Projective modules in the category $\mathcal{O}$ for the Cherednik algebra

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Abstract

We study projective objects in the category $\mathcal{O}_{\mathsf{H}_c}(0)$ of the Cherednik algebra introduced recently by Berest, Etingof and Ginzburg. We prove that it has enough projectives and that it is a highest weight category in the sense of Cline, Parshall and Scott, and therefore satisfies an analog of the BGG-reciprocity formula for a semisimple Lie algebra.

1 Introduction

In the representation theory of a semisimple Lie algebra $\mathfrak{g}$, a well studied category of modules is the category $\mathcal{O}_{\mathfrak{g}}$ of Bernstein,Gelfand, and Gelfand [3]. Generalizations of this category have been extensively investigated since the publication of their seminal paper. In this paper, we consider a category $\mathcal{O}_{\mathsf{H}_c}(0)$ of modules over the Cherednik algebra H_c, defined in [8] (see also [7]), which can be considered as an analog of the category $\mathcal{O}_{\mathfrak{g}}$.

The category $\mathcal{O}_{\mathsf{H}_c}$ is defined in the first section, where we recall also some of its basic properties and the definition of the standard modules introduced in [2] and [7]. These standard modules play the same role as the Verma modules in $\mathcal{O}_{\mathfrak{g}}$. A certain full subcategory $\mathcal{O}_{\mathsf{H}_c}(0)$ of $\mathcal{O}_{\mathsf{H}_c}$ is of particular interest and we are going to be mostly exclusively concerned with it. To each standard module in $\mathcal{O}_{\mathsf{H}_c}(0)$ corresponds a simple H_c-module, and these simple modules are in bijection with the irreducible representations of a given Weyl group W. The first main result we prove is the following theorem.

Theorem 1.1. *The category $\mathcal{O}_{\mathsf{H}_c}(0)$ has enough projective objects.*

In particular, to each simple module, we can associate its projective cover. Given an irreducible representation σ of W, we denote by $\Delta(\sigma)$ the corresponding standard module with its unique simple quotient $L(\sigma)$ whose projective indecomposable cover is $P(\sigma)$. We show that $P(\sigma)$ admits a finite filtration whose successive quotients are standard modules. We denote the multiplicity of a standard module $\Delta(\tau)$ in such a filtration by $[P(\sigma):\Delta(\tau)]$. Furthermore, since modules in $\mathcal{O}_{\mathsf{H}_c}(0)$ have finite length, we can define the multiplicity of a simple module $L(\sigma)$ in a module $M \in \mathcal{O}_{\mathsf{H}_c}(0)$, which we denote by $[M : L(\sigma)]$. Our second main result is that a BGG-type reciprocity formula [3] holds in $\mathcal{O}_{\mathsf{H}_c}(0)$.

Theorem 1.2. $[P(\sigma):\Delta(\tau)] = [\Delta(\tau):L(\sigma)]$.

In [5], the authors introduced a general notion of a highest weight category, which includes some classical examples from Lie theory, for instance $\mathcal{O}_{\mathfrak{g}}$ and certain categories of representations of reductive algebraic groups over fields of positive characteristic. In subsequent papers, they generalized to those categories many results about $\mathcal{O}_{\mathfrak{g}}$. In particular, our second main theorem is actually a corollary of the next result.

Theorem 1.3. *The category $\mathcal{O}_{\mathsf{H}_c}(0)$ is a highest weight category.*

In section 2, we recall some definitions and set up the notation. The main results are proved in sections 3 and 4. At the end of section 4, we point out how our second main result yields a necessary and sufficient condition for the semisimplicity of the category $\mathcal{O}_{\mathsf{H}_c}(0)$. In section 5, we introduce a decomposition of $\mathcal{O}_{\mathsf{H}_c}(0)$ into a direct sum of subcategories, which we call $\mathbb{Z}$-strings, and study their cohomological dimension.

Remark 1.1. *Similar results have been obtained independently by E. Opdam and R. Rouquier [9].*

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2 Category $\mathcal{O}_{\mathsf{H}_c}$ for the Cherednik algebra

The Cherednik algebra H_c is defined in the following way. Let $\mathfrak{h}$ be a complex vector space and W a finite Weyl group acting on $\mathfrak{h}$ with corresponding root system $R \subset \mathfrak{h}^*$. We denote by $\mathbb{C}[R]^W$ the vector space of W-invariant functions $c : R \to \mathbb{C}, \alpha \mapsto c_\alpha$. Given such a function c, we define the Cherednik algebra H_c (denoted $\mathsf{H}_{1,c}$ in [8]) as the algebra generated by $\mathfrak{h}$, $\mathfrak{h}^*$, and $\mathbb{C}[W]$ subject to the following relations:

$$w \cdot x \cdot w^{-1} = w(x) \ , \ \ w \cdot y \cdot w^{-1} = w(y) \, , \forall y \in \mathfrak{h} \, , \, x \in \mathfrak{h}^* \, , \, w \in W$$

$$[x_1, x_2] = 0 = [y_1, y_2] \, , \forall y_1, y_2 \in \mathfrak{h}, \ x_1 \, , \, x_2 \in \mathfrak{h}^*$$

$$[y, x] = \langle y, x \rangle - \sum_{\alpha \in R_+} c_\alpha \cdot \langle y, \alpha \rangle \langle \alpha^\vee, x \rangle \cdot s_\alpha \, , \forall y \in \mathfrak{h} \, , \, x \in \mathfrak{h}^*.$$

The main reason why H_c admits a theory of modules with certain similarities with the case of a semisimple Lie algebra is that an analog of the Poincaré-Birkhoff-Witt decomposition holds for H_c: the multiplication map induces a vector space isomorphism $\mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}} \mathbb{C}[W] \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{h}^*] \xrightarrow{\sim} \mathsf{H}_c$ [8].

A central object of study in the representation theory of H_c is the following category of modules.

Definition 2.1 ([2] 2.4). *Let $\mathcal{O}_{\mathsf{H}_c}$ denote the abelian category of finitely generated left H_c-modules M such that the action on M of the subalgebra $\mathbb{C}[\mathfrak{h}^*] \subset \mathsf{H}_c$ is locally finite, i.e.* $\dim_{\mathbb{C}} \mathbb{C}[\mathfrak{h}^*] \cdot m < \infty$ *for any $m \in M$.*

As in the Lie algebra case, the category $\mathcal{O}_{\mathsf{H}_c}$ affords a block decomposition. More precisely, it splits into the direct sum $\bigoplus_{\bar{\lambda}\in\mathfrak{h}^*/W}\mathcal{O}_{\mathsf{H}_c}(\bar{\lambda})$ where $\mathcal{O}_{\mathsf{H}_c}(\bar{\lambda})$ is the category of modules on which $P - P(\bar{\lambda})$ acts locally nilpotently for all $P \in \mathbb{C}[\mathfrak{h}^*]^W$.

