Cherednik algebras and Yangians

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Abstract

We construct a functor from the category of modules over the trigonometric (resp. rational) Cherednik algebra of type $\mathfrak{gl}_l$ to the category of integrable modules of level l over a Yangian for the loop algebra $\mathfrak{sl}_n$ (resp. over a subalgebra of this loop Yangian) and we establish that it is an equivalence of categories if $l+2<n$. Finally, we treat the case of the rational Cherednik algebras of type A_{l-1}.

1 Introduction

Affine Hecke algebras are very important in representation theory and have been studied extensively over the past few decades, along with their degenerate version introduced in [Dr1] and in [Lu]. About fifteen years ago, I. Cherednik introduced the notion of double affine Hecke algebra [Ch2], abbreviated DAHA, which he used to prove some important conjectures of I. Macdonald. His algebra also admits degenerate versions, the trigonometric one and the rational one, which are called Cherednik algebras.

One of the most important classical results in representation theory is an equivalence, often called Schur-Weyl duality, between the category of modules over the symmetric group S_l and the category of modules of level l over the Lie algebra $\mathfrak{sl}_n$ for $n \geq l+1$. When quantum groups were invented in the 1980's, it became an interesting problem to generalize the Schur-Weyl correspondence and similar equivalences were obtained between finite Hecke algebras and quantized enveloping algebras [Ji], between degenerate affine Hecke algebras and Yangians [Dr1, Ch1], between affine Hecke algebras and quantized affine Lie algebras [GRV, ChPr1], and between double affine Hecke algebras and toroidal quantum algebras [VaVa1]. In this paper, we prove a similar equivalence of categories between the trigonometric (resp. rational) Cherednik algebra associated to the symmetric group S_l and a (resp. subalgebra $\mathbb{L}$ of a) Yangian LY for the loop algebra $L\mathfrak{sl}_n = \mathfrak{sl}_n \otimes_{\mathbb{C}} \mathbb{C}[u,u^{-1}]$. The loop Yangians are barely known. They were mentioned briefly in [Va], [VaVa2].(See also [BoLe] for the $\widehat{\mathfrak{sl}}_2$ case.) The (sub-)algebra on the other side of our equivalence from the rational Cherednik algebra has never been considered before. By contrast, there has been a recent surge of interest in the representation theory of Cherednik algebras and their relations to the geometry of Hilbert schemes, integrable systems and other important mathematical objects. (See [BEG1, GGOR, GoSt] among others.) Our duality theorem indicates a new route to those questions via a careful study of LY and $\mathbb{L}$ and makes the study of these algebras more relevant and interesting.

On one hand, the rational Cherednik algebra is simpler than the DAHA which, a priori, makes it look less appealing. On the other hand, there are several interesting features in the rational case that do not have counterparts (at least for the moment) for the DAHA. We hope that the same can be said about Yangian-deformed double loop algebras and quantum toroidal algebras, whose representation theory is still very mysterious. The former have a simpler structure and one can hope that this will make them easier to study and that it will have some special, interesting features that do not exist for toroidal quantum algebras.

The trigonometric (resp. rational) DAHA is generated by two subalgebras, one isomorphic to a degenerate affine Hecke algebra and the other one isomorphic to the group algebra of an affine Weyl group (resp. both isomorphic to the smash product of S_l with a polynomial ring). For this reason, and because of the results mentioned above, we can expect its Schur-Weyl dual to be built from one copy of the Yangian Y for $\mathfrak{sl}_n$ and from one copy of the loop algebra $L\mathfrak{sl}_n$ (resp. two copies of $\mathfrak{sl}_n \otimes_{\mathbb{C}} \mathbb{C}[u]$). This is indeed true for LY (resp. $\mathbb{L}$).

An epimorphic image of $\mathbb{L}$, defined in terms of operators acting on a certain space, appeared for the first time in [BHW]; this was known to P. Etingof and V. Ginzburg. However, the algebra considered in that paper is not described in a very precise way and no equivalence of categories is established. One motivation for the present article comes from our desire to find exactly the relations between the generators of the Schur-Weyl dual of a Cherednik algebra of type $\mathfrak{gl}_l$.

In the next three sections, we define Cherednik algebras and Yangians and explore some of their basic properties, in particular their connections with double affine Hecke algebras and quantum toroidal algebras. The fifth section states the main result (theorem 5.2) for the trigonometric case, which is proved in the following one. After that, we look more closely at the action of certain elements of LY since this is useful in the last section, which concerns the rational case (theorem 8.1). Most of our results in the rational case follows from the observation that the rational Cherednik algebra of type $\mathfrak{gl}_l$ is contained in the trigonometric one. Our results are first proved for Cherednik algebras of type $\mathfrak{gl}_l$, but we are able to obtain similar ones in type A_{l-1} also. Furthermore, our equivalence restricts to an equivalence between two categories of BGG-type (theorem 8.3).

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2 Hecke algebras and Cherednik algebras

The definitions given in this section could be stated for any Weyl group W. However, in this paper, we will be concerned only with the symmetric group S_l, so we will restrict our definitions to the case $W = S_l$. We set $\mathfrak{h} = \mathbb{C}^l$. The symmetric group S_l acts on $\mathfrak{h}$ by permuting the coordinates. Associated to $\mathfrak{h}$ are two polynomial algebras: $\mathbb{C}[\mathfrak{h}] = \mathrm{Sym}(\mathfrak{h}^*) = \mathbb{C}[x_1, \ldots, x_l]$ and $\mathbb{C}[\mathfrak{h}^*] = \mathrm{Sym}(\mathfrak{h}) = \mathbb{C}[y_1, \ldots, y_l]$, where $\{x_1, \ldots, x_l\}$ and $\{y_1, \ldots, y_l\}$ are dual bases of $\mathfrak{h}^*$ and $\mathfrak{h}$, respectively. For $i \neq j$, we set $\alpha_{ij} = x_i - x_j, \alpha_{ij}^\vee = y_i - y_j, R = \{\alpha_{ij} | 1 \leq i \neq j \leq l\}$ and $R^+ = \{\alpha_{ij} | 1 \leq i < j \leq l\}$. The set $\Pi = \{x_i - x_{i+1} | 1 \leq i \leq n-1\}$ is a basis of simple roots. The reflection in $\mathfrak{h}$ with respect to the hyperplane $\alpha = 0$ is denoted s_α, so $s_\alpha(y) = y - \langle \alpha, y \rangle \alpha^\vee$, where $\langle\, , \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$ is the canonical pairing. We set $s_{ij} = s_{\alpha_{ij}}$.

The finite Hecke algebra $\mathcal{H}_q$ associated to S_l is a deformation of the group algebra $\mathbb{C}[S_l]$ and the affine Hecke algebra $\widetilde{\mathcal{H}}_q$ is a deformation of the group algebra of the (extended) affine Weyl group $\widetilde{S}_l = P \rtimes S_l$ where P is the lattice $\oplus_{i=1}^l \mathbb{Z} x_i \subset \mathfrak{h}^*$ (so $\mathbb{C}[\widetilde{S}_l] = \mathbb{C}[x_1^\pm, \ldots, x_l^\pm] \rtimes S_l$). The algebra $\widetilde{\mathcal{H}}_q$ admits a degenerate form $\mathtt{H}_c$ first introduced by Drinfeld [Dr1] and by Lusztig [Lu].

