End}_{\mathbb{C}(q)}(\mathbb{C}_q(n|n))$. It is enough to check the following congruence in $\mathrm{End}_{\mathbb{C}(q)}(M \otimes_{\mathbb{C}(q)} \mathbb{C}_q(n|n)^{\otimes l})$ modulo W, the right ideal generated by $T_j \otimes 1 - 1 \otimes P_{j,j+1}S_{j,j+1}, 1 \le j \le l-1$ and $\mathsf{c}_j \otimes 1 - 1 \otimes J^{(j)}, 1 \le j \le l$:

$$\begin{aligned}&T_kX_k \otimes E_{-n,n}^{(k)}h_{-n}^{(k+1)} + T_kX_k^{-1} \otimes E_{n,-n}^{(k)}h_{-n}^{(k+1)} + T_kX_{k+1} \otimes h_n^{(k)}E_{-n,n}^{(k+1)} + T_kX_{k+1}^{-1} \otimes h_n^{(k)}E_{n,-n}^{(k+1)} \\ &\equiv X_k \otimes E_{-n,n}^{(k)}h_{-n}^{(k+1)}P_{k,k+1}S_{k,k+1} + X_k^{-1} \otimes E_{n,-n}^{(k)}h_{-n}^{(k+1)}P_{k,k+1}S_{k,k+1} \\ &+ X_{k+1} \otimes h_n^{(k)}E_{-n,n}^{(k+1)}P_{k,k+1}S_{k,k+1} + X_{k+1}^{-1} \otimes h_n^{(k)}E_{n,-n}^{(k+1)}P_{k,k+1}S_{k,k+1}\end{aligned} \tag{51}$$

Using the relations in definition 4.1, we can rewrite the left-hand side as:

$$\begin{aligned}&(X_{k+1}T_k - \epsilon(X_{k+1} - \mathsf{c}_k\mathsf{c}_{k+1}X_k)) \otimes E_{-n,n}^{(k)}h_{-n}^{(k+1)} + (X_{k+1}^{-1}T_k + \epsilon(X_k^{-1} - \mathsf{c}_k\mathsf{c}_{k+1}X_{k+1})) \otimes E_{n,-n}^{(k)}h_{-n}^{(k+1)} \\ &+ (X_kT_k + \epsilon(1 + \mathsf{c}_k\mathsf{c}_{k+1})X_{k+1}) \otimes h_n^{(k)}E_{-n,n}^{(k+1)} + (X_k^{-1}T_k - \epsilon(X_k^{-1} + \mathsf{c}_k\mathsf{c}_{k+1}X_k)) \otimes h_n^{(k)}E_{n,-n}^{(k+1)}\end{aligned}$$

Therefore, the congruence (51) holds if and only if the following expression is congruent to 0 modulo W:

$$\begin{aligned}&X_{k+1} \otimes \big((P_{k,k+1}S_{k,k+1} - \epsilon)E_{-n,n}^{(k)}h_{-n}^{(k+1)} - h_n^{(k)}E_{-n,n}^{(k+1)}(P_{k,k+1}S_{k,k+1} - \epsilon)\big) \\ &+ X_k \otimes \big(P_{k,k+1}S_{k,k+1}h_n^{(k)}E_{-n,n}^{(k+1)} - E_{-n,n}^{(k)}h_{-n}^{(k+1)}P_{k,k+1}S_{k,k+1}\big)\end{aligned}$$

[1]The explicit calculations are available in the appendix of the version of this paper available on the second author's web page.

$$+ X_{k+1}^{-1} \otimes \Big(\big(P_{k,k+1}S_{k,k+1} - \epsilon J^{(k)}J^{(k+1)}\big) E_{n,-n}^{(k)} h_{-n}^{(k+1)} - h_n^{(k)} E_{n,-n}^{(k+1)} \big(P_{k,k+1}S_{k,k+1} - \epsilon J^{(k)}J^{(k+1)}\big)\Big)$$
$$+ X_k^{-1} \otimes \Big(\big(P_{k,k+1}S_{k,k+1} - \epsilon(1 + J^{(k)}J^{(k+1)})\big) h_n^{(k)} E_{n,-n}^{(k+1)}$$
$$- E_{n,-n}^{(k)} h_{-n}^{(k+1)} \big(P_{k,k+1}S_{k,k+1} - \epsilon(1 + J^{(k)}J^{(k+1)})\big)\Big)$$

Using that the inverse of $P_{k,k+1}S_{k,k+1} - \epsilon J^{(k)}J^{(k+1)}$ is $P_{k,k+1}S_{k,k+1} - \epsilon(1 + J^{(k)}J^{(k+1)})$ and the inverse of $P_{k,k+1}S_{k,k+1}$ is $P_{k,k+1}S_{k,k+1} - \epsilon$, we have reduced the problem to showing that

$$P_{k,k+1}S_{k,k+1}h_n^{(k)}E_{-n,n}^{(k+1)} - E_{-n,n}^{(k)}h_{-n}^{(k+1)}P_{k,k+1}S_{k,k+1} = 0$$

and

$$\big(P_{k,k+1}S_{k,k+1} - \epsilon(1 + J^{(k)}J^{(k+1)})\big) h_n^{(k)} E_{n,-n}^{(k+1)} = E_{n,-n}^{(k)} h_{-n}^{(k+1)} \big(P_{k,k+1}S_{k,k+1} - \epsilon(1 + J^{(k)}J^{(k+1)})\big)$$

Using formula (39) for $S_{k,k+1}$, we can compute directly that

$$P_{k,k+1}S_{k,k+1}h_n^{(k)}E_{-n,n}^{(k+1)} - E_{-n,n}^{(k)}h_{-n}^{(k+1)}P_{k,k+1}S_{k,k+1} = P_{k,k+1}(E_{-n,n}^{(k+1)} - E_{-n,n}^{(k+1)}) = 0.$$

As for the second equality, it follows from the first one:

$$\begin{aligned}
&(P_{k,k+1}S_{k,k+1} - \epsilon(1 + J^{(k)}J^{(k+1)}))h_n^{(k)}E_{n,-n}^{(k+1)} - E_{n,-n}^{(k)}h_{-n}^{(k+1)}(P_{k,k+1}S_{k,k+1} - \epsilon(1 + J^{(k)}J^{(k+1)}))\\
&= -(P_{k,k+1}S_{k,k+1} - \epsilon(1 + J^{(k)}J^{(k+1)}))J^{(k)}J^{(k+1)}h_n^{(k)}E_{-n,n}^{(k+1)}J^{(k)}J^{(k+1)}\\
&+ J^{(k)}J^{(k+1)}E_{-n,n}^{(k)}h_{-n}^{(k+1)}J^{(k)}J^{(k+1)}(P_{k,k+1}S_{k,k+1} - \epsilon(1 + J^{(k)}J^{(k+1)}))\\
&= -(P_{k,k+1}S_{k,k+1}J^{(k)}J^{(k+1)} - \epsilon(J^{(k)}J^{(k+1)} - 1))h_n^{(k)}E_{-n,n}^{(k+1)}J^{(k)}J^{(k+1)}\\
&+ J^kJ^{(k+1)}E_{-n,n}^{(k)}h_{-n}^{(k+1)}(J^{(k)}J^{(k+1)}P_{k,k+1}S_{k,k+1} - \epsilon(J^{(k)}J^{(k+1)} - 1))\\
&= -J^{(k+1)}J^{(k)}P_{k,k+1}S_{k,k+1}h_n^{(k)}E_{-n,n}^{(k+1)}J^{(k)}J^{(k+1)} + J^{(k)}J^{(k+1)}E_{-n,n}^{(k)}h_{-n}^{(k+1)}P_{k,k+1}S_{k,k+1}J^{(k+1)}J^{(k)}\\
&= J^{(k)}J^{(k+1)}P_{k,k+1}(S_{k,k+1}h_n^{(k)}E_{-n,n}^{(k+1)} - h_{-n}^{(k)}E_{-n,n}^{(k+1)}S_{k,k+1})J^{(k)}J^{(k+1)} = 0
\end{aligned}$$

We have thus proved that the action of $t_{n,-n}^{(1)}$ on $M \otimes_{\mathbb{C}(q)} \mathbb{C}_q(n|n)^{\otimes l}$ descends to the quotient $M \otimes_{\mathcal{H}_{q,l}} \mathbb{C}_q(n|n)^{\otimes l}$. Therefore, from theorem 5.2 in [Ol], we deduce that $M \otimes_{\mathcal{H}_{q,l}} \mathbb{C}_q(n|n)^{\otimes l}$ is a left module over $\widetilde{\mathfrak{U}}_q$.

To obtain a functor $\mathsf{SW} : \mathrm{mod}_R^{gr} - \mathbf{H}_{q,l} \longrightarrow \mathrm{mod}_{L,l}^{gr} - \mathfrak{U}_q\mathcal{L}_{tw}\mathfrak{q}_n$, we are left to specify how SW acts on morphisms. If $M_1, M_2 \in \mathrm{mod}_R^{gr} - \mathbf{H}_{q,l}$ and $f \in \mathrm{Hom}_{\mathbf{H}_{q,l}}(M_1, M_2)$, then $\mathsf{SW}(f) : \mathsf{SW}(M_1) \longrightarrow \mathsf{SW}(M_2)$ is defined on pure tensors as $\mathsf{SW}(f)(m \otimes \mathbf{v}) = f(m) \otimes \mathbf{v}$.

In the second part of the proof, we have to show that, given a module $N \in \mathrm{mod}_{L,l}^{gr,0} - \widetilde{\mathfrak{U}}_{\mathbb{F}}$, there exists $M \in \mathrm{mod}_R^{gr} - \mathbf{H}_{\mathbb{F},l}$ such that $N = \mathsf{SW}(M)$. We must also explain why SW is bijective on morphisms. By hypothesis, N is a direct sum of submodules of $\mathbb{F}(n|n)^{\otimes l}$ when viewed as a module over $\mathfrak{U}_{\mathbb{F}}\mathfrak{q}_n$. It follows from the explanations given before theorem 7.1 that $N = M_0 \otimes_{\mathcal{H}_{\mathbb{F},l}} \mathbb{F}(n|n)^{\otimes l}$ for some $M_0 \in \mathrm{mod}_R^{gr} - \mathcal{H}_{\mathbb{F},l}$. We now have to explain how to turn M_0 into a right module over $\mathbf{H}_{\mathbb{F},l}$; for the affine Hecke algebra of S_l, this was done in [ChPr] and we follow similar ideas. We will need the next lemma, which follows from its non-quantum (classical) analog.

Lemma 7.1. *Suppose that $v_{i_1}, \ldots, v_{i_l} \in \mathbb{F}(n|n)$ are such that $v_{i_j} \neq v_{\pm i_k}$ for distinct j, k. Then $\mathbf{v}$ generates $\mathbb{F}(n|n)^{\otimes l}$ as a module over $\mathfrak{U}_{\mathbb{F}}\mathfrak{q}_n$. Consequently, if M_0 is a right irreducible module over $\mathcal{H}_{\mathbb{F},l}$ such that $M_0 \otimes_{\mathcal{H}_{\mathbb{F},l}} \mathbb{F}(n|n)^{\otimes l} \neq 0$ and $m \in M_0$, then $m \otimes \mathbf{v} = 0 \Rightarrow m = 0$.*

For $1 \leq j \leq l$, set $\mathbf{v}^{(j)} = v_1 \otimes \cdots \otimes v_{j-1} \otimes v_n \otimes v_{j+1} \otimes \cdots \otimes v_l$ (v_n is in the j^{th} spot), and $\mathbf{w}^{(j)} = v_1 \otimes \cdots \otimes v_{j-1} \otimes v_{-n} \otimes v_{j+1} \otimes \cdots \otimes v_l$. For any $\tau \in \Lambda^{\otimes l} \rtimes S_l$, let $\mathbf{w}_\tau^{(j)}$ be the vector obtained from $\mathbf{w}^{(j)}$ by applying τ to it. (S_l acts by permuting the factors in $\mathbb{F}(n|n)^{\otimes l}$, whereas the non-trivial element in the k^{th}

copy of Λ in $\Lambda^{\otimes l}$ acts by $J^{(k)}$.) By considering the weights of $t_{ii}^{(0)}$ on $\mathbb{F}(n|n)^{\otimes l}$ for $1 \leq i \leq n$, we see that we can write $t_{n,-n}^{(1)}(m \otimes \mathbf{v}^{(j)}) = \sum_{\tau \in \Lambda^{\otimes l} \rtimes S_l} m_\tau \otimes \mathbf{w}_\tau^{(j)}$ for some $m_\tau \in M$. (Here, we are also using the relation $t_{ii}^{(0)} t_{n,-n}^{(1)} = t_{n,-n}^{(1)} t_{ii}^{(0)}$ for $1 \leq i \leq n$, which can be deduced from (45).) $\mathbf{w}_\tau^{(j)}$ can be obtained from $\mathbf{w}^{(j)}$ by applying to it the action of an element of $\mathcal{H}_{\mathbb{F},l}$. Therefore, $t_{n,-n}^{(1)}(m \otimes \mathbf{v}^{(j)}) = \widetilde{m} \otimes \mathbf{w}^{(j)}$ for some $\widetilde{m} \in M$. Since $\mathbf{w}^{(j)}$ satisfies the condition of the previous lemma, we can see that the assignment $m \mapsto \widetilde{m}$ determines a linear function $\zeta_j \in \mathrm{End}_{\mathbb{C}}(M_0)$ such that $\widetilde{m} = (-1)^{p(m)} \zeta_j(m)$. Similarly, one can define a function $\xi_j \in \mathrm{End}_{\mathbb{C}}(M_0)$ by considering the action of $t_{n,-n}^{(1)}$ on $\mathbf{w}^{(j)}$; more precisely, $t_{n,-n}^{(1)}(m \otimes \mathbf{w}^{(j)}) = (-1)^{p(m)} \xi_j(m) \otimes \mathbf{v}^{(j)}$.