This category contains analogs of the Verma modules of a semisimple Lie algebra. We call them standard modules and they are defined in the following way ([2],[7]). Fix $\lambda \in \mathfrak{h}^*$, and let W_λ be its stabilizer in W. We can form the smash product $\mathbb{C}[\mathfrak{h}^*]\#W_\lambda$ since W acts on $\mathbb{C}[\mathfrak{h}^*]$. Let τ be an irreducible representation of W_λ on a vector space $V(\tau)$, and let I_λ be the maximal ideal of $\mathbb{C}[\mathfrak{h}^*]$ generated by $y-\lambda(y)$ with $y \in \mathfrak{h}$; set $\lambda\#\tau = (\mathbb{C}[\mathfrak{h}^*]/I_\lambda)\otimes_{\mathbb{C}} V(\tau)$. Then $\lambda\#\tau$ is a module over $\mathbb{C}[\mathfrak{h}^*]\#W_\lambda$, W_λ acting diagonally and $\mathbb{C}[\mathfrak{h}^*]$ by left multiplication, which we can induce to H_c, and we set $\Delta(\lambda,\tau) = \mathrm{Ind}_{\mathbb{C}[\mathfrak{h}^*]\#W_\lambda}^{\mathsf{H}_c}(\lambda\#\tau)$, and $\Delta(\tau) = \Delta(0,\tau)$. These are the standard modules.

We recall here some basic properties of the category $\mathcal{O}_{\mathsf{H}_c}$.

Lemma 2.1 ([2] lemma 2.5). 1. *Any object $M \in \mathcal{O}_{\mathsf{H}_c}$ is finitely generated over the subalgebra $\mathbb{C}[\mathfrak{h}] \subset \mathsf{H}_c$, hence $\mathcal{O}_{\mathsf{H}_c}$ is an abelian category.*

2. *For any $\lambda \in \mathfrak{h}^*$ and $\tau \in \mathrm{Irrep}(W_\lambda)$, we have $\Delta(\lambda,\tau) \in \mathcal{O}_{\mathsf{H}_c}$.*

3. *For any $M \in \mathcal{O}_{\mathsf{H}_c}$, there exists a non-zero homomorphism $\Delta(\lambda,\tau) \to M$ for certain $\lambda \in \mathfrak{h}^*$ and $\tau \in \mathrm{Irrep}(W_\lambda)$.*

4. *Every object of the category $\mathcal{O}_{\mathsf{H}_c}(0)$ has finite length.*

In the remaining sections, we will be only concerned with the category $\mathcal{O}_{\mathsf{H}_c}(0)$, which appears to be the most interesting case among all the $\mathcal{O}_{\mathsf{H}_c}(\bar{\lambda})$. By a highest weight vector in a module $M \in \mathcal{O}_{\mathsf{H}_c}(0)$, we will mean a vector which is annihilated by any element of $\mathfrak{h}$. In our analysis of that category, we will use very often the canonical element $\mathbf{h} = -\frac{1}{2}\sum_{i=1}^n x_iy_i + y_ix_i$ which was introduced in [2]. Here, $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ are dual bases of $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively. Any other choice of dual bases gives the same element $\mathbf{h}$, so in particular $\mathbf{h}$ is W-invariant. In [2], the element $-\mathbf{h}$ was used; the reason for the minus sign is that it allows us to work with highest weight vectors and modules instead of lowest weight ones. This element $\mathbf{h}$ differs also from the Euler element $\mathbf{E} = \sum_{i=1}^n x_iy_i$ introduced in [7]. $\mathbf{h}+\mathbf{E} = -\frac{1}{2}\sum_{i=1}^n [y_i,x_i] \in \mathbb{C}[W]$ and $[\mathbf{h},\mathbf{E}] = [\mathbf{h},\mathbf{E}+\mathbf{h}] = 0$ since $\mathbf{h}$ commutes with elements of $\mathbb{C}[W]$.

The next lemma will be used very frequently.

Lemma 2.2 ([2] 2.6). *$[\mathbf{h},x] = -x, [\mathbf{h},y] = y$ for all $x \in \mathfrak{h}^*, y \in \mathfrak{h}$, that is, $\mathbf{h}x = x(\mathbf{h}-1), \mathbf{h}y = y(\mathbf{h}+1)$.*

Every module M in $\mathcal{O}_{\mathsf{H}_c}(0)$ splits into a direct sum of finite dimensional generalized weight spaces for the action of $\mathbf{h}$. This follows by induction from the fact that M has finite length and the action of $\mathbf{h}$ on a simple (or standard) module is diagonalizable, which is an easy consequence of the previous lemma. Indeed, the action of $\mathbf{h}$ on a standard module $\Delta(\sigma)$ can be described quite explicitly. Let $z(c)$ be the following element in the center of W: $z(c) = 2\sum_{\alpha\in R_+} c_\alpha s_\alpha$.

That $z(c)$ lies in the center of W follows from the invariance of the coefficients c_α. $z(c)$ acts on the irreducible representation σ of W by multiplication by a constant which we denote $k(\sigma, c)$. Moreover, the following relation holds: $\sum_{i=1}^n c_\alpha \langle y_i, \alpha\rangle\langle\alpha^\vee, x_i\rangle = c_\alpha\langle\sum_{i=1}^n y_i\langle x_i, \alpha^\vee\rangle, \alpha\rangle = c_\alpha\langle\alpha, \alpha^\vee\rangle = 2c_\alpha$, so that $\sum_{i=1}^n [y_i, x_i] = \sum_{i=1}^n \big(\langle y_i, x_i\rangle - \sum_{\alpha\in R_+} c_\alpha\langle y_i, \alpha\rangle\langle\alpha^\vee, x_i\rangle s_\alpha\big) = n - z(c)$.

Now, suppose that $v \in V(\sigma)$, and that $p \in \mathbb{C}[\mathfrak{h}]$ is homogeneous of degree m. We think of $p \otimes v$ as an element of the H_c-module $\Delta(\sigma)$, which isomorphic to $\mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}} V(\sigma)$ as a $\mathbb{C}[\mathfrak{h}]$-module. Elements of $1 \otimes V(\sigma)$ are highest weight vectors. $\mathbf{h}$ acts on $p \otimes v$ by multiplication by the constant $-m + \frac{1}{2}\big(k(\sigma, c) - n\big)$. Indeed, $[\mathbf{h}, x_{i_1}\cdots x_{i_m}] = \sum_{j=0}^{m-1} x_{i_1}\cdots x_{i_j}[\mathbf{h}, x_{i_{j+1}}]x_{i_{j+2}}\cdots x_{i_m} = -m x_{i_1}\cdots x_{i_m}$ for $x_{i_j} \in \mathfrak{h}^*$, so

$$\begin{aligned}
\mathbf{h}(p\otimes v) &= [\mathbf{h}, p]\otimes v + p\mathbf{h}\otimes v \;=\; -mp\otimes v - \frac{p}{2}\sum_{i=1}^n y_i x_i \otimes v \\
&= -mp\otimes v - \frac{p}{2}\sum_{i=1}^n [y_i, x_i]\otimes v \;=\; -(m+\frac{n}{2})p\otimes v + \frac{1}{2}p\otimes z(c)v \\
&= \big(-m - \frac{n}{2} + \frac{1}{2}k(\sigma, c)\big)p\otimes v
\end{aligned}$$

This shows that each standard module acquires a grading from the action of $\mathbf{h}$, which is simply the opposite of the grading coming from the order grading on $\mathbb{C}[\mathfrak{h}]$ shifted by the constant $\frac{1}{2}(k(\sigma, c) - n)$. Every quotient and submodule of a standard module is similarly graded, but this is not necessarily true for extensions of those modules.

