Almost cellular algebras
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Abstract

We obtain results on algebras which have many of the properties of cellular algebras except for the existence of a certain anti-involution. We show that they are applicable to $q$-walled Brauer-Clifford superalgebras.

Introduction

Cellular algebras were introduced by J. Graham and G. Lehrer in [GrLe]. Their definition captures some essential features of the Kazhdan-Lusztig bases of Hecke algebras and is applicable to other families of algebras like cyclotomic $q$-Schur algebras [DJM], cyclotomic Temperley-Lieb algebras [GrLe, RuXi] and cyclotomic Birman-Murakami-Wenzl algebras [En, Xi, Go]. An equivalent definition of cellularity which avoids specifying an explicit basis was given by S. König and C. C. Xi in terms of cell ideals and a filtration by two-sided ideals [KoXi1]. One of the advantages of the concept of cellularity is that it provides a way to parametrize irreducible modules.

The definition of cellular algebras requires the existence of an anti-involution with the property that it fixes isomorphism classes of irreducible modules (see section 5 of [KoXi1]). The motivation for this paper came from [BGJKW] where new algebras called $q$-walled Brauer-Clifford superalgebras were introduced. (These are $q$-analogues of the walled Brauer superalgebras studied in [JuKa].) Because of their similarity with Hecke algebras, it would be natural to expect them to be cellular. However, this cannot be the case because of the absence of a proper anti-involution: see the paragraph after Definition 2.2. Therefore, we wanted to see what kind of theory could be developed without taking into account an anti-involution. The approach we follow is similar to the one expounded in [KoXi1]: we consider so-called sandwich filtration algebras which are reminiscent of the inflation algebras in [KoXi1, KoXi2]. Sandwich filtration algebras equipped with a compatible anti-involution are cellular. The same can be said about the standardly based algebras introduced in [DuRu], but our approach in terms of ideals does not require any explicit basis as in Definition 1.2.1 in [DuRu].

The first section starts with the definition of a sandwich filtration. Our main result is Theorem 1.3 which gives properties of such an algebra. In particular, it provides a way to obtain a classification of all its irreducible modules and gives a necessary and sufficient condition for semisimplicity. In the second section, we show that a $q$-walled Brauer-Clifford superalgebra admits a sandwich filtration built from Hecke-Clifford superalgebras and spaces of linear endomorphisms of mixed tensor products.

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1 Sandwich filtrations

Broadly speaking, a basis for an algebra is cellular if multiplication with respect to the basis takes a certain form. A cellular algebra is an algebra with a cellular basis. This notion, introduced in [GrLe], allows certain properties of the algebra to be easily obtained; for example it produces a complete set of irreducible modules and gives a simple condition for the algebra to be semisimple. There is a procedure [KoXi2] for constructing a cellular basis for an algebra which is “built” from smaller cellular algebras in a certain sense. In the case that motivated this paper, the $q$-walled Brauer-Clifford superalgebra $BC_{r,s}(q)$ is built from Hecke-Clifford algebras in this same sense. ($BC_{r,s}(q)$ is defined in section 2.) Since Brauer algebras and walled Brauer algebras are known to be cellular [KoXi3, CdVDM], one might hope to apply this procedure to exhibit a cellular algebra structure on $BC_{r,s}(q)$.

The definition of a cellular basis requires a compatible anti-involution, but $BC_{r,s}(q)$ and $HC_ℓ(q)$ do not seem to admit a one. (See section 2.) Therefore, we remove the anti-involution from the definition of cellularity to obtain a weaker notion in Definition 1.1 which produces the same information about irreducible modules and semisimplicity. We also extend the definition to include algebras which are not split-semisimple over their base field.

**Definition 1.1.** Suppose $A, A_1, A_2, \ldots, A_k$ are unital associative rings. We say $A$ has a sandwich filtration over $A_1, \ldots, A_k$ if it has a filtration by two-sided ideals

$$0 = J_0 \subseteq J_1 \subseteq \ldots \subseteq J_k = A$$

such that $J_i/J_{i-1} \cong V_i \otimes_{A_i} W_i$ as an $(A, A)$-bimodule, for some nonzero $(A, A_i)$-bimodule $V_i$ and $(A_i, A)$-bimodule $W_i$, both free of finite rank over $A_i$. We call $V_i \otimes_{A_i} W_i$ the factors of $A$. If the rings $A_1, \ldots, A_k$ all coincide, we simply say that $A$ has a sandwich filtration over $A_1$.

The terminology was chosen due to the following construction: For any matrix $P$ in the space $M_{nm}(R)$ of $n \times m$ matrices over a ring $R$, the space $M_{mn}(R)$ becomes an associative ring with operation $A \circ B = APB$. We call $P$ the sandwich matrix. This construction is similar to that of Rees matrix semigroups [Re] as well as the generalised matrix algebras of Brown [Br]. Lemma 1.2 below shows that the factors of $A$ may be obtained via this construction, though we state the result without an explicit basis.

The relationship between our terminology and cellular algebras is as follows: a cellular algebra admits a sandwich filtration over the base field, and has a compatible anti-involution. Since we allow the $A_i$ to be arbitrary rings, this definition is analogous to the iterated inflations of [KoXi2].