The next lemma is fundamental. Let $\eta_k^{\pm} \in \mathrm{End}_{\mathbb{F}}(\mathbb{F}(n|n)^{\otimes l})$ be $\eta_k^{\pm}(\mathbf{v}) = q^{\alpha_k(\mathbf{v}) - \beta_k(\mathbf{v})} E_{\mp n, \pm n}^{(k)}(\mathbf{v})$.

Lemma 7.2. *For all homogeneous $m \in M$ and $\mathbf{v} \in \mathbb{F}(n|n)^{\otimes l}$, $t_{n,-n}^{(1)}(m \otimes \mathbf{v}) = (-1)^{p(m)} \sum_{j=1}^{l} \zeta_j(m) \otimes \eta_j^{+}(\mathbf{v}) + (-1)^{p(m)} \sum_{j=1}^{l} \xi_j(m) \otimes \eta_j^{-}(\mathbf{v})$*

Proof. As in the proof of lemma 4.5 in [ChPr], we will proceed via a double induction. First of all, though, if $\mathbf{v} = v_{i_1} \otimes \cdots \otimes v_{i_l}$ and if $i_j \neq \pm n$ for any $1 \leq j \leq l$, then $t_{n,-n}^{(1)}(m \otimes \mathbf{v}) = 0$. This is where we need the technical assumption in our theorem. From (45), we deduce that $(t_{n,-n}^{(1)})^2 = \frac{\epsilon}{q+q^{-1}} \big((t_{nn}^{(0)})^2 - (t_{-n,-n}^{(0)})^2 \big)$. Since $t_{nn}^{(0)}(\mathbf{v}) = t_{-n,-n}^{(0)}(\mathbf{v}) = \mathbf{v}$ for such a $\mathbf{v}$, $(t_{n,-n}^{(1)})^2 (m \otimes \mathbf{v}) = 0$, hence by hypothesis it must be the case that $t_{n,-n}^{(1)}(m \otimes \mathbf{v}) = 0$.

Let $r \geq 0, s \geq 1$, $1 \leq j_1 < \cdots < j_r \leq l$, $1 \leq j'_1 < \cdots < j'_s \leq l$, and suppose that $\{j_1, \ldots, j_r\} \cap \{j'_1, \ldots, j'_s\} = \phi$. Set $\mathbf{j} = (j_1, \ldots, j_r)$, $\mathbf{j}' = (j'_1, \ldots, j'_s)$, and denote by $\mathbb{F}(n|n)^{\mathbf{j},\mathbf{j}'}$ the subspace of $\mathbb{F}(n|n)^{\otimes l}$ spanned by the vectors $\mathbf{v} = v_1 \otimes \cdots \otimes v_l$ with v_{-n} in positions $j_1, \ldots, j_r$, v_n in positions $j'_1, \ldots j'_s$ and vectors $\{v_1, v_{-1}, \ldots, v_{n-1}, v_{-(n-1)}\}$ in the other positions. By the previous paragraph, we can now suppose that $r + s \geq 1$.

It is enough to consider an element $\mathbf{v} \in \mathbb{F}(n|n)^{\mathbf{j},\mathbf{j}'}$ which has the vectors $v_1, \ldots, v_{n-2}$ in positions other than $\mathbf{j}, \mathbf{j}'$. (This is possible since we are assuming that $n \geq l+1$.) The lemma is true for $r = 1, s = 0$ or $r = 0, s = 1$ by the definition of ζ_j and ξ_j. Let us prove by induction that it is true also for any $r \geq 1$ when $s = 0$, so let's assume it's true for $r - 1$ and that $r \geq 2$. Let $\mathbf{v}'$ be obtained from $\mathbf{v}$ by replacing v_{-n} in positions j_{r-1}, j_r with $v_{-(n-1)}$, so that $\mathbf{v} = \frac{q^{-1}}{(q-q^{-1})^2 (q+q^{-1})} (t_{n-1,n}^{(0)})^2 \mathbf{v}'$. We now need one more lemma.

Lemma 7.3. *The following relation holds in $\mathfrak{U}_q \mathcal{L}_{tw} \mathfrak{q}_n$:*

$$t_{n,-n}^{(1)} (t_{n-1,n}^{(0)})^2 = (q + q^{-1}) t_{n-1,n}^{(0)} t_{n,-n}^{(1)} t_{n-1,n}^{(0)} - (t_{n-1,n}^{(0)})^2 t_{n,-n}^{(1)} \tag{52}$$

Proof. ¿From (45) with $i = n, j = -n, r = 1, k = n-1, l = n, s = 0$, we deduce that $q t_{n,-n}^{(1)} t_{n-1,n}^{(0)} = t_{n-1,n}^{(0)} t_{n,-n}^{(1)} + \epsilon t_{nn}^{(0)} t_{n-1,-n}^{(1)}$. Multiplying by $t_{n-1,n}^{(0)}$ on the right on both sides and using $t_{n-1,-n}^{(1)} t_{n-1,n}^{(0)} = t_{n-1,n}^{(0)} t_{n-1,-n}^{(1)}$, we obtain $q t_{n,-n}^{(1)} (t_{n-1,n}^{(0)})^2 = t_{n-1,n}^{(0)} t_{n,-n}^{(1)} t_{n-1,n}^{(0)} + \epsilon t_{nn}^{(0)} t_{n-1,n}^{(0)} t_{n-1,-n}^{(1)}$. We need also the equation $q^{-1} t_{nn}^{(0)} t_{n-1,n}^{(0)} = t_{n-1,n}^{(0)} t_{nn}^{(0)}$ which, when substituted into the previous one, yields $q t_{n,-n}^{(1)} (t_{n-1,n}^{(0)})^2 = t_{n-1,n}^{(0)} t_{n,-n}^{(1)} t_{n-1,n}^{(0)} + \epsilon q t_{n-1,n}^{(0)} t_{nn}^{(0)} t_{n-1,-n}^{(1)}$. Finally, we substitute $\epsilon t_{nn}^{(0)} t_{n-1,-n}^{(1)} = q t_{n,-n}^{(1)} t_{n-1,n}^{(0)} - t_{n-1,n}^{(0)} t_{n,-n}^{(1)}$ in the previous equation and simplify to obtain (52). □

Using this lemma, we can now write that

$$t_{n,-n}^{(1)}(m \otimes \mathbf{v}) = \frac{q^{-1}}{(q-q^{-1})^2} t_{n-1,n}^{(0)} t_{n,-n}^{(1)} t_{n-1,n}^{(0)} (m \otimes \mathbf{v}') - \frac{q}{(q-q^{-1})^2 (q+q^{-1})} (t_{n-1,n}^{(0)})^2 t_{n,-n}^{(1)} (m \otimes \mathbf{v}').$$

Let $\mathbf{v}''$ (resp. $\mathbf{v}'''$) be obtained from $\mathbf{v}'$ by replacing $v_{-(n-1)}$ in position j_{r-1} (resp. j_r) by v_{-n}. By the inductive assumption, $(-1)^{p(m)}t^{(1)}_{n,-n}(m\otimes \mathbf{v})$ equals

$$
\begin{aligned}
& \frac{(-1)^{p(m)}q^{-1}}{q-q^{-1}}t^{(0)}_{n-1,n}t^{(1)}_{n,-n}(m\otimes \mathbf{v}''+qm\otimes \mathbf{v}''')-\frac{q^{-1}}{(q-q^{-1})^2(q+q^{-1})}(t^{(0)}_{n-1,n})^2\sum_{k=1}^{r-2}\xi_{j_k}(m)\otimes \eta^-_{j_k}(\mathbf{v}') \\
= \;& \frac{q^{-1}}{q-q^{-1}}t^{(0)}_{n-1,n}\sum_{k=1}^{r-1}\xi_{j_k}(m)\otimes \eta^-_{j_k}(\mathbf{v}'')+\frac{1}{q-q^{-1}}t^{(0)}_{n-1,n}\sum_{\substack{k=1\\ k\neq r-1}}^{r}\xi_{j_k}(m)\otimes \eta^-_{j_k}(\mathbf{v}''') \\
& -\frac{q^{-1}}{q^2-q^{-2}}\sum_{k=1}^{r-2}t^{(0)}_{n-1,n}\xi_{j_k}(m)\otimes q^{\alpha_{j_k}(\mathbf{v}')-\beta_{j_k}(\mathbf{v}')}(E^{(j_k)}_{n,-n}(\mathbf{v}'')+qE^{(j_k)}_{n,-n}(\mathbf{v}''')) \\
= \;& q^{-1}\sum_{k=1}^{r-1}q^{\alpha_{j_k}(\mathbf{v}'')-\beta_{j_k}(\mathbf{v}'')}\xi_{j_k}(m)\otimes E^{(j_k)}_{n,-n}(\mathbf{v})+q\sum_{\substack{k=1\\ k\neq r-1}}^{r-2}q^{\alpha_{j_k}(\mathbf{v}''')-\beta_{j_k}(\mathbf{v}''')}\xi_{j_k}(m)\otimes E^{(j_k)}_{n,-n}(\mathbf{v}) \\
& -\frac{q^{-1}}{q+q^{-1}}\sum_{k=1}^{r-2}\xi_{j_k}(m)\otimes q^{\alpha_{j_k}(\mathbf{v}')-\beta_{j_k}(\mathbf{v}')}(E^{(j_k)}_{n,-n}(\mathbf{v})+q^2E^{(j_k)}_{n,-n}(\mathbf{v})) \\
= \;& \sum_{k=1}^{r-1}q^{\alpha_{j_k}(\mathbf{v})-\beta_{j_k}(\mathbf{v})}\xi_{j_k}(m)\otimes E^{(j_k)}_{n,-n}(\mathbf{v})+q^{\alpha_{j_r}(\mathbf{v})-\beta_{j_r}(\mathbf{v})}\xi_{j_r}(m)\otimes E^{(j_r)}_{n,-n}(\mathbf{v})=\sum_{j=1}^{l}\xi_j(m)\otimes \eta^-_j(\mathbf{v})
\end{aligned}
$$

This proves the lemma when $s=0$. The case $r=0, s\geq 1$ is very similar. Let us now assume that $r,s\geq 1$; we use relation (52) again and argue by induction.

