Almost cellular algebras

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Abstract

We obtain results on algebras which have many of the properties of cellular algebras except for the existence of a certain anti-involution. We show that they are applicable to q-walled Brauer-Clifford superalgebras.

Introduction

Cellular algebras were introduced by J. Graham and G. Lehrer in [GrLe]. Their definition captures some essential features of the Kazhdan-Lusztig bases of Hecke algebras and is applicable to other families of algebras like cyclotomic q-Schur algebras [DJM], cyclotomic Temperley-Lieb algebras [GrLe, RuXi] and cyclotomic Birman-Murakami-Wenzl algebras [En, Xi, Go]. An equivalent definition of cellularity which avoids specifying an explicit basis was given by S. König and C. C. Xi in terms of cell ideals and a filtration by two-sided ideals [KoXi1]. One of the advantages of the concept of cellularity is that it provides a way to parametrize irreducible modules.

The definition of cellular algebras requires the existence of an anti-involution with the property that it fixes isomorphism classes of irreducible modules (see section 5 of [KoXi1]). The motivation for this paper came from [BGJKW] where new algebras called q-walled Brauer-Clifford superalgebras were introduced. (These are q-analogues of the walled Brauer superalgebras studied in [JuKa].) Because of their similarity with Hecke algebras, it would be natural to expect them to be cellular. However, this cannot be the case because of the absence of a proper anti-involution: see the paragraph after Definition 2.2. Therefore, we wanted to see what kind of theory could be developed without taking into account an anti-involution. The approach we follow is similar to the one expounded in [KoXi1]: we consider so-called sandwich filtration algebras which are reminiscent of the inflation algebras in [KoXi1, KoXi2]. Sandwich filtration algebras equipped with a compatible anti-involution are cellular. The same can be said about the standardly based algebras introduced in [DuRu], but our approach in terms of ideals does not require any explicit basis as in Definition 1.2.1 in [DuRu].

The first section starts with the definition of a sandwich filtration. Our main result is Theorem 1.3 which gives properties of such an algebra. In particular, it provides a way to obtain a classification of all its irreducible modules and gives a necessary and sufficient condition for semisimplicity. In the second section, we show that a q-walled Brauer-Clifford superalgebra admits a sandwich filtration built from Hecke-Clifford superalgebras and spaces of linear endomorphisms of mixed tensor products.

Acknowledgments

The first author gratefully acknowledges the financial support received through the Discovery Grant program of the Natural Sciences and Engineering Research Council of Canada. The second author held a Postdoctoral Fellowship of the Pacific Institute for the Mathematical Sciences.

1 Sandwich filtrations

Broadly speaking, a basis for an algebra is *cellular* if multiplication with respect to the basis takes a certain form. A *cellular algebra* is an algebra with a cellular basis. This notion, introduced in [GrLe], allows certain properties of the algebra to be easily obtained; for example it produces a complete set of irreducible modules and gives a simple condition for the algebra to be semisimple. There is a procedure [KoXi2] for constructing a cellular basis for an algebra which is “built” from smaller cellular algebras in a certain sense. In the case that motivated this paper, the q-walled Brauer-Clifford superalgebra $\mathsf{BC}_{r,s}(q)$ is built from Hecke-Clifford algebras in this same sense. ($\mathsf{BC}_{r,s}(q)$ is defined in section 2.) Since Brauer algebras and walled Brauer algebras are known to be cellular [KoXi3, CdVDM], one might hope to apply this procedure to exhibit a cellular algebra structure on $\mathsf{BC}_{r,s}(q)$.

The definition of a cellular basis requires a compatible anti-involution, but $\mathsf{BC}_{r,s}(q)$ and $\mathsf{HC}_{\ell}(q)$ do not seem to admit a one. (See section 2.) Therefore, we remove the anti-involution from the definition of cellularity to obtain a weaker notion in Definition 1.1 which produces the same information about irreducible modules and semisimplicity. We also extend the definition to include algebras which are not split-semisimple over their base field.

Definition 1.1. Suppose A, A_1, $\mathsf{A}_2, \ldots, \mathsf{A}_k$ are unital associative rings. We say A has a *sandwich filtration* over $\mathsf{A}_1, \ldots \mathsf{A}_k$ if it has a filtration by two-sided ideals

$$0 = \mathsf{J}_0 \subsetneq \mathsf{J}_1 \subsetneq \ldots \subsetneq \mathsf{J}_k = \mathsf{A}$$

such that $\mathsf{J}_i/\mathsf{J}_{i-1} \cong \mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{W}_i$ as an (A, A)-bimodule, for some nonzero $(\mathsf{A}, \mathsf{A}_i)$-bimodule V_i and $(\mathsf{A}_i, \mathsf{A})$-bimodule W_i, both free of finite rank over A_i. We call $\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{W}_i$ the *factors* of A. If the rings $\mathsf{A}_1, \ldots, \mathsf{A}_k$ all coincide, we simply say that A has a *sandwich filtration* over A_1.

The terminology was chosen due to the following construction: For any matrix P in the space $M_{nm}(R)$ of $n \times m$ matrices over a ring R, the space $M_{mn}(R)$ becomes an associative ring with operation $A \circ B = APB$. We call P the *sandwich matrix.* This construction is similar to that of Rees matrix semigroups [Re] as well as the generalised matrix algebras of Brown [Br]. Lemma 1.2 below shows that the factors of A may be obtained via this construction, though we state the result without an explicit basis.

The relationship between our terminology and cellular algebras is as follows: a cellular algebra admits a sandwich filtration over the base field, and has a compatible anti-involution. Since we allow the A_i to be arbitrary rings, this definition is analogous to the iterated inflations of [KoXi2].

Lemma 1.2. Suppose A and R are associative rings. Moreover suppose we have an (A, A)-bimodule injection $\mathsf{V} \otimes_{\mathsf{R}} \mathsf{W} \hookrightarrow \mathsf{A}$, where V is an (A, R)-bimodule, W is an (R, A)-bimodule, and V and W are both free over R. Then the multiplication map

$$(\mathsf{V} \otimes_{\mathsf{R}} \mathsf{W}) \otimes (\mathsf{V} \otimes_{\mathsf{R}} \mathsf{W}) \to \mathsf{V} \otimes_{\mathsf{R}} \mathsf{W}$$

induced by this injection is given by

$$(v' \otimes_{\mathsf{R}} w)(v \otimes_{\mathsf{R}} w') = v' \vartheta(w \otimes v) \otimes_{\mathsf{R}} w'$$

where $\vartheta : \mathsf{W} \otimes_{\mathsf{A}} \mathsf{V} \to \mathsf{R}$ is an (R, R)-bimodule homomorphism uniquely determined by this formula.