**Lemma 1.2.** Suppose $A$ and $R$ are associative rings. Moreover suppose we have an $(A, A)$-bimodule injection $V \otimes_R W \hookrightarrow A$, where $V$ is an $(A, R)$-bimodule, $W$ is an $(R, A)$-bimodule, and $V$ and $W$ are both free over $R$. Then the multiplication map

$$(V \otimes_R W) \otimes (V \otimes_R W) \rightarrow V \otimes_R W$$

induced by this injection is given by

$$(v' \otimes_R w')(v \otimes_R w') = v'\vartheta(w \otimes v) \otimes_R w'$$

where $\vartheta : W \otimes_A V \rightarrow R$ is an $(R, R)$-bimodule homomorphism uniquely determined by this formula.
Proof. Let \( \{v_i\} \) and \( \{w_j\} \) be \( R \)-bases for \( V \) and \( W \) respectively, so
\[
V \otimes_R W = \bigoplus_{i,j} v_i R \otimes_R w_j.
\]

Consider any \( v \in V \) and \( w \in W \), and let \( a, b \in A \) correspond to \( v \otimes_R w \) and \( v \otimes_R w_j \) respectively. We have
\[
(aw) \otimes_R w_j = (v_i \otimes_R w)(v \otimes_R w_j) = v_i \otimes_R (wb).
\]
The left-hand side lies in \( V \otimes_R w_j \) and the right-hand side in \( v_i \otimes_R W \), so the above must equal \( v_i r \otimes_R w_j \) for some \( r \in R \). Moreover \( r \) is independent of \( w_j \) (from the left-hand side) and of \( v_i \) (from the right-hand side), so we may write \( r = \vartheta(w, v) \). Note that \( r \) is uniquely determined. From this uniqueness it is easy to see that \( \vartheta \) is \((R, R)\)-bilinear. Finally for \( x \in A \) we have
\[
v_i \vartheta(wx, v) \otimes_R w_j = (v_i \otimes_R wx)(v \otimes_R w_j) = (v_i \otimes_R w)(xv \otimes_R w_j) = v_i \vartheta(w, xv) \otimes_R w_j.
\]
Therefore \( \vartheta \) factors through \( W \otimes_A V \), as required. \( \square \)

Many useful statements about cellular algebras generalise to our situation as follows. We also have a Morita equivalence result analogous to that of [KoXi3].

**Theorem 1.3.** Suppose \( A \) has a sandwich filtration over the rings \( A_i \) with factors \( V_i \otimes_A W_i \). For statements 3 and 6 we additionally suppose each \( A_i \) is Artinian with unit.

1. The induced multiplication on \( V_i \otimes_A W_i \) is determined by an \((A_i, A_i)\)-bimodule homomorphism \( \vartheta_i : W_i \otimes_A V_i \to A_i \). We call \( \vartheta_i \) the sandwich bilinear form.

2. Let
\[
\tilde{\vartheta}_i : V_i \to W^*_i = \text{Hom}_{A_i}(W_i, A_i)
\]
be the \((A_i, A_i)\)-bimodule homomorphism induced by \( \vartheta_i \). Suppose \( L \) is an irreducible left \( A_i \)-module, and let \( \text{rad}(V_i \otimes_A L) \) be the kernel of the map
\[
\tilde{\vartheta}_i \otimes_{A_i} \text{id}_L : V_i \otimes_A L \to W^*_i \otimes_{A_i} L.
\]
Then \( \text{rad}(V_i \otimes_A L) \) either equals \( V_i \otimes_A L \), or is the unique maximal \( A \)-submodule of \( V_i \otimes_A L \).

3. Each irreducible left \( A \)-module is of the form \( V_i \otimes_A L/\text{rad}(V_i \otimes_A L) \) for some \( i \) and some irreducible \( A_i \)-module \( L \).

4. If \( L \) and \( M \) are irreducible \( A_i \)- and \( A_j \)-modules, respectively, such that \( V_i \otimes_A L/\text{rad}(V_i \otimes_A L) \) and \( V_j \otimes_A M/\text{rad}(V_j \otimes_A M) \) are both non-zero and isomorphic, then \( i = j \) and \( L \cong M \).

5. If \( \tilde{\vartheta}_i \) is an isomorphism (in particular this requires that \( V_i \) and \( W_i \) have the same rank over \( A_i \)) for each \( i \), then \( A \) is Morita equivalent to \( \bigoplus_{i=1}^k A_i \).

6. The ring \( A \) is semisimple if and only if each \( \tilde{\vartheta}_i \) is an isomorphism and each \( A_i \) is semisimple.

Proof. Let \( J_i \) denote the two-sided ideals of \( A \) as in Definition 1.1. Note that since \( J_i/J_{i-1} \) is annihilated by \( J_{i-1} \) as a left \( A \)-module, the same is true of \( V_i \otimes W_i \), and therefore of \( V_i \). Similarly \( W_i J_{i-1} = 0 \).
1. Identifying $J_i/J_{i-1}$ with a two-sided ideal in $A/J_{i-1}$, the statement follows directly from Lemma 1.2.

2. Since $\bar{\mathcal{L}}$ is an $(A, A_i)$-bimodule homomorphism, $\text{rad} (V_i \otimes_{A_i} L)$ is an $A$-submodule of $V_i \otimes_{A_i} L$. It remains to prove that any element outside of $\text{rad} (V_i \otimes_{A_i} L)$ generates $V_i \otimes_{A_i} L$ over $A$. Suppose $x \in V_i \otimes_{A_i} L$ does not lie in $\text{rad} (V_i \otimes_{A_i} L)$. Then its image in

$$W_i^* \otimes_{A_i} L \cong \text{Hom}_{A_i}(W_i, L)$$

is nonzero. Therefore the image of $W_i \otimes_A x$ under the map

$$W_i \otimes_A V_i \otimes_{A_i} L \xrightarrow{\varrho_i \otimes \text{id}} A_i \otimes_{A_i} L \xrightarrow{\sim} L$$

is a nonzero $A_i$-submodule of $L$. Since $L$ is irreducible, it must equal $L$. Thus $V_i \otimes_{A_i} W_i \otimes_A x$ surjects onto $V_i \otimes_{A_i} L$ under the map

$$V_i \otimes_{A_i} W_i \otimes_A V_i \otimes_{A_i} L \to V_i \otimes_{A_i} L.$$

That is, $J_i x = V_i \otimes_{A_i} L$, so $x$ generates $V_i \otimes_{A_i} L$ over $A$.