Let $\mathbf{v}$ be as above and assume, without loss of generality that $j'_{s-1}<j_r<j'_s$. Let $\mathbf{v}'$ be obtained from $\mathbf{v}$ by replacing v_{-n} in position j_r by $v_{-(n-1)}$ and v_n in position j'_s by v_{n-1}. We need also $\mathbf{v}''$ and $\mathbf{v}'''$ which are obtained from $\mathbf{v}'$ by replacing, respectively, v_{n-1} in position j'_s and $v_{-(n-1)}$ in position j_r by v_n and v_{-n}. Starting from the observation that $\mathbf{v}=\frac{1}{(1+q^2)\epsilon^2}(t^{(0)}_{n-1,n})^2\mathbf{v}'$, we now get that $(-1)^{p(m)}t^{(1)}_{n,-n}(m\otimes\mathbf{v})$ equals

$$
\begin{aligned}
& \frac{(-1)^{p(m)}(q+q^{-1})}{(1+q^2)\epsilon^2}t^{(0)}_{n-1,n}t^{(1)}_{n,-n}t^{(0)}_{n-1,n}(m\otimes\mathbf{v}')-\frac{(-1)^{p(m)}}{(1+q^2)\epsilon^2}(t^{(0)}_{n-1,n})^2t^{(1)}_{n,-n}(m\otimes\mathbf{v}') \\
= \;& \frac{(-1)^{p(m)}(q+q^{-1})}{(1+q^2)\epsilon}t^{(0)}_{n-1,n}t^{(1)}_{n,-n}(qm\otimes\mathbf{v}''+m\otimes\mathbf{v}''')-\frac{1}{(1+q^2)\epsilon^2}(t^{(0)}_{n-1,n})^2\sum_{k=1}^{r-1}\xi_{j_k}(m)\otimes\eta^-_{j_k}(\mathbf{v}') \\
& -\frac{1}{(1+q^2)\epsilon^2}(t^{(0)}_{n-1,n})^2\sum_{k=1}^{s-1}\zeta_{j_k}(m)\otimes\eta^+_{j'_k}(\mathbf{v}') \\
= \;& \frac{1}{\epsilon}t^{(0)}_{n-1,n}\left(\sum_{k=1}^{r-1}\xi_{j_k}(m)\otimes\eta^-_{j_k}(\mathbf{v}'')+\sum_{k=1}^{s}\zeta_{j'_k}(m)\otimes\eta^+_{j'_k}(\mathbf{v}'')\right) \\
& +\frac{q^{-1}}{\epsilon}t^{(0)}_{n-1,n}\left(\sum_{k=1}^{r}\xi_{j_k}(m)\otimes\eta^-_{j_k}(\mathbf{v}''')+\sum_{k=1}^{s-1}\zeta_{j'_k}(m)\otimes\eta^+_{j'_k}(\mathbf{v}''')\right) \\
& -\frac{1}{(1+q^2)\epsilon}t^{(0)}_{n-1,n}\sum_{k=1}^{r-1}\xi_{j_k}(m)\otimes(q^2\eta^-_{j_k}(\mathbf{v}'')+q\eta^-_{j_k}(\mathbf{v}''')) \\
& -\frac{1}{(1+q^2)\epsilon}t^{(0)}_{n-1,n}\sum_{k=1}^{s-1}\zeta_{j'_k}(m)\otimes(q^2\eta^+_{j'_k}(\mathbf{v}'')+q\eta^+_{j'_k}(\mathbf{v}''')) \\
= \;& \left(\sum_{k=1}^{r-1}q^2\xi_{j_k}(m)\otimes\eta^-_{j_k}(\mathbf{v})+\sum_{k=1}^{s-1}q^2\zeta_{j'_k}(m)\otimes\eta^+_{j'_k}(\mathbf{v})+\zeta_{j'_s}(m)\otimes\eta^+_{j'_s}(\mathbf{v})\right)
\end{aligned}
$$

$$
\begin{aligned}
& +q^{-1}\left(\sum_{k=1}^{r} q\xi_{j_k}(m)\otimes\eta^-_{j_k}(\mathbf{v})+\sum_{k=1}^{s-1} q\zeta_{j'_k}(m)\otimes\eta^+_{j'_k}(\mathbf{v})\right)\\
& -\frac{1}{1+q^2}\sum_{k=1}^{r-1}\xi_{j_k}(m)\otimes(q^4\eta^-_{j_k}(\mathbf{v})+q^2\eta^-_{j_k}(\mathbf{v}))-\frac{1}{1+q^2}\sum_{k=1}^{s-1}\zeta_{j'_k}(m)\otimes(q^4\eta^+_{j'_k}(\mathbf{v})+q^2\eta^+_{j'_k}(\mathbf{v}))\\
= & \sum_{k=1}^{r}\xi_{j_k}(m)\otimes\eta^-_{j_k}(\mathbf{v})+\sum_{k=1}^{s}\zeta_{j'_k}(m)\otimes\eta^+_{j'_k}(\mathbf{v})
\end{aligned}
$$

This completes by induction the proof of lemma 7.2. □

Now that lemma 7.2 has been proved, we can complete the proof of theorem 7.1. We have that M_0 is a $\mathbb{Z}/2\mathbb{Z}$-graded right module over $\mathcal{H}_{\mathbb{F},l}$ and we have to explain how to turn it into a right module over $\mathbf{H}_{\mathbb{F},l}$. Set $mX_j = \frac{1}{\epsilon}\zeta_j(m)$ and $mX_j^{-1} = \frac{1}{\epsilon}\xi_j(m)$. We have to check that $mX_jX_j^{-1} = mX_j^{-1}X_j = m$, $mX_iX_j = mX_jX_i$, and the other relations coming from definition 4.1.

To prove $X_jX_j^{-1} = X_j^{-1}X_j = 1$, we need the following relation: $(q+q^{-1})(t^{(1)}_{n,-n})^2 = \epsilon((t^{(0)}_{nn})^2 - (t^{(0)}_{-n,-n})^2)$. Applying both sides to $m\otimes\mathbf{v}$ with $\mathbf{v} = v_1\otimes\cdots\otimes v_{j-1}\otimes v_n\otimes v_{j+1}\otimes\cdots\otimes v_l$, we deduce, since $n > l$, that $(q+q^{-1})\epsilon^2 mX_jX_j^{-1}\otimes\mathbf{v} = \epsilon(q^2-q^{-2})m\otimes\mathbf{v}$. Therefore, by lemma 7.1, $mX_jX_j^{-1} = m$. Replacing v_n by v_{-n} yields $mX_j^{-1}X_j = m$. The proof that $mX_iX_j = mX_jX_i$ for $i\neq j$ uses the same relation and a $\mathbf{v}$ with v_n as its i^{th} and j^{th} factors.

Since $N = M_0\otimes_{\mathcal{H}_{\mathbb{F},l}}\mathbb{F}(n|n)^{\otimes l}$ and $t^{(1)}_{n,-n}$ acts on this space, $t^{(1)}_{n,-n}(m\mathsf{c}_j\otimes\mathbf{v}) = t^{(1)}_{n,-n}(m\otimes J^{(j)}\mathbf{v})$; letting $\mathbf{v}$ be as in the previous paragraph, we find that $(-1)^{p(m)+1}m\mathsf{c}_jX_j\otimes E^{(j)}_{-n,n}\mathbf{v} = (-1)^{p(m)}mX_j^{-1}\otimes E^{(j)}_{n,-n}J^{(j)}\mathbf{v}$, so $m\mathsf{c}_jX_j\otimes E^{(j)}_{-n,n}\mathbf{v} = mX_j^{-1}\mathsf{c}_j\otimes E^{(j)}_{-n,n}\mathbf{v}$. It follows, by lemma 7.1, that $m\mathsf{c}_jX_j = mX_j^{-1}\mathsf{c}_j$. The proof that $m\mathsf{c}_kX_j = mX_j\mathsf{c}_k$ if $k\neq j$ is similar.

Finally, we also need to verify that $mT_iX_i = m\big(X_{i+1}T_i - \epsilon(X_{i+1}-\mathsf{c}_i\mathsf{c}_{i+1}X_i)\big)$ and $mT_iX_j = mX_jT_i$ if $j\neq i, i+1$. We prove only the former, the latter being easier. Since $t^{(1)}_{n,-n}(mT_i\otimes\mathbf{v}) = t^{(1)}_{n,-n}(m\otimes(PS)_{i,i+1}\mathbf{v})$, if $\mathbf{v} = v_1\otimes\cdots\otimes v_{i-1}\otimes v_n\otimes v_{i+1}\otimes\cdots\otimes v_l$ then $(PS)_{i,i+1}\mathbf{v} = v_1\otimes\cdots\otimes v_{i-1}\otimes v_{i+1}\otimes v_n\otimes\cdots\otimes v_l - \epsilon v_1\otimes\cdots\otimes v_{i-1}\otimes v_{-n}\otimes v_{-i-1}\otimes\cdots\otimes v_l$, and hence $(-1)^{p(m)}t^{(1)}_{n,-n}(m\otimes(PS)_{i,i+1}\mathbf{v})$ equals

$$
\begin{aligned}
& mX_{i+1}\otimes v_1\otimes\cdots\otimes v_{i-1}\otimes v_{i+1}\otimes v_{-n}\otimes\cdots\otimes v_l + \epsilon mX_i^{-1}\otimes v_1\otimes\cdots\otimes v_{i-1}\otimes v_n\otimes v_{-i-1}\otimes\cdots\otimes v_l\\
= \; & mX_{i+1}\otimes v_1\otimes\cdots\otimes v_{i-1}\otimes v_{i+1}\otimes v_{-n}\otimes\cdots\otimes v_l\\
& +\epsilon mX_i^{-1}\mathsf{c}_i\mathsf{c}_{i+1}\otimes v_1\otimes\cdots\otimes v_{i-1}\otimes v_{-n}\otimes v_{i+1}\otimes\cdots\otimes v_l\\
= \; & m(X_{i+1}-\epsilon X_{i+1})T_i\otimes v_1\otimes\cdots\otimes v_{i-1}\otimes v_{-n}\otimes v_{i+1}\otimes\cdots\otimes v_l\\
& +\epsilon m\mathsf{c}_i\mathsf{c}_{i+1}X_i\otimes v_1\otimes\cdots\otimes v_{i-1}\otimes v_{-n}\otimes v_{i+1}\otimes\cdots\otimes v_l
\end{aligned}
$$

Since $(-1)^{p(m)}t^{(1)}_{n,-n}(mT_i\otimes\mathbf{v}) = mT_iX_i\mathbf{v}\otimes\cdots\otimes v_{i-1}\otimes v_{-n}\otimes v_{i+1}\otimes\cdots\otimes v_l$, lemma 7.1 implies that $mT_iX_i = mX_{i+1}T_i - \epsilon m(X_{i+1}-\mathsf{c}_i\mathsf{c}_{i+1}X_i)$, as desired.

Finally, to complete the proof of theorem 7.1, we should explain why SW is bijective on sets of morphisms. Injectivity follows from the finite case: see the remarks at the beginning of this section. (Here, the assumption that n is large enough, e.g. $n\geq l+1$, is needed.) Now let $g:\mathsf{SW}(M_1)\longrightarrow\mathsf{SW}(M_2)$ be a homomorphism of graded left $\widetilde{\mathfrak{U}}_{\mathbb{F}}$-modules. Then g is also a homomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded $\mathfrak{U}_{\mathbb{F}}\mathfrak{q}_n$-modules of level l, so is equal to $\mathsf{SW}(f)$ for some $f\in\mathrm{Hom}_{\mathcal{H}_{\mathbb{F},l}}(M_1,M_2)$. We have to see that f is actually a homomorphism of $\mathbf{H}_{\mathbb{F},l}$-modules. Fix $1\leq j\leq l$ and set $\mathbf{v} = v_1\otimes\cdots\otimes v_{j-1}\otimes v_n\otimes v_{j+1}\otimes\cdots\otimes v_l$. Since $g(t^{(1)}_{n,-n}(m\otimes\mathbf{v})) = t^{(1)}_{n,-n}(g(m\otimes\mathbf{v}))$ because g is a homomorphism of $\widetilde{\mathfrak{U}}_{\mathbb{F}}$-modules, it follows that $f(mX_j)\otimes\widetilde{\mathbf{v}} = f(m)X_j\otimes\widetilde{\mathbf{v}}$ where $\widetilde{\mathbf{v}} = v_1\otimes\cdots\otimes v_{j-1}\otimes v_{-n}\otimes v_{j+1}\otimes\cdots\otimes v_l$; lemma 7.1 implies that $f(mX_j) = f(m)X_j$. Therefore, f is a homomorphism of right $\mathbf{H}_{\mathbb{F},l}$-modules. □

The following proposition gives us a property of the Schur-Weyl functor that is also true in the previously studied affine cases ([Dr, ChPr, Na2]). Note that, if $l_1+l_2=l$, then $\mathbf{H}_{q,l_1}\otimes_{\mathbb{C}(q)}\mathbf{H}_{q,l_2}$ embeds into $\mathbf{H}_{q,l}$.

Suppose that M_1 and M_2 are right modules over $\mathbf{H}_{q,l_1}$ and $\mathbf{H}_{q,l_2}$. Then $(M_1 \otimes_{\mathbb{C}(q)} M_2) \otimes_{(\mathbf{H}_{q,l_1} \otimes_{\mathbb{C}(q)} \mathbf{H}_{q,l_2})} \mathbf{H}_{q,l}$ is a right module over $\mathbf{H}_{q,l}$, so we can apply the Schur-Weyl functor to it. On the other hand, $\mathsf{SW}(M_1)$ and $\mathsf{SW}(M_2)$ are modules over $\mathfrak{U}_q\mathcal{L}_{tw}\mathfrak{q}_n$, which is a Hopf superalgebra, so $\mathsf{SW}(M_1) \otimes_{\mathbb{C}(q)} \mathsf{SW}(M_2)$ is also a module over $\mathfrak{U}_q\mathcal{L}_{tw}\mathfrak{q}_n$.