Proof. Let $\{v_i\}$ and $\{w_j\}$ be R-bases for V and W respectively, so

$$\mathsf{V} \otimes_{\mathsf{R}} \mathsf{W} = \bigoplus_{i,j} v_i \mathsf{R} \otimes_{\mathsf{R}} w_j.$$

Consider any $v \in \mathsf{V}$ and $w \in \mathsf{W}$, and let $a, b \in \mathsf{A}$ correspond to $v_i \otimes_{\mathsf{R}} w$ and $v \otimes_{\mathsf{R}} w_j$ respectively. We have

$$(av) \otimes_{\mathsf{R}} w_j = (v_i \otimes_{\mathsf{R}} w)(v \otimes_{\mathsf{R}} w_j) = v_i \otimes_{\mathsf{R}} (wb).$$

The left-hand side lies in $\mathsf{V} \otimes_{\mathsf{R}} w_j$ and the right-hand side in $v_i \otimes_{\mathsf{R}} \mathsf{W}$, so the above must equal $v_i r \otimes_{\mathsf{R}} w_j$ for some $r \in \mathsf{R}$. Moreover r is independent of w_j (from the left-hand side) and of v_i (from the right-hand side), so we may write $r = \vartheta(w, v)$. Note that r is uniquely determined. From this uniqueness it is easy to see that ϑ is (R, R)-bilinear. Finally for $x \in \mathsf{A}$ we have

$$v_i \vartheta(wx, v) \otimes_{\mathsf{R}} w_j = (v_i \otimes_{\mathsf{R}} wx)(v \otimes_{\mathsf{R}} w_j) = (v_i \otimes_{\mathsf{R}} w)(xv \otimes_{\mathsf{R}} w_j) = v_i \vartheta(w, xv) \otimes_{\mathsf{R}} w_j.$$

Therefore ϑ factors through $\mathsf{W} \otimes_{\mathsf{A}} \mathsf{V}$, as required. $\square$

Many useful statements about cellular algebras generalise to our situation as follows. We also have a Morita equivalence result analogous to that of [KoXi3].

Theorem 1.3. Suppose A has a sandwich filtration over the rings A_i with factors $\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{W}_i$. For statements 3 and 6 we additionally suppose each A_i is Artinian with unit.

1. The induced multiplication on $\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{W}_i$ is determined by an $(\mathsf{A}_i, \mathsf{A}_i)$-bimodule homomorphism $\vartheta_i : \mathsf{W}_i \otimes_{\mathsf{A}} \mathsf{V}_i \to \mathsf{A}_i$. We call ϑ_i the *sandwich bilinear form*.

2. Let
$$\bar{\vartheta}_i : \mathsf{V}_i \to \mathsf{W}_i^* = \mathsf{Hom}_{\mathsf{A}_i}(\mathsf{W}_i, \mathsf{A}_i)$$
be the $(\mathsf{A}, \mathsf{A}_i)$-bimodule homomorphism induced by ϑ_i. Suppose L is an irreducible left A_i-module, and let $\mathsf{rad}\,(\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L})$ be the kernel of the map
$$\bar{\vartheta}_i \otimes_{\mathsf{A}_i} \mathsf{id}_{\mathsf{L}} : \mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L} \to \mathsf{W}_i^* \otimes_{\mathsf{A}_i} \mathsf{L}.$$
Then $\mathsf{rad}\,(\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L})$ either equals $\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L}$, or is the unique maximal A-submodule of $\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L}$.

3. Each irreducible left A-module is of the form $\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L}/\mathsf{rad}\,(\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L})$ for some i and some irreducible A_i-module L.

4. If L and M are irreducible A_i- and A_j-modules, respectively, such that $\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L}/\mathsf{rad}\,(\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L})$ and $\mathsf{V}_j \otimes_{\mathsf{A}_j} \mathsf{M}/\mathsf{rad}\,(\mathsf{V}_j \otimes_{\mathsf{A}_j} \mathsf{M})$ are both non-zero and isomorphic, then $i = j$ and $\mathsf{L} \cong \mathsf{M}$.

5. If $\bar{\vartheta}_i$ is an isomorphism (in particular this requires that V_i and W_i have the same rank over A_i) for each i, then A is Morita equivalent to $\bigoplus_{i=1}^k \mathsf{A}_i$.

6. The ring A is semisimple if and only if each $\bar{\vartheta}_i$ is an isomorphism and each A_i is semisimple.

Proof. Let J_i denote the two-sided ideals of A as in Definition 1.1. Note that since $\mathsf{J}_i/\mathsf{J}_{i-1}$ is annihilated by J_{i-1} as a left A-module, the same is true of $\mathsf{V}_i \otimes \mathsf{W}_i$, and therefore of V_i. Similarly $\mathsf{W}_i \mathsf{J}_{i-1} = 0$.

1. Identifying $\mathsf{J}_i/\mathsf{J}_{i-1}$ with a two-sided ideal in $\mathsf{A}/\mathsf{J}_{i-1}$, the statement follows directly from Lemma 1.2.

2. Since $\bar{\vartheta}_i$ is an $(\mathsf{A}, \mathsf{A}_i)$-bimodule homomorphism, $\mathsf{rad}\,(\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L})$ is an A-submodule of $\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L}$. It remains to prove that any element outside of $\mathsf{rad}\,(\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L})$ generates $\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L}$ over A.

   Suppose $x \in \mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L}$ does not lie in $\mathsf{rad}\,(\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L})$. Then its image in

   $$\mathsf{W}_i^* \otimes_{\mathsf{A}_i} \mathsf{L} \cong \mathsf{Hom}_{\mathsf{A}_i}(\mathsf{W}_i, \mathsf{L})$$

   is nonzero. Therefore the image of $\mathsf{W}_i \otimes_{\mathsf{A}} x$ under the map

   $$\mathsf{W}_i \otimes_{\mathsf{A}} \mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L} \xrightarrow{\vartheta_i \otimes \mathsf{id}} \mathsf{A}_i \otimes_{\mathsf{A}_i} \mathsf{L} \xrightarrow{\sim} \mathsf{L}$$

   is a nonzero A_i-submodule of L. Since L is irreducible, it must equal L. Thus $\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{W}_i \otimes_{\mathsf{A}} x$ surjects onto $\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L}$ under the map

   $$\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{W}_i \otimes_{\mathsf{A}} \mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L} \to \mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L}.$$

   That is, $\mathsf{J}_i x = \mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L}$, so x generates $\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L}$ over A.

3. The previous statement shows that the nonzero modules of the form $\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L}/\mathsf{rad}\,(\mathsf{V}_i \otimes_{A_i} \mathsf{L})$ are irreducible, so we need only show that each irreducible is of this form.

   Let U be an irreducible left A-module, and pick i minimal such that $\mathsf{J}_i\mathsf{U} \neq 0$. Choose any $u \in U$ with $\mathsf{J}_i u \neq 0$. The map $a \mapsto au$ gives a surjection of left A-modules

   $$\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{W}_i \cong \mathsf{J}_i/\mathsf{J}_{i-1} \twoheadrightarrow \mathsf{U}.$$

   Since A_i is Artinian, we may choose a surjection of the form

   $$\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L} \twoheadrightarrow \mathsf{U}$$

   such that L is a left A_i-module with minimal length. Suppose L has a proper nonzero submodule L'. The image of $\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L}'$ in U is an A-submodule, so it either equals U or 0. In the latter case our surjection factors through a surjection

   $$\mathsf{V}_i \otimes_{\mathsf{A}_i} (\mathsf{L}/\mathsf{L}') \twoheadrightarrow \mathsf{U}.$$

   Thus both cases contradict the minimality of L, so L must be irreducible. Also since $\mathsf{J}_i\mathsf{U} \neq 0$, we have $\mathsf{J}_i(\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L}) \neq 0$, so $\mathsf{rad}\,(\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L})$ does not equal $\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L}$. Now the previous statement implies that

   $$\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L}/\mathsf{rad}\,(\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L})$$

   is the unique irreducible quotient of $\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L}$, so it must be isomorphic to U, as required.