3. The previous statement shows that the nonzero modules of the form $V_i \otimes_{A_i} L/\text{rad} (V_i \otimes_{A_i} L)$ are irreducible, so we need only show that each irreducible is of this form. Let $U$ be an irreducible left $A$-module, and pick $i$ minimal such that $J_i U \neq 0$. Choose any $u \in U$ with $J_i u \neq 0$. The map $a \mapsto au$ gives a surjection of left $A$-modules

$$V_i \otimes_{A_i} W_i \cong J_i/J_{i-1} \to U.$$  

Since $A_i$ is Artinian, we may choose a surjection of the form

$$V_i \otimes_{A_i} L \to U$$

such that $L$ is a left $A_i$-module with minimal length. Suppose $L$ has a proper nonzero submodule $L'$. The image of $V_i \otimes_{A_i} L'$ in $U$ is an $A$-submodule, so it either equals $U$ or $0$. In the latter case our surjection factors through a surjection

$$V_i \otimes_{A_i} (L/L') \to U.$$  

Thus both cases contradict the minimality of $L$, so $L$ must be irreducible. Also since $J_i U \neq 0$, we have $J_i (V_i \otimes_{A_i} L) \neq 0$, so $\text{rad} (V_i \otimes_{A_i} L)$ does not equal $V_i \otimes_{A_i} L$. Now the previous statement implies that

$$V_i \otimes_{A_i} L/\text{rad} (V_i \otimes_{A_i} L)$$

is the unique irreducible quotient of $V_i \otimes_{A_i} L$, so it must be isomorphic to $U$, as required.

4. Suppose $L$ and $M$ are irreducible $A_i$- and $A_j$- modules respectively, such that $L' = V_i \otimes_{A_i} L/\text{rad} (V_i \otimes_{A_i} L)$ and $M' = V_j \otimes_{A_j} M/\text{rad} (V_j \otimes_{A_j} M)$ are isomorphic $A$-modules. We have shown above that $J_i L' = L'$ and $J_{i-1} L' = 0$, so $i = j$. We have also shown that the map

$$W_i \otimes_A V_i \otimes_{A_i} L \xrightarrow{\varrho_i \otimes \text{id}} L$$

is surjective. This induces a surjection $f : W_i \otimes_A L' \to L$. We claim that $L$ is the unique irreducible quotient of $W_i \otimes_A L'$. It suffices to show that there is no proper $A_i$ submodule
If $A \subseteq W_i \otimes_A L'$ such that $f(N) \neq 0$. Suppose such $N$ exists. Then $f(N) = L$, so the bottom map in the following commutative diagram is surjective.

\[
\begin{array}{cccccc}
W_i \otimes_A V_i \otimes_{A_i} W_i \otimes_A V_i \otimes_{A_i} L & \rightarrow & W_i \otimes_A V_i \otimes_{A_i} L \\
\downarrow & & \downarrow & & \downarrow \\
W_i \otimes_A V_i \otimes_{A_i} W_i \otimes_A V_i \otimes_{A_i} L' & \rightarrow & W_i \otimes_A L' \\
\downarrow & & \downarrow \\
W_i \otimes_A V_i \otimes_{A_i} N & & & & & & \\
\end{array}
\]

The topmost map in the above diagram is given by

\[
w_1 \otimes_A v_1 \otimes_{A_i} w_2 \otimes_A v_2 \otimes_{A_i} l \mapsto w_1 \otimes_A v_1 \otimes_{A_i} d_i(w_2, v_2)l = w_1 \otimes_A (v_1 \otimes_{A_i} w_2) v_2 \otimes_{A_i} l = d_i(w_1, v_1) w_2 \otimes_A v_2 \otimes_{A_i} l
\]

where we identify $(v_1 \otimes_{A_i} w_2) \in V_i \otimes_{A_i} W_i$ with an element of $J_i/J_{i-1}$. Therefore the surjection $W_i \otimes_A V_i \otimes_{A_i} N \rightarrow W_i \otimes_A L'$ is given by $w \otimes_A v \otimes_{A_i} N \mapsto d_i(w, v)n$. Since $N$ is an $A_i$ submodule, $N = W_i \otimes_A L'$ as required. Therefore $L$ and $M$ are the unique irreducible quotients of $W_i \otimes A L'$ and $W_i \otimes A M'$ respectively, but the latter are isomorphic, so $L \cong M$.

5. We recall the following simple fact: If $A$ is a unital associative ring, and $J \subseteq A$ is a two sided ideal which, considered as a ring, is also unital, then we have a ring isomorphism $A \cong J \oplus (A/J)$. Indeed let $1 \in A$ and $e \in J$ be the units of these rings. For any $a \in A$ we have $ae = eae = ea$, so $e$ is a central idempotent. Thus $A \cong eA \oplus (1 - e)A$. Since $eA = J$ and $(1 - e)A \cong (A/J)$, this proves the claim.

Fix $i$, and suppose $\tilde{d}_i$ induces an isomorphism. We may choose bases $\{v_1, \ldots, v_m\}$ for $V_i$ and $\{w_1, \ldots, w_m\}$ for $W_i$ such that $d_i(w_a \otimes_A v_b) = \delta_{ab}$. We have a ring isomorphism

\[
\text{Mat}_m(A_i) \cong V_i \otimes_{A_i} W_i
\]

given by

\[
X \mapsto \sum_{a,b} v_a \otimes_{A_i} X_{ab} w_b.
\]

We also have a ring isomorphism $V_i \otimes_{A_i} W_i \cong J_i/J_{i-1}$, so $J_i/J_{i-1}$ is unital and Morita equivalent to $A_i$. Applying the above observation repeatedly, it follows that

\[
A \cong \bigoplus_{i=1}^k J_i/J_{i-1}
\]

is Morita equivalent to $\bigoplus_{i=1}^k A_i$.

6. If the conditions hold, then $A$ is semisimple by the previous statement.

Conversely suppose $A$ is semisimple. We must prove that $A_i$ is semisimple and $\tilde{d}_i$ is an isomorphism, for each $i$. By the above observation and induction, it suffices to prove this for $i = 1$. 