Proposition 7.1. *The left modules* $\mathsf{SW}((M_1 \otimes_{\mathbb{C}(q)} M_2) \otimes_{(\mathbf{H}_{q,l_1} \otimes_{\mathbb{C}(q)} \mathbf{H}_{q,l_2})} \mathbf{H}_{q,l})$ *and* $\mathsf{SW}(M_1) \otimes_{\mathbb{C}(q)} \mathsf{SW}(M_2)$ *over* $\mathfrak{U}_q\mathcal{L}_{tw}\mathfrak{q}_n$ *are isomorphic.*

Proof. Since

$$\mathbf{H}_{q,l} \cong \mathbb{C}(q)[X_1^{\pm 1}, \ldots, X_l^{\pm 1}] \otimes_{\mathbb{C}(q)} \mathcal{H}_{q,l} \cong (\mathbf{H}_{q,l_1} \otimes_{\mathbb{C}(q)} \mathbf{H}_{q,l_2}) \otimes_{(\mathcal{H}_{q,l_1} \otimes_{\mathbb{C}(q)} \mathcal{H}_{1,l_2})} \mathcal{H}_{q,l},$$

we have an isomorphism of super vector spaces

$$\begin{aligned}(M_1 \otimes_{\mathbb{C}(q)} M_2) \otimes_{(\mathbf{H}_{q,l_1} \otimes_{\mathbb{C}(q)} \mathbf{H}_{q,l_2})} \mathbf{H}_{q,l} \otimes_{\mathcal{H}_{q,l}} \mathbb{C}_q(n|n)^{\otimes l} &\cong (M_1 \otimes_{\mathbb{C}(q)} M_2) \otimes_{(\mathcal{H}_{q,l_1} \otimes_{\mathbb{C}(q)} \mathcal{H}_{q,l_2})} \mathbb{C}_q(n|n)^{\otimes l} \\ &\cong M_1 \otimes_{\mathcal{H}_{q,l_1}} \mathbb{C}_q(n|n)^{\otimes l_1} \otimes_{\mathbb{C}} M_2 \otimes_{\mathcal{H}_{q,l_2}} \mathbb{C}_q(n|n)^{\otimes l_2}\end{aligned}$$

which is also an isomorphism of left modules over $\mathfrak{U}_q\mathcal{L}_{tw}\mathfrak{q}_n$: this follows from (50) and the observation that $\prod_{1\le p\le l}^{\rightarrow} S(X_p,z)_{p,l+1} = \left(\prod_{1\le p\le l_1}^{\rightarrow} S(X_p,z)_{p,l+1}\right)\left(\prod_{l_1+1\le \widetilde{p}\le l}^{\rightarrow} S(X_{\widetilde{p}},z)_{\widetilde{p},l+1}\right)$. □

We end this section by giving information about $\mathsf{SW}(M)$ when M is irreducible. See [Na2] for a similar result for Yangians of type Q.

Proposition 7.2. *Let M be a finite dimensional right module over $\mathbf{H}_{\mathbb{F},l}$ which is irreducible as a $\mathbb{Z}/2\mathbb{Z}$-graded module. If $\mathsf{SW}(M) \neq \{0\}$, then $\mathsf{SW}(M)$ is an irreducible module over $\mathfrak{U}_{\mathbb{F}}\mathcal{L}_{tw}\mathfrak{q}_n$.*

Proof. The first half of the proof consists in reducing the problem to the classical (non quantum) setting using proposition 6.1. Suppose that $\mathsf{SW}(M) \neq 0$ and let $V_0 \subset \mathsf{SW}(M)$ be a non-zero $\mathbb{Z}/2\mathbb{Z}$-graded subspace invariant under $\mathfrak{U}_{\mathbb{F}}\mathfrak{q}_n$.

Corollary 6.6 in [GJKK] states that $\mathbb{C}_q(n|n)^{\otimes l}$ is completely reducible as a module over $\mathfrak{U}_q\mathfrak{q}_n$; moreover, $\mathcal{H}_{\mathbb{F},l}$ is a semisimple algebra over $\mathbb{F}$ [JoNa]. These two facts, combined with theorem 5.3 in [Ol], imply that $\mathbb{F}(n|n)^{\otimes l}$ can be decomposed as a direct sum of irreducible $\mathcal{H}_{\mathbb{F},l} \otimes_{\mathbb{F}} \mathfrak{U}_{\mathbb{F}}\mathfrak{q}_n$-modules. It follows that V_0 equals $\mathsf{SW}(M_0)$ where $M_0 \subset M$ is an $\mathcal{H}_{\mathbb{F},l}$-submodule.

Furthermore, since M is irreducible, it is a quotient of the induced module $M_0 \otimes_{\mathcal{H}_{\mathbb{F},l}} \mathbf{H}_{\mathbb{F},l}$. We have to prove that V_0 generates $\mathsf{SW}(M)$ as a module over $\mathfrak{U}_{\mathbb{F}}\mathcal{L}_{tw}\mathfrak{q}_n$; by the previous observation, we can replace M by the induced module $\widetilde{M} = M_0 \otimes_{\mathcal{H}_{\mathbb{F},l}} \mathbf{H}_{\mathbb{F},l}$ and M_0 by $\widetilde{M_0} = M_0 \otimes_{\mathcal{H}_{\mathbb{F},l}} \mathcal{H}_{\mathbb{F},l} \subset \widetilde{M}$. We thus have to show that $\mathsf{SW}(\widetilde{M_0})$ generates $\mathsf{SW}(\widetilde{M})$ as a module over $\mathfrak{U}_{\mathbb{F}}\mathcal{L}_{tw}\mathfrak{q}_n$.

Without loss of generality, we can assume that M_0 and $\widetilde{M_0}$ are irreducible. Let $\mathcal{R}$ be the $\mathbb{C}$-subalgebra of $\mathbb{F}$ generated by $\mathbb{C}[q,q^{-1}]$ and by the square roots added to $\mathbb{C}(q)$ to obtain $\mathbb{F}$ (see [JoNa]); let $\mathbb{A}$ be the localization of $\mathcal{R}$ at the ideal generated by $q-1$. Let $\mathcal{H}_{\mathbb{A},l}$ (resp. $\mathbf{H}_{\mathbb{A},l}$) be the subalgebra of $\mathcal{H}_{\mathbb{F},l}$ (resp. $\mathbf{H}_{\mathbb{F},l}$) which is equal to the $\mathbb{A}$-submodule generated by $T_1, \ldots, T_{l-1}, \mathsf{c}_1, \ldots, \mathsf{c}_l$ (resp. and by $X_1^{\pm 1}, \ldots, X_l^{\pm 1}$). Let $\{m_1, \ldots, m_d\}$ be a basis of M_0 over $\mathbb{F}$ and let $M_{0,\mathbb{A}}$ be the $\mathcal{H}_{\mathbb{A},l}$-submodule of M_0 generated by $m_1, \ldots, m_d$. Let $\widetilde{M}_{\mathbb{A}}$ be the induced module $M_{0,\mathbb{A}} \otimes_{\mathcal{H}_{\mathbb{A},l}} \mathbf{H}_{\mathbb{A},l}$ and let $\widetilde{M}_{0,\mathbb{A}}$ be $M_{0,\mathbb{A}} \otimes_{\mathcal{H}_{\mathbb{A},l}} \mathcal{H}_{\mathbb{A},l} \subset \widetilde{M}_{\mathbb{A}}$. SW can be defined as before simply by replacing $\mathbb{F}(n|n)^{\otimes l}$ by $\mathbb{A} \otimes_{\mathbb{C}} \mathbb{C}(n|n)^{\otimes l}$. $\mathsf{SW}(\widetilde{M}_{\mathbb{A}})$ is a module over $\mathfrak{U}_{\mathbb{A}}\mathcal{L}_{tw}\mathfrak{q}_n$ and it is enough to show that it is generated by $\mathsf{SW}(\widetilde{M}_{0,\mathbb{A}})$. This will follow if we can show that $\mathsf{SW}(\widetilde{M}_{0,\mathbb{A}})/(q-1)\mathsf{SW}(\widetilde{M}_{0,\mathbb{A}})$ generates all of $\mathsf{SW}(\widetilde{M}_{\mathbb{A}})/(q-1)\mathsf{SW}(\widetilde{M}_{\mathbb{A}})$ as a module over $\mathfrak{U}\mathcal{L}_{tw}\mathfrak{q}_n$ (see theorem 6.1).

Note that $\mathsf{SW}(\widetilde{M}_{0,\mathbb{A}})/(q-1)\mathsf{SW}(\widetilde{M}_{0,\mathbb{A}}) \cong \overline{M}_0 \otimes_{\overline{\mathcal{H}}_l} \mathbb{C}(n|n)^{\otimes l}$ where $\overline{M}_0 \cong M_{0,\mathbb{A}}/(q-1)M_{0,\mathbb{A}}$ and also $\mathsf{SW}(\widetilde{M}_{\mathbb{A}})/(q-1)\mathsf{SW}(\widetilde{M}_{\mathbb{A}}) \cong \overline{M}_0 \otimes_{\mathbb{C}} \mathbb{C}[X_1^{\pm 1}, \ldots, X_l^{\pm l}] \otimes_{\overline{\mathcal{H}}_l} \mathbb{C}(n|n)^{\otimes l}$. We can deduce our claim from section 4 in [Ar] after making the following observation. $\overline{M}_0$ can be viewed as a module over the subalgebra $\mathbb{C}[S_l]$ of

$\overline{\mathcal{H}}_l$ and splits into a direct sum $\overline{M}_0 = \oplus_{i=1}^k \overline{M}_{0,i}$ of $\mathbb{C}[S_l]$-modules. Set $N_i = \overline{M}_{0,i} \otimes_{\mathbb{C}} \mathbb{C}[X_1^{\pm 1}, \dots, X_l^{\pm l}] \otimes_{\mathbb{C}[S_l]} (\mathbb{C}^n)^{\otimes l}$ and $N = \oplus_{i=1}^k N_i$; both are modules over $\mathfrak{U}\mathfrak{gl}_n(\mathbb{C}[u,u^{-1}])$. It follows from the proofs in section 4 in [Ar] that N_i is generated by $\overline{M}_{0,i} \otimes_{\mathbb{C}[S_l]} (\mathbb{C}^n)^{\otimes l}$ as a module over $\mathfrak{U}\mathfrak{gl}_n(\mathbb{C}[u,u^{-1}])$. Therefore, N is generated by $\overline{M}_0 \otimes_{\mathbb{C}[S_l]} (\mathbb{C}^n)^{\otimes l}$ as a module over $\mathfrak{U}\mathfrak{gl}_n(\mathbb{C}[u,u^{-1}])$. Consider the natural linear map $N \longrightarrow \overline{M}_0 \otimes_{\mathbb{C}} \mathbb{C}[X_1^{\pm 1}, \dots, X_l^{\pm l}] \otimes_{\overline{\mathcal{H}}_l} \mathbb{C}(n|n)^{\otimes l}$ (we view $\mathbb{C}^n$ as the even part of $\mathbb{C}(n|n)$): it is onto and also a homomorphism of $\mathfrak{U}\mathfrak{gl}_n(\mathbb{C}[u,u^{-1}])$-modules. (We view $\mathfrak{gl}_n(\mathbb{C}[u,u^{-1}])$ as the even part of $\mathfrak{gl}_n(\mathbf{\Lambda})$, which is isomorphic to $\mathcal{L}_{tw}\mathfrak{q}_n$: see lemma 3.2.) It follows that $\overline{M}_0 \otimes_{\overline{\mathcal{H}}_l} \mathbb{C}(n|n)^{\otimes l}$ generates $\overline{M}_0 \otimes_{\mathbb{C}} \mathbb{C}[X_1^{\pm 1}, \dots, X_l^{\pm l}] \otimes_{\overline{\mathcal{H}}_l} \mathbb{C}(n|n)^{\otimes l}$. $\square$