4. Suppose L and M are irreducible A_i- and A_j- modules respectively, such that $\mathsf{L}' = \mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L}/\mathsf{rad}\,(\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L})$ and $\mathsf{M}' = \mathsf{V}_j \otimes_{\mathsf{A}_j} \mathsf{M}/\mathsf{rad}\,(\mathsf{V}_j \otimes_{\mathsf{A}_j} \mathsf{M})$ are isomorphic A-modules. We have shown above that $\mathsf{J}_i\mathsf{L}' = \mathsf{L}'$ and $\mathsf{J}_{i-1}\mathsf{L}' = 0$, so $i = j$. We have also shown that the map

   $$\mathsf{W}_i \otimes_{\mathsf{A}} \mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L} \xrightarrow{\vartheta_i \otimes \mathsf{id}} \mathsf{L}$$

   is surjective. This induces a surjection $f : \mathsf{W}_i \otimes_{\mathsf{A}} \mathsf{L}' \to \mathsf{L}$. We claim that L is the unique irreducible quotient of $\mathsf{W}_i \otimes_{\mathsf{A}} \mathsf{L}'$. It suffices to show that there is no proper A_i submodule

$\mathsf{N} \subseteq \mathsf{W}_i \otimes_{\mathsf{A}} \mathsf{L}'$ such that $f(\mathsf{N}) \neq 0$. Suppose such N exists. Then $f(\mathsf{N}) = \mathsf{L}$, so the bottom map in the following commutative diagram is surjective.

$$\begin{tikzcd}
\mathsf{W}_i \otimes_{\mathsf{A}} \mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{W}_i \otimes_{\mathsf{A}} \mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L} \arrow[r] \arrow[d, two heads] & \mathsf{W}_i \otimes_{\mathsf{A}} \mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L} \arrow[d, two heads] \\
\mathsf{W}_i \otimes_{\mathsf{A}} \mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{W}_i \otimes_{\mathsf{A}} \mathsf{L}' \arrow[r, two heads] \arrow[ur, two heads] & \mathsf{W}_i \otimes_{\mathsf{A}} \mathsf{L}' \\
\mathsf{W}_i \otimes_{\mathsf{A}} \mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{N} \arrow[u] \arrow[ur, bend right, two heads] &
\end{tikzcd}$$

The topmost map in the above diagram is given by

$$\begin{aligned}
w_1 \otimes_{\mathsf{A}} v_1 \otimes_{\mathsf{A}_i} w_2 \otimes_{\mathsf{A}} v_2 \otimes_{\mathsf{A}_i} l \quad &\mapsto \quad w_1 \otimes_{\mathsf{A}} v_1 \otimes_{\mathsf{A}_i} \vartheta_i(w_2, v_2) l \\
&= \quad w_1 \otimes_{\mathsf{A}} (v_1 \otimes_{\mathsf{A}_i} w_2) v_2 \otimes_{\mathsf{A}_i} l \\
&= \quad \vartheta_i(w_1, v_1) w_2 \otimes_{\mathsf{A}} v_2 \otimes_{\mathsf{A}_i} l
\end{aligned}$$

where we identify $(v_1 \otimes_{\mathsf{A}_i} w_2) \in \mathsf{V}_i \otimes_{A_i} \mathsf{W}_i$ with an element of $\mathsf{J}_i/\mathsf{J}_{i-1}$. Therefore the surjection $\mathsf{W}_i \otimes_{\mathsf{A}} \mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{N} \twoheadrightarrow \mathsf{W}_i \otimes_{\mathsf{A}} \mathsf{L}'$ is given by $w \otimes_{\mathsf{A}} v \otimes_{\mathsf{A}_i} \mathsf{N} \mapsto \vartheta_i(w, v) n$. Since N is an A_i submodule, $\mathsf{N} = \mathsf{W}_i \otimes_{\mathsf{A}} \mathsf{L}'$ as required. Therefore L and M are the unique irreducible quotients of $\mathsf{W}_i \otimes_{\mathsf{A}} \mathsf{L}'$ and $\mathsf{W}_i \otimes_{\mathsf{A}} \mathsf{M}'$ respectively, but the latter are isomorphic, so $\mathsf{L} \cong \mathsf{M}$.

5. We recall the following simple fact: If A is a unital associative ring, and $\mathsf{J} \subseteq \mathsf{A}$ is a two sided ideal which, considered as a ring, is also unital, then we have a ring isomorphism $\mathsf{A} \cong \mathsf{J} \oplus (\mathsf{A}/\mathsf{J})$. Indeed let $1 \in \mathsf{A}$ and $e \in \mathsf{J}$ be the units of these rings. For any $a \in \mathsf{A}$ we have $ae = eae = ea$, so e is a central idempotent. Thus $\mathsf{A} \cong e\mathsf{A} \oplus (1-e)\mathsf{A}$. Since $e\mathsf{A} = \mathsf{J}$ and $(1-e)\mathsf{A} \cong (\mathsf{A}/\mathsf{J})$, this proves the claim.

   Fix i, and suppose $\bar{\vartheta}_i$ induces an isomorphism. We may choose bases $\{v_1, \dots v_m\}$ for V_i and $\{w_1, \dots, w_m\}$ for W_i such that $\vartheta_i(w_a \otimes_{\mathsf{A}} v_b) = \delta_{ab}$. We have a ring isomorphism

$$\mathrm{Mat}_m(\mathsf{A}_i) \xrightarrow{\sim} \mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{W}_i$$

   given by

$$X \mapsto \sum_{a,b} v_a \otimes_{\mathsf{A}_i} X_{ab} w_b.$$

   We also have a ring isomorphism $\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{W}_i \cong \mathsf{J}_i/\mathsf{J}_{i-1}$, so $\mathsf{J}_i/\mathsf{J}_{i-1}$ is unital and Morita equivalent to A_i. Applying the above observation repeatedly, it follows that

$$\mathsf{A} \cong \bigoplus_{i=1}^{k} \mathsf{J}_i/\mathsf{J}_{i-1}$$

   is Morita equivalent to $\bigoplus_{i=1}^{k} \mathsf{A}_i$.

6. If the conditions hold, then A is semisimple by the previous statement.

   Conversely suppose A is semisimple. We must prove that A_i is semisimple and $\bar{\vartheta}_i$ is an isomorphism, for each i. By the above observation and induction, it suffices to prove this for $i = 1$.

Let J denote the Jacobson radical of A_1. Since A_1 is Artinian, J is nilpotent. For each $j \geq 1$,

$$\mathsf{V}_1 \otimes_{\mathsf{A}_1} \mathsf{J}^j \mathsf{W}_1$$

is a sub-(A, A)-bimodule of $\mathsf{V}_1 \otimes_{\mathsf{A}_1} \mathsf{W}_1$, so it corresponds to a two sided ideal of A. Moreover

$$\begin{aligned}(\mathsf{V}_1 \otimes_{\mathsf{A}_1} \mathsf{J}\mathsf{W}_1)(\mathsf{V}_1 \otimes_{\mathsf{A}_1} \mathsf{J}^j \mathsf{W}_1) &\subseteq \mathsf{V}_1 \otimes_{\mathsf{A}_1} \mathsf{J}\vartheta_1(\mathsf{W}_1 \otimes_{\mathsf{A}} \mathsf{V}_1)\mathsf{J}^j \mathsf{W}_1 \\ &\subseteq \mathsf{V}_1 \otimes_{\mathsf{A}_1} \mathsf{J}^{j+1} \mathsf{W}_1.\end{aligned}$$

Thus $\mathsf{V}_1 \otimes_{\mathsf{A}_1} \mathsf{J}\mathsf{W}_1$ corresponds to a nilpotent ideal. Since A is semisimple, $\mathsf{V}_1 \otimes_{\mathsf{A}_1} \mathsf{J}\mathsf{W}_1 = 0$. But V_1 and W_1 are nonzero and free over A_1, so $\mathsf{J} = 0$. That is, A_1 is semisimple.