3 Projective objects

In this section, we prove our first main theorem.

Theorem 3.1. *The category $\mathcal{O}_{\mathsf{H}_c}(0)$ has enough projective objects.*

However, before we proceed with the proof, we need to introduce a duality on the category $\mathcal{O}_{\mathsf{H}_c}(0)$. To define it, we first have to put an H_c-module structure on the dual $M^* = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ of a module $M \in \mathcal{O}_{\mathsf{H}_c}(0)$. To achieve this, choose a non-degenerate W-invariant symmetric bilinear form $(\ ,\ )$ on $\mathfrak{h}^*$, and let $\theta : \mathfrak{h} \to \mathfrak{h}^*$ be the (W-equivariant) isomorphism such that $\langle y, x\rangle = x(y) = \big(x, \theta(y)\big)$ for $x \in \mathfrak{h}^*, y \in \mathfrak{h}$. Then $\theta(\alpha^\vee) = \frac{2}{(\alpha,\alpha)}\alpha$.

We define ζ on the generators of H_c by $\zeta(x) = \theta^{-1}(x), \zeta(y) = \theta(y), \zeta(w) = w^{-1}$. ζ can be extended to an anti-involution of the free algebra $\mathcal{A}$ on generators $x \in \mathfrak{h}^*, y \in \mathfrak{h}, w \in W$, hence to an algebra anti-homomorphism $\mathcal{A} \to \mathsf{H}_c$. To prove that ζ indeed defines an anti-involution of H_c, we have to show that if $x \in \mathfrak{h}^*, y \in \mathfrak{h}$, then $\zeta\big([y, x] - \langle y, x\rangle + \sum_{\alpha\in R_+} c_\alpha\langle y, \alpha\rangle\langle\alpha^\vee, x\rangle s_\alpha\big) = 0$ and $\zeta\big(wxw^{-1} - w(x)\big) = \zeta\big(wyw^{-1} - w(y)\big) = 0$. The last two relations are obvious from the equivariance of θ, so let us prove only the first one.

$$
\begin{aligned}
\zeta\big([y,x]\big) &= [\zeta(x),\zeta(y)] = \langle\zeta(x),\zeta(y)\rangle - \sum_{\alpha\in R_+} c_\alpha\langle\zeta(x),\alpha\rangle\langle\alpha^\vee,\zeta(y)\rangle s_\alpha \\
&= \big\langle\theta^{-1}(x),\theta(y)\big\rangle - \sum_{\alpha\in R_+} c_\alpha(\alpha,x)\big(\theta(y),\frac{2}{(\alpha,\alpha)}\alpha\big)s_\alpha \\
&= \big(\theta(y),x\big) - \sum_{\alpha\in R_+} c_\alpha\big(x,\frac{2}{(\alpha,\alpha)}\alpha\big)\big(\alpha,\theta(y)\big)s_\alpha \\
&= \zeta\big(\langle y,x\rangle - \sum_{\alpha\in R_+} c_\alpha\langle\alpha^\vee,x\rangle\langle y,\alpha\rangle s_\alpha\big).
\end{aligned}
$$

Notice that $\zeta(\mathbf{h})=\mathbf{h}$ since $\{\zeta(x_i)\}_{i=1}^n$ and $\{\zeta(y_i)\}_{i=1}^n$ are also dual bases.

Using the anti-involution ζ, we can give the dual M^* of a module $M\in\mathcal{O}_{\mathsf{H}_c}(0)$ the structure of a module over $\mathcal{O}_{\mathsf{H}_c}(0)$: given $m\in M, f\in M^*, z\in\mathsf{H}_c$, set $(zf)(m)=f(\zeta(z)m)$. We can now define the "restricted" dual $M^\vee$ for an arbitrary $M\in\mathcal{O}_{\mathsf{H}_c}(0)$ to be the submodule of M^* generated (spanned) by its generalized $\mathbf{h}$-weight vectors. We let $\nabla(\tau)$ be $\Delta(\tau)^\vee$.

If M is a standard or simple module, then the dimension of the $\mathbf{h}$-weight space $M(a)$, corresponding to the weight $a\in\mathbb{C}$, is the same as the dimension of $M^\vee(a)$. This follows from the fact that, in this case, $\mathbf{h}$ acts semisimply on M (and on $M^\vee$). Using induction on the length, we can show that this equality of dimension is true for any module $M\in\mathcal{O}_{\mathsf{H}_c}(0)$.

We claim that $M^\vee$ is in the category $\mathcal{O}_{\mathsf{H}_c}(0)$. Before proving this, we will need one of the following properties of $(\cdot)^\vee$, which is, a priori, only a functor from $\mathcal{O}_{\mathsf{H}_c}(0)$ to $\mathrm{mod}_L-\mathsf{H}_c$, the category of all left modules over H_c.

Lemma 3.1. *1. The functor $M\mapsto M^\vee$ is exact.*

2. $(M^\vee)^\vee\cong M$ canonically.

3. The canonical map $\mathrm{Hom}_{\mathsf{H}_c}(M_1,M_2)\to\mathrm{Hom}_{\mathsf{H}_c}(M_2^\vee,M_1^\vee)$ is an isomorphism.

Proof. (1) If $0\to M_1\to M_2\to M_3\to 0$ is an exact sequence, then $0\to M_3^*\to M_2^*\to M_1^*\to 0$ is also exact; taking the sum of the generalized $\mathbf{h}$-weight spaces, we get an exact sequence $0\to M_3^\vee\to M_2^\vee\to M_1^\vee\to 0$. That this last sequence is exact at $M_3^\vee$ and $M_2^\vee$ is obvious; surjectivity of the homomorphism $M_2^\vee\to M_1^\vee$ follows by considering the dimensions of the generalized $\mathbf{h}$-weight spaces.

(2) The inclusion $M\hookrightarrow(M^\vee)^\vee$ and the equality of the dimension of the weight spaces imply the isomophism $M\cong(M^\vee)^\vee$.

(3) This follows from part 2 and the natural embeddings $\mathrm{Hom}_{\mathsf{H}_c}(M_1,M_2)\hookrightarrow\mathrm{Hom}_{\mathsf{H}_c}(M_2^\vee,M_1^\vee)$ and $\mathrm{Hom}_{\mathsf{H}_c}(M_2^\vee,M_1^\vee)\hookrightarrow\mathrm{Hom}_{\mathsf{H}_c}\big((M_1^\vee)^\vee,(M_2^\vee)^\vee\big)$. □

We can now prove that $M^\vee\in\mathcal{O}_{\mathsf{H}_c}(0)$.