5
Let $J$ denote the Jacobson radical of $A_1$. Since $A_1$ is Artinian, $J$ is nilpotent. For each $j \geq 1$,

$$V_1 \otimes_{A_1} J^j W_1$$

is a sub-$(A, A)$-bimodule of $V_1 \otimes_{A_1} W_1$, so it corresponds to a two sided ideal of $A$. Moreover

$$(V_1 \otimes_{A_1} J W_1)(V_1 \otimes_{A_1} J^j W_1) \subseteq V_1 \otimes_{A_1} J^j (W_1 \otimes_{A} V_1) J^j W_1 \subseteq V_1 \otimes_{A_1} J^{j+1} W_1.$$

Thus $V_1 \otimes_{A_1} J W_1$ corresponds to a nilpotent ideal. Since $A$ is semisimple, $V_1 \otimes_{A_1} J W_1 = 0$. But $V_1$ and $W_1$ are nonzero and free over $A_1$, so $J = 0$. That is, $A_1$ is semisimple.

By symmetry, we may suppose the rank of $V_1$ is at least the rank of $W_1$. Let $K \subseteq V_1$ be the kernel of $\vartheta_1 : V_1 \rightarrow W_1^*$. This is a sub-$(A, A)$-bimodule of $V_1$, so $K \otimes_{A_1} W_1$ corresponds to a two sided ideal in $A$. Moreover

$$(K \otimes_{A_1} W_1)(K \otimes_{A_1} W_1) \subseteq K \otimes_{A_1} \vartheta_1(W_1 \otimes_{A} K) W_1 = 0.$$

Thus $K \otimes_{A_1} W_1 = 0$. Again since $W_1$ is nonzero and free over $A_1$, it follows that $K = 0$. On the other hand the length of $V_1$ is at least the length of $W_1^*$ as right $A_1$-modules. Therefore $\vartheta_i$ is an isomorphism.

\[\square\]

It is easier to determine whether the above conditions hold using more explicit descriptions. Suppose the assumptions of the theorem hold, and fix $i$. Moreover suppose $A_i$ is an algebra over a field $K$. Let $\{v_a\}$ and $\{w_b\}$ be $A_i$-bases of $V_i$ and $W_i$, and let $\{w^*_b\}$ denote the corresponding dual basis of $W_i^*$. Let $x^b_a = \vartheta_i(v_a \otimes w_b) \in A_i$. Let $L$ be an irreducible $A_i$ module with $K$-basis $\{\ell_c\}$. Then $\{v_a \otimes \ell_c\}$ and $\{w^*_b \otimes \ell_d\}$ are $K$-bases for $V_i \otimes_{A_i} L$ and $W_i^* \otimes_{A_i} L$, and with respect to these bases the map $\vartheta_i \otimes_{A_i} \text{id}_L$ corresponds to the matrix $[\gamma_{ac}^{bd}]$, where

$$x^b_a v_c = \sum_d \gamma_{ac}^{bd} \ell_d.$$  

Now determining $\text{rad}(V_i \otimes_{A_i} L)$ corresponds to finding the kernel of this matrix over $K$.

Note that with respect to the bases $\{v_a\}$ and $\{w^*_b\}$, the map $\vartheta_i$ corresponds to the matrix $[x^b_a]$ over $A_i$. Therefore $\vartheta_i$ is an isomorphism if and only if this matrix is invertible. Let $J$ be the Jacobson radical of $A_i$, and let $\tilde{x}^b_a$ be the image of $x^b_a$ in $A_i/J$. Since $J$ is nilpotent, $\vartheta_i$ is an isomorphism if and only if the matrix $[\tilde{x}^b_a]$ is invertible over $A_i/J$. Finally if $L$ is a complete set of irreducible $A_i$-modules, then the map

$$A_i/J \rightarrow \bigoplus_{L \in \mathcal{L}} \text{End}_K(L)$$

is injective. Therefore $\vartheta_i$ is an isomorphism if and only if the matrix $[\gamma_{ac}^{bd}]$ constructed above is invertible for each irreducible $A_i$-module $L$.

In [KoXi2], a procedure is described for constructing cellular algebras from other cellular algebras via “inflation”. The following theorem is the analogue of this procedure for sandwich filtration algebras. It follows immediately from the definition.
Definition 2.2. but we first provide a concrete definition in terms of generators and relations and then recall the \( \ell \)-generated by the elements \( t_1, \ldots, t_\ell \). In this second section, we show that the \( q \)-walled Brauer-Clifford superalgebras. They are built from two finite Hecke-Clifford algebras, whose definition we recall below, and one additional generator.

Theorem 1.4. Suppose a ring \( A \) has a sandwich filtration over \( A_1, \ldots, A_k \) with factors \( V_i \otimes_{A_i} W_i \). Moreover suppose each \( A_i \) has a sandwich filtration over \( A_{i1}, A_{i2}, \ldots, A_{i_k} \) with factors \( V_{ij} \otimes_{A_{ij}} W_{ij} \). Then \( A \) has a sandwich filtration over the \( A_{ij} \) (ordered lexicographically) with factors \( (V_i \otimes_{A_i} V_{ij}) \otimes_{A_{ij}} (W_{ij} \otimes_{A_i} W_i) \). The sandwich bilinear form \( \vartheta_{ij}^{\ell} : (W_{ij} \otimes_{A_i} W_i) \otimes_A (V_i \otimes_{A_i} V_{ij}) \to A_{ij} \)

is simply the composite

\[
\begin{array}{c}
W_{ij} \otimes_{A_i} W_i \otimes_A V_i \otimes_{A_i} V_{ij} \\
\xrightarrow{id_{W_{ij}} \otimes_A, id_{V_i} \otimes_{A_i} id_{V_{ij}}} \\
W_{ij} \otimes_{A_i} A_i \otimes_{A_i} V_{ij} \\
\xrightarrow{\vartheta_{ij}} \\
W_{ij} \otimes_{A_i} V_{ij} \\
\xrightarrow{\vartheta_{ij}} \\
A_{ij}.
\end{array}
\]

2 \( q \)-walled Brauer-Clifford superalgebras

In this second section, we show that the \( q \)-walled Brauer-Clifford superalgebras studied in [BGJKW] are sandwich filtration algebras. They are built from two finite Hecke-Clifford algebras, whose definition we recall below, and one additional generator.