Definition 2.1. *The degenerate affine Hecke algebra* $\mathtt{H}_c$ *of type* $\mathfrak{gl}_l$ *is the algebra generated by the* *polynomial algebra* $\mathrm{Sym}(\mathfrak{h}) = \mathbb{C}[z_1, \dots, z_l]$ *and the group algebra* $\mathbb{C}[S_l]$ *with the relations*

$$s_\alpha \cdot z - s_\alpha(z)s_\alpha = -c\langle \alpha, z\rangle \ \ \forall z \in \mathfrak{h}, \forall \alpha \in \Pi$$

The double affine Hecke algebra $\mathbb{H}_{q,\kappa}$ (defined in section 4) introduced by I. Cherednik [Ch2] also admits degenerate versions: the trigonometric one and the rational one. Recall that the group $\widetilde{S}_l$ is generated by s_α $\forall \alpha \in R$ and by the element $\pi = x_1 s_{12} s_{23} \cdots s_{l-1,l}$.

Definition 2.2 (Cherednik). *Let* $t, c \in \mathbb{C}$. *The degenerate (trigonometric) double affine Hecke algebra of type* $\mathfrak{gl}_l$ *is the algebra* $\mathbf{H}_{t,c}$ *generated by the group algebra of the (extended) affine Weyl group* $\mathbb{C}[\widetilde{S}_l]$ *and the polynomial algebra* $\mathbb{C}[z_1, \dots, z_l] = \mathrm{Sym}(\mathfrak{h})$ *subject to the following relations:*

$$s_\alpha \cdot z - s_\alpha(z)s_\alpha = -c\langle \alpha, z\rangle \ \ \forall z \in \mathfrak{h}, \forall \alpha \in \Pi$$

$$\pi z_i = z_{i+1}\pi, \ \ 1 \leq i \leq l-1 \quad \pi z_l = (z_1 - t)\pi$$

Remark 2.1. *The subalgebra generated by* $\mathbb{C}[S_l]$ *and the polynomial algebra* $\mathbb{C}[z_1, \dots, z_l]$ *is isomorphic to the degenerate affine Hecke algebra* $\mathtt{H}_c$.

Definition 2.3. *Let* $t, c \in \mathbb{C}$. *The rational Cherednik algebra* $\mathsf{H}_{t,c}$ *of type* $\mathfrak{gl}_l$ *is the algebra generated by* $\mathbb{C}[\mathfrak{h}], \mathbb{C}[\mathfrak{h}^*]$ *and* $\mathbb{C}[S_l]$ *subject to the following relations:*

$$w \cdot x \cdot w^{-1} = w(x) \qquad w \cdot y \cdot w^{-1} = w(y) \quad \forall x \in \mathfrak{h}^*, \ \forall y \in \mathfrak{h}$$

$$[y, x] = yx - xy = t\langle y, x\rangle + c\sum_{\alpha \in R^+} \langle \alpha, y\rangle\langle x, \alpha^\vee\rangle s_\alpha$$

Remark 2.2. *The rational Cherednik algebra* $\widetilde{\mathsf{H}}_{t,c}$ *of type* A_{l-1} *is the subalgebra of* $\mathsf{H}_{t,c}$ *generated by* $\mathbb{C}[x_i - x_j] \subset \mathbb{C}[x_1, \dots, x_l]$, *by* $\mathbb{C}[y_i - y_j] \subset \mathbb{C}[y_1, \dots, y_l]$ *and by* $\mathbb{C}[S_l]$.

The exists a simple relation between $\mathbf{H}_{t,c}$ and $\mathsf{H}_{t,c}$.

Proposition 2.1 ([Su]). *The algebra* $\mathbb{C}[x_1^{\pm}, \dots, x_l^{\pm}] \otimes_{\mathbb{C}[\mathfrak{h}]} \mathsf{H}_{t,c}$ *is isomorphic to* $\mathbf{H}_{t,c}$.

Before giving a proof of this proposition, we need to introduce elements in $\mathsf{H}_{t,c}$ which will be very useful later. For $1 \leq i \leq l$, set $\mathcal{U}_i = \frac{t}{2} + x_i y_i + c\sum_{j<i} s_{ij}$ and $\mathcal{Y}_i = \mathcal{U}_i + \frac{c}{2}\sum_{j \neq i} \mathrm{sign}(j-i)s_{ij} = \frac{t}{2} + x_i y_i + \frac{c}{2}\sum_{j \neq i} s_{ij}$.

Proposition 2.2. *[DuOp],[EtGi]*

1. $\mathcal{Y}_i = \frac{1}{2}(x_i y_i + y_i x_i)$.
2. $\mathcal{U}_i \mathcal{U}_j = \mathcal{U}_j \mathcal{U}_i$ *for any* i, j.
3. $w \cdot \mathcal{Y}_i \cdot w^{-1} = \mathcal{Y}_{w(i)}$.
4. *The elements* $\mathcal{U}_i, 1 \leq i \leq l$, *and* $\mathbb{C}[S_l]$ *generate a subalgebra of* $\mathsf{H}_{t,c}$ *isomorphic to the degenerate affine Hecke algebra* $\mathtt{H}_c$.

Remark 2.3. *The elements $\mathcal{Y}_i$ are not pairwise commutative if $c \neq 0$:*

$$[\mathcal{Y}_j, \mathcal{Y}_k] = \frac{c^2}{4} \sum_{\substack{i=1 \\ i \neq j,k}}^{l} [s_{ij}, s_{jk}].$$

Proof. The first statement follows from the equality

$$y_i x_i - x_i y_i \;=\; t + c \sum_{j=1, j\neq i}^{l} \langle x_i - x_j, y_i \rangle \langle x_i, y_i - y_j \rangle s_{ij} = t + c \sum_{j=1, j\neq i}^{l} s_{ij}.$$

The second part is proved in [DuOp]. The third part is obvious, so we prove only the fourth one. If $|k-i| > 1$, then $s_{k,k+1}\mathcal{U}_i = \mathcal{U}_i s_{k,k+1}$, so the non-trivial relations that we have to check involve $s_{i-1,i}$ and $s_{i,i+1}$:

$$s_{i-1,i}\mathcal{U}_i = (\frac{t}{2} + x_{i-1}y_{i-1})s_{i-1,i} + c \sum_{j<i-1} s_{i-1,j}s_{i-1,i} + c = \mathcal{U}_{i-1}s_{i-1,i} + c$$

$$s_{i,i+1}\mathcal{U}_i = (\frac{t}{2} + x_{i+1}y_{i+1})s_{i,i+1} + c \sum_{j<i+1} s_{i+1,j}s_{i+1,i} - c = \mathcal{U}_{i+1}s_{i,i+1} - c$$

These two equalities, combined with the PBW-property of $\mathsf{H}_{t,c}$ [EtGi] and of $\mathtt{H}_c$, complete the proof of part 4. □

In the proof of the two main theorems, we will need the following identities.

Proposition 2.3. *1. If $i \neq j$, then $[y_j, x_i] = -cs_{ij}$ and $[x_i^{-1}, y_j] = -cx_i^{-1}x_j^{-1}s_{ij}$.*

2. $[y_i, x_i] = t + c\sum_{k\neq i} s_{ij}$ and $[x_i^{-1}, y_i] = tx_i^{-2} + c\sum_{j\neq i} x_i^{-1}x_j^{-1}s_{ij}$.

3. If $i \neq j$, then $[\mathcal{Y}_j, x_i] = -\frac{c}{2}(x_i + x_j)s_{ij}$ and $[x_i^{-1}, \mathcal{Y}_j] = -\frac{c}{2}(x_i^{-1} + x_j^{-1})s_{ij}$.

4. $[\mathcal{Y}_i, x_i] = tx_i + \frac{c}{2}\sum_{j\neq i}(x_i + x_j)s_{ij}$ and $[x_i^{-1}, \mathcal{Y}_i] = tx_i^{-1} + \frac{c}{2}\sum_{j\neq i}(x_i^{-1} + x_j^{-1})s_{ij}$.