By symmetry, we may suppose the rank of V_1 is at least the rank of W_1. Let $\mathsf{K} \subseteq \mathsf{V}_1$ be the kernel of $\bar{\vartheta}_1 : \mathsf{V}_1 \to \mathsf{W}_1^*$. This is a sub-$(\mathsf{A}, \mathsf{A}_1)$-bimodule of V_1, so $\mathsf{K} \otimes_{\mathsf{A}_1} \mathsf{W}_1$ corresponds to a two sided ideal in A. Moreover

$$(\mathsf{K} \otimes_{\mathsf{A}_1} \mathsf{W}_1)(\mathsf{K} \otimes_{\mathsf{A}_1} \mathsf{W}_1) \subseteq \mathsf{K} \otimes_{\mathsf{A}_1} \vartheta_1(\mathsf{W}_1 \otimes_{\mathsf{A}} \mathsf{K})\mathsf{W}_1 = 0.$$

Thus $\mathsf{K} \otimes_{\mathsf{A}_1} \mathsf{W}_1 = 0$. Again since W_1 is nonzero and free over A_1, it follows that $\mathsf{K} = 0$. On the other hand the length of V_i is at least the length of W_i^* as right A_i-modules. Therefore $\bar{\vartheta}_i$ is an isomorphism.

□

It is easier to determine whether the above conditions hold using more explicit descriptions. Suppose the assumptions of the theorem hold, and fix i. Moreover suppose A_i is an algebra over a field K. Let $\{v_a\}$ and $\{w_b\}$ be A_i-bases of V_i and W_i, and let $\{w_b^*\}$ denote the corresponding dual basis of W_i^*. Let $x_a^b = \vartheta_i(v_a \otimes w_b) \in \mathsf{A}_i$. Let L be an irreducible A_i module with K-basis $\{\ell_c\}$. Then $\{v_a \otimes \ell_c\}$ and $\{w_b^* \otimes \ell_d\}$ are K-bases for $\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{L}$ and $\mathsf{W}_i^* \otimes_{\mathsf{A}_i} \mathsf{L}$, and with respect to these bases the map $\bar{\vartheta}_i \otimes_{\mathsf{A}_i} \mathsf{id}_{\mathsf{L}}$ corresponds to the matrix $[\gamma_{ac}^{bd}]$, where

$$x_a^b v_c = \sum_d \gamma_{ac}^{bd} \ell_d.$$

Now determining $\mathsf{rad}\,(\mathsf{V}_i \otimes_{\mathsf{A}_i} L)$ corresponds to finding the kernel of this matrix over K.

Note that with respect to the bases $\{v_a\}$ and $\{w_b^*\}$, the map $\bar{\vartheta}_i$ corresponds to the matrix $[x_a^b]$ over A_i. Therefore $\bar{\vartheta}_i$ is an isomorphism if and only if this matrix is invertible. Let J be the Jacobson radical of A_i, and let $\bar{x}_a^b$ be the image of x_a^b in A_i/J. Since J is nilpotent, $\bar{\vartheta}_i$ is an isomorphism if and only if the matrix $[\bar{x}_a^b]$ is invertible over A_i/J. Finally if $\mathcal{L}$ is a complete set of irreducible A_i-modules, then the map

$$\mathsf{A}_i/\mathsf{J} \to \bigoplus_{\mathsf{L} \in \mathcal{L}} \mathsf{End}_{\mathsf{K}}(\mathsf{L})$$

is injective. Therefore $\bar{\vartheta}_i$ is an isomorphism if and only if the matrix $[\gamma_{ac}^{bd}]$ constructed above is invertible for each irreducible A_i-module L.

In [KoXi2], a procedure is described for constructing cellular algebras from other cellular algebras via "inflation". The following theorem is the analogue of this procedure for sandwich filtration algebras. It follows immediately from the definition.

Theorem 1.4. Suppose a ring A has a sandwich filtration over $\mathsf{A}_1, \ldots \mathsf{A}_k$ with factors $\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{W}_i$. Moreover suppose each A_i has a sandwich filtration over $\mathsf{A}_{i1}, \mathsf{A}_{i2}, \ldots, \mathsf{A}_{ik_i}$ with factors $\mathsf{V}_{ij} \otimes_{\mathsf{A}_{ij}} \mathsf{W}_{ij}$. Then A has a sandwich filtration over the A_{ij} (ordered lexicographically) with factors $(\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{V}_{ij}) \otimes_{\mathsf{A}_{ij}} (\mathsf{W}_{ij} \otimes_{\mathsf{A}_i} \mathsf{W}_i)$. The sandwich bilinear form

$$\vartheta'_{ij} : (\mathsf{W}_{ij} \otimes_{\mathsf{A}_i} \mathsf{W}_i) \otimes_A (\mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{V}_{ij}) \to \mathsf{A}_{ij}$$

is simply the composite

$$\begin{tikzcd}
\mathsf{W}_{ij} \otimes_{\mathsf{A}_i} \mathsf{W}_i \otimes_A \mathsf{V}_i \otimes_{\mathsf{A}_i} \mathsf{V}_{ij} \arrow[d, "\mathsf{id}_{\mathsf{W}_{ij}} \otimes_{\mathsf{A}_i} \vartheta_i \otimes_{\mathsf{A}_i} \mathsf{id}_{\mathsf{V}_{ij}}"] \\
\mathsf{W}_{ij} \otimes_{\mathsf{A}_i} \mathsf{A}_i \otimes_{\mathsf{A}_i} \mathsf{V}_{ij} \arrow[d, "\wr"] \\
\mathsf{W}_{ij} \otimes_{\mathsf{A}_i} \mathsf{V}_{ij} \arrow[d, "\vartheta_{ij}"] \\
\mathsf{A}_{ij}.
\end{tikzcd}$$

2 q-walled Brauer-Clifford superalgebras

In this second section, we show that the q-walled Brauer-Clifford superalgebras studied in [BGJKW] are sandwich filtration algebras. They are built from two finite Hecke-Clifford algebras, whose definition we recall below, and one additional generator.

Definition 2.1. [Ol] The finite Hecke-Clifford algebra $\mathsf{HC}_\ell(q)$ is the unital associative $\mathbb{C}(q)$-algebra generated by elements $\mathsf{t}_1, \ldots, \mathsf{t}_{\ell-1}$ and anti-commuting elements $\mathsf{c}_1, \ldots, \mathsf{c}_\ell$ which satisfy the relations:

$$(\mathsf{t}_i - q)(\mathsf{t}_i + q^{-1}) = 0, \ i = 1, \ldots, \ell-1, \qquad \mathsf{t}_i\mathsf{t}_{i+1}\mathsf{t}_i = \mathsf{t}_{i+1}\mathsf{t}_i\mathsf{t}_{i+1}, \ i = 1, \ldots, \ell-2;$$

$$\mathsf{t}_i\mathsf{t}_j = \mathsf{t}_j\mathsf{t}_i \ \text{ if } \ j \neq i-1, i+1, \qquad \mathsf{c}_i^2 = -1, \ \ \mathsf{c}_i\mathsf{c}_j = -\mathsf{c}_j\mathsf{c}_i \ \text{ if } \ 1 \leq i \neq j \leq \ell;$$

$$\mathsf{t}_i\mathsf{c}_i = \mathsf{c}_{i+1}\mathsf{t}_i, \qquad \mathsf{t}_i\mathsf{c}_j = \mathsf{c}_j\mathsf{t}_i \ \text{ if } \ j \neq i, i+1.$$

$\mathsf{HC}_\ell(q)$ becomes a $\mathbb{Z}_2$-graded algebra if we declare t_i to be even and c_j to be odd for all $1 \leq i \leq \ell-1, \ 1 \leq j \leq \ell$.