Definition 2.1. [Ol] The finite Hecke-Clifford algebra \( HC_1(q) \) is the unital associative \( \mathbb{C}(q) \)-algebra generated by elements \( t_1, \ldots, t_{\ell-1} \) and anti-commuting elements \( c_1, \ldots, c_\ell \) which satisfy the relations:

\[
(t_i - q)(t_i + q^{-1}) = 0, \quad i = 1, \ldots, \ell - 1, \quad t_it_{i+1}t_i = t_{i+1}t_it_i, \quad i = 1, \ldots, \ell - 2; \\
t_i t_j = t_j t_i \quad \text{if} \quad j \neq i - 1, i + 1, \quad c_i^2 = -1, \quad c_ic_j = -c_jc_i \quad \text{if} \quad 1 \leq i \neq j \leq \ell; \\
t_i c_i = c_{i+1} t_i, \quad t_i c_j = c_j t_i \quad \text{if} \quad j \neq i, i + 1.
\]

\( HC_1(q) \) becomes a \( \mathbb{Z}_2 \)-graded algebra if we declare \( t_i \) to be even and \( c_j \) to be odd for all \( 1 \leq i \leq \ell - 1, 1 \leq j \leq \ell \).

The quickest way to define the \( q \)-walled Brauer-Clifford algebra is as a centralizer algebra, but we first provide a concrete definition in terms of generators and relations and then recall the equivalence of the two definitions proved in [BGJKW].

Definition 2.2. The \( q \)-walled Brauer-Clifford algebra \( BC_{\ell,s}(q) \) is a unital associative \( \mathbb{C}(q) \)-algebra generated by the elements \( t_1, t_2, \ldots, t_{\ell-1}, c_1, c_2, \ldots, c_r, t_1^*, t_2^*, \ldots, t_{s-1}^*, c_1^*, c_2^*, \ldots, c_s^* \), and *e*. The elements \( t_1, \ldots, t_{\ell-1}, c_1, \ldots, c_r \) satisfy the relations of the Hecke-Clifford algebra \( HC_1(q) \), while \( t_s^*, \ldots, t_{s-1}^*, c_1^*, \ldots, c_s^* \) satisfy those of \( HC_s(q) \) except that \( (c_s^*)^2 = 1 \). Moreover, \( t_1, \ldots, t_{r-1}, c_1, \ldots, c_r \) supercommute with \( t_1^*, \ldots, t_{s-1}^*, c_1^*, \ldots, c_s^* \). The generator *e* commutes with

\[
t_1, \ldots, t_{r-2}, c_1, \ldots, c_{r-1}, t_2^*, \ldots, t_{s-1}^*, c_2^*, \ldots, c_s^*,
\]
and satisfies

\[ e^2 = 0, \quad e t_{r-1} e = e = e t_i^* e, \quad c_r e = c_i^* e \quad \text{and} \quad e c_r = e c_i^*, \]

\[ ec_r e = 0, \quad e t_{r-1}^* t_i^* e t_{r-1} = e t_{r-1}^* t_i^* t_{r-1}^* e, \quad t_{r-1} e t_{r-1}^* t_i^* e = t_i^* e t_{r-1}^* t_{r-1} e. \]

\( BC_{r,s}(q) \) becomes a \( \mathbb{Z}_2 \)-graded algebra if we declare \( t_i, t_i^* , e \) to be even and \( c_j, c_j^* \) to be odd for all \( 1 \leq i \leq \ell - 1, \ 1 \leq j \leq \ell \).

**Remark 2.3.** The walled Brauer superalgebra \( BC_{r,s} \) in [JuKa] is obtained from \( BC_{r,s}(q) \) by setting \( q = 1 \). In [JuKa, Thm. 5.1], \( e \) is denoted by \( e_{r,r+1} \), \( t_i \) by \( s_i \), and the two five-term equations above are replaced by a single six-term relation, which can be shown to be equivalent to the two relations above.

The algebra \( BC_{r,s}(q) \) has a natural anti-involution defined on the generators by

\[ t_i \mapsto -t_i^{-1}, \quad t_i^* \mapsto -(t_i^*)^{-1}, \quad c_i \mapsto c_i, \quad c_i^* \mapsto c_i^*, \quad e \mapsto e. \]

This induces an anti-involution on \( HC_\ell(q) \) which also sends \( t_i \) to \(-t_i^{-1}\) and \( c_i \) to \( c_i \). Unfortunately the centre of \( HC_\ell(q) \) (ignoring the \( \mathbb{Z}_2 \) grading) is not fixed by this anti-involution. Indeed, the specialisation of \( HC_\ell(q) \) at \( q = 1 \) is the Sergeev algebra. By Lemma 5 of [Se], there are odd elements in the centre of the Sergeev algebra in the \(-1\)-eigenspace of the anti-involution. Also Lemma 6 of [Se] and Theorem 6.7 of [JoNa] show that the dimension of the centre doesn’t change upon specialisation, so the same is true of \( HC_\ell(q) \). It follows that \( HC_\ell(q) \) cannot be cellular with respect to this anti-involution. It may be possible to find a different anti-involution on \( BC_{r,s}(q) \), or to extend the notion of cellularity to super algebras in a way which accounts for this behaviour, but we have not pursued this direction.

We will need the following theorem, which is one of the main results in [BGJKW]. Set \( V = C(n/n), \quad V_{r,s} = (V) \otimes (V^*) \otimes s, \quad V_q = C(q) \otimes C V \) and \( V_{q,r,s} = (V_q) \otimes (V^q) \otimes s \).