Proof. These are all immediate consequences of the definition of $\mathsf{H}_{t,c}$. □

Proof. (of proposition 2.1. See also [Su].) Because of proposition 2.2, part 4, and the PBW-property of $\mathsf{H}_{t,c}$ and $\mathbf{H}_{t,c}$, we only have to check the relation involving π in definition 2.2. First, assume that $i \neq l$.

$$\begin{aligned}
\pi\mathcal{U}_i &= (x_1 s_{12}\cdots s_{l-1,l})(\mathcal{U}_i) = x_1(\mathcal{U}_{i+1} - cs_{1,i+1})s_{12}\cdots s_{l-1,l} \\
&= ([x_1, \mathcal{U}_{i+1}] + \mathcal{U}_{i+1}x_1 - cx_1 s_{1,i+1})s_{12}\cdots s_{l-1,l} \\
&= \big(cx_{i+1}s_{1,i+1} + c[x_1, s_{1,i+1}] + \mathcal{U}_{i+1}x_1 - cx_1 s_{1,i+1}\big)s_{12}\cdots s_{l-1,l} \\
&= (\mathcal{U}_{i+1})x_1 s_{12}\cdots s_{l-1,l} = \mathcal{U}_{i+1}\pi
\end{aligned}$$

If $i=l$, we obtain:

$$\begin{aligned}\pi\mathcal{U}_l &= x_1(\mathcal{U}_1+c\sum_{j=2}^{l}s_{1,j})s_{12}\cdots s_{l-1,l} = \big(x_1[x_1,y_1]+cx_1\sum_{j=2}^{l}s_{1,j}+\mathcal{U}_1x_1\big)s_{12}\cdots s_{l-1,l}\\ &= \big(x_1(-t-c\sum_{i\neq 1}s_{i,1})+cx_1\sum_{j=2}^{l}s_{1,j}+\mathcal{U}_1x_1\big)s_{12}\cdots s_{l-1,l} \;=\; (\mathcal{U}_1-t)\pi\end{aligned}$$

□

Corollary 2.1 (of proposition 2.1). *The algebra $\mathbf{H}_{t,c}$ can also be defined as the algebra generated by the elements $x_1^{\pm},\dots,x_l^{\pm}$, $\mathcal{Y}_1,\dots,\mathcal{Y}_l$ and S_l with the relations*

$$w\cdot x_i\cdot w^{-1}=x_{w(i)}\quad w\cdot\mathcal{Y}_i\cdot w^{-1}=\mathcal{Y}_{w(i)}\quad [\mathcal{Y}_j,\mathcal{Y}_k]=\frac{c^2}{4}\sum_{\substack{i=1\\ i\neq j,k}}^{l}(s_{jk}s_{ik}-s_{kj}s_{ij})$$

$$\mathcal{Y}_jx_i-x_i\mathcal{Y}_j=t\delta_{ij}x_i+\frac{c}{2}\sum_{\alpha\in R^+}\langle\alpha,y_j\rangle\langle x_i,\alpha^\vee\rangle(x_is_\alpha+s_\alpha x_i).$$

3 Finite and loop Yangians

The Yangians of finite type are quantum groups, introduced by V. Drinfeld in [Dr1], which are quantizations of the enveloping algebra of the polynomial loop algebra $\mathfrak{g}\otimes_{\mathbb{C}}\mathbb{C}[u]$ of a semisimple Lie algebra $\mathfrak{g}$. The second definition in [Dr2] is given in terms of a finite Cartan matrix. If we replace it with a Cartan matrix of affine type, we obtain algebras that we call loop Yangians $LY_{\beta,\lambda}$. Let $C_{n-1}=(c_{ij})_{1\le i,j\le n-1}$ ($\widehat{C}_{n-1}=(c_{ij})_{0\le i,j\le n-1}$) be a Cartan matrix of finite (resp. affine) type A_{n-1} (resp. $A_{n-1}^{(1)}$). If $n\geq 3$:

$$\widehat{C}_{n-1}=\begin{pmatrix}2&-1&0&\cdots&\cdots&0&-1\\-1&2&-1&0&\cdots&\cdots&0\\0&-1&2&-1&0&\cdots&0\\ \vdots&&\vdots&&\vdots&&\vdots\\0&\cdots&0&-1&2&-1&0\\0&\cdots&\cdots&0&-1&2&-1\\-1&0&\cdots&\cdots&0&-1&2\end{pmatrix}$$

Definition 3.1. *[Dr2], [ChPr2] Let $\lambda\in\mathbb{C}$. The Yangian Y_λ associated to C_{n-1} is the algebra generated by $X_{i,r}^{\pm},H_{i,r},i=1,\dots n-1,r\in\mathbb{Z}_{\geq 0}$, which satisfy the following relations :*

$$[H_{i,r},H_{j,s}]=0,\quad [H_{i,0},X_{j,s}^{\pm}]=\pm c_{ij}X_{j,s}^{\pm} \tag{1}$$

$$[H_{i,r+1},X_{j,s}^{\pm}]-[H_{i,r},X_{j,s+1}^{\pm}]=\pm\frac{\lambda}{2}c_{ij}(H_{i,r}X_{j,s}^{\pm}+X_{j,s}^{\pm}H_{i,r}) \tag{2}$$

$$[X_{i,r}^{+},X_{j,s}^{-}]=\delta_{ij}H_{i,r+s}\quad [X_{i,r}^{\pm},X_{j,s}^{\pm}]=0 \text{ if } 1<|j-i|<n-1 \tag{3}$$

$$[X^{\pm}_{i,r+1}, X^{\pm}_{j,s}] - [X^{\pm}_{i,r}, X^{\pm}_{j,s+1}] = \pm\frac{\lambda}{2} c_{ij}(X^{\pm}_{i,r}X^{\pm}_{j,s} + X^{\pm}_{j,s}X^{\pm}_{i,r}) \tag{4}$$

$$\left[X^{\pm}_{i,r_1}, [X^{\pm}_{i,r_2}, X^{\pm}_{j,s}]\right] + \left[X^{\pm}_{i,r_2}, [X^{\pm}_{i,r_1}, X^{\pm}_{j,s}]\right] = 0 \ \forall r_1, r_2, s \geq 0 \ \text{if}\ j - i \equiv \pm 1 \ \text{mod}\ n \tag{5}$$

Remark 3.1. *The Yangian Y_{λ_1} is isomorphic to Y_{λ_2} if $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$.*