The quickest way to define the q-walled Brauer-Clifford algebra is as a centralizer algebra, but we first provide a concrete definition in terms of generators and relations and then recall the equivalence of the two definitions proved in [BGJKW].

Definition 2.2. The q-walled Brauer-Clifford algebra $\mathsf{BC}_{r,s}(q)$ is a unital associative $\mathbb{C}(q)$-algebra generated by the elements $\mathsf{t}_1, \mathsf{t}_2, \ldots, \mathsf{t}_{r-1}, \mathsf{c}_1, \mathsf{c}_2, \ldots, \mathsf{c}_r, \mathsf{t}_1^*, \mathsf{t}_2^*, \ldots, \mathsf{t}_{s-1}^*, \mathsf{c}_1^*, \mathsf{c}_2^*, \ldots, \mathsf{c}_s^*$, and e. The elements $\mathsf{t}_1, \ldots, \mathsf{t}_{r-1}, \mathsf{c}_1, \ldots, \mathsf{c}_r$ satisfy the relations of the Hecke-Clifford algebra $\mathsf{HC}_r(q)$, while $\mathsf{t}_1^*, \ldots, \mathsf{t}_{s-1}^*, \mathsf{c}_1^*, \ldots, \mathsf{c}_s^*$ satisfy those of $\mathsf{HC}_s(q)$ except that $(\mathsf{c}_i^*)^2 = 1$. Moreover, $\mathsf{t}_1, \ldots, \mathsf{t}_{r-1}, \mathsf{c}_1, \ldots, \mathsf{c}_r$ supercommute with $\mathsf{t}_1^*, \ldots, \mathsf{t}_{s-1}^*, \mathsf{c}_1^*, \ldots, \mathsf{c}_s^*$. The generator e commutes with

$$\mathsf{t}_1, \ldots, \mathsf{t}_{r-2}, \mathsf{c}_1, \ldots, \mathsf{c}_{r-1}, \ \mathsf{t}_2^*, \ldots, \mathsf{t}_{s-1}^*, \mathsf{c}_2^*, \ldots, \mathsf{c}_s^*$$

and satisfies

$$\mathsf{e}^2 = 0,\ \mathsf{e}\mathsf{t}_{r-1}\mathsf{e} = \mathsf{e} = \mathsf{e}\mathsf{t}_1^*\mathsf{e},\ \mathsf{c}_r\mathsf{e} = \mathsf{c}_1^*\mathsf{e} \ \text{ and } \mathsf{e}\mathsf{c}_r = \mathsf{e}\mathsf{c}_1^*,$$
$$\mathsf{e}\mathsf{c}_r\mathsf{e} = 0,\ \mathsf{e}\mathsf{t}_{r-1}^{-1}\mathsf{t}_1^*\mathsf{e}\mathsf{t}_{r-1} = \mathsf{e}\mathsf{t}_{r-1}^{-1}\mathsf{t}_1^*\mathsf{e}\mathsf{t}_1^*,\ \mathsf{t}_{r-1}\mathsf{e}\mathsf{t}_{r-1}^{-1}\mathsf{t}_1^*\mathsf{e} = \mathsf{t}_1^*\mathsf{e}\mathsf{t}_{r-1}^{-1}\mathsf{t}_1^*\mathsf{e}.$$

$\mathsf{BC}_{r,s}(q)$ becomes a $\mathbb{Z}_2$-graded algebra if we declare $\mathsf{t}_i, \mathsf{t}_i^*, \mathsf{e}$ to be even and $\mathsf{c}_j, \mathsf{c}_j^*$ to be odd for all $1 \le i \le \ell - 1,\ 1 \le j \le \ell$.

Remark 2.3. The walled Brauer superalgebra $\mathsf{BC}_{r,s}$ in [JuKa] is obtained from $\mathsf{BC}_{r,s}(q)$ by setting $q = 1$. In [JuKa, Thm. 5.1], e is denoted by $e_{r,r+1}$, t_i by s_i, and the two five-term equations above are replaced by a single six-term relation, which can be shown to be equivalent to the two relations above.

The algebra $\mathsf{BC}_{r,s}(q)$ has a natural anti-involution defined on the generators by

$$\mathsf{t}_i \mapsto -\mathsf{t}_i^{-1}, \quad \mathsf{t}_i^* \mapsto -(\mathsf{t}_i^*)^{-1}, \quad \mathsf{c}_i \mapsto \mathsf{c}_i, \quad \mathsf{c}_i^* \mapsto \mathsf{c}_i^*, \quad \mathsf{e} \mapsto \mathsf{e}.$$

This induces an anti-involution on $\mathsf{HC}_\ell(q)$ which also sends t_i to $-\mathsf{t}_i^{-1}$ and c_i to c_i. Unfortunately the centre of $\mathsf{HC}_\ell(q)$ (ignoring the $\mathbb{Z}_2$ grading) is not fixed by this anti-involution. Indeed, the specialisation of $\mathsf{HC}_\ell(q)$ at $q = 1$ is the Sergeev algebra. By Lemma 5 of [Se], there are odd elements in the centre of the Sergeev algebra in the -1-eigenspace of the anti-involution. Also Lemma 6 of [Se] and Theorem 6.7 of [JoNa] show that the dimension of the centre doesn't change upon specialisation, so the same is true of $\mathsf{HC}_\ell(q)$. It follows that $\mathsf{HC}_\ell(q)$ cannot be cellular with respect to this anti-involution. It may be possible to find a different anti-involution on $\mathsf{BC}_{r,s}(q)$, or to extend the notion of cellularity to super algebras in a way which accounts for this behaviour, but we have not pursued this direction.

We will need the following theorem, which is one of the main results in [BGJKW]. Set $\mathbf{V} = \mathbb{C}(n|n)$, $\mathbf{V}^{r,s} = (\mathbf{V})^{\otimes r} \otimes (\mathbf{V}^*)^{\otimes s}$, $\mathbf{V}_q = \mathbb{C}(q) \otimes_{\mathbb{C}} \mathbf{V}$ and $\mathbf{V}_q^{r,s} = (\mathbf{V}_q)^{\otimes r} \otimes (\mathbf{V}_q^*)^{\otimes s}$.

Theorem 2.4. There is an action of $\mathsf{BC}_{r,s}(q)$ on $\mathbf{V}_q^{r,s}$ with commutes with the action of the quantized enveloping superalgebra $\mathfrak{U}_q\mathfrak{q}_n$ of type Q [Ol]. Let $\rho_{n,q} : \mathsf{BC}_{r,s}(q) \to \mathsf{End}_{\mathfrak{U}_q\mathfrak{q}_n}(\mathbf{V}_q^{r,s})$ denote the homomorphism coming from this representation. Then $\rho_{n,q}$ is surjective, and when $n \ge r + s$, it is an isomorphism.

q-walled Brauer-Clifford algebras thus fit into the old topic of centralizer algebras. Walled Brauer algebras were studied in [BCHLLS] and their q-analogues in [KoMu]; a version of Theorem 2.4 holds for them if $\mathfrak{U}_q\mathfrak{q}_n$ is replaced by the quantized enveloping algebra of $\mathfrak{gl}_n$.