**Theorem 2.4.** There is an action of \( BC_{r,s}(q) \) on \( V_{q,r,s} \) with commutes with the action of the quantized enveloping superalgebra \( U_q q_n \) of type \( Q \) [Ol]. Let \( \rho_{n,q} : BC_{r,s}(q) \rightarrow End_{U_q q_n}(V_{q,r,s}) \) denote the homomorphism coming from this representation. Then \( \rho_{n,q} \) is surjective, and when \( n \geq r + s \), it is an isomorphism.

\( q \)-walled Brauer-Clifford algebras thus fit into the old topic of centralizer algebras. Walled Brauer algebras were studied in [BCHLLS] and their \( q \)-analogues in [KoMu]; a version of Theorem 2.4 holds for them if \( U_q q_n \) is replaced by the quantized enveloping algebra of \( gl_n \).

In the proof of the main theorem of this section, we will work over the local ring \( C[q,q^{-1}](q^{-1}) \) and we will need the next lemma.

**Lemma 2.5.** Suppose \( R \) is a Noetherian local integral domain whose maximal ideal is generated by a single element \( x \in R \). Let \( \psi : A \rightarrow B \) be a homomorphism of finitely generated \( R \)-modules, and consider the corresponding induced homomorphism

\[ \overline{\psi} : A/xA \rightarrow B/xB, \quad \overline{\psi}(a + xA) = \psi(a) + xB. \]

1. If \( \overline{\psi} \) is surjective, then \( \psi \) is surjective.

2. If \( B \) is torsion free and \( \overline{\psi} \) is injective, then \( \psi \) is injective, and its cokernel is also torsion free.
We now show that $BC_{r,s}(q)$ has a sandwich filtration over the Hecke-Clifford algebras $HC_t(q)$. To allow specialisation at $q = 1$, it is convenient to do this over the local ring $\mathcal{R} = \mathbb{C}[q, q^{-1}]_{(q-1)}$ rather than $\mathbb{C}(q)$. Denote by $BC_{r,s}(\mathcal{R})$ the $\mathcal{R}$-subalgebra of $BC_{r,s}(q)$ generated by the generators in Definition 2.2. Set $V_\mathcal{R} = \mathcal{R} \otimes \mathbb{C}(n|n)$ and $V_{\mathcal{R}}^{r,s} = (V_\mathcal{R})^{\otimes r} \otimes (V_\mathcal{R})^{\otimes s}$. We adopt the convention that the generator $e$ is excluded if $r = 0$ or $s = 0$; thus $\mathbb{C}(q) \otimes_\mathcal{R} BC_{r,0}(\mathcal{R})$ is exactly the Hecke-Clifford algebra $HC_r(q)$.

**Theorem 2.6.** The algebra $BC_{r,s}(\mathcal{R})$ has a sandwich filtration over the algebras $A_i(\mathcal{R})$ given by $A_i(\mathcal{R}) = BC_{r-i,0}(\mathcal{R}) \otimes_\mathcal{R} BC_{0,s-i}(\mathcal{R})$ for $0 \leq i \leq \min(r, s)$.

**Proof.** Fix $n \geq r+s$. In the proof of Theorem 2.4 above provided in [BGJKW], we introduced a subspace $\text{End}_G(V_{\mathcal{R}}^{r,s})$ of $\text{End}_\mathcal{R}(V_{\mathcal{R}}^{r,s})$ whose tensor products with $\mathcal{R}/(q-1)\mathcal{R}$ and $\mathbb{C}(q)$ are, respectively, $\text{End}_{\mathcal{R}}(V_{\mathcal{R}}^{r,s})$ and $\text{End}_{\mathcal{R}(q)}(V_{\mathcal{R}}^{r,s})$. Similarly, we may construct a space

$$\text{Hom}_G(V_{\mathcal{R}}^{a,b}, V_{\mathcal{R}}^{c,d}) \subseteq \text{Hom}_\mathcal{R}(V_{\mathcal{R}}^{a,b}, V_{\mathcal{R}}^{c,d})$$

for any nonnegative integers $a$, $b$, $c$, and $d$. For convenience, we use the following notation throughout this proof.

$$N_\mathcal{R}(i, j) = \text{Hom}_G(V_{\mathcal{R}}^{r-j,s-j}, V_{\mathcal{R}}^{r-i,s-i}),
N_\mathcal{C}(i, j) = \text{Hom}_{\mathcal{R}(q)}(V_{\mathcal{R}}^{r-j,s-j}, V_{\mathcal{R}}^{r-i,s-i}).$$

Thus $\mathcal{R}/(q-1) \otimes \mathcal{R} N_\mathcal{R}(i, j) \cong N_\mathcal{C}(i, j)$. We have composition maps

$$N_\mathcal{C}(l, j) \otimes \mathcal{C} N_\mathcal{C}(j, i) \to N_\mathcal{C}(l, i).$$

In particular, $N_\mathcal{C}(i, i)$ is an algebra and $N_\mathcal{C}(i, j)$ is an $(N_\mathcal{C}(i, i), N_\mathcal{C}(j, j))$-bimodule. Similar statements hold for $N_\mathcal{R}(i, j)$. By Theorem 3.5 of [JuKa], we have $BC_{r,s} \cong N_\mathcal{C}(0,0)$.