(i) $M^\vee$ is finitely generated: It is enough to show that $M^\vee$ has finite length - actually, the same length as M. This is clearly true when M is simple, for then $M^\vee$ must be simple by (2) and in fact $M \cong M^\vee$ because the isomorphism $V(\sigma) \cong V(\sigma)^*$ holds for any representation of a finite Weyl group. The general case follows by induction from lemma 3.1 (1).

(ii) $\mathbb{C}[\mathfrak{h}^*]$ acts locally nilpotently on $M^\vee$: By using induction on the length, we can reduce to proving this for simple modules, and then for standard modules; in this case, this is clear because they are graded by the degree of polynomials in $\mathbb{C}[\mathfrak{h}]$ and the action of the elements of $\mathfrak{h}$ decreases the degree by one - their action is given by the Dunkl operators [7]. Alternatively, that $\mathbb{C}[\mathfrak{h}^*]$ acts locally nilpotently on $M^\vee$ also follows from the fact that the generalized weights of $\mathbf{h}$ on $M^\vee$, which are the same as those of M, are bounded from above. □

Before proceeding any further, we also need to define a partial order $\leq$ on Σ, Σ being the finite set of irreducible representations of W or, equivalently, the set of simple modules in $\mathcal{O}_{\mathsf{H}_c}(0)$. We define $\tau_1 \leq \tau_2$ if and only if $k(\tau_2, c) - k(\tau_1, c)$ is a non-negative number. We will assume also, from now on, that if τ_1 is not isomorphic to τ_2, then the constants $k(\tau_1, c)$ and $k(\tau_2, c)$ are distinct. This excludes the case, for instance, $c \equiv 0$.

Proof. PROOF of theorem 3.1. To prove that $\mathcal{O}_{\mathsf{H}_c}(0)$ has enough projective objects, we use a criterion given in theorem 3.2.1 of [4] - see also the important third remark following that theorem. It is proved there that an abelian $\mathbb{C}$-category which satisfies five precise properties has enough projective objects. We translate below these properties in the context of the category $\mathcal{O}_{\mathsf{H}_c}(0)$ and we verify that they are indeed satisfied.

1. Every object has finite length: This is lemma 2.1 (4).

2. There are only finitely many isomorphism classes of simple objects in $\mathcal{O}_{\mathsf{H}_c}(0)$: We know already that simple objects in $\mathcal{O}_{\mathsf{H}_c}(0)$ are in bijection with the irreducible representations of W.

3. The endomorphism ring of a simple object reduces to the scalars: This is simply Schur's lemma.

4. Choose a closed subset $T \subset \Sigma$ and $\tau \in T$ a maximal element. We denote by $\mathcal{O}_{\mathsf{H}_c}(0)^T$ the full subcategory of $\mathcal{O}_{\mathsf{H}_c}(0)$ consisting of all objects whose simple subquotients are in T; the functor $(\cdot)^\vee$ induces also an equivalence of this category. What we have to prove is that $\Delta(\tau)$ is a projective cover of $L(\tau)$ inside $\mathcal{O}_{\mathsf{H}_c}(0)^T$ and that $\nabla(\tau)(= \Delta(\tau)^\vee)$ is an injective hull of $L(\tau)$ in $\mathcal{O}_{\mathsf{H}_c}(0)^T$. The second assertion follows by duality, so we will only prove the first one.

Let $\psi : M \to N$ be an epimorphism, $N, M \in \mathcal{O}_{\mathsf{H}_c}(0)^T$. We want to show that a homomorphism $\varphi : \Delta(\tau) \to N$ factors through M, and that the kernel of $\Delta(\tau) \to L(\tau)$ is superfluous – this last assertion follows from 5 below.

Let $v \in \Delta(\tau)$ be a generator of maximal weight, so $\mathbf{h}$ acts on it by the constant $\frac{1}{2}(k(\tau, c) - n)$ and $yv = 0\ \forall y \in \mathfrak{h}$. $\varphi(v)$, which we can assume to be non-zero, is also a highest weight vector and $\mathbf{h}$ acts on it by multiplication by the same constant. Choose $w \in \psi^{-1}\big(\varphi(v)\big)$. We can assume that $\big(h - \frac{1}{2}(k(\tau, c) - n)\big)^r w = 0$ for some $r \geq 1$, with r minimal, and that w generates an irreducible W-submodule of type τ, so to construct a map $\Delta(\tau) \to M$ which lifts φ, we only have to show that $yw = 0\ \forall y \in \mathfrak{h}$.

Choose a monomial $P(y_1,\dots,y_n)$ which does not annihilate w, but such that $y_iP(y_1,\dots,y_n)w$ vanishes for $i=1,\dots,n$. Let $d=\deg\big(P(y_1,\dots,y_n)\big)$, and assume $d\geq 1$. Consider the vector $\tilde{w}=\big(\mathbf{h}-(\frac{1}{2}(k(\tau,c)-n)+d)\big)^tP(y_1,\dots,y_n)w$, with t maximal such that this is non-zero. $\tilde{w}$ is a highest weight vector and it belongs to the ν-isotypic component of M with $\nu>\tau$ because the constant by which $\mathbf{h}$ acts on it is strictly larger than $\frac{1}{2}(k(\tau,c)-n)$. This contradicts our assumption that $M\in\mathcal{O}_{\mathsf{H}_c}(0)^T$. It follows that $d=0$ and $yw=0$ $\forall y\in\mathfrak{h}$. Incidentally, r must be equal to 1.

5. $\ker\big(\Delta(\tau)\to L(\tau)\big)$ and $\operatorname{coker}\big(L(\tau)\to\nabla(\tau)\big)$ lie in the subcategory $\mathcal{O}_{\mathsf{H}_c}(0)^{<\tau}$ where $\mathcal{O}_{\mathsf{H}_c}(0)^{<\tau}$ is the category of modules whose simple subquotients $L(\nu)$ satisfy $\nu<\tau$: It is enough to consider only the assertion concerning the kernel. Suppose that u is a highest weight vector of type μ inside $\Delta(\tau)$. Then $\mathbf{h}$ acts on it by multiplication by $\frac{1}{2}(k(\mu,c)-n)$. On the other hand, if u is a sum of elements of the form $p\otimes v$ with p of degree m,$v\in V(\tau)$, then we know already that $\mathbf{h}$ acts on it by multiplication by $-m+\frac{1}{2}(k(\tau,c)-n)$, so if these two constant are equal and $m>0$, it must be the case that $k(\mu,c)<k(\tau,c)$, so $\mu<\tau$. $\square$

Remark 3.1. *The hypothesis of theorem 3.2.1 in [4] includes a sixth condition which, in our case, is the vanishing of the group* $\operatorname{Ext}^2_{\mathcal{O}_{\mathsf{H}_c}(0)}\big(\Delta(\sigma),\nabla(\tau)\big)$, $\forall\sigma,\tau\in\Sigma$. *This extra condition yields a stronger conclusion, namely that the projectives have standard filtrations in the sense of the next section. We will argue in the opposite direction: we will prove the vanishing of certain* Ext*-groups (corollary 4.2) only after showing that the projective cover $P(\sigma)$ of $L(\sigma)$ possesses a standard filtration.*

4 Standard filtrations and reciprocity

The goal of this section is to prove an analog of the classical BGG-reciprocity [3] for the category $\mathcal{O}_{\mathsf{H}_c}(0)$ (theorem 4.2), but first, we have to construct filtrations on the projective covers $P(\sigma)$ of $L(\sigma)$ which are "standard" in the sense that the quotient of two successive modules in such a filtration is a standard module. This is achieved in proposition 4.5 below.