In the proof of the main theorem of this section, we will work over the local ring $\mathbb{C}[q, q^{-1}]_{(q-1)}$ and we will need the next lemma.

Lemma 2.5. Suppose R is a Noetherian local integral domain whose maximal ideal is generated by a single element $x \in \mathsf{R}$. Let $\psi : \mathsf{A} \to \mathsf{B}$ be a homomorphism of finitely generated R-modules, and consider the corresponding induced homomorphism

$$\overline{\psi} : \mathsf{A}/x\mathsf{A} \to \mathsf{B}/x\mathsf{B}, \qquad \overline{\psi}(a + x\mathsf{A}) = \psi(a) + x\mathsf{B}.$$

1. If $\overline{\psi}$ is surjective, then ψ is surjective.

2. If B is torsion free and $\overline{\psi}$ is injective, then ψ is injective, and its cokernel is also torsion free.

We now show that $\mathsf{BC}_{r,s}(q)$ has a sandwich filtration over the Hecke-Clifford algebras $\mathsf{HC}_\ell(q)$. To allow specialisation at $q=1$, it is convenient to do this over the local ring $\mathcal{R} = \mathbb{C}[q,q^{-1}]_{(q-1)}$ rather than $\mathbb{C}(q)$. Denote by $\mathsf{BC}_{r,s}(\mathcal{R})$ the $\mathcal{R}$-subalgebra of $\mathsf{BC}_{r,s}(q)$ generated by the generators in Definition 2.2. Set $\mathbf{V}_\mathcal{R} = \mathcal{R} \otimes_\mathbb{C} \mathbb{C}(n|n)$ and $\mathbf{V}_\mathcal{R}^{r,s} = (\mathbf{V}_\mathcal{R})^{\otimes r} \otimes (\mathbf{V}_\mathcal{R}^*)^{\otimes s}$. We adopt the convention that the generator e is excluded if $r=0$ or $s=0$; thus $\mathbb{C}(q) \otimes_\mathcal{R} \mathsf{BC}_{r,0}(\mathcal{R})$ is exactly the Hecke-Clifford algebra $\mathsf{HC}_r(q)$.

Theorem 2.6. The algebra $\mathsf{BC}_{r,s}(\mathcal{R})$ has a sandwich filtration over the algebras $\mathsf{A}_i(\mathcal{R})$ given by $\mathsf{A}_i(\mathcal{R}) = \mathsf{BC}_{r-i,0}(\mathcal{R}) \otimes_\mathcal{R} \mathsf{BC}_{0,s-i}(\mathcal{R})$ for $0 \le i \le \min(r,s)$.

Proof. Fix $n \ge r+s$. In the proof of Theorem 2.4 above provided in [BGJKW], we introduced a subspace $\mathsf{End}_{\tilde{U}}(\mathbf{V}_\mathcal{R}^{r,s})$ of $\mathsf{End}_\mathcal{R}(\mathbf{V}_\mathcal{R}^{r,s})$ whose tensor products with $\mathcal{R}/(q-1)\mathcal{R}$ and $\mathbb{C}(q)$ are, respectively, $\mathsf{End}_{\mathfrak{q}_n}(\mathbf{V}^{r,s})$ and $\mathsf{End}_{\mathfrak{U}_q\mathfrak{q}_n}(\mathbf{V}_q^{r,s})$. Similarly, we may construct a space

$$\mathsf{Hom}_{\tilde{U}}(\mathbf{V}_\mathcal{R}^{a,b}, \mathbf{V}_\mathcal{R}^{c,d}) \subseteq \mathsf{Hom}_\mathcal{R}(\mathbf{V}_\mathcal{R}^{a,b}, \mathbf{V}_\mathcal{R}^{c,d})$$

for any nonnegative integers a, b, c and d. For convenience, we use the following notation throughout this proof.

$$\begin{aligned} \mathsf{N}_\mathcal{R}(i,j) &= \mathsf{Hom}_{\tilde{U}}(\mathbf{V}_\mathcal{R}^{r-j,s-j}, \mathbf{V}_\mathcal{R}^{r-i,s-i}), \\ \mathsf{N}_\mathbb{C}(i,j) &= \mathsf{Hom}_{\mathfrak{q}_n}(\mathbf{V}^{r-j,s-j}, \mathbf{V}^{r-i,s-i}). \end{aligned}$$

Thus $\mathcal{R}/(q-1) \otimes_\mathcal{R} \mathsf{N}_\mathcal{R}(i,j) \cong \mathsf{N}_\mathbb{C}(i,j)$. We have composition maps

$$\mathsf{N}_\mathbb{C}(l,j) \otimes_\mathbb{C} \mathsf{N}_\mathbb{C}(j,i) \to \mathsf{N}_\mathbb{C}(l,i).$$

In particular, $\mathsf{N}_\mathbb{C}(i,i)$ is an algebra and $\mathsf{N}_\mathbb{C}(i,j)$ is an $(\mathsf{N}_\mathbb{C}(i,i), \mathsf{N}_\mathbb{C}(j,j))$-bimodule. Similar statements hold for $\mathsf{N}_\mathcal{R}(i,j)$. By Theorem 3.5 of [JuKa], we have $\mathsf{BC}_{r,s} \cong \mathsf{N}_\mathbb{C}(0,0)$.

The composition map

$$\mathsf{N}_\mathcal{R}(0,i) \otimes_\mathcal{R} \mathsf{N}_\mathcal{R}(i,0) \to \mathsf{N}_\mathcal{R}(0,0)$$

factors through $\mathsf{N}_\mathcal{R}(0,i) \otimes_{\mathsf{N}_\mathcal{R}(i,i)} \mathsf{N}_\mathcal{R}(i,0)$. It is also an $(\mathsf{N}_\mathcal{R}(0,0), \mathsf{N}_\mathcal{R}(0,0))$-bimodule homomorphism, so its image $\mathsf{J}_i(\mathcal{R})$ is a two-sided ideal in $\mathsf{N}_\mathcal{R}(0,0)$. Now define the $(\mathsf{N}_\mathcal{R}(0,0), \mathsf{N}_\mathcal{R}(i,i))$-bimodule $\mathsf{V}_i(\mathcal{R})$ and the $(\mathsf{N}_\mathcal{R}(i,i), \mathsf{N}_\mathcal{R}(0,0))$-bimodule $\mathsf{W}_i(\mathcal{R})$ by the following right exact sequences:

$$\begin{aligned} \mathsf{N}_\mathcal{R}(0,i+1) \otimes_\mathcal{R} \mathsf{N}_\mathcal{R}(i+1,i) &\to \mathsf{N}_\mathcal{R}(0,i) \twoheadrightarrow \mathsf{V}_i(\mathcal{R}), \\ \mathsf{N}_\mathcal{R}(i,i+1) \otimes_\mathcal{R} \mathsf{N}_\mathcal{R}(i+1,0) &\to \mathsf{N}_\mathcal{R}(i,0) \twoheadrightarrow \mathsf{W}_i(\mathcal{R}). \end{aligned} \tag{2.7}$$