The composition map

$$N_\mathcal{R}(0, i) \otimes_\mathcal{R} N_\mathcal{R}(i, 0) \to N_\mathcal{R}(0, 0)$$

factors through $N_\mathcal{R}(0, i) \otimes N_\mathcal{R}(i, i) N_\mathcal{R}(i, 0)$. It is also an $(N_\mathcal{R}(0, 0), N_\mathcal{R}(0, 0))$-bimodule homomorphism, so its image $J_1(\mathcal{R})$ is a two-sided ideal in $N_\mathcal{R}(0, 0)$. Now define the $(N_\mathcal{R}(0, 0), N_\mathcal{R}(i, i))$-bimodule $V_i(\mathcal{R})$ and the $(N_\mathcal{R}(i, i), N_\mathcal{R}(0, 0))$-bimodule $W_i(\mathcal{R})$ by the following right exact sequences:

$$N_\mathcal{R}(0, i + 1) \otimes_\mathcal{R} N_\mathcal{R}(i + 1, i) \to N_\mathcal{R}(0, i) \to V_i(\mathcal{R}),$$

$$N_\mathcal{R}(i, i + 1) \otimes_\mathcal{R} N_\mathcal{R}(i + 1, 0) \to N_\mathcal{R}(i, 0) \to W_i(\mathcal{R}).$$

Here we adopt the convention that $N_\mathcal{R}(i, j) = 0$ if $i$ or $j$ is greater than $r$ or $s$. Clearly the images of

$$N_\mathcal{R}(0, i + 1) \otimes_\mathcal{R} N_\mathcal{R}(i + 1, i) \otimes_\mathcal{R} N_\mathcal{R}(i, 0)$$

and

$$N_\mathcal{R}(0, i) \otimes_\mathcal{R} N_\mathcal{R}(i, i + 1) \otimes_\mathcal{R} N_\mathcal{R}(i + 1, 0)$$

in $N_\mathcal{R}(0, 0)$ are contained in $J_{i+1}(\mathcal{R})$, so the surjective map

$$N_\mathcal{R}(0, i) \otimes_{N_\mathcal{R}(i, i)} N_\mathcal{R}(i, 0) \to J_i(\mathcal{R})$$

factors to give a map

$$V_i(\mathcal{R}) \otimes_{N_\mathcal{R}(i, i)} W_i(\mathcal{R}) \to J_i(\mathcal{R})/(J_i(\mathcal{R}) \cap J_{i+1}(\mathcal{R})).$$
Finally let $A_i(\mathcal{R}) = BC_{r-i,0}(\mathcal{R}) \otimes_{\mathcal{R}} BC_{0,s-i}(\mathcal{R})$. We have a natural inclusion $A_i(\mathcal{R}) \hookrightarrow N(\mathcal{R}, (r, s))$, so we obtain a surjective homomorphism

$$\pi_i : V_i(\mathcal{R}) \otimes_{A_i(\mathcal{R})} W_i(\mathcal{R}) \twoheadrightarrow J_i(\mathcal{R})/(J_i(\mathcal{R}) \cap J_{i+1}(\mathcal{R}))$$

of $(N(0,0), N(0,0))$-bimodules. We will prove that the various objects we have constructed satisfy the following properties:

1. The $J_i(\mathcal{R})$ form a filtration of $N(0,0)$; that is, $J_{i+1}(\mathcal{R}) \subseteq J_i(\mathcal{R})$.
2. Both $V_i(\mathcal{R})$ and $W_i(\mathcal{R})$ are free over $A_i(\mathcal{R})$ (as right and left modules respectively).
3. The map $\pi_i$ is an isomorphism.

All three claims will follow from the corresponding claims over $\mathcal{R}/(q-1)\mathcal{R} \cong \mathbb{C}$, which we will in turn deduce from explicit diagrammatic bases of these spaces.

Theorem 3.5 and Proposition 4.3 of [JuKa] show that $N(\mathcal{C}, (0,0))$ has a basis indexed by $(r, s)$-diagrams, and multiplication of these basis elements corresponds to concatenation of the diagrams. By the same argument, $N(\mathcal{C}, (r, s))$ has a basis indexed by diagrams from $(r-i) + (s-i)$ dots to $(r-j) + (s-j)$ dots, and the above composition maps correspond to concatenation of diagrams. If $1 \leq i \leq r, s$ then any diagram in $N(\mathcal{C}, (i, 0))$ has at least one horizontal edge. By moving part of this edge to the bottom of the diagram, we may express the diagram as a concatenation of a diagram in $N(\mathcal{C}, (i, i-1))$ and one in $N(\mathcal{C}, (i-1, 0))$. Thus,

$$N(\mathcal{C}, (i, i-1)) \otimes_{\mathcal{C}} N(\mathcal{C}, (i-1, 0)) \twoheadrightarrow N(\mathcal{C}, (i, 0))$$

is surjective. By Lemma 2.5,

$$N(\mathcal{R}, (i, i-1)) \otimes_{\mathcal{R}} N(\mathcal{R}, (i-1, 0)) \twoheadrightarrow N(\mathcal{R}, (i, 0))$$

is surjective. Now the commutative diagram

$$\begin{array}{ccc}
N(\mathcal{R}, (0, i)) \otimes_{\mathcal{R}} N(\mathcal{R}, (i, i-1)) \otimes_{\mathcal{R}} N(\mathcal{R}, (i-1, 0)) & \twoheadrightarrow & N(\mathcal{R}, (0, i)) \otimes_{\mathcal{R}} N(\mathcal{R}, (i, 0)) \\
\downarrow & & \downarrow \\
N(\mathcal{R}, (0, i-1)) \otimes_{\mathcal{R}} N(\mathcal{R}, (i-1, 0)) & & N(\mathcal{R}, (0, 0))
\end{array}$$

shows that $J_i(\mathcal{R}) \subseteq J_{i-1}(\mathcal{R})$.