8 Appendix

To prove the existence of the functor $\mathsf{SW} : \mathrm{mod}_R - \mathbf{H}_{q,l} \longrightarrow \mathrm{mod}_L - \mathfrak{U}_q\mathcal{L}_{tw}\mathfrak{q}_n$, we have to show that

$$\begin{aligned}
&(T_1 \otimes 1 \otimes 1 \otimes 1)\left(1 \otimes S_{13} + \frac{\epsilon}{(X_1^{-1}z-1)} \otimes P_{13} + \frac{\epsilon}{(X_1 z-1)} \otimes P_{13}J_1J_3\right)\\
&\cdot\left(1 \otimes S_{23} + \frac{\epsilon}{(X_2^{-1}z-1)} \otimes P_{23} + \frac{\epsilon}{(X_2 z-1)} \otimes P_{23}J_2J_3\right)\\
&-\left(1 \otimes S_{13} + \frac{\epsilon}{(X_1^{-1}z-1)} \otimes P_{13} + \frac{\epsilon}{(X_1 z-1)} \otimes P_{13}J_1J_3\right)\\
&\cdot\left(1 \otimes S_{23} + \frac{\epsilon}{(X_2^{-1}z-1)} \otimes P_{23} + \frac{\epsilon}{(X_2 z-1)} \otimes P_{23}J_2J_3\right)(1 \otimes P_{12}S_{12} \otimes 1)\\
&\equiv 0
\end{aligned}$$

where the symbol $\equiv$ denotes equivalence modulo the right ideal in $\mathrm{End}^{\mathrm{opp}}_{\mathbb{C}(q)}(M) \otimes_{\mathbb{C}(q)} \mathrm{End}_{\mathbb{C}(q)}(\mathbb{C}_q(n|n))^{\otimes 3}$ generated by $T_1 \otimes 1 \otimes 1 \otimes 1 - 1 \otimes (PS)_{12} \otimes 1$ and by $\mathsf{c}_i \otimes 1 \otimes 1 \otimes 1 - 1 \otimes J^{(i)} \otimes 1, i = 1,2$. Expanding the products, we find that the difference above is equal to:

$$\begin{aligned}
&T_1 \otimes S_{13}S_{23} + \epsilon T_1 \frac{1}{(X_2^{-1}z-1)} \otimes S_{13}P_{23}\\
&+\epsilon T_1 \frac{1}{(X_2 z-1)} \otimes S_{13}P_{23}J_2J_3 + \epsilon T_1 \frac{1}{(X_1^{-1}z-1)} \otimes P_{13}S_{23}\\
&+\epsilon^2 T_1 \frac{1}{(X_1^{-1}z-1)}\frac{1}{(X_2^{-1}z-1)} \otimes P_{13}P_{23} + \epsilon^2 T_1 \frac{1}{(X_1^{-1}z-1)}\frac{1}{(X_2 z-1)} \otimes P_{13}P_{23}J_2J_3\\
&+\epsilon^2 T_1 \frac{1}{(X_1 z-1)} \otimes P_{13}J_1J_3S_{23} + \epsilon^2 T_1 \frac{1}{(X_1 z-1)}\frac{1}{(X_2^{-1}z-1)} \otimes P_{13}J_1J_3P_{23}\\
&+\epsilon^2 T_1 \frac{1}{(X_1 z-1)}\frac{1}{(X_2 z-1)} \otimes P_{13}J_1J_3P_{23}J_2J_3 - 1 \otimes S_{13}S_{23}P_{12}S_{12}\\
&-\epsilon\frac{1}{(X_2^{-1}z-1)} \otimes S_{13}P_{23}P_{12}S_{12} - \epsilon\frac{1}{(X_2 z-1)} \otimes S_{13}P_{23}J_2J_3P_{12}S_{12}\\
&-\epsilon\frac{1}{(X_1^{-1}z-1)} \otimes P_{13}S_{23}P_{12}S_{12} - \epsilon^2\frac{1}{(X_1^{-1}z-1)}\frac{1}{(X_2^{-1}z-1)} \otimes P_{13}P_{23}P_{12}S_{12}\\
&-\epsilon^2\frac{1}{(X_1^{-1}z-1)}\frac{1}{(X_2 z-1)} \otimes P_{13}P_{23}J_2J_3P_{12}S_{12} - \epsilon\frac{1}{(X_1 z-1)} \otimes P_{13}J_1J_3S_{23}P_{12}S_{12}\\
&-\frac{\epsilon^2}{(X_1 z-1)}\frac{1}{(X_2^{-1}z-1)} \otimes P_{13}J_1J_3P_{23}P_{12}S_{12} - \frac{\epsilon^2}{(X_1 z-1)}\frac{1}{(X_2 z-1)} \otimes P_{13}J_1J_3P_{23}J_2J_3P_{12}S_{12}\\
&\equiv\\
&1 \otimes P_{12}S_{12}S_{13}S_{23} + \epsilon\left(\frac{1}{(X_1^{-1}z-1)}T_1 + z\epsilon\frac{1}{(zX_1^{-1}-1)}X_1^{-1}(1+\mathsf{c}_1\mathsf{c}_2)\frac{1}{(zX_2^{-1}-1)}\right) \otimes S_{13}P_{23}\\
&+\epsilon\left(\frac{1}{(zX_1-1)}T_1 - \epsilon z\frac{1}{(zX_1-1)}(1+\mathsf{c}_1\mathsf{c}_2)X_2\frac{1}{(zX_2-1)}\right) \otimes S_{13}P_{23}J_2J_3\\
&+\epsilon\left(\frac{1}{(zX_2^{-1}-1)}T_1 - \epsilon z\frac{1}{(zX_2^{-1}-1)}(X_1^{-1} - \mathsf{c}_1\mathsf{c}_2X_2)\frac{1}{(zX_1^{-1}-1)}\right) \otimes P_{13}S_{23}\\
&+\epsilon^2\left(\frac{1}{(zX_2^{-1}-1)}T_1 - \epsilon z\frac{1}{(zX_2^{-1}-1)}(X_1^{-1} - \mathsf{c}_1\mathsf{c}_2X_2)\frac{1}{(zX_1^{-1}-1)}\right)\frac{1}{(X_2^{-1}z-1)} \otimes P_{13}P_{23}\\
&+\epsilon^2\left(\frac{1}{(zX_2^{-1}-1)}T_1 - \epsilon z\frac{1}{(zX_2^{-1}-1)}(X_1^{-1} - \mathsf{c}_1\mathsf{c}_2X_2)\frac{1}{(zX_1^{-1}-1)}\right)\frac{1}{(X_2 z-1)} \otimes P_{13}P_{23}J_2J_3
\end{aligned}$$

$$+\epsilon\left(\frac{1}{(zX_2-1)}T_1+z\epsilon\frac{1}{(zX_2-1)}(X_2-\mathsf{c}_1\mathsf{c}_2X_1)\frac{1}{(zX_1-1)}\right)\otimes P_{13}J_1J_3S_{23}$$
$$+\epsilon^2\left(\frac{1}{(zX_2-1)}T_1+z\epsilon\frac{1}{(zX_2-1)}(X_2-\mathsf{c}_1\mathsf{c}_2X_1)\frac{1}{(zX_1-1)}\right)\frac{1}{(X_2^{-1}z-1)}\otimes P_{13}J_1J_3P_{23}$$
$$+\epsilon^2\left(\frac{1}{(zX_2-1)}T_1+z\epsilon\frac{1}{(zX_2-1)}(X_2-\mathsf{c}_1\mathsf{c}_2X_1)\frac{1}{(zX_1-1)}\right)\frac{1}{(X_2z-1)}\otimes P_{13}J_1J_3P_{23}J_2J_3$$
$$-1\otimes S_{13}S_{23}P_{12}S_{12}-\epsilon\frac{1}{(X_2^{-1}z-1)}\otimes S_{13}P_{23}P_{12}S_{12}$$
$$-\epsilon\frac{1}{(X_2z-1)}\otimes S_{13}P_{23}J_2J_3P_{12}S_{12}-\epsilon\frac{1}{(X_1^{-1}z-1)}\otimes P_{13}S_{23}P_{12}S_{12}$$
$$-\epsilon^2\frac{1}{(X_1^{-1}z-1)}\frac{1}{(X_2^{-1}z-1)}\otimes P_{13}P_{23}P_{12}S_{12}-\epsilon^2\frac{1}{(X_1^{-1}z-1)}\frac{1}{(X_2z-1)}\otimes P_{13}P_{23}J_2J_3P_{12}S_{12}$$
$$-\epsilon\frac{1}{(X_1z-1)}\otimes P_{13}J_1J_3S_{23}P_{12}S_{12}-\epsilon^2\frac{1}{(X_1z-1)}\frac{1}{(X_2^{-1}z-1)}\otimes P_{13}J_1J_3P_{23}P_{12}S_{12}$$
$$-\epsilon^2\frac{1}{(X_1z-1)}\frac{1}{(X_2z-1)}\otimes P_{13}J_1J_3P_{23}J_2J_3P_{12}S_{12}$$
$$\equiv$$
$$+\epsilon\frac{1}{(X_1^{-1}z-1)}T_1\otimes S_{13}P_{23}+\epsilon^2z\frac{1}{(zX_1^{-1}-1)}X_1^{-1}\frac{1}{(zX_2^{-1}-1)}\otimes S_{13}P_{23}$$
$$+\epsilon^2z\frac{1}{(zX_1^{-1}-1)}X_1^{-1}\mathsf{c}_1\mathsf{c}_2\frac{1}{(zX_2^{-1}-1)}\otimes S_{13}P_{23}+\epsilon\frac{1}{(zX_1-1)}T_1\otimes S_{13}P_{23}J_2J_3$$
$$-\epsilon^2z\frac{1}{(zX_1-1)}X_2\frac{1}{(zX_2-1)}\otimes S_{13}P_{23}J_2J_3-\epsilon^2z\frac{1}{(zX_1-1)}\mathsf{c}_1\mathsf{c}_2X_2\frac{1}{(zX_2-1)}\otimes S_{13}P_{23}J_2J_3$$
$$+\epsilon\frac{1}{(zX_2^{-1}-1)}T_1\otimes P_{13}S_{23}-\epsilon^2z\frac{1}{(zX_2^{-1}-1)}X_1^{-1}\frac{1}{(zX_1^{-1}-1)}\otimes P_{13}S_{23}$$
$$+\epsilon^2z\frac{1}{(zX_2^{-1}-1)}\mathsf{c}_1\mathsf{c}_2X_2\frac{1}{(zX_1^{-1}-1)}\otimes P_{13}S_{23}+\epsilon^2\frac{1}{(zX_2^{-1}-1)}T_1\frac{1}{(X_2^{-1}z-1)}\otimes P_{13}P_{23}$$
$$-\epsilon^3z\frac{1}{(zX_2^{-1}-1)}X_1^{-1}\frac{1}{(zX_1^{-1}-1)}\frac{1}{(X_2^{-1}z-1)}\otimes P_{13}P_{23}+\epsilon^2\frac{1}{(zX_2^{-1}-1)}T_1\frac{1}{(X_2z-1)}\otimes P_{13}P_{23}J_2J_3$$
$$+\epsilon^3z\frac{1}{(zX_2^{-1}-1)}\mathsf{c}_1\mathsf{c}_2X_2\frac{1}{(zX_1^{-1}-1)}\frac{1}{(X_2^{-1}z-1)}\otimes P_{13}P_{23}$$
$$-\epsilon^3z\frac{1}{(zX_2^{-1}-1)}X_1^{-1}\frac{1}{(zX_1^{-1}-1)}\frac{1}{(X_2z-1)}\otimes P_{13}P_{23}J_2J_3$$
$$+\epsilon^3z\frac{1}{(zX_2^{-1}-1)}\mathsf{c}_1\mathsf{c}_2X_2\frac{1}{(zX_1^{-1}-1)}\frac{1}{(X_2z-1)}\otimes P_{13}P_{23}J_2J_3+\epsilon\frac{1}{(zX_2-1)}T_1\otimes P_{13}J_1J_3S_{23}$$
$$+\epsilon^2z\frac{1}{(zX_2-1)}X_2\frac{1}{(zX_1-1)}\otimes P_{13}J_1J_3S_{23}-\epsilon^2z\frac{1}{(zX_2-1)}\mathsf{c}_1\mathsf{c}_2X_1\frac{1}{(zX_1-1)}\otimes P_{13}J_1J_3S_{23}$$
$$+\epsilon^2\frac{1}{(zX_2-1)}T_1\frac{1}{(X_2^{-1}z-1)}\otimes P_{13}J_1J_3P_{23}+\epsilon^3z\frac{1}{(zX_2-1)}X_2\frac{1}{(zX_1-1)}\frac{1}{(X_2^{-1}z-1)}\otimes P_{13}J_1J_3P_{23}$$
$$-\epsilon^3z\frac{1}{(zX_2-1)}\mathsf{c}_1\mathsf{c}_2X_1\frac{1}{(zX_1-1)}\frac{1}{(X_2^{-1}z-1)}\otimes P_{13}J_1J_3P_{23}$$
$$+\epsilon^2\frac{1}{(zX_2-1)}T_1\frac{1}{(X_2z-1)}\otimes P_{13}J_1J_3P_{23}J_2J_3+\epsilon^3z\frac{1}{(zX_2-1)}X_2\frac{1}{(zX_1-1)}\frac{1}{(X_2z-1)}\otimes P_{13}J_1J_3P_{23}J_2J_3$$
$$-\epsilon^3z\frac{1}{(zX_2-1)}\mathsf{c}_1\mathsf{c}_2X_1\frac{1}{(zX_1-1)}\frac{1}{(X_2z-1)}\otimes P_{13}J_1J_3P_{23}J_2J_3$$
$$-\epsilon\frac{1}{(X_2^{-1}z-1)}\otimes S_{13}P_{23}P_{12}S_{12}-\epsilon\frac{1}{(X_2z-1)}\otimes S_{13}P_{23}J_2J_3P_{12}S_{12}$$