Here we adopt the convention that $\mathsf{N}_\mathcal{R}(i,j) = 0$ if i or j is greater than r or s. Clearly the images of

$$\mathsf{N}_\mathcal{R}(0,i+1) \otimes_\mathcal{R} \mathsf{N}_\mathcal{R}(i+1,i) \otimes_\mathcal{R} \mathsf{N}_\mathcal{R}(i,0) \quad \text{and} \quad \mathsf{N}_\mathcal{R}(0,i) \otimes_\mathcal{R} \mathsf{N}_\mathcal{R}(i,i+1) \otimes_\mathcal{R} \mathsf{N}_\mathcal{R}(i+1,0)$$

in $\mathsf{N}_\mathcal{R}(0,0)$ are contained in $\mathsf{J}_{i+1}(\mathcal{R})$, so the surjective map

$$\mathsf{N}_\mathcal{R}(0,i) \otimes_{\mathsf{N}_\mathcal{R}(i,i)} \mathsf{N}_\mathcal{R}(i,0) \twoheadrightarrow \mathsf{J}_i(\mathcal{R})$$

factors to give a map

$$\mathsf{V}_i(\mathcal{R}) \otimes_{\mathsf{N}_\mathcal{R}(i,i)} \mathsf{W}_i(\mathcal{R}) \twoheadrightarrow \mathsf{J}_i(\mathcal{R})/(\mathsf{J}_i(\mathcal{R}) \cap \mathsf{J}_{i+1}(\mathcal{R})).$$

Finally let $\mathsf{A}_i(\mathcal{R}) = \mathsf{BC}_{r-i,0}(\mathcal{R}) \otimes_{\mathcal{R}} \mathsf{BC}_{0,s-i}(\mathcal{R})$. We have a natural inclusion $\mathsf{A}_i(\mathcal{R}) \hookrightarrow \mathsf{N}_{\mathcal{R}}(i,i)$, so we obtain a surjective homomorphism

$$\pi_i : \mathsf{V}_i(\mathcal{R}) \otimes_{\mathsf{A}_i(\mathcal{R})} \mathsf{W}_i(\mathcal{R}) \twoheadrightarrow \mathsf{J}_i(\mathcal{R})/(\mathsf{J}_i(\mathcal{R}) \cap \mathsf{J}_{i+1}(\mathcal{R}))$$

of $(\mathsf{N}_{\mathcal{R}}(0,0), \mathsf{N}_{\mathcal{R}}(0,0))$-bimodules. We will prove that the various objects we have constructed satisfy the following properties:

1. The $\mathsf{J}_i(\mathcal{R})$ form a filtration of $\mathsf{N}_{\mathcal{R}}(0,0)$; that is, $\mathsf{J}_{i+1}(\mathcal{R}) \subseteq \mathsf{J}_i(\mathcal{R})$.
2. Both $\mathsf{V}_i(\mathcal{R})$ and $\mathsf{W}_i(\mathcal{R})$ are free over $\mathsf{A}_i(\mathcal{R})$ (as right and left modules respectively).
3. The map π_i is an isomorphism.

All three claims will follow from the corresponding claims over $\mathcal{R}/(q-1)\mathcal{R} \cong \mathbb{C}$, which we will in turn deduce from explicit diagrammatic bases of these spaces.

Theorem 3.5 and Proposition 4.3 of [JuKa] show that $\mathsf{N}_{\mathbb{C}}(0,0)$ has a basis indexed by (r,s)-diagrams, and multiplication of these basis elements corresponds to concatenation of the diagrams. By the same argument, $\mathsf{N}_{\mathbb{C}}(i,j)$ has a basis indexed by diagrams from $(r-i)+(s-i)$ dots to $(r-j)+(s-j)$ dots, and the above composition maps correspond to concatenation of diagrams. If $1 \le i \le r, s$ then any diagram in $\mathsf{N}_{\mathbb{C}}(i,0)$ has at least one horizontal edge. By moving part of this edge to the bottom of the diagram, we may express the diagram as a concatenation of a diagram in $\mathsf{N}_{\mathbb{C}}(i,i-1)$ and one in $\mathsf{N}_{\mathbb{C}}(i-1,0)$. Thus,

$$\mathsf{N}_{\mathbb{C}}(i,i-1) \otimes_{\mathbb{C}} \mathsf{N}_{\mathbb{C}}(i-1,0) \to \mathsf{N}_{\mathbb{C}}(i,0)$$

is surjective. By Lemma 2.5,

$$\mathsf{N}_{\mathcal{R}}(i,i-1) \otimes_{\mathcal{R}} \mathsf{N}_{\mathcal{R}}(i-1,0) \to \mathsf{N}_{\mathcal{R}}(i,0)$$

is surjective. Now the commutative diagram

$$\begin{tikzcd}
\mathsf{N}_{\mathcal{R}}(0,i) \otimes_{\mathcal{R}} \mathsf{N}_{\mathcal{R}}(i,i-1) \otimes_{\mathcal{R}} \mathsf{N}_{\mathcal{R}}(i-1,0) \arrow[r, two heads] \arrow[d] & \mathsf{N}_{\mathcal{R}}(0,i) \otimes_{\mathcal{R}} \mathsf{N}_{\mathcal{R}}(i,0) \arrow[d] \\
\mathsf{N}_{\mathcal{R}}(0,i-1) \otimes_{\mathcal{R}} \mathsf{N}_{\mathcal{R}}(i-1,0) \arrow[r] & \mathsf{N}_{\mathcal{R}}(0,0)
\end{tikzcd}$$

shows that $\mathsf{J}_i(\mathcal{R}) \subseteq \mathsf{J}_{i-1}(\mathcal{R})$.

Recall that $\mathsf{V}_i(\mathcal{R})$ and $\mathsf{W}_i(\mathcal{R})$ were defined by the right exact sequences in (2.7). Tensoring with $\mathcal{R}/(q-1)\mathcal{R}$, we obtain right exact sequences

$$\mathsf{N}_{\mathbb{C}}(0,i+1) \otimes_{\mathbb{C}} \mathsf{N}_{\mathbb{C}}(i+1,i) \to \mathsf{N}_{\mathbb{C}}(0,i) \twoheadrightarrow \mathsf{V}_i(\mathbb{C}) = \mathcal{R}/(q-1)\mathcal{R} \otimes_{\mathcal{R}} \mathsf{V}_i(\mathcal{R}),$$
$$\mathsf{N}_{\mathbb{C}}(i,i+1) \otimes_{\mathbb{C}} \mathsf{N}_{\mathbb{C}}(i+1,0) \to \mathsf{N}_{\mathbb{C}}(i,0) \twoheadrightarrow \mathsf{W}_i(\mathbb{C}) = \mathcal{R}/(q-1)\mathcal{R} \otimes_{\mathcal{R}} \mathsf{W}_i(\mathcal{R}).$$

A diagram in $\mathsf{N}_{\mathbb{C}}(0,i)$ has at least i horizontal edges. Moreover, it has more than i horizontal edges if and only if it can be expressed as a concatenation of a diagram in $\mathsf{N}_{\mathbb{C}}(0,i+1)$ and one in $\mathsf{N}_{\mathbb{C}}(i+1,i)$. Therefore the diagrams with exactly i horizontal edges map to a basis for $\mathsf{V}_i(\mathbb{C})$.