Recall that $V_i(\mathcal{R})$ and $W_i(\mathcal{R})$ were defined by the right exact sequences in (2.7). Tensoring with $\mathcal{R}/(q-1)\mathcal{R}$, we obtain right exact sequences

$$N(\mathcal{C}, (0, i+1)) \otimes_{\mathcal{C}} N(\mathcal{C}, (i+1, i)) \twoheadrightarrow N(\mathcal{C}, (0, i)) \twoheadrightarrow V_i(\mathcal{C}) = \mathcal{R}/(q-1)\mathcal{R} \otimes_{\mathcal{R}} V_i(\mathcal{R}),$$

$$N(\mathcal{C}, (i, i+1)) \otimes_{\mathcal{C}} N(\mathcal{C}, (i+1, 0)) \twoheadrightarrow N(\mathcal{C}, (i, 0)) \twoheadrightarrow W_i(\mathcal{C}) = \mathcal{R}/(q-1)\mathcal{R} \otimes_{\mathcal{R}} W_i(\mathcal{R}).$$

A diagram in $N(\mathcal{C}, (0, i))$ has at least $i$ horizontal edges. Moreover, it has more than $i$ horizontal edges if and only if it can be expressed as a concatenation of a diagram in $N(\mathcal{C}, (0, i+1))$ and one in $N(\mathcal{C}, (i+1, i))$. Therefore the diagrams with exactly $i$ horizontal edges map to a basis for $V_i(\mathcal{C})$.

Let $A_i(\mathcal{C}) = \mathcal{R}/(q-1)\mathcal{R} \otimes_{\mathcal{R}} A_i(\mathcal{R}) = BC_{r-i,0} \otimes_{\mathcal{C}} BC_{0,s-i}$. We have bases for $BC_{r-i,0}$ and $BC_{0,s-i}$ indexed by diagrams on $(r-i) + 0$ dots and $0 + (s-i)$ dots respectively. The map

$$A_i(\mathcal{C}) \rightarrow BC_{r-i,s-i} = N(\mathcal{C}, (i, i))$$
sends a pair of diagrams to the diagram obtained by putting them next to each other, up to sign. Therefore its image is exactly the span of those diagrams with no horizontal edges.

Let $X_i$ be the set of diagrams in $\mathbb{N}_C(0, i)$ with exactly $i$ horizontal edges, such that the vertical edges don’t cross each other and are not marked. Any diagram in our basis for $V_i(C)$ is uniquely expressible as a concatenation of a diagram in $X_i$ with a diagram without horizontal edges. That is, concatenation gives an isomorphism

$$\text{span}_C(X_i) \otimes_C A_i(C) \to V_i(C).$$

Thus, $V_i(C)$ is free over $A_i(C)$. Similarly, $W_i(C)$ is freely generated over $A_i(C)$ by $X_i^*$, the set of diagrams in $X_i$ reflected about the horizontal middle axis.

Let $J_i(C)$ denote the image of $J_i(R)$ in $\mathbb{N}_C(0, 0)$. (Note that we have not shown that $R/(q - 1)R \otimes_R J_i(R) \to J_i(C)$ is injective.) By definition, $J_i(C)$ is the image of the composition map

$$\mathbb{N}_C(0, i) \otimes_C \mathbb{N}_C(i, 0) \to \mathbb{N}_C(0, 0).$$

Thus $J_i(C)$ is exactly the span of those diagrams with at least $i$ horizontal edges. The diagrams with exactly $i$ horizontal edges map to a basis for $J_i(C)/J_{i+1}(C)$. Each such diagram is uniquely expressible as a concatenation of a diagram in $X_i$, a diagram in $\mathbb{N}_C(i, i)$ without horizontal edges, and a diagram in $X_i^*$. Therefore, the map

$$V_i(C) \otimes_{A_i(C)} W_i(C) \to J_i(C)/J_{i+1}(C)$$

is an isomorphism.

We will prove by descending induction on $i$ that $\mathbb{N}_R(0, 0)/J_i(R)$ is torsion free over $R$. Since $J_i(R) = 0$ for $i > r, s$ and $\mathbb{N}_R(0, 0)$ is a free $R$-module, the base case is trivial. Suppose $i \leq r, s$ and that $\mathbb{N}_R(0, 0)/J_{i+1}(R)$ is torsion free. The map

$$\eta : V_i(R) \otimes_{A_i(R)} W_i(R) \to \mathbb{N}_R(0, 0)/J_{i+1}(R)$$

becomes injective when tensored with $R/(q - 1)R$, so Lemma 2.5 shows that $\eta$ is injective and its cokernel is torsion free. However, the map

$$\pi_i : V_i(R) \otimes_{A_i(R)} W_i(R) \to J_i(R)/J_{i+1}(R)$$

is surjective by its construction, so the cokernel of $\eta$ is exactly $\mathbb{N}_R(0, 0)/J_i(R)$. This completes the induction.

Moreover, since $\eta$ is injective, $\pi_i$ is an isomorphism. It remains to show that $V_i(R)$ and $W_i(R)$ are free over $A_i(R)$. We have an $A_i(C)$-basis $X_i$ for $V_i(C)$. Let $\tilde{V}_i(R)$ be the free right $A_i(R)$-module generated by $X_i$. By lifting $X_i$ arbitrarily to $V_i(R)$, we obtain a right $A_i(R)$-module homomorphism $\tilde{V}_i(R) \to V_i(R)$ whose tensor product with $R/(q - 1)R$ is an isomorphism. We construct $\tilde{W}_i(R) \to W_i(R)$ similarly. The resulting map

$$\tilde{V}_i(R) \otimes_{A_i(R)} \tilde{W}_i(R) \to \tilde{V}_i(R) \otimes_{A_i(R)} W_i(R)$$

becomes an isomorphism when tensored with $R/(q - 1)R$. We have shown above that $V_i(R) \otimes_{A_i(R)} W_i(R)$ is torsion free, so Lemma 2.5 shows that this map is itself an isomorphism. In particular,

$$\tilde{V}_i(R) \otimes_{A_i(R)} \tilde{W}_i \to V_i(R) \otimes_{A_i(R)} W_i$$

is injective. Since $\tilde{W}_i$ is free over $A_i(R)$, this implies that $\tilde{V}_i(R) \to V_i(R)$ is injective. It is also surjective by Lemma 2.5, so it is an isomorphism. Thus, $V_i(R)$ is free over $A_i(R)$, and the same argument applies to $W_i(R)$.

\end{proof}
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