$$-\epsilon\frac{1}{(X_1^{-1}z-1)}\otimes P_{13}S_{23}P_{12}S_{12}-\epsilon^2\frac{1}{(X_1^{-1}z-1)}\frac{1}{(X_2^{-1}z-1)}\otimes P_{13}P_{23}P_{12}S_{12}$$
$$-\epsilon^2\frac{1}{(X_1^{-1}z-1)}\frac{1}{(X_2z-1)}\otimes P_{13}P_{23}J_2J_3P_{12}S_{12}-\epsilon\frac{1}{(X_1z-1)}\otimes P_{13}J_1J_3S_{23}P_{12}S_{12}$$
$$-\epsilon^2\frac{1}{(X_1z-1)}\frac{1}{(X_2^{-1}z-1)}\otimes P_{13}J_1J_3P_{23}P_{12}S_{12}-\epsilon^2\frac{1}{(X_1z-1)}\frac{1}{(X_2z-1)}\otimes P_{13}J_1J_3P_{23}J_2J_3P_{12}S_{12}$$
$$\equiv$$
$$+\epsilon\frac{1}{(X_1^{-1}z-1)}\otimes P_{12}S_{12}S_{13}P_{23}+\epsilon z\epsilon\frac{1}{(zX_1^{-1}-1)}X_1^{-1}\frac{1}{(zX_2^{-1}-1)}\otimes S_{13}P_{23}$$
$$+\epsilon z\epsilon\frac{1}{(zX_1^{-1}-1)}X_1^{-1}\frac{1}{(zX_2-1)}\otimes J_1J_2S_{13}P_{23}+\epsilon\frac{1}{(zX_1-1)}\otimes P_{12}S_{12}S_{13}P_{23}J_2J_3$$
$$-\epsilon^2z\frac{1}{(zX_1-1)}X_2\frac{1}{(zX_2-1)}\otimes S_{13}P_{23}J_2J_3-\epsilon^2z\frac{1}{(zX_1-1)}X_2^{-1}\frac{1}{(zX_2^{-1}-1)}\otimes J_1J_2S_{13}P_{23}J_2J_3$$
$$+\epsilon\frac{1}{(zX_2^{-1}-1)}\otimes P_{12}S_{12}P_{13}S_{23}-\epsilon^2z\frac{1}{(zX_2^{-1}-1)}X_1^{-1}\frac{1}{(zX_1^{-1}-1)}\otimes P_{13}S_{23}$$
$$+\epsilon^2z\frac{1}{(zX_2^{-1}-1)}X_2^{-1}\frac{1}{(zX_1-1)}\otimes J_1J_2P_{13}S_{23}$$
$$+\epsilon^2\frac{1}{(zX_2^{-1}-1)}\left(\frac{1}{(zX_1^{-1}-1)}T_1+z\epsilon\frac{1}{(zX_1^{-1}-1)}X_1^{-1}(1+\mathsf{c}_1\mathsf{c}_2)\frac{1}{(zX_2^{-1}-1)}\right)\otimes P_{13}P_{23}$$
$$-\epsilon^3zX_1^{-1}\frac{1}{(zX_1^{-1}-1)}\frac{1}{(X_2^{-1}z-1)^2}\otimes P_{13}P_{23}+\epsilon^3z\frac{1}{(zX_2^{-1}-1)}X_2^{-1}\frac{1}{(zX_1-1)}\frac{1}{(X_2z-1)}\otimes J_1J_2P_{13}P_{23}$$
$$+\epsilon^2\frac{1}{(zX_2^{-1}-1)}\left(\frac{1}{(zX_1-1)}T_1-\epsilon z\frac{1}{(zX_1-1)}(1+\mathsf{c}_1\mathsf{c}_2)X_2\frac{1}{(zX_2-1)}\right)\otimes P_{13}P_{23}J_2J_3$$
$$-\epsilon^3z\frac{1}{(zX_2^{-1}-1)}X_1^{-1}\frac{1}{(zX_1^{-1}-1)}\frac{1}{(X_2z-1)}\otimes P_{13}P_{23}J_2J_3$$
$$+\epsilon^3zX_2^{-1}\frac{1}{(zX_1-1)}\frac{1}{(X_2^{-1}z-1)^2}\otimes J_1J_2P_{13}P_{23}J_2J_3+\epsilon\frac{1}{(zX_2-1)}\otimes P_{12}S_{12}P_{13}J_1J_3S_{23}$$
$$+\epsilon^2z\frac{1}{(zX_2-1)}X_2\frac{1}{(zX_1-1)}\otimes P_{13}J_1J_3S_{23}$$
$$-\epsilon^2z\frac{1}{(zX_2-1)}X_1^{-1}\frac{1}{(zX_1^{-1}-1)}\otimes J_1J_2P_{13}J_1J_3S_{23}$$
$$+\epsilon^2\frac{1}{(zX_2-1)}\left(\frac{1}{(zX_1^{-1}-1)}T_1+z\epsilon\frac{1}{(zX_1^{-1}-1)}X_1^{-1}(1+\mathsf{c}_1\mathsf{c}_2)\frac{1}{(zX_2^{-1}-1)}\right)\otimes P_{13}J_1J_3P_{23}$$
$$+\epsilon^3z\frac{1}{(zX_2-1)}X_2\frac{1}{(zX_1-1)}\frac{1}{(X_2^{-1}z-1)}\otimes P_{13}J_1J_3P_{23}$$
$$-\epsilon^3z\frac{1}{(zX_2-1)^2}X_1^{-1}\frac{1}{(zX_1^{-1}-1)}\otimes J_1J_2P_{13}J_1J_3P_{23}+\epsilon^3z\frac{1}{(zX_2-1)^2}X_2\frac{1}{(zX_1-1)}\otimes P_{13}J_1J_3P_{23}J_2J_3$$
$$+\epsilon^2\frac{1}{(zX_2-1)}\left(\frac{1}{(zX_1-1)}T_1-\epsilon z\frac{1}{(zX_1-1)}(1+\mathsf{c}_1\mathsf{c}_2)X_2\frac{1}{(zX_2-1)}\right)\otimes P_{13}J_1J_3P_{23}J_2J_3$$
$$-\epsilon^3z\frac{1}{(zX_2-1)}X_1^{-1}\frac{1}{(zX_1^{-1}-1)}\frac{1}{X_2^{-1}z-1}\otimes J_1J_2P_{13}J_1J_3P_{23}J_2J_3-\epsilon\frac{1}{(X_2^{-1}z-1)}\otimes S_{13}P_{23}P_{12}S_{12}$$
$$-\epsilon\frac{1}{(X_2z-1)}\otimes S_{13}P_{23}J_2J_3P_{12}S_{12}-\epsilon\frac{1}{(X_1^{-1}z-1)}\otimes P_{13}S_{23}P_{12}S_{12}$$
$$-\epsilon^2\frac{1}{(X_1^{-1}z-1)}\frac{1}{(X_2^{-1}z-1)}\otimes P_{13}P_{23}P_{12}S_{12}-\epsilon^2\frac{1}{(X_1^{-1}z-1)}\frac{1}{(X_2z-1)}\otimes P_{13}P_{23}J_2J_3P_{12}S_{12}$$
$$-\epsilon\frac{1}{(X_1z-1)}\otimes P_{13}J_1J_3S_{23}P_{12}S_{12}-\epsilon^2\frac{1}{(X_1z-1)}\frac{1}{(X_2^{-1}z-1)}\otimes P_{13}J_1J_3P_{23}P_{12}S_{12}$$

$$
\begin{aligned}
&-\epsilon^2 \frac{1}{(X_1 z-1)} \frac{1}{(X_2 z-1)} \otimes P_{13} J_1 J_3 P_{23} J_2 J_3 P_{12} S_{12} \\
&\equiv \\
&+\epsilon \frac{1}{(X_1^{-1} z-1)} \otimes P_{12} S_{12} S_{13} P_{23}+\epsilon z \epsilon \frac{1}{(z X_1^{-1}-1)} X_1^{-1} \frac{1}{(z X_2^{-1}-1)} \otimes S_{13} P_{23} \\
&+\epsilon z \epsilon \frac{1}{(z X_1^{-1}-1)} X_1^{-1} \frac{1}{(z X_2-1)} \otimes J_1 J_2 S_{13} P_{23}+\epsilon \frac{1}{(z X_1-1)} \otimes P_{12} S_{12} S_{13} P_{23} J_2 J_3 \\
&-\epsilon^2 z \frac{1}{(z X_1-1)} X_2 \frac{1}{(z X_2-1)} \otimes S_{13} P_{23} J_2 J_3-\epsilon^2 z \frac{1}{(z X_1-1)} X_2^{-1} \frac{1}{(z X_2^{-1}-1)} \otimes J_1 J_2 S_{13} P_{23} J_2 J_3 \\
&+\epsilon \frac{1}{(z X_2^{-1}-1)} \otimes P_{12} S_{12} P_{13} S_{23}-\epsilon^2 z \frac{1}{(z X_2^{-1}-1)} X_1^{-1} \frac{1}{(z X_1^{-1}-1)} \otimes P_{13} S_{23} \\
&+\epsilon^2 z \frac{1}{(z X_2^{-1}-1)} X_2^{-1} \frac{1}{(z X_1-1)} \otimes J_1 J_2 P_{13} S_{23}+\epsilon^2 \frac{1}{(z X_2^{-1}-1)} \frac{1}{(z X_1^{-1}-1)} T_1 \otimes P_{13} P_{23} \\
&+\epsilon^3 z \frac{1}{(z X_1^{-1}-1)} z X_1^{-1} \frac{1}{(z X_2^{-1}-1)^2} \otimes P_{13} P_{23} \\
&+\epsilon^3 \frac{1}{(z X_2^{-1}-1)} \frac{1}{(z X_1^{-1}-1)} z X_1^{-1} \mathsf{c}_1 \mathsf{c}_2 \frac{1}{(z X_2^{-1}-1)} \otimes P_{13} P_{23} \\
&-\epsilon^3 z X_1^{-1} \frac{1}{(z X_1^{-1}-1)} \frac{1}{(X_2^{-1} z-1)^2} \otimes P_{13} P_{23}+\epsilon^3 z \frac{1}{(z X_2^{-1}-1)} X_2^{-1} \frac{1}{(z X_1-1)} \frac{1}{(X_2 z-1)} \otimes J_1 J_2 P_{13} P_{23} \\
&+\epsilon^2 \frac{1}{(z X_2^{-1}-1)} \frac{1}{(z X_1-1)} T_1 \otimes P_{13} P_{23} J_2 J_3-\epsilon^3 \frac{1}{(z X_2^{-1}-1)} z \frac{1}{(z X_1-1)} X_2 \frac{1}{(z X_2-1)} \otimes P_{13} P_{23} J_2 J_3 \\
&-\epsilon^3 \frac{1}{(z X_2^{-1}-1)} z \frac{1}{(z X_1-1)} \mathsf{c}_1 \mathsf{c}_2 X_2 \frac{1}{(z X_2-1)} \otimes P_{13} P_{23} J_2 J_3 \\
&-\epsilon^3 z \frac{1}{(z X_2^{-1}-1)} X_1^{-1} \frac{1}{(z X_1^{-1}-1)} \frac{1}{(X_2 z-1)} \otimes P_{13} P_{23} J_2 J_3 \\
&+\epsilon^3 z X_2^{-1} \frac{1}{(z X_1-1)} \frac{1}{(X_2^{-1} z-1)^2} \otimes J_1 J_2 P_{13} P_{23} J_2 J_3+\epsilon \frac{1}{(z X_2-1)} \otimes P_{12} S_{12} P_{13} J_1 J_3 S_{23} \\
&+\epsilon^2 z \frac{1}{(z X_2-1)} X_2 \frac{1}{(z X_1-1)} \otimes P_{13} J_1 J_3 S_{23}-\epsilon^2 z \frac{1}{(z X_2-1)} X_1^{-1} \frac{1}{(z X_1^{-1}-1)} \otimes J_1 J_2 P_{13} J_1 J_3 S_{23} \\
&+\epsilon^2 \frac{1}{(z X_2-1)} \frac{1}{(z X_1^{-1}-1)} T_1 \otimes P_{13} J_1 J_3 P_{23}+\epsilon^3 \frac{z}{(z X_2-1)} \frac{1}{(z X_1^{-1}-1)} X_1^{-1} \frac{1}{(z X_2^{-1}-1)} \otimes P_{13} J_1 J_3 P_{23} \\
&+\epsilon^3 \frac{1}{(z X_2-1)} z \frac{1}{(z X_1^{-1}-1)} X_1^{-1} \mathsf{c}_1 \mathsf{c}_2 \frac{1}{(z X_2^{-1}-1)} \otimes P_{13} J_1 J_3 P_{23} \\
&+\epsilon^3 z \frac{1}{(z X_2-1)} X_2 \frac{1}{(z X_1-1)} \frac{1}{(X_2^{-1} z-1)} \otimes P_{13} J_1 J_3 P_{23} \\
&-\epsilon^3 z \frac{1}{(z X_2-1)^2} X_1^{-1} \frac{1}{(z X_1^{-1}-1)} \otimes J_1 J_2 P_{13} J_1 J_3 P_{23}+\epsilon^2 \frac{1}{(z X_2-1)} \frac{1}{(z X_1-1)} T_1 \otimes P_{13} J_1 J_3 P_{23} J_2 J_3 \\
&-\epsilon^3 \frac{1}{(z X_2-1)^2} z \frac{1}{(z X_1-1)} X_2 \otimes P_{13} J_1 J_3 P_{23} J_2 J_3+\epsilon^3 z \frac{1}{(z X_2-1)^2} X_2 \frac{1}{(z X_1-1)} \otimes P_{13} J_1 J_3 P_{23} J_2 J_3 \\
&-\epsilon^3 \frac{1}{(z X_2-1)} z \frac{1}{(z X_1-1)} \mathsf{c}_1 \mathsf{c}_2 X_2 \frac{1}{(z X_2-1)} \otimes P_{13} J_1 J_3 P_{23} J_2 J_3 \\
&-\epsilon^3 z \frac{1}{(z X_2-1)} X_1^{-1} \frac{1}{(z X_1^{-1}-1)} \frac{1}{X_2^{-1} z-1} \otimes J_1 J_2 P_{13} J_1 J_3 P_{23} J_2 J_3 \\
&-\epsilon \frac{1}{(X_2^{-1} z-1)} \otimes S_{13} P_{23} P_{12} S_{12}-\epsilon \frac{1}{(X_2 z-1)} \otimes S_{13} P_{23} J_2 J_3 P_{12} S_{12} \\
&-\epsilon \frac{1}{(X_1^{-1} z-1)} \otimes P_{13} S_{23} P_{12} S_{12}-\epsilon^2 \frac{1}{(X_1^{-1} z-1)} \frac{1}{(X_2^{-1} z-1)} \otimes P_{13} P_{23} P_{12} S_{12}
\end{aligned}
$$