We start by considering certain subcategories of $\mathcal{O}_{\mathsf{H}_c}(0)$; although this is not essential for our argument, the result of the preceeding section leads to an interesting conclusion regarding them (proposition 4.1).

Fix $k\in\mathbb{N}$ and define $\mathcal{O}^k_{\mathsf{H}_c}(0)$ to be the full subcategory of $\mathcal{O}_{\mathsf{H}_c}(0)$ consisting of the modules M such that if $m\in M$ is a generalized weight vector of $\mathbf{h}$ with weight a, then $(\mathbf{h}-a)^km=0$. We have a functor $F_k:\mathcal{O}_{\mathsf{H}_c}(0)\to\mathcal{O}^k_{\mathsf{H}_c}(0)$ given by $F_k(M)=\operatorname{span}\{m\in M|(\mathbf{h}-a)^km=0\text{ for some }a\in\mathbb{C}\}$. Note that lemma 2.2 and the invariance of $\mathbf{h}$ under W imply that $F_k(M)$ is indeed an H_c-submodule. These subcategories provide a "filtration" of $\mathcal{O}_{\mathsf{H}_c}(0)$ which is exhaustive in the sense that any $M\in\mathcal{O}_{\mathsf{H}_c}(0)$ is an object of $\mathcal{O}^k_{\mathsf{H}_c}(0)$ for k large enough – actually, k can be taken to be one if M is a quotient of a standard module and, in general, it can be proved by induction that the least such k is less than or equal to the length of M.

Proposition 4.1. *$\mathcal{O}_{\mathsf{H}_c}(0) = \mathcal{O}^k_{\mathsf{H}_c}(0)$ if k is large enough.*

Proof. Every module in $\mathcal{O}_{\mathsf{H}_c}(0)$ is an epimorphic image of a finite direct sum of copies of the indecomposable projectives $P(\sigma), \sigma \in \Sigma$, so we have to choose k such that $P(\sigma) \in \mathcal{O}^k_{\mathsf{H}_c}(0)$ for every $\sigma \in \Sigma$, e.g. $k \geq \max_{\sigma\in\Sigma} \operatorname{length}\big(P(\sigma)\big)$. $\square$

It can be checked that the conditions 1–5 in the proof of theorem 3.1 are satisfied for any of the categories $\mathcal{O}^k_{\mathsf{H}_c}(0)$ – the same argument as in that proof applies – so these categories contain also enough projectives. However, we will give a new proof of this assertion by giving an explicit construction similar to the original one in [3]. This does not provide a new proof that $\mathcal{O}_{\mathsf{H}_c}(0)$ has enough projectives, because proposition 4.1 depends on this fact. Afterwards, using some modules that we will construct, we will deduce that projective modules in $\mathcal{O}_{\mathsf{H}_c}(0)$ have filtrations by standard modules.

We define now some modules in $\mathcal{O}_{\mathsf{H}_c}(0)$ which are extensions of standard modules and possess standard filtrations; we show also below that certain quotients of these modules are projective in $\mathcal{O}^k_{\mathsf{H}_c}(0)$. Fix $k \in \mathbb{N}$, $a \in \mathbb{C}$. There exists a minimal integer $N(a)$ such that if $M \in \mathcal{O}_{\mathsf{H}_c}(0)$ and $m \in M$ is a generalized weight vector of $\mathbf{h}$ of weight a, then $\mathbb{C}[\mathfrak{h}^*]_+^{N(a)} m = 0$, where $\mathbb{C}[\mathfrak{h}^*]_+$ is the space of polynomials with constant term equal to zero. This is simply because there is an upper bound on the possible weights of $\mathbf{h}$ on an H_c-module which are in the set $\{a + \mathbb{Z}\}$ (i.e. this upper bound is valid for any module). We choose $N(a)$ minimal with this property. For $\sigma \in \Sigma$, let $Q(a,\sigma)$ be the H_c-module obtained by induction from the $\mathbb{C}[\mathfrak{h}^*]\#W$-module $\mathbb{C}[\mathfrak{h}^*]/(\mathbb{C}[\mathfrak{h}^*]_+^{N(a)}) \otimes_{\mathbb{C}} V(\sigma)$. ($\mathbb{C}[\mathfrak{h}^*]$ acts by left multiplication and W acts diagonally on this module.) As a $\mathbb{C}[\mathfrak{h}]$-module, $Q(a,\sigma) \cong \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{h}^*]/(\mathbb{C}[\mathfrak{h}^*]_+^{N(a)}) \otimes_{\mathbb{C}} V(\sigma)$: this follows from the PBW-property of H_c. Let $Q(k,a,\sigma)$ be the quotient of $Q(a,\sigma)$ by the left submodule generated by $(\mathbf{h}-a)^k \otimes 1 \otimes V(\sigma) \in \mathsf{H}_c \otimes_{\mathbb{C}[\mathfrak{h}^*]\#W} \mathbb{C}[\mathfrak{h}^*]/(\mathbb{C}[\mathfrak{h}^*]_+^{N(a)}) \otimes_{\mathbb{C}} V(\sigma)$. Let also $R(a)$ be the H_c-module induced from the $\mathbb{C}[\mathfrak{h}^*]\#W$-module $\mathbb{C}[\mathfrak{h}^*]/(\mathbb{C}[\mathfrak{h}^*]_+^{N(a)}) \otimes_{\mathbb{C}} \mathbb{C}[W]$. Let $R(a,k)$ be the quotient of $R(a)$ by the submodule generated by $(\mathbf{h}-a)^k 1$; $1 \in R(a)$ denotes the generator $1 \otimes 1 \otimes 1 \in \mathsf{H}_c \otimes_{\mathbb{C}[\mathfrak{h}^*]\#W} \mathbb{C}[\mathfrak{h}^*]/(\mathbb{C}[\mathfrak{h}^*]_+^{N(a)}) \otimes_{\mathbb{C}} \mathbb{C}[W]$, and we denote its image in $R(a,k)$ also by 1. Viewed as a left W-module, by Wedderburn's theorem, $\mathbb{C}[W] = \oplus_{\sigma\in\Sigma} V(\sigma)^{d(\sigma)}$ where $d(\sigma)$ is the dimension of $V(\sigma)$, so $R(a) = \oplus_{\sigma\in\Sigma} Q(a,\sigma)^{d(\sigma)}$ and $R(a,k) = \oplus_{\sigma\in\Sigma} Q(a,k,\sigma)^{d(\sigma)}$.