Let $\mathsf{A}_i(\mathbb{C}) = \mathcal{R}/(q-1)\mathcal{R} \otimes_{\mathcal{R}} \mathsf{A}_i(\mathcal{R}) = \mathsf{BC}_{r-i,0} \otimes_{\mathbb{C}} \mathsf{BC}_{0,s-i}$. We have bases for $\mathsf{BC}_{r-i,0}$ and $\mathsf{BC}_{0,s-i}$ indexed by diagrams on $(r-i)+0$ dots and $0+(s-i)$ dots respectively. The map

$$\mathsf{A}_i(\mathbb{C}) \to \mathsf{BC}_{r-i,s-i} = \mathsf{N}_{\mathbb{C}}(i,i)$$

sends a pair of diagrams to the diagram obtained by putting them next to each other, up to sign. Therefore its image is exactly the span of those diagrams with no horizontal edges.

Let X_i be the set of diagrams in $\mathsf{N}_{\mathbb{C}}(0,i)$ with exactly i horizontal edges, such that the vertical edges don't cross each other and are not marked. Any diagram in our basis for $\mathsf{V}_i(\mathbb{C})$ is uniquely expressible as a concatenation of a diagram in X_i with a diagram without horizontal edges. That is, concatenation gives an isomorphism

$$\mathsf{span}_{\mathbb{C}}(\mathsf{X}_i) \otimes_{\mathbb{C}} \mathsf{A}_i(\mathbb{C}) \to \mathsf{V}_i(\mathbb{C}).$$

Thus, $\mathsf{V}_i(\mathbb{C})$ is free over $\mathsf{A}_i(\mathbb{C})$. Similarly, $\mathsf{W}_i(\mathbb{C})$ is freely generated over $\mathsf{A}_i(\mathbb{C})$ by X_i^*, the set of diagrams in X_i reflected about the horizontal middle axis.

Let $\mathsf{J}_i(\mathbb{C})$ denote the image of $\mathsf{J}_i(\mathcal{R})$ in $\mathsf{N}_{\mathbb{C}}(0,0)$. (Note that we have not shown that $\mathcal{R}/(q-1)\mathcal{R} \otimes_{\mathcal{R}} J_i(\mathcal{R}) \to \mathsf{J}_i(\mathbb{C})$ is injective.) By definition, $\mathsf{J}_i(\mathbb{C})$ is the image of the composition map

$$\mathsf{N}_{\mathbb{C}}(0,i) \otimes_{\mathbb{C}} \mathsf{N}_{\mathbb{C}}(i,0) \to \mathsf{N}_{\mathbb{C}}(0,0).$$

Thus $\mathsf{J}_i(\mathbb{C})$ is exactly the span of those diagrams with at least i horizontal edges. The diagrams with exactly i horizontal edges map to a basis for $\mathsf{J}_i(\mathbb{C})/\mathsf{J}_{i+1}(\mathbb{C})$. Each such diagram is uniquely expressible as a concatenation of a diagram in X_i, a diagram in $\mathsf{N}_{\mathbb{C}}(i,i)$ without horizontal edges, and a diagram in X_i^*. Therefore, the map

$$\mathsf{V}_i(\mathbb{C}) \otimes_{\mathsf{A}_i(\mathbb{C})} \mathsf{W}_i(\mathbb{C}) \to \mathsf{J}_i(\mathbb{C})/\mathsf{J}_{i+1}(\mathbb{C})$$

is an isomorphism.

We will prove by descending induction on i that $\mathsf{N}_{\mathcal{R}}(0,0)/\mathsf{J}_i(\mathcal{R})$ is torsion free over $\mathcal{R}$. Since $\mathsf{J}_i(\mathcal{R}) = 0$ for $i > r, s$ and $\mathsf{N}_{\mathcal{R}}(0,0)$ is a free $\mathcal{R}$-module, the base case is trivial. Suppose $i \le r, s$ and that $\mathsf{N}_{\mathcal{R}}(0,0)/J_{i+1}(\mathcal{R})$ is torsion free. The map

$$\eta : \mathsf{V}_i(\mathcal{R}) \otimes_{\mathsf{A}_i(\mathcal{R})} \mathsf{W}_i(\mathcal{R}) \to \mathsf{N}_{\mathcal{R}}(0,0)/\mathsf{J}_{i+1}(\mathcal{R})$$

becomes injective when tensored with $\mathcal{R}/(q-1)\mathcal{R}$, so Lemma 2.5 shows that η is injective and its cokernel is torsion free. However, the map

$$\pi_i : \mathsf{V}_i(\mathcal{R}) \otimes_{\mathsf{A}_i(\mathcal{R})} \mathsf{W}_i(\mathcal{R}) \twoheadrightarrow \mathsf{J}_i(\mathcal{R})/\mathsf{J}_{i+1}(\mathcal{R})$$

is surjective by its construction, so the cokernel of η is exactly $\mathsf{N}_{\mathcal{R}}(0,0)/\mathsf{J}_i(\mathcal{R})$. This completes the induction.

Moreover, since η is injective, π_i is an isomorphism. It remains to show that $\mathsf{V}_i(\mathcal{R})$ and $\mathsf{W}_i(\mathcal{R})$ are free over $\mathsf{A}_i(\mathcal{R})$. We have an $\mathsf{A}_i(\mathbb{C})$-basis X_i for $\mathsf{V}_i(\mathbb{C})$. Let $\bar{\mathsf{V}}_i(\mathcal{R})$ be the free right $\mathsf{A}_i(\mathcal{R})$-module generated by X_i. By lifting X_i arbitrarily to $\mathsf{V}_i(\mathcal{R})$, we obtain a right $\mathsf{A}_i(\mathcal{R})$-module homomorphism $\bar{\mathsf{V}}_i(\mathcal{R}) \to \mathsf{V}_i(\mathcal{R})$ whose tensor product with $\mathcal{R}/(q-1)\mathcal{R}$ is an isomorphism. We construct $\bar{\mathsf{W}}_i(\mathcal{R}) \to \mathsf{W}_i(\mathcal{R})$ similarly. The resulting map

$$\bar{\mathsf{V}}_i(\mathcal{R}) \otimes_{\mathsf{A}_i(\mathcal{R})} \bar{\mathsf{W}}_i(\mathcal{R}) \to \mathsf{V}_i(\mathcal{R}) \otimes_{\mathsf{A}_i(\mathcal{R})} \mathsf{W}_i(\mathcal{R})$$

becomes an isomorphism when tensored with $\mathcal{R}/(q-1)\mathcal{R}$. We have shown above that $\mathsf{V}_i(\mathcal{R}) \otimes_{\mathsf{A}_i(\mathcal{R})} \mathsf{W}_i(\mathcal{R})$ is torsion free, so Lemma 2.5 shows that this map is itself an isomorphism. In particular,

$$\bar{\mathsf{V}}_i(\mathcal{R}) \otimes_{\mathsf{A}_i(\mathcal{R})} \bar{\mathsf{W}}_i \to \mathsf{V}_i(\mathcal{R}) \otimes_{\mathsf{A}_i(\mathcal{R})} \bar{\mathsf{W}}_i$$

is injective. Since $\bar{\mathsf{W}}_i$ is free over $\mathsf{A}_i(\mathcal{R})$, this implies that $\bar{\mathsf{V}}_i(\mathcal{R}) \to \mathsf{V}_i(\mathcal{R})$ is injective. It is also surjective by Lemma 2.5, so it is an isomorphism. Thus, $\mathsf{V}_i(\mathcal{R})$ is free over $\mathsf{A}_i(\mathcal{R})$, and the same argument applies to $\mathsf{W}_i(\mathcal{R})$. $\square$

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