Definition 3.2. *Let $\beta, \lambda \in \mathbb{C}$. The Yangian $LY_{\beta,\lambda}$ associated to $\widehat{C}_{n-1}$ is the algebra generated by $X^{\pm}_{i,r}, H_{i,r}, i = 0, \ldots n-1, r \in \mathbb{Z}_{\geq 0}$, which satisfy the relations of definition 3.1 for $i, j \in \{0, \ldots, n-1\}$ except that the relations (2),(4) must be modified for $i = 0$ and $j = 1, n-1$ in the following way when $n \geq 3$:*

$$[H_{1,r+1}, X^{\pm}_{0,s}] - [H_{1,r}, X^{\pm}_{0,s+1}] = (\beta - \frac{\lambda}{2} \mp \frac{\lambda}{2})H_{1,r}X^{\pm}_{0,s} + (\frac{\lambda}{2} \mp \frac{\lambda}{2} - \beta)X^{\pm}_{0,s}H_{1,r} \tag{6}$$

$$[H_{0,r+1}, X^{\pm}_{1,s}] - [H_{0,r}, X^{\pm}_{1,s+1}] = (\frac{\lambda}{2} \mp \frac{\lambda}{2} - \beta)H_{0,r}X^{\pm}_{1,s} + (\beta - \frac{\lambda}{2} \mp \frac{\lambda}{2})X^{\pm}_{1,s}H_{0,r} \tag{7}$$

$$[H_{0,r+1}, X^{\pm}_{n-1,s}] - [H_{0,r}, X^{\pm}_{n-1,s+1}] = (\beta - \frac{\lambda}{2} \mp \frac{\lambda}{2})H_{0,r}X^{\pm}_{n-1,s} + (\frac{\lambda}{2} \mp \frac{\lambda}{2} - \beta)X^{\pm}_{n-1,s}H_{0,r} \tag{8}$$

$$[H_{n-1,r+1}, X^{\pm}_{0,s}] - [H_{n-1,r}, X^{\pm}_{0,s+1}] = (\frac{\lambda}{2} \mp \frac{\lambda}{2} - \beta)H_{n-1,r}X^{\pm}_{0,s} + (\beta - \frac{\lambda}{2} \mp \frac{\lambda}{2})X^{\pm}_{0,s}H_{n-1,r} \tag{9}$$

$$[X^{\pm}_{1,r+1}, X^{\pm}_{0,s}] - [X^{\pm}_{1,r}, X^{\pm}_{0,s+1}] = (\beta - \frac{\lambda}{2} \mp \frac{\lambda}{2})X^{\pm}_{1,r}X^{\pm}_{0,s} + (\frac{\lambda}{2} \mp \frac{\lambda}{2} - \beta)X^{\pm}_{0,s}X^{\pm}_{1,r} \tag{10}$$

$$[X^{\pm}_{0,r+1}, X^{\pm}_{n-1,s}] - [X^{\pm}_{0,r}, X^{\pm}_{n-1,s+1}] = (\beta - \frac{\lambda}{2} \mp \frac{\lambda}{2})X^{\pm}_{0,r}X^{\pm}_{n-1,s} + (\frac{\lambda}{2} \mp \frac{\lambda}{2} - \beta)X^{\pm}_{n-1,s}X^{\pm}_{0,r} \tag{11}$$

We will also impose the relation $\sum_{i=0}^{n-1} H_{i,0} = 0$.

Remark 3.2. *We set $X^{\pm}_i = X^{+}_{i,0}, H_i = H_{i,0}$. If $\beta = \frac{\lambda}{2}$, the relations defining $LY_{\beta,\lambda}$ are the same as those in definition 3.1 with $i, j \in \{0, \ldots, n-1\}$. Note also that the relations (6),(7),(8) and (9) all follow from (10) and (11) using relation (3); they were added above as a convenient reference since they will be useful later in our computations. We should also note that $LY_{\beta_1,\lambda_1} \cong LY_{\beta_2,\lambda_2}$ if $\beta_2 = \eta\beta_1$ and $\lambda_2 = \eta\lambda_1$ for some $\eta \neq 0$. When no confusion is possible, we will write LY and Y instead of $LY_{\beta,\lambda}$ and Y_λ. Without the relation $\sum_{i=0}^{n-1} H_i = 0$, we obtain the affine Yangian $\widehat{Y}_{\lambda,\beta}$.*

Let $\Delta = \{\epsilon_{ij}, 1 \leq i \neq j \leq n\}$ be the root system of type A_{n-1}, $\Delta \subset E = \text{span}\{\epsilon_i, i = 1, \ldots, n\}, \epsilon_{ij} = \epsilon_i - \epsilon_j$. We denote by $(\cdot, \cdot)$ the non-degenerate bilinear form on E given by $(\epsilon_i, \epsilon_j) = \delta_{ij}$. For a positive root $\epsilon \in \Delta^+$, we denote by $X^{\pm}_\epsilon$ the corresponding standard root vector of $\mathfrak{sl}_n$. If $\epsilon = \epsilon_{ij}, i < j$, then $X^+_\epsilon = E_{ij}$ and $X^-_\epsilon = E_{ji}$, where E_{rs} is the matrix with 1 in the (r, s)-entry and zeros everywhere else. In particular, $X^+_\theta = E_{1n}$ and $X^-_\theta = E_{n1}$, where θ is the longest root of $\mathfrak{sl}_n$. If $\epsilon = \epsilon_{i,i+1}$, then $X^{\pm}_\epsilon = X^{\pm}_i$.

One useful observation is that these two Yangians are generated by $X^{\pm}_{i,r}, H_{i,r}, i = 1, \ldots n-1$ (resp. $i = 0, \ldots, n-1$) with $r = 0, 1$ only. The other elements are obtained inductively by the formulas:

$$X^{\pm}_{i,r+1} = \pm\frac{1}{2}[H_{i,1}, X^{\pm}_{i,r}] - \frac{1}{2}(H_iX^{\pm}_{i,r} + X^{\pm}_{i,r}H_i), \quad H_{i,r+1} = [X^+_{i,r}, X^-_{i,1}]. \tag{12}$$

Furthermore, the subalgebra generated by the elements with $r = 0$ is a quotient of (actually, is isomorphic to) the enveloping algebra of the Lie (resp. loop) algebra $\mathfrak{sl}_n$ (resp. $L\mathfrak{sl}_n = \mathfrak{sl}_n \otimes_{\mathbb{C}}$

$\mathbb{C}[u,u^{-1}]$). The subalgebra Y_λ^0 generated by the elements with $i\neq 0$ is a quotient of the Yangian Y_λ.

The two subalgebras Y_λ^0 and $\mathfrak{U}(L\mathfrak{sl}_n)$ generate $LY_{\beta,\lambda}$. Indeed, combining the observations in the previous two paragraphs, we see that we only have to show that the subalgebra they generate contains $X_{0,1}^{\pm}$. From the relation (1) in definition 3.2 with $i=1$, we know that $[H_1,X_{0,1}^{\pm}]=\mp X_{0,1}^{\pm}$, so, substituting into equation (6), we obtain

$$[H_{1,1},X_0^{\pm}]\pm X_{0,1}^{\pm}=(\beta-\frac{\lambda}{2}\mp\frac{\lambda}{2})H_1X_0^{\pm}+(\frac{\lambda}{2}\mp\frac{\lambda}{2}-\beta)X_0^{\pm}H_1.$$

Thus $X_{0,1}^{\pm}$ (hence also $H_{0,1}$) belongs to the subalgebra of $LY_{\beta,\lambda}$ generated by Y_λ^0 and $\mathfrak{U}(L\mathfrak{sl}_n)$.