$$-\epsilon^2\frac{1}{(X_1^{-1}z-1)}\frac{1}{(X_2z-1)}\otimes P_{13}P_{23}J_2J_3P_{12}S_{12}-\epsilon\frac{1}{(X_1z-1)}\otimes P_{13}J_1J_3S_{23}P_{12}S_{12}$$
$$-\epsilon^2\frac{1}{(X_1z-1)}\frac{1}{(X_2^{-1}z-1)}\otimes P_{13}J_1J_3P_{23}P_{12}S_{12}-\epsilon^2\frac{1}{(X_1z-1)}\frac{1}{(X_2z-1)}\otimes P_{13}J_1J_3P_{23}J_2J_3P_{12}S_{12}$$
$$\equiv$$
$$\epsilon\frac{1}{(X_1^{-1}z-1)}\otimes P_{12}S_{12}S_{13}P_{23}+\epsilon^2z\frac{1}{(zX_1^{-1}-1)}X_1^{-1}\frac{1}{(zX_2^{-1}-1)}\otimes S_{13}P_{23}$$
$$+\epsilon^2z\frac{1}{(zX_1^{-1}-1)}X_1^{-1}\frac{1}{(zX_2-1)}\otimes S_{13}P_{23}J_1J_3+\epsilon\frac{1}{(zX_1-1)}\otimes P_{12}S_{12}S_{13}P_{23}J_2J_3$$
$$-\epsilon^2z\frac{1}{(zX_1-1)}X_2\frac{1}{(zX_2-1)}\otimes S_{13}P_{23}J_2J_3-\epsilon^2z\frac{1}{(zX_1-1)}X_2^{-1}\frac{1}{(zX_2^{-1}-1)}\otimes S_{13}P_{23}J_1J_2$$
$$+\epsilon\frac{1}{(zX_2^{-1}-1)}\otimes P_{12}S_{12}P_{13}S_{23}-\epsilon^2z\frac{1}{(zX_2^{-1}-1)}X_1^{-1}\frac{1}{(zX_1^{-1}-1)}\otimes P_{13}S_{23}$$
$$+\epsilon^2z\frac{1}{(zX_2^{-1}-1)}X_2^{-1}\frac{1}{(zX_1-1)}\otimes P_{13}J_3J_2S_{23}+\epsilon^2\frac{1}{(zX_2^{-1}-1)}\frac{1}{(zX_1^{-1}-1)}\otimes P_{12}S_{12}P_{13}P_{23}$$
$$+\epsilon^3\frac{z}{(zX_1^{-1}-1)}zX_1^{-1}\frac{1}{(zX_2^{-1}-1)^2}\otimes P_{13}P_{23}+\epsilon^3\frac{z}{(zX_2^{-1}-1)}\frac{1}{(zX_1^{-1}-1)}X_1^{-1}\frac{1}{(zX_2-1)}\otimes P_{13}P_{23}J_2J_3$$
$$-\epsilon^3zX_1^{-1}\frac{1}{(zX_1^{-1}-1)}\frac{1}{(X_2^{-1}z-1)^2}\otimes P_{13}P_{23}+\epsilon^3\frac{z}{(zX_2^{-1}-1)}X_2^{-1}\frac{1}{(zX_1-1)}\frac{1}{(X_2z-1)}\otimes P_{13}P_{23}J_2J_3$$
$$+\epsilon^2\frac{1}{(zX_2^{-1}-1)}\frac{1}{(zX_1-1)}\otimes P_{12}S_{12}P_{13}P_{23}J_2J_3-\epsilon^3\frac{z}{(zX_2^{-1}-1)}\frac{1}{(zX_1-1)}X_2\frac{1}{(zX_2-1)}\otimes P_{13}P_{23}J_2J_3$$
$$+\epsilon^3z\frac{1}{(zX_1-1)}X_2^{-1}\frac{1}{(zX_2^{-1}-1)^2}\otimes P_{13}P_{23}-\epsilon^3\frac{z}{(zX_2^{-1}-1)}X_1^{-1}\frac{1}{(zX_1^{-1}-1)}\frac{1}{(X_2z-1)}\otimes P_{13}P_{23}J_2J_3$$
$$-\epsilon^3zX_2^{-1}\frac{1}{(zX_1-1)}\frac{1}{(X_2^{-1}z-1)^2}\otimes P_{13}P_{23}+\epsilon\frac{1}{(zX_2-1)}\otimes P_{12}S_{12}P_{13}J_1J_3S_{23}$$
$$+\epsilon^2z\frac{1}{(zX_2-1)}X_2\frac{1}{(zX_1-1)}\otimes P_{13}J_1J_3S_{23}-\epsilon^2z\frac{1}{(zX_2-1)}X_1^{-1}\frac{1}{(zX_1^{-1}-1)}\otimes P_{13}J_1J_2S_{23}$$
$$+\epsilon^2\frac{1}{(zX_2-1)}\frac{1}{(zX_1^{-1}-1)}\otimes P_{12}S_{12}P_{13}P_{23}J_1J_2$$
$$+\epsilon^3\frac{1}{(zX_2-1)}z\frac{1}{(zX_1^{-1}-1)}X_1^{-1}\frac{1}{(zX_2^{-1}-1)}\otimes P_{13}P_{23}J_1J_2$$
$$+\epsilon^3\frac{1}{(zX_2-1)^2}z\frac{1}{(zX_1^{-1}-1)}X_1^{-1}\otimes P_{13}P_{23}J_1J_3+\epsilon^3z\frac{1}{(zX_2-1)}X_2\frac{1}{(zX_1-1)}\frac{1}{(X_2^{-1}z-1)}\otimes P_{13}P_{23}J_1J_2$$
$$-\epsilon^3z\frac{1}{(zX_2-1)^2}X_1^{-1}\frac{1}{(zX_1^{-1}-1)}\otimes P_{13}P_{23}J_1J_3-\epsilon^2\frac{1}{(zX_2-1)}\frac{1}{(zX_1-1)}\otimes P_{12}S_{12}P_{13}P_{23}J_1J_3$$
$$+\epsilon^3\frac{1}{(zX_2-1)^2}z\frac{1}{(zX_1-1)}X_2\otimes P_{13}P_{23}J_1J_3-\epsilon^3\frac{1}{(zX_2-1)}z\frac{1}{(zX_1-1)}X_2^{-1}\frac{1}{(zX_2^{-1}-1)}\otimes P_{13}P_{23}J_1J_2$$
$$-\epsilon^3z\frac{1}{(zX_2-1)^2}X_2\frac{1}{(zX_1-1)}\otimes P_{13}P_{23}J_1J_3-\epsilon\frac{1}{(X_2^{-1}z-1)}\otimes S_{13}P_{23}P_{12}S_{12}$$
$$+\epsilon^3z\frac{1}{(zX_2-1)}X_1^{-1}\frac{1}{(zX_1^{-1}-1)}\frac{1}{X_2^{-1}z-1}\otimes J_1J_2P_{13}P_{23}J_1J_3-\epsilon\frac{1}{(X_2z-1)}\otimes S_{13}P_{23}J_2J_3P_{12}S_{12}$$
$$-\epsilon\frac{1}{(X_1^{-1}z-1)}\otimes P_{13}S_{23}P_{12}S_{12}-\epsilon^2\frac{1}{(X_1^{-1}z-1)}\frac{1}{(X_2^{-1}z-1)}\otimes P_{13}P_{23}P_{12}S_{12}$$
$$-\epsilon^2\frac{1}{(X_1^{-1}z-1)}\frac{1}{(X_2z-1)}\otimes P_{13}P_{23}J_2J_3P_{12}S_{12}-\epsilon\frac{1}{(X_1z-1)}\otimes P_{13}J_1J_3S_{23}P_{12}S_{12}$$
$$-\epsilon^2\frac{1}{(X_1z-1)}\frac{1}{(X_2^{-1}z-1)}\otimes P_{13}J_1J_3P_{23}P_{12}S_{12}-\epsilon^2\frac{1}{(X_1z-1)}\frac{1}{(X_2z-1)}\otimes P_{13}J_1J_3P_{23}J_2J_3P_{12}S_{12}$$