Proposition 4.2. *For any $M \in \mathcal{O}^k_{\mathsf{H}_c}(0)$, $\operatorname{Hom}_{\mathsf{H}_c}\big(R(a,k), M\big) \xrightarrow{\sim} M(a)$ given by $\varphi \mapsto \varphi(1)$ is an isomorphism. Here, $M(a)$ is the generalized weightspace of $\mathbf{h}$ for the weight a.*

Proof. The map is clearly injective, so choose $m \in M(a)$. Then the homomorphism $\mathbb{C}[\mathfrak{h}^*] \otimes_{\mathbb{C}} \mathbb{C}[W] \to M$ that we get by sending $(p \otimes w)$ to pwm descends to a homomorphism $\mathbb{C}[\mathfrak{h}^*]/(\mathbb{C}[\mathfrak{h}^*]_+^{N(a)}) \otimes_{\mathbb{C}} \mathbb{C}[W] \to M$. By the universal property of induced modules, we obtain a homomorphism $R(a) \to M$. Since this homomorphism maps 1 to an $\mathbf{h}$-generalized weight vector of M with weight a, it sends the submodule generated by $(\mathbf{h}-a)^k 1$ to zero, thus it descends to a homomorphism $\varphi : R(a,k) \to M$ satisfying $\varphi(1) = m$. $\square$

This proposition and the decomposition $R(a,k) = \oplus_{\sigma\in\Sigma} Q(a,k,\sigma)^{d(\sigma)}$ imply that $M(a,\sigma)$, the σ-isotypic component of $M(a)$, is isomorphic to the space $\mathrm{Hom}_{\mathsf{H}_c}\big(Q(a,k,\sigma)^{d(\sigma)}, M\big)$, this isomorphism being given by $\varphi \mapsto \varphi(1_\sigma)$ where 1_σ is the projection of $1 \in R(a)$ in $Q(a,k,\sigma)^{d(\sigma)}$ in the decomposition above.

Proposition 4.3. *The modules $R(a,k)$ and $Q(a,k,\sigma)$ are projective in $\mathcal{O}^k_{\mathsf{H}_c}(0)$.*

Proof. The previous proposition says that the functor $\mathrm{Hom}\big(R(a,k),\cdot\big)$ is exact because it is isomorphic to the exact functor $M \to M(a)$. Since $Q(a,k,\sigma)$ is a direct summand of $R(a,k)$, $Q(a,k,\sigma)$ is also projective. $\square$

Remark 4.1. *This last proposition also implies that the category $\mathcal{O}^k_{\mathsf{H}_c}(0)$ has enough projectives because any module in $\mathcal{O}_{\mathsf{H}_c}(0)$ is generated by finitely many generalized $\mathbf{h}$-weight vectors.*

We will need the next proposition to prove the main theorem of this section.

Proposition 4.4. *The module $Q(a,\sigma)$ admits a finite filtration whose successive quotients are standard modules.*

Proof. Let M_j be the H_c-submodule of $Q(a,\sigma)$ which is generated by the space $\mathbb{C}[\mathfrak{h}^*]^j_+/\mathbb{C}[\mathfrak{h}^*]^{N(a)}_+ \otimes_{\mathbb{C}} V(\sigma)$. Then $Q(a,\sigma) = M_0 \supset M_1 \supset \ldots \supset M_{N(a)} = 0$. The quotient M_j/M_{j+1} can be identified with the H_c-module induced from the $\mathbb{C}[\mathfrak{h}^*]\#W$-module $\big(\mathbb{C}[\mathfrak{h}^*]^j_+/\mathbb{C}[\mathfrak{h}^*]^{j+1}_+\big) \otimes_{\mathbb{C}} V(\sigma)$. $\mathbb{C}[\mathfrak{h}^*]$ acts trivially on this module; decompose it as a sum of irreducible W-modules: $\mathbb{C}[\mathfrak{h}^*]^j_+/\mathbb{C}[\mathfrak{h}^*]^{j+1}_+ \otimes_{\mathbb{C}} V(\sigma) \cong \oplus_{\tau\in\Sigma} V(\tau)^{\oplus n(\tau,\sigma,j)}$, where $n(\tau,\sigma,j)$ is the corresponding multiplicity. It follows that the quotient M_j/M_{j+1} is isomorphic to the direct sum $\oplus_{\tau\in\Sigma}\Delta(\tau)^{\oplus n(\tau,\sigma,j)}$. $\square$

Consider the projection $\pi : P(\sigma) \to L(\sigma)$ where $P(\sigma)$ is the projective cover of $L(\sigma)$. Let $v \in L(\sigma)$ be a generator of highest weight, which is thus annihilated by $\mathbb{C}[\mathfrak{h}^*]$, belongs to a simple W-module isomorphic to $V(\sigma)$ and is a weight vector for $\mathbf{h}$ of weight $a = \frac{1}{2}(k(\sigma,c) - n)$. Then we can choose a $w \in \pi^{-1}(v)$ which is also a generalized weight vector for $\mathbf{h}$ with the same weight and is in an irreducible W-submodule isomorphic to $V(\sigma)$. Therefore, we can find an epimorphism $Q(a,\sigma) \to P(\sigma)$. Since $P(\sigma)$ is projective, this epimorphism splits.

Proposition 4.5. *The projective cover $P(\sigma)$ has a standard filtration $P(\sigma) = F^0_\sigma \supset F^1_\sigma \supset \ldots \supset F^{r(\sigma)}_\sigma = 0$ with the property that $F^0_\sigma/F^1_\sigma \cong \Delta(\sigma)$ and $F^i_\sigma/F^{i+1}_\sigma \cong \Delta(\tau_{i,\sigma})$ for some $\tau_{i,\sigma} > \sigma$ if $i \geq 1$.*

Proof. Let $\tilde{P} \in \mathcal{O}_{\mathsf{H}_c}(0)$ be a complement of $P(\sigma)$ inside $Q(a,\sigma)$, i.e. $Q(a,\sigma) = P(\sigma) \oplus \tilde{P}$. Let us denote by $F^\bullet_{a,\sigma}$ the submodules of a standard filtration for

$Q(a,\sigma)$. Choose a highest weight vector $v \in Q(a,\sigma)$ which is in an irreducible W-module of type $\mu \in \Sigma$. We can further assume, without loss of generality, that $v \in P(\sigma)$ – the same argument applies if $v \in \tilde{P}$. Suppose that $v \in F^l_{a,\sigma} \smallsetminus F^{l+1}_{a,\sigma}$. Then we have a non-zero homomorphism $\Delta(\mu) \to F^l_{a,\sigma}/F^{l+1}_{a,\sigma}$. By the maximality of the weight of v, this must be an epimorphism. Since $F^l_{a,\sigma}/F^{l+1}_{a,\sigma}$ is a standard module, hence free over $\mathbb{C}[\mathfrak{h}]$, this epimorphism must be an isomorphism. Therefore, we have a splitting $F^l_{a,\sigma} = F^{l+1}_{a,\sigma} \oplus \mathsf{H}_c v$, $\mathsf{H}_c v \cong \Delta(\mu)$, and since $F^l_{a,\sigma}/\mathsf{H}_c v \cong F^{l+1}_{a,\sigma}$, $Q(a,\sigma)/\mathsf{H}_c v$ also has a standard filtration, so we can repeat this argument and apply it to $P(\sigma)/\mathsf{H}_c v \oplus \tilde{P}$.