For $1\le i\le n-1$, set $J(X_i^{\pm})=X_{i,1}^{\pm}+\lambda\omega_i^{\pm}$ and $\omega_i^{\pm}=\omega_{i,1}^{\pm}-\omega_{i,2}^{\pm}$ where

$$\omega_{i,1}^{\pm}=\pm\frac{1}{4}\sum_{\epsilon\in\Delta^+}\left([X_i^{\pm},X_\epsilon^{\pm}]X_\epsilon^{\mp}+X_\epsilon^{\mp}[X_i^{\pm},X_\epsilon^{\pm}]\right),\ \omega_{i,2}^{\pm}=\frac{1}{4}(X_i^{\pm}H_i+H_iX_i^{\pm})$$

$$J(H_i)=H_{i,1}+\lambda\nu_i \text{ where } \nu_i=\frac{1}{4}\sum_{\epsilon\in\Delta^+}(\epsilon,\epsilon_{i,i+1})(X_\epsilon^+X_\epsilon^-+X_\epsilon^-X_\epsilon^+)-\frac{1}{2}H_i^2.$$

More explicitly, since $X_i^+=E_{i,i+1}$, $X_i^-=E_{i+1,i}$ and $H_i=E_{ii}-E_{i+1,i+1}$ for $1\le i\le n-1$, we can write

$$\omega_i^+=\frac{1}{4}\sum_{\substack{j=1\\ j\neq i,i+1}}^{n}\operatorname{sign}(j-i)(E_{j,i+1}E_{ij}+E_{ij}E_{j,i+1})-\frac{1}{4}(E_{i,i+1}H_i+H_iE_{i,i+1})\tag{13}$$

$$\omega_i^-=\frac{1}{4}\sum_{\substack{j=1\\ j\neq i,i+1}}^{n}\operatorname{sign}(j-i)(E_{i+1,j}E_{ji}+E_{ji}E_{i+1,j})-\frac{1}{4}(E_{i+1,i}H_i+H_iE_{i+1,i})\tag{14}$$

It is possible to define elements $J(z)\in Y$ for any $z\in\mathfrak{sl}_n$ in such a way that $[J(z_1),z_2]=J([z_1,z_2])$: this follows from the isomorphism given in [Dr2] between two different realizations of the Yangian Y_λ: the one given above and the one first given in [Dr1] in terms of generators $z,J(z)\ \forall z\in\mathfrak{sl}_n$ (the $J(z)$'s satisfy a "deformed" Jacobi identity).

In the proof of our first main theorem, the following algebra automorphism will be very important.

Lemma 3.1. *It is possible to define an algebra automorphism ρ of LY by setting*

$$\rho(H_{i,r})=\sum_{s=0}^{r}\binom{r}{s}\left(\frac{\lambda}{2}\right)^{r-s}H_{i-1,s},\quad \rho(X_{i,r}^{\pm})=\sum_{s=0}^{r}\binom{r}{s}\left(\frac{\lambda}{2}\right)^{r-s}X_{i-1,s}^{\pm}\quad \textit{for } i\neq 0,1$$

$$\rho(H_{i,r})=\sum_{s=0}^{r}\binom{r}{s}\beta^{r-s}H_{i-1,s},\quad \rho(X_{i,r}^{\pm})=\sum_{s=0}^{r}\binom{r}{s}\beta^{r-s}X_{i-1,s}^{\pm}\quad \textit{for } i=0,1$$

We use the convention that $X_{-1,r}^{\pm}=X_{n-1,r}^{\pm}$ and $H_{-1,r}=H_{n-1,r}$. Note that, in particular, $\rho(X_i^{\pm})=X_{i-1}^{\pm},\rho(H_i)=H_{i-1}\ \forall i$ and $\rho(X_{i,1}^{\pm})=X_{i-1,1}^{\pm}+\frac{\lambda}{2}X_{i-1}^{\pm},\rho(H_{i,1})=H_{i-1,1}+\frac{\lambda}{2}H_{i-1}$ if $i\neq 0,1$, whereas $\rho(X_{i,1}^{\pm})=X_{i-1,1}^{\pm}+\beta X_{i-1}^{\pm},\rho(H_{i,1})=H_{i-1,1}+\beta H_{i-1}$ if $i=0,1$. The automorphism ρ is very similar to the automorphism $\tau_{\frac{\lambda}{2}}$ (or τ_β) in [ChPr2] followed by a decrement of the first index.

Proof of lemma 3.1. We have to verify that ρ is indeed an automorphism of LY, that is, that it respects the defining relations of LY. In the case when $i,j \neq 0,1$ in the relations (1)-(5), this follows from the fact that ρ is the same as the automorphism $\tau_{\frac{\lambda}{2}}$ from [ChPr2] followed by a decrement of the indices. A short verification shows that ρ preserves the relations (1),(3) and (5) when $i = 0,1$ or $j = 0,1$. (In the case of equation (3) and $i = j$, , one has to use the identity $\sum_{a+b=k} \begin{pmatrix} r \\ a \end{pmatrix} \begin{pmatrix} s \\ b \end{pmatrix} = \begin{pmatrix} r+s \\ k \end{pmatrix}$.) Since the relations (6)-(9) follow from (10) and (11) by applying $[\cdot, X^{\mp}_{?,0}], ? = 0,1,n-1$, there are three cases left that require a more detailed verification. We will use the identity $\begin{pmatrix} r \\ a \end{pmatrix} = \begin{pmatrix} r-1 \\ a \end{pmatrix} + \begin{pmatrix} r-1 \\ a-1 \end{pmatrix}$.

Case 1: With $i = 2, j = 1$ in relation (4), we find that $\rho\big([X^{\pm}_{2,r+1}, X^{\pm}_{1,s}] - [X^{\pm}_{2,r}, X^{\pm}_{1,s+1}]\big)$ is equal to