$$
\begin{aligned}
&\equiv \epsilon \frac{1}{(X_1^{-1}z-1)} \otimes P_{12}P_{23}S_{13}S_{12} + \epsilon^2 \frac{1}{(zX_2^{-1}-1)} \otimes P_{23}S_{12} \\
&+\epsilon^2 \frac{1}{(zX_1^{-1}-1)} \frac{1}{(zX_2^{-1}-1)} \otimes P_{23}S_{12} + \epsilon^2 \frac{1}{(zX_2-1)} \otimes P_{23}S_{12}J_1J_3 \\
&+\epsilon^2 \frac{1}{(zX_1^{-1}-1)} \frac{1}{(zX_2-1)} \otimes P_{23}S_{12}J_1J_3 + \epsilon \frac{1}{(zX_1-1)} \otimes P_{12}P_{23}S_{13}S_{12}J_2J_3 \\
&-\epsilon^2 \frac{1}{(zX_1-1)} \otimes P_{23}S_{12}J_2J_3 - \epsilon^2 \frac{1}{(zX_1-1)} \frac{1}{(zX_2-1)} \otimes P_{23}S_{12}J_2J_3 \\
&-\epsilon^2 \frac{1}{(zX_1-1)} \otimes P_{23}S_{12}J_1J_2 - \epsilon^2 \frac{1}{(zX_1-1)} \frac{1}{(zX_2^{-1}-1)} \otimes P_{23}S_{12}J_1J_2 \\
&+\epsilon \frac{1}{(zX_2^{-1}-1)} \otimes P_{23}P_{12} + \epsilon^2 \frac{1}{(zX_2^{-1}-1)} \otimes P_{12}P_{23}P_{12}S_{23} \\
&-\epsilon^2 \frac{1}{(zX_2^{-1}-1)} \otimes P_{13}S_{23} - \epsilon^2 \frac{1}{(zX_2^{-1}-1)} \frac{1}{(zX_1^{-1}-1)} \otimes P_{13}S_{23} \\
&+\epsilon^2 \frac{1}{(zX_1-1)} \otimes P_{13}S_{23}J_3J_2 + \epsilon^2 \frac{1}{(zX_2^{-1}-1)} \frac{1}{(zX_1-1)} \otimes P_{13}S_{23}J_3J_2 \\
&-\epsilon^3 \frac{1}{(zX_1-1)} \otimes P_{13}P_{23} - \epsilon^3 \frac{1}{(zX_2^{-1}-1)} \frac{1}{(zX_1-1)} \otimes P_{13}P_{23} \\
&+\epsilon^3 \frac{1}{(zX_1-1)} \otimes P_{13}P_{23}J_2J_3 + \epsilon^3 \frac{1}{(zX_2^{-1}-1)} \frac{1}{(zX_1-1)} \otimes P_{13}P_{23}J_2J_3 \\
&+\epsilon^2 \frac{1}{(zX_2^{-1}-1)} \frac{1}{(zX_1^{-1}-1)} \otimes P_{13}S_{23} + \epsilon^3 \frac{1}{(zX_2^{-1}-1)^2} \otimes P_{13}P_{23} \\
&+\epsilon^3 \frac{1}{(zX_1^{-1}-1)} \frac{1}{(zX_2^{-1}-1)^2} \otimes P_{13}P_{23} + \epsilon^3 \frac{1}{(zX_2^{-1}-1)} \frac{1}{(zX_2-1)} \otimes P_{13}P_{23}J_2J_3 \\
&+\epsilon^3 \frac{1}{(zX_2^{-1}-1)} \frac{1}{(zX_1^{-1}-1)} \frac{1}{(zX_2-1)} \otimes P_{13}P_{23}J_2J_3 - \epsilon^3 \frac{1}{(X_2^{-1}z-1)^2} \otimes P_{13}P_{23} \\
&-\epsilon^3 \frac{1}{(zX_1^{-1}-1)} \frac{1}{(X_2^{-1}z-1)^2} \otimes P_{13}P_{23} + \epsilon^3 \frac{1}{(zX_1-1)} \frac{1}{(X_2z-1)} \otimes P_{13}P_{23}J_2J_3 \\
&+\epsilon^3 \frac{1}{(zX_2^{-1}-1)} \frac{1}{(zX_1-1)} \frac{1}{(X_2z-1)} \otimes P_{13}P_{23}J_2J_3 + \epsilon^2 \frac{1}{(zX_2^{-1}-1)} \frac{1}{(zX_1-1)} \otimes P_{13}S_{23}J_2J_3 \\
&-\epsilon^3 \frac{1}{(zX_2^{-1}-1)} \frac{1}{(zX_1-1)} \otimes P_{13}P_{23}J_2J_3 - \epsilon^3 \frac{1}{(zX_2^{-1}-1)} \frac{1}{(zX_1-1)} \frac{1}{(zX_2-1)} \otimes P_{13}P_{23}J_2J_3 \\
&+\epsilon^3 \frac{1}{(zX_1-1)} \frac{1}{(zX_2^{-1}-1)} \otimes P_{13}P_{23} + \epsilon^3 \frac{1}{(zX_1-1)} \frac{1}{(zX_2^{-1}-1)^2} \otimes P_{13}P_{23} \\
&-\epsilon^3 \frac{1}{(zX_2^{-1}-1)} \frac{1}{(X_2z-1)} \otimes P_{13}P_{23}J_2J_3 - \epsilon^3 \frac{1}{(zX_2^{-1}-1)} \frac{1}{(zX_1^{-1}-1)} \frac{1}{(X_2z-1)} \otimes P_{13}P_{23}J_2J_3 \\
&-\epsilon^3 \frac{1}{(zX_1-1)} \frac{1}{X_2^{-1}z-1} \otimes P_{13}P_{23} - \epsilon^3 \frac{1}{(zX_1-1)} \frac{1}{(X_2^{-1}z-1)^2} \otimes P_{13}P_{23} \\
&-\epsilon \frac{1}{(zX_2-1)} \otimes P_{23}P_{12}J_3J_1 + \epsilon^2 \frac{1}{(zX_2-1)} \otimes P_{13}S_{23}J_1J_2 \\
&-\epsilon^3 \frac{1}{(zX_2-1)} \otimes P_{23}P_{12}J_1J_2 - \epsilon^2 \frac{1}{(zX_2-1)} \otimes P_{13}S_{23}J_1J_3 \\
&+\epsilon^3 \frac{1}{(zX_2-1)} \otimes P_{23}P_{12}J_1J_3 - \epsilon^2 \frac{1}{(zX_2-1)} \otimes P_{13}S_{23}J_3J_1 \\
&-\epsilon^3 \frac{1}{(zX_2-1)} \otimes P_{13}P_{23}J_2J_1 - \epsilon^3 \frac{1}{(zX_2-1)} \otimes P_{13}P_{23}J_1J_3 \\
&-\epsilon^2 \frac{1}{(zX_1-1)} \otimes P_{13}S_{23}J_3J_1 - \epsilon^3 \frac{1}{(zX_1-1)} \otimes P_{13}P_{23}J_2J_1
\end{aligned}
$$

$$
\begin{aligned}
&-\epsilon^3 \frac{1}{(zX_1-1)} \otimes P_{13}P_{23}J_1J_3 - \epsilon^2 \frac{1}{(zX_2-1)} \frac{1}{(zX_1-1)} \otimes P_{13}S_{23}J_3J_1 \\
&-\epsilon^3 \frac{1}{(zX_2-1)} \frac{1}{(zX_1-1)} \otimes P_{13}P_{23}J_2J_1 - \epsilon^3 \frac{1}{(zX_2-1)} \frac{1}{(zX_1-1)} \otimes P_{13}P_{23}J_1J_3 \\
&-\epsilon^2 \frac{1}{(zX_2-1)} \otimes P_{13}S_{23}J_1J_2 - \epsilon^2 \frac{1}{(zX_2-1)} \frac{1}{(zX_1^{-1}-1)} \otimes P_{13}S_{23}J_1J_2 \\
&+\epsilon^2 \frac{1}{(zX_2-1)} \frac{1}{(zX_1^{-1}-1)} \otimes P_{13}S_{23}J_1J_2 + \epsilon^3 \frac{1}{(zX_2-1)} \frac{1}{(zX_2^{-1}-1)} \otimes P_{13}P_{23}J_1J_2 \\
&+\epsilon^3 \frac{1}{(zX_2-1)} \frac{1}{(zX_1^{-1}-1)} \frac{1}{(zX_2^{-1}-1)} \otimes P_{13}P_{23}J_1J_2 + \epsilon^3 \frac{1}{(zX_2-1)^2} \otimes P_{13}P_{23}J_1J_3 \\
&+\epsilon^3 \frac{1}{(zX_2-1)^2} \frac{1}{(zX_1^{-1}-1)} \otimes P_{13}P_{23}J_1J_3 + \epsilon^3 \frac{1}{(zX_1-1)} \frac{1}{(X_2^{-1}z-1)} \otimes P_{13}P_{23}J_1J_2 \\
&+\epsilon^3 \frac{1}{(zX_2-1)} \frac{1}{(zX_1-1)} \frac{1}{(X_2^{-1}z-1)} \otimes P_{13}P_{23}J_1J_2 - \epsilon^3 \frac{1}{(zX_2-1)^2} \otimes P_{13}P_{23}J_1J_3 \\
&-\epsilon^3 \frac{1}{(zX_2-1)^2} \frac{1}{(zX_1^{-1}-1)} \otimes P_{13}P_{23}J_1J_3 - \epsilon^2 \frac{1}{(zX_2-1)} \frac{1}{(zX_1-1)} \otimes P_{13}S_{23}J_1J_3 \\
&+\epsilon^3 \frac{1}{(zX_2-1)} \frac{1}{(zX_1-1)} \otimes P_{13}P_{23}J_1J_3 + \epsilon^3 \frac{1}{(zX_2-1)^2} \frac{1}{(zX_1-1)} \otimes P_{13}P_{23}J_1J_3 \\
&-\epsilon^3 \frac{1}{(zX_2-1)} \frac{1}{(zX_1-1)} \otimes P_{13}P_{23}J_1J_2 - \epsilon^3 \frac{1}{(zX_2-1)} \frac{1}{(zX_1-1)} \frac{1}{(zX_2^{-1}-1)} \otimes P_{13}P_{23}J_1J_2 \\
&-\epsilon^3 \frac{1}{(zX_2-1)} \frac{1}{(zX_1-1)} \otimes P_{13}P_{23}J_1J_3 - \epsilon^3 \frac{1}{(zX_2-1)^2} \frac{1}{(zX_1-1)} \otimes P_{13}P_{23}J_1J_3 \\
&+\epsilon^3 \frac{1}{(zX_2-1)} \frac{1}{X_2^{-1}z-1} \otimes P_{13}P_{23}J_2J_1 + \epsilon^3 \frac{1}{(zX_2-1)} \frac{1}{(zX_1^{-1}-1)} \frac{1}{X_2^{-1}z-1} \otimes P_{13}P_{23}J_2J_1 \\
&-\epsilon^2 \frac{1}{(X_2^{-1}z-1)} \otimes P_{23}S_{12} - \epsilon \frac{1}{(X_2^{-1}z-1)} \otimes P_{23}P_{12} \\
&-\epsilon \frac{1}{(X_2z-1)} \otimes S_{13}P_{23}P_{12}S_{12}J_1J_3 - \epsilon \frac{1}{(X_1^{-1}z-1)} \otimes P_{13}P_{12}S_{13}S_{12} \\
&-\epsilon^2 \frac{1}{(X_1^{-1}z-1)} \frac{1}{(X_2^{-1}z-1)} \otimes P_{23}S_{12} - \epsilon^2 \frac{1}{(X_1^{-1}z-1)} \frac{1}{(X_2z-1)} \otimes P_{23}S_{12}J_1J_3 \\
&-\epsilon \frac{1}{(X_1z-1)} \otimes P_{13}P_{12}S_{13}S_{12}J_2J_3 - \epsilon^2 \frac{1}{(X_1z-1)} \otimes P_{13}S_{23}J_1J_3 \\
&+\epsilon^2 \frac{1}{(X_1z-1)} \otimes P_{13}S_{23}J_2J_3 - \epsilon^2 \frac{1}{(X_1z-1)} \otimes P_{23}S_{12}J_2J_1 \\
&+\epsilon^2 \frac{1}{(X_1z-1)} \otimes P_{23}S_{12}J_2J_3 + \epsilon^3 \frac{1}{(X_1z-1)} \otimes P_{23}P_{12}J_1J_3 \\
&-\epsilon^3 \frac{1}{(X_1z-1)} \otimes P_{23}P_{12}J_2J_3 + \epsilon^3 \frac{1}{(X_1z-1)} \otimes P_{13}P_{23}J_2J_1 \\
&+\epsilon^3 \frac{1}{(X_1z-1)} \otimes P_{13}P_{23} - \epsilon^2 \frac{1}{(X_1z-1)} \frac{1}{(X_2^{-1}z-1)} \otimes P_{23}S_{12}J_2J_1 \\
&+\epsilon^3 \frac{1}{(X_1z-1)} \frac{1}{(X_2^{-1}z-1)} \otimes P_{23}P_{12} - \epsilon^3 \frac{1}{(X_1z-1)} \frac{1}{(X_2^{-1}z-1)} \otimes P_{23}P_{12}J_1J_2 \\
&+\epsilon^2 \frac{1}{(X_1z-1)} \frac{1}{(X_2z-1)} \otimes P_{23}S_{12}J_2J_3 + \epsilon^3 \frac{1}{(X_1z-1)} \frac{1}{(X_2z-1)} \otimes P_{23}J_1J_3 \\
&-\epsilon^3 \frac{1}{(X_1z-1)} \frac{1}{(X_2z-1)} \otimes P_{23}J_2J_3 \\
&\equiv 0
\end{aligned}
$$

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