What we have to check now is that the standard filtration $F^\bullet_\sigma$ of $P(\sigma)$ thus obtained satisfies the condition of the proposition. If the first subquotient of this filtration is $\Delta(\nu)$, then $F^0_\sigma/F^1_\sigma \cong \Delta(\nu)$ and F^1_σ is contained in the unique maximal submodule $\mathrm{rad}\big(P(\sigma)\big)$ of $P(\sigma)$. The image $\mathrm{rad}\big(P(\sigma)\big)/F^1_\sigma$ of this maximal submodule in F^0_σ/F^1_σ is the unique maximal submodule $\mathrm{rad}\big(\Delta(\nu)\big)$ of $\Delta(\nu)$; the quotient of $\Delta(\nu)$ by $\mathrm{rad}\big(\Delta(\nu)\big)$ is $L(\nu)$, so $L(\nu) \cong P(\sigma)/\mathrm{rad}\big(P(\sigma)\big)$ and $\nu = \sigma$.

To prove the second assertion, we use the lemma below. Assume that, in our standard filtration of $P(\sigma)$, two consecutive terms satisfy $F^{l-1}_\sigma/F^l_\sigma \cong \Delta(\mu_1), F^l_\sigma/F^{l+1}_\sigma \cong \Delta(\mu_2)$ and $\mu_2 \not> \mu_1$. Then $\mathrm{Ext}^1\big(\Delta(\mu_1),\Delta(\mu_2)\big) = 0$, so that we can define a new filtration $\tilde{F}^\bullet_\sigma$ with $\tilde{F}^q_\sigma = F^q_\sigma$ if $q \neq l$ and $\tilde{F}^{l-1}_\sigma/\tilde{F}^l_\sigma \cong \Delta(\mu_2), \tilde{F}^l_\sigma/\tilde{F}^{l+1}_\sigma \cong \Delta(\mu_1)$. Now $F^1_\sigma/F^2_\sigma \cong \Delta(\zeta)$ for some $\zeta \in \Sigma$, so ζ must be strictly greater than σ because $P(\sigma)$ has a unique maximal submodule. Combining these two observations, we get the desired conclusion. $\square$

Lemma 4.1. *Suppose that $\nu \not> \mu$. Then $\mathrm{Ext}^1_{\mathcal{O}_{\mathsf{H}_c}(0)}\big(\Delta(\mu),\Delta(\nu)\big) = 0$.*

Proof. Suppose that $0 \to \Delta(\nu) \to M \xrightarrow{\pi} \Delta(\mu) \to 0$ is an extension of $\Delta(\mu)$ by $\Delta(\nu)$. Let $v \in \Delta(\mu)$ be a generator of highest weight, that is, of weight $\frac{1}{2}(k(\mu,c)-n)$. In $\pi^{-1}(v)$, we can choose a vector m which is in an irreducible representation of W of type μ and which is a generalized weight vector of $\mathbf{h}$ of weight $\frac{1}{2}(k(\mu,c)-n)$ also. Furthermore, since $\frac{1}{2}(k(\mu,c)-n)$ is a maximal weight of M under the action of $\mathbf{h}$ by assumption, $\mathfrak{h}m = 0$, and therefore $\mathbf{h}m = -\frac{1}{2}\sum_{i=1}^n [y_i,x_i]m = \frac{1}{2}(k(\mu,c)-n)m$.

Since the H_c-submodule generated by m is a quotient of a standard module and surjects onto $\Delta(\mu)$, $\pi|_{\mathsf{H}_c m}$ must be an isomorphism onto $\Delta(\mu)$ (since $\Delta(\mu)$ is a free module over $\mathbb{C}[\mathfrak{h}]$), thus its intersection with $\Delta(\nu)$ is trivial. This gives us an H_c-splitting $M \cong \Delta(\nu) \oplus \Delta(\mu)$. $\square$

Remark 4.2. *Using induction, the long exact sequence of* Ext*-groups along with proposition 4.5, it is possible to prove more general vanishing results for* Ext*-groups between standard modules or between standard and simple modules. For instance, the previous lemma generalizes to all higher* Ext*-groups. Similar results which are true in any highest weight category can be found in [5].*

Remark 4.3. *This proposition allows us to characterize the standard module $\Delta(\sigma)$ as being the largest quotient of $P(\sigma)$ supported on $\mathcal{O}_{\mathsf{H}_c}(0)^{\leq \sigma}$, i.e. whose composition factors $L(\lambda)$ all satisfy $\lambda \leq \sigma$ (cf.[5]). Indeed, suppose that $M \subset$*

$P(\sigma), M \neq P(\sigma)$, and $P(\sigma)/M$ has this property. Let i be maximal such that $F^i_\sigma \not\subset M$. We want to show that $i = 0$, since this means that $F^1_\sigma \subset M$ and therefore $\Delta(\sigma) \cong P(\sigma)/F^1_\sigma \twoheadrightarrow P(\sigma)/M$, proving our claim. $F^i_\sigma/F^i_\sigma \cap M \hookrightarrow P(\sigma)/M$, so $F^i_\sigma/F^i_\sigma \cap M$ is supported on $\mathcal{O}_{\mathsf{H}_c}(0)^{\leq\sigma}$ because so is $P(\sigma)/M$ by assumption. Let N be the largest proper submodule of F^i_σ containing F^{i+1}_σ, so that $F^i_\sigma/N \cong L(\tau_{i,\sigma})$ since $F^i_\sigma/F^{i+1}_\sigma \cong \Delta(\tau_{i,\sigma})$ for some $\tau_i \in \Sigma$. Then $F^i_\sigma \cap M \subset N$ because $F^{i+1}_\sigma \subset M$ by the maximality of i, so $F^i_\sigma/F^i_\sigma \cap M$ surjects onto F^i_σ/N; this implies that $L(\tau_{i,\sigma}) \cong F^i_\sigma/N$ is also supported on $\mathcal{O}_{\mathsf{H}_c}(0)^{\leq\sigma}$. We already know that $\tau_{i,\sigma} \geq \sigma$ (by proposition 4.5), so the only possibility is that $\tau_{i,\sigma} = \sigma$ and $i = 0$.