$$
\begin{aligned}
&= \sum_{a=0}^{r+1}\sum_{b=0}^{s} \begin{pmatrix} r+1 \\ a \end{pmatrix} \begin{pmatrix} s \\ b \end{pmatrix} \left[\left(\frac{\lambda}{2}\right)^{r+1-a} X^{\pm}_{1,a}, \beta^{s-b} X^{\pm}_{0,b}\right] \\
&\quad - \sum_{a=0}^{r}\sum_{b=0}^{s+1} \begin{pmatrix} r \\ a \end{pmatrix} \begin{pmatrix} s+1 \\ b \end{pmatrix} \left[\left(\frac{\lambda}{2}\right)^{r-a} X^{\pm}_{1,a}, \beta^{s+1-b} X^{\pm}_{0,b}\right] \\
&= \sum_{a=0}^{r+1}\sum_{b=0}^{s} \left(\begin{pmatrix} r \\ a \end{pmatrix} + \begin{pmatrix} r \\ a-1 \end{pmatrix}\right) \begin{pmatrix} s \\ b \end{pmatrix} \left(\frac{\lambda}{2}\right)^{r+1-a} \beta^{s-b} [X^{\pm}_{1,a}, X^{\pm}_{0,b}] \\
&\quad - \sum_{a=0}^{r}\sum_{b=0}^{s+1} \begin{pmatrix} r \\ a \end{pmatrix} \left(\begin{pmatrix} s \\ b \end{pmatrix} + \begin{pmatrix} s \\ b-1 \end{pmatrix}\right) \left(\frac{\lambda}{2}\right)^{r-a} \beta^{s+1-b} [X^{\pm}_{1,a}, X^{\pm}_{0,b}] \\
&= \sum_{a=0}^{r}\sum_{b=0}^{s} \begin{pmatrix} r \\ a \end{pmatrix} \begin{pmatrix} s \\ b \end{pmatrix} \left(\frac{\lambda}{2}\right)^{r-a} \beta^{s-b} \left(\frac{\lambda}{2} - \beta\right) [X^{\pm}_{1,a}, X^{\pm}_{0,b}] \\
&\quad + \sum_{a=0}^{r+1}\sum_{b=0}^{s} \begin{pmatrix} r \\ a-1 \end{pmatrix} \begin{pmatrix} s \\ b \end{pmatrix} \left(\frac{\lambda}{2}\right)^{r-a+1} \beta^{s-b} [X^{\pm}_{1,a}, X^{\pm}_{0,b}] \\
&\quad - \sum_{a=0}^{r}\sum_{b=0}^{s+1} \begin{pmatrix} r \\ a \end{pmatrix} \begin{pmatrix} s \\ b-1 \end{pmatrix} \left(\frac{\lambda}{2}\right)^{r-a} \beta^{s-b+1} [X^{\pm}_{1,a}, X^{\pm}_{0,b}] \\
&= \sum_{a=0}^{r}\sum_{b=0}^{s} \begin{pmatrix} r \\ a \end{pmatrix} \begin{pmatrix} s \\ b \end{pmatrix} \left(\frac{\lambda}{2}\right)^{r-a} \beta^{s-b} \left(\frac{\lambda}{2} - \beta\right) [X^{\pm}_{1,a}, X^{\pm}_{0,b}] \\
&\quad + \sum_{\tilde{a}=0}^{r}\sum_{b=0}^{s} \begin{pmatrix} r \\ \tilde{a} \end{pmatrix} \begin{pmatrix} s \\ b \end{pmatrix} \left(\frac{\lambda}{2}\right)^{r-\tilde{a}} \beta^{s-b} [X^{\pm}_{1,\tilde{a}+1}, X^{\pm}_{0,b}] \\
&\quad - \sum_{a=0}^{r}\sum_{\tilde{b}=0}^{s} \begin{pmatrix} r \\ a \end{pmatrix} \begin{pmatrix} s \\ \tilde{b} \end{pmatrix} \left(\frac{\lambda}{2}\right)^{r-a} \beta^{s-\tilde{b}} [X^{\pm}_{1,a}, X^{\pm}_{0,\tilde{b}+1}] \\
&= \sum_{a=0}^{r}\sum_{b=0}^{s} \begin{pmatrix} r \\ a \end{pmatrix} \begin{pmatrix} s \\ b \end{pmatrix} \left(\frac{\lambda}{2}\right)^{r-a} \beta^{s-b} \left(\frac{\lambda}{2} - \beta\right) [X^{\pm}_{1,a}, X^{\pm}_{0,b}] \\
&\quad + \sum_{\tilde{a}=0}^{r}\sum_{\tilde{b}=0}^{s} \begin{pmatrix} r \\ \tilde{a} \end{pmatrix} \begin{pmatrix} s \\ \tilde{b} \end{pmatrix} \left(\frac{\lambda}{2}\right)^{r-\tilde{a}} \beta^{s-\tilde{b}} \big([X^{\pm}_{1,\tilde{a}+1}, X^{\pm}_{0,\tilde{b}}] - [X^{\pm}_{1,\tilde{a}}, X^{\pm}_{0,\tilde{b}+1}]\big)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\tilde a=0}^{r}\sum_{\tilde b=0}^{s}\begin{pmatrix} r\\ \tilde a\end{pmatrix}\begin{pmatrix} s\\ \tilde b\end{pmatrix}\left(\frac{\lambda}{2}\right)^{r-\tilde a}\beta^{s-\tilde b}(\frac{\lambda}{2}-\beta)[X^{\pm}_{1,\tilde a},X^{\pm}_{0,\tilde b}]\\
&\quad +\sum_{\tilde a=0}^{r}\sum_{\tilde b=0}^{s}\begin{pmatrix} r\\ \tilde a\end{pmatrix}\begin{pmatrix} s\\ \tilde b\end{pmatrix}\left(\frac{\lambda}{2}\right)^{r-\tilde a}\beta^{s-\tilde b}((\beta-\frac{\lambda}{2}\mp\frac{\lambda}{2})X^{\pm}_{1,\tilde a}X^{\pm}_{0,\tilde b}+(\frac{\lambda}{2}\mp\frac{\lambda}{2}-\beta)X^{\pm}_{0,\tilde b}X^{\pm}_{1,\tilde a}))\\
&= \sum_{\tilde a=0}^{r}\sum_{\tilde b=0}^{s}\begin{pmatrix} r\\ \tilde a\end{pmatrix}\begin{pmatrix} s\\ \tilde b\end{pmatrix}\left(\frac{\lambda}{2}\right)^{r-\tilde a}\beta^{s-\tilde b}\left(\mp\frac{\lambda}{2}\right)(X^{\pm}_{1,\tilde a}X^{\pm}_{0,\tilde b}+X^{\pm}_{0,\tilde b}X^{\pm}_{1,\tilde a})\\
&= \rho\left(\left(\mp\frac{\lambda}{2}\right)(X^{\pm}_{2,r}X^{\pm}_{1,s}+X^{\pm}_{1,s}X^{\pm}_{2,r})\right)
\end{aligned}
$$

Case 2: $i=1, j=0$. We have to prove that

$$\rho\big([X^{\pm}_{1,r+1},X^{\pm}_{0,s}]-[X^{\pm}_{1,r},X^{\pm}_{0,s+1}]\big)=\rho\big((\beta-\frac{\lambda}{2}\mp\frac{\lambda}{2})X^{\pm}_{1,r}X^{\pm}_{0,s}+(\frac{\lambda}{2}\mp\frac{\lambda}{2}-\beta)X^{\pm}_{0,s}X^{\pm}_{1,r}\big).$$

This case is analogous to case 1, but a little bit simpler since the term $(\frac{\lambda}{2}-\beta)$ above becomes $(\beta-\beta)=0$.

Case 3: $i=0, j=n-1$. We have to show that

$$\rho\big([X^{\pm}_{0,r+1},X^{\pm}_{n-1,s}]-[X^{\pm}_{0,r},X^{\pm}_{n-1,s+1}]\big)=\rho\big((\beta-\frac{\lambda}{2}\mp\frac{\lambda}{2})X^{\pm}_{0,r}X^{\pm}_{n-1,s}+(\frac{\lambda}{2}\mp\frac{\lambda}{2}-\beta)X^{\pm}_{n-1,s}X^{\pm}_{0,r}\big).$$

The computations are again very similar to those of case 1: the main difference is that the factor $(\frac{\lambda}{2}-\beta)$ gets replaced by $(\beta-\frac{\lambda}{2})$. □

4 Relations with DAHA's and toroidal quantum algebras

It is known that the Yangians of finite type can be obtained from quantum loop algebras via a limiting procedure [Dr3] and that the same is true about the trigonometric Cherednik algebra and the double affine Hecke algebra (or elliptic Cherednik algebra), see [Ch3] for instance. We will recall these results and explain how the loop Yangians introduced in section 3 can be obtained from toroidal quantum algebras.

Definition 4.1 (Cherednik). *Let $q,\kappa\in\mathbb{C}^{\times}$. The double affine Hecke algebra $\mathbb{H}_{q,\kappa}$ of type $\mathfrak{gl}_l$ is the unital associative algebra over $\mathbb{C}$ with generators $T_i^{\pm1},X_j^{\pm1},Y_j^{\pm}$ for $i\in\{1,\dots,l-1\}$ and $j\in\{1,\dots,l\}$ satisfying the following relations:*

$$(T_i+1)(T_i-q^2)=0,\quad T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1}$$

$$T_iT_j=T_jT_i \text{ if } |i-j|>1,\quad X_0Y_1=\kappa Y_1X_0,\quad X_2Y_1^{-1}X_2^{-1}Y_1=q^{-2}T_1^2$$

$$X_iX_j=X_jX_i,\quad Y_iY_j=Y_jY_i,\quad T_iX_iT_i=q^2X_{i+1},\quad T_i^{-1}Y_iT_i^{-1}=q^{-2}Y_{i+1},$$

$$X_jT_i=T_iX_j, Y_jT_i=T_iY_j \text{ if } j\neq i,i+1$$

where $X_0=X_1X_2\cdots X_l$.

Remark 4.1. *We set $y = 1$ in definition 1.1 in [VaVa1].*

The trigonometric Cherednik algebra can be viewed as a limit (degenerate) version of the double affine Hecke algebra. We sketch here a few computations which illustrate this fact. We extend the scalars from $\mathbb{C}$ to $\mathbb{C}[[h]]$ and consider the completed algebra $\mathbb{H}_{q,\kappa}[[h]]$ with $q = e^{\frac{c}{2}h}, \kappa = e^{th}$. Setting $Y_i = e^{-h\mathcal{U}_i}$, the equality $X_2 Y_1^{-1} X_2^{-1} Y_1 = q^{-2} T_1^2$ becomes $X_2(1 + h\mathcal{U}_1)X_2^{-1}(1 - h\mathcal{U}_1) = (1 - ch)((chT_1 + 1 + ch) + o(h^2)$, where $o(h^2)$ is in $h^2\mathbb{H}_{q,\kappa}[[h]]$. Cancelling the constant term 1 on both sides, dividing by h and then letting $h \to 0$ gives $x_2\mathcal{U}_1 x_2^{-1} - \mathcal{U}_1 = cs_{12}$, which implies that $[\mathcal{U}_1, X_2] = -cs_{12}x_2$. (In this limit, the finite Hecke algebra identifies with $\mathbb{C}[S_l]$ and T_1 with s_{12}.) This is indeed the relation between $\mathcal{U}_1$ and x_2 in $\mathbf{H}_{t,c}$, as can be seen from the third relation in proposition 2.3 with $j = 1, i = 2$.

If we make the same substitution in the relation $X_0 Y_1 = \kappa Y_1 X_0$, we obtain $X_1 \cdots X_l(1-h\mathcal{U}_1) = (1+th)(1 - h\mathcal{U}_1)X_1 \cdots X_l + o(h^2)$. Subtracting $X_1 \cdots X_l$ on both sides, dividing by h and letting $h \to 0$ gives $[\mathcal{U}_1, x_1 \cdots x_l] = tx_1 \cdots x_l$, which implies that $\sum_{k=1}^{l} x_1 \cdots x_{k-1}[\mathcal{U}_1, x_k]x_{k+1} \cdots x_l = tx_1 \cdots x_l$. Since $[\mathcal{U}_1, x_k] = -cs_{1k}x_k$ for $k \neq 1$, $[\mathcal{U}_1, x_1]x_2 \cdots x_l - c\sum_{k=2}^{l} s_{1k}x_2 \cdots x_l = tx_1 \cdots x_l$, which leads to $[\mathcal{U}_1, x_1] = tx_1 + c\sum_{k=2}^{l} s_{1k}x_k$: after some simplifications, we obtain the fourth relation in proposition 2.3.

The rational, trigonometric and elliptic Cherednik algebras of S_l are all isomorphic after completion: see [Ch3] for a detailed discussion. This implies that, for generic values of the deformation parameters, these three algebras have equivalent categories of finite dimensional representations. For modules which are not finite dimensional, we don't have such an equivalence in general. However, it is sometimes possible to lift a module over $\mathbf{H}$ or H to one over $\mathbb{H}$ if the parameters satisfy certain technical conditions: see [Ch3] section 2.12 for more on this subject.

Definition 4.2. *Let $q_1, q_2 \in \mathbb{C}^\times$. The toroidal quantum algebra $\ddot{\mathbf{U}}_{q_1,q_2}$ of type A_{n-1} is the unital associative algebra over $\mathbb{C}$ with generators $e_{i,r}, f_{i,r}, k_{i,r}k_{i,0}^{-1}, i \in \{0, \ldots, n-1\}, r \in \mathbb{Z}$ which satisfy the following relations:*

$$[k_{i,r}, k_{j,s}] \; \forall i, j \in \{0, \ldots, n-1\}, \forall r, s \in \mathbb{Z} \tag{15}$$

$$k_{i,0}e_{j,r} = q_1^{c_{ij}} e_{j,r}k_{i,0}, \;\; k_{i,0}f_{j,r} = q_1^{-c_{ij}} f_{j,r}k_{i,0}, \;\; (q_1 - q_1^{-1})[e_{i,r}, f_{j,s}] = \delta_{ij}(k_{i,r+s}^{+} - k_{i,r+s}^{-}) \tag{16}$$

(Here, $k_{i,r+s}^{\pm} = k_{i,r+s}$ if $\pm(r+s) \geq 0$ and $= 0$ otherwise.)

The next three relations hold $\forall i, j \in \{0, \ldots, n-1\}, \forall r, s \in \mathbb{Z}$ except for $\{i, j\} = \{n-1, 0\}, \{0, 1\}$:

$$k_{i,r+1}e_{j,s} - q_1^{c_{ij}} k_{i,r}e_{j,s+1} = q_1^{c_{ij}} e_{j,s}k_{i,r+1} - e_{j,s+1}k_{i,r} \tag{17}$$

$$e_{i,r+1}e_{j,s} - q_1^{c_{ij}} e_{i,r}e_{j,s+1} = q_1^{c_{ij}} e_{j,s}e_{i,r+1} - e_{j,s+1}e_{i,r} \tag{18}$$

$$\{e_{i,r}e_{i,s}e_{j,t} - (q_1 + q_1^{-1})e_{i,r}e_{j,t}e_{i,s} + e_{j,t}e_{i,r}e_{i,s}\} + \{r \leftrightarrow s\} = 0 \text{ if } i - j \equiv \pm 1 \text{ mod } n-1 \tag{19}$$

The same relations hold with $e_{i,r}$ replaced by $f_{i,r}$ and $q_1^{c_{ij}}$ by $q_1^{-c_{ij}}$.

In the cases $\{i, j\} = \{n-1, 0\}, \{0, 1\}$, we must modify the relations (17)-(19) above in the following way: we introduce a second parameter q_2 in such a way that we obtain an algebra isomorphism Ψ of $\ddot{\mathbf{U}}_{q_1,q_2}$ given by $e_{i,r}, f_{i,r}, k_{i,r} \mapsto q_1^r e_{i-1,r}, q_1^r f_{i-1,r}, q_1^r k_{i-1,r}$ for $2 \leq i \leq n-1$ and $e_{i,r}, f_{i,r}, k_{i,r} \mapsto$

$q_2^r e_{i-1,r}, q_2^r f_{i-1,r}, q_2^r k_{i-1,r}$ if $i = 0, 1$. (We identify $e_{-1,r}$ with $e_{n-1,r}$, etc.) For instance, relation (18) for $i = 0, j = 1$ becomes

$$q_2 e_{0,r+1} e_{1,s} - e_{0,r} e_{1,s+1} = q_1^{-1} q_2 e_{1,s} e_{0,r+1} - q_1 e_{1,s+1} e_{0,r},$$

and with $i = n-1, j = 0$ we have the very similar identity:

$$q_2 e_{n-1,r+1} e_{0,s} - e_{n-1,r} e_{0,s+1} = q_1^{-1} q_2 e_{0,s} e_{n-1,r+1} - q_1 e_{0,s+1} e_{n-1,r}.$$

Remark 4.2. *We could have expressed the relations above (and also those for Yangians) using power series as in [VaVa1]. The definition in [VaVa1] involves a central parameter c which we have taken to be equal to 1. The subalgebra $\dot{\mathbf{U}}^{hor}_{q_1,q_2}$ generated by the elements $e_{i,0}, f_{i,0}, k_{i,0}^{\pm 1}, i \in \{0, \ldots n-1\}$ is a quotient the quantum affine algebra of type $\widehat{A}_{n-1}$. The subalgebra $\dot{\mathbf{U}}^{ver}_{q_1,q_2}$ generated by the elements $e_{i,r}, f_{i,r}, k_{i,r}, k_{i,0}^{-}, i \in \{1, \ldots n-1\}, r \in \mathbb{Z}$ is a quotient of the quantum loop algebra of type $\widehat{A}_{n-1}$.*

The connection between the representation theory of the quantum affine (resp. toroidal) algebras and the Yangians of finite (resp. affine) type is less direct than in the case of Hecke algebras. However, in view of the relation between $\ddot{\mathbf{U}}$ and LY explained below, which is an extension of a result of Drinfeld in the finite case, one can often expects that results which are true for quantum affine (or toroidal) algebras have analogs for Yangians which can be proved using similar arguments. It is known that the Yangians of finite type and the quantum affine algebras have the same finite dimensional representation theory: this was proved using geometrical methods in [Va]. More general equivalences are not known at the moment between these two types of algebras.