Theorem 4.1. *Every projective module in $\mathcal{O}_{\mathsf{H}_c}(0)$ has a filtration with standard subquotients.*

Proof. Every projective module is a direct sum of copies of the indecomposable modules $P(\sigma)$, so this is clear from the previous proposition. □

Proposition 4.5 actually shows that the category $\mathcal{O}_{\mathsf{H}_c}(0)$ is a highest weight category with duality in the sense of [5],[6]. We can therefore state the following corollary, which is true in any highest weight category.

Theorem 4.2. *BGG-reciprocity holds also in the category $\mathcal{O}_{\mathsf{H}_c}(0)$, that is, the multiplicity of $\Delta(\sigma)$ in a standard filtration of $P(\tau)$ is independent of the choice of such a filtration and is equal to the multiplicity of $L(\tau)$ in $\Delta(\sigma)$.*

Proof. See [5] Theorem 3.11, using remark 4.3. □

Furthermore, this corollary imply the following.

Corollary 4.1. *The category $\mathcal{O}_{\mathsf{H}_c}(0)$ is semisimple if and only if all the standard modules $\Delta(\sigma)$, $\sigma \in \Sigma$, are simple.*

Proof. By theorem 4.2, if all the standard modules are simple, then the simple modules are projective, so any module is isomorphic to the sum of the simple modules which occur in any of its composition series. Conversely, if the category $\mathcal{O}_{\mathsf{H}_c}(0)$ is semisimple, then $\Delta(\sigma)$ splits into a direct sum of simple modules, and this can occur only if $\Delta(\sigma) = L(\sigma)$. □

This occurs, for instance, when $c \in \mathbb{C}[R]^W_{\mathrm{reg}}$. (As explained in [2], $\mathbb{C}[R]^W_{\mathrm{reg}}$ is the set of functions c such that the corresponding Hecke algebra $H_W(e^{2\pi i c})$ is semisimple.)

We can now recover the sixth condition alluded to after the proof of theorem 3.1. Actually, we even obtain a stronger vanishing statement.

Corollary 4.2. $\mathrm{Ext}^j_{\mathcal{O}_{\mathsf{H}_c}(0)}\big(\Delta(\sigma), \nabla(\tau)\big) = 0$ *for all $\sigma, \tau \in \Sigma$ and any $j \in \mathbb{Z}_{\geq 1}$.*

Proof. As noted in the proof of theorem 3.11 in [5], this is true in any highest weight category. □

5 $\mathbb{Z}$-strings and cohomological dimension

In this section, we introduce a decomposition of the category $\mathcal{O}_{\mathsf{H}_c}(0)$ and use it to obtain an upper bound on its cohomological dimension, which we know is finite by corollary 3.2.2 of [4].

Definition 5.1. *By the $\mathbb{Z}$-string through $\sigma \in \Sigma$, we will mean the set $\tau \in \Sigma$ such that $k(\tau, c) - k(\sigma, c) \in \mathbb{Z}$. The $\mathbb{Z}_{>0}$-string and the $\mathbb{Z}_{<0}$-string through σ are defined similarly. The length of a $\mathbb{Z}$-string is the number of elements in that set.*

The $\mathbb{Z}$-strings, denoted $\mathcal{S}_1, \ldots, \mathcal{S}_t$, give us a partition of Σ; we choose an arbitrary set of representatives $\sigma_1, \ldots, \sigma_t$ of these $\mathbb{Z}$-strings, i.e. $\sigma_j \in \mathcal{S}_j$. Given a module $M \in \mathcal{O}_{\mathsf{H}_c}(0)$ and $\sigma \in \Sigma$, let M^σ be the set of elements $m \in M$ such that m is a generalized weight vector of $\mathbf{h}$ of weight b with $b \in \mathbb{Z} + k(\sigma, c)$. If $k(\tau, c)$ and $k(\sigma, c)$ are in the same $\mathbb{Z}$-string, then the spaces M^τ and M^σ are the same, so we can define unambiguously $M^{\mathcal{S}_j}$ to be M^{σ_j}. The direct sum $M = \oplus_{j=1,\ldots,t} M^{\mathcal{S}_j}$ is actually a decomposition into H_c-submodules: this is a consequence of lemma 2.2.

Definition 5.2. *We denote by $\mathfrak{S}_j$ the full subcategory of modules $M \in \mathcal{O}_{\mathsf{H}_c}(0)$ such that $M = M^{\mathcal{S}_j}$.*

Therefore, we obtain the following proposition.

Proposition 5.1. *$\mathcal{O}_{\mathsf{H}_c}(0)$ is the direct sum of its strings $\mathfrak{S}_j, j = 1, \ldots, t$.*

It follows from this that the cohomological dimension of $\mathcal{O}_{\mathsf{H}_c}(0)$ is bounded above by the maximum of the cohomological dimensions of its strings, and an upper bound for these is the content of our next result.

Proposition 5.2. *The cohomological dimension of $\mathfrak{S}_j$ is bounded above by $2(L_j - 1)$, L_j being the length of $\mathcal{S}_j$.*

Before giving the proof, note that if σ is minimal in its $\mathbb{Z}$-string, so that $\frac{1}{2}(k(\sigma, c) - n)$ is smaller than $\frac{1}{2}(k(\tau, c) - n)$ for any other τ in the same $\mathbb{Z}$-string, then the standard module $\Delta(\sigma)$ is irreducible. This follows from the argument used to prove lemma 2.31 (i) in [7]. If it is maximal, $\Delta(\sigma)$ is projective: this follows from the argument used to prove theorem 3.1. These two observations provide the starting point for the two induction arguments in the following proof.

Proof. We claim that, for a standard module $\Delta(\sigma)$, an upper bound for its projective dimension is given by the length $L_j^{>\sigma}$ of the $\mathbb{Z}_{>0}$-string through σ. Consider the exact sequence $0 \to F_\sigma^1 \to P(\sigma) \to \Delta(\sigma) \to 0$. The standard modules occuring in a standard filtration of F_σ^1 are in the $\mathbb{Z}_{>0}$-string of σ by proposition 4.5, so by induction the cohomological dimension of F_σ^1 is strictly smaller than $L_j^{>\sigma}$, hence the cohomological dimension of $\Delta(\sigma)$ is less than or equal to $L_j^{>\sigma}$.

We now turn to simple modules and claim that the cohomological dimension of $L(\sigma)$ with $\sigma \in \mathcal{S}_j$, is bounded above by $2L_j - L_j^{\geq\sigma} - 1$ where $L_j^{\geq 0}$ is the length of the $\mathbb{Z}_{\geq 0}$-string through σ. Consider the exact sequence $0 \to \mathrm{rad}(\Delta(\sigma)) \to \Delta(\sigma) \to L(\sigma) \to 0$. The simple modules which occur in a composition series for $\mathrm{rad}(\Delta(\sigma))$ are in the $\mathbb{Z}_{<0}$-string through σ, so induction and the result of the preceding paragraph gives us also that the cohomological dimension of $L(\sigma)$ is less than or equal to $2L_j - L_j^{\geq\sigma} - 1$. In particular, the cohomological dimension of any simple module is less than or equal to $2(L-1)$, $L = \max_{j=1,\ldots,t} L_j$. $\square$

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