It is possible to view the Yangian Y_λ as a limit version of the quantum affine algebra $\dot{\mathbf{U}}_q$. The same is true for $\widehat{Y}_{\lambda,\beta}$ and $\ddot{\mathbf{U}}_{q_1,q_2}$. Let $\ddot{\mathbf{U}}[[h]]$ be the completed algebra with parameters $q_1 = e^{\frac{\lambda}{2}h}, q_2 = e^{\beta h}$. Consider the kernel K of the composite map $\ddot{\mathbf{U}}[[h]] \xrightarrow{h\mapsto 0} \ddot{\mathbf{U}}_{h=0} \twoheadrightarrow \mathfrak{U}(\widehat{\mathfrak{sl}}_n) = \dot{\mathbf{U}}^{hor}|_{h=0}$. Let A be the $\mathbb{C}[[h]]$-subalgebra of $\ddot{\mathbf{U}}((h))$ generated by $\ddot{\mathbf{U}}[[h]]$ and $\frac{K}{h}$. Then the quotient A/hA is isomorphic to $\widehat{Y}_{\lambda,\beta}$. To see this, let A^1 be the subalgebra of A generated by $\dot{\mathbf{U}}^{ver}$ and $\frac{K\cap\dot{\mathbf{U}}^{ver}}{h}$. Since $\dot{\mathbf{U}}^{ver}$ is a quotient of the quantum loop algebra, A^1/hA^1 is a quotient of the Yangian Y_λ (see [Dr3]), that is, we have an epimorphism $\zeta : Y_\lambda \longrightarrow A^1/hA^1$. The automorphism Ψ of $\ddot{\mathbf{U}}[[h]]$ induces an automorphism, also denoted Ψ, on A. It is related to the automorphism ρ of $\widehat{Y}_{\lambda,\beta}$ in the following way for $2 \le i \le n$:

$$\Psi(\zeta(X^{\pm}_{i,r})) = \zeta(\rho(X^{\pm}_{i,r})), \quad \Psi(\zeta(H_{i,r})) = \zeta(\rho(H_{i,r}))$$

$$\Psi^2(\zeta(X^{\pm}_{1,r})) = \zeta(\rho^2(X^{\pm}_{1,r})), \quad \Psi^2(\zeta(H_{1,r})) = \zeta(\rho^2(H_{1,r}))$$

From these relations, one concludes that it is possible to extend ζ to $\widehat{Y}_{\lambda,\beta}$ by setting $\zeta(X^{\pm}_{0,r}) = \Psi(\zeta(\rho^{-1}(X^{\pm}_{0,r})))$ and similarly for $H_{0,r}$. One can show that we have thus obtained an isomorphism.

5 Schur-Weyl duality functor

The Schur-Weyl duality established by M. Varagnolo and E. Vasserot [VaVa1] involves, on one side, a toroidal quantum algebra and, on the other side, a double affine Hecke algebra for S_l. Theorem

5.2 establishes a similar type of duality between the trigonometric DAHA $\mathbf{H}_{t,c}$ and the loop Yangian $LY_{\beta,\lambda}$, which extends the duality for the Yangian of finite type due to V. Drinfeld [Dr1].

Before stating the more classical results on the theme of Schur-Weyl duality, we have to define the notion of module of level l over $\mathfrak{sl}_n$ and over the quantized enveloping algebra $\mathfrak{U}_q\mathfrak{sl}_n$. Fix a positive integer n and set $V = \mathbb{C}^n$.

Definition 5.1. *A finite dimensional representation of $\mathfrak{sl}_n$ or $\mathfrak{U}_q\mathfrak{sl}_n$ (q not a root of unity) is of level l if each of its irreducible components is isomorphic to a direct summand of $V^{\otimes l}$.*

Theorem 5.1. *[Ji, Dr1, ChPr1] Fix $l \geq 1, n \geq 2$ and assume that $q \in \mathbb{C}^\times$ is not a root of unity. Let A be one of the algebras $\mathbb{C}[S_l], \mathcal{H}_q(S_l), \mathtt{H}_{c=1}(S_l), \widetilde{\mathcal{H}}_q(S_l)$, and let B be the corresponding one (in the same order) among $\mathfrak{U}\mathfrak{sl}_n, \mathfrak{U}_q\mathfrak{sl}_n, Y_{\lambda=1}(\mathfrak{sl}_n), \mathfrak{U}_q L\mathfrak{sl}_n$. There exists a functor $\mathcal{F}$ from the category of finite dimensional right A-modules to the category of finite dimensional left B-modules which are of level l as $\mathfrak{sl}_n$-modules in the first and third case (and as $\mathfrak{U}_q\mathfrak{sl}_n$-modules in the second and fourth case) which is given by*

$$\mathcal{F}(M) = M \otimes_C V^{\otimes l}$$

where $C = \mathbb{C}[S_l]$ (first and third case) or $C = \mathcal{H}_q(S_l)$ (second and fourth case). Furthermore, this functor is an equivalence of categories if $l \leq n-1$.

The $\mathfrak{sl}_n$ module structure on $V^{\otimes l}$ commutes with the S_l-module structure obtained by simply permuting the factors in the tensor product. Let M be a right module over $\mathbf{H}_{t,c}$. Since $\mathbb{C}[S_l] \subset \mathbf{H}_{t,c}$, we can form the tensor product $\mathcal{F}(M) = M \otimes_{\mathbb{C}[S_l]} V^{\otimes l}$.

On one hand, since $\mathbf{H}_{t,c}$ contains the degenerate affine Hecke algebra $\mathtt{H}_c$, M can be viewed as a right module over $\mathtt{H}_c$, so it follows from [Dr1] that $\mathcal{F}(M)$ is a module of level l over the Yangian Y_λ of $\mathfrak{sl}_n$ with $\lambda = c$. On the other hand, $\mathbf{H}_{t,c}$ also contains a copy of the group algebra of the extended affine Weyl group $\widetilde{S}_l$, so it follows from [ChPr1] (the case $q = 1$) that $\mathcal{F}(M)$ is also a module of level l over the loop algebra $L\mathfrak{sl}_n$. These two module structures can be glued together to obtain a module over LY. This is the content of our first main theorem. Before stating it, we need one definition.

Definition 5.2. *A module M over $LY_{\lambda,\beta}$ (or over $\mathbb{L}_{\lambda,\beta}$) is called integrable if it is the direct sum of its integral weight spaces under the action of $\mathfrak{h}$ and if each generator $X_{i,r}^{\pm}$ acts locally nilpotently on M.*

Theorem 5.2. *Suppose that $l \geq 1, n \geq 3$ and set $\lambda = c, \beta = \frac{t}{2} - \frac{nc}{4} + \frac{c}{2}$. The functor $\mathcal{F} : M \mapsto M \otimes_{\mathbb{C}[S_l]} V^{\otimes l}$ sends a right $\mathbf{H}_{t,c}$-module to an integrable $LY_{\beta,\lambda}$-module of level l (as $\mathfrak{sl}_n$-module). Furthermore, if $l + 2 < n$, this functor is an equivalence.*

Remark 5.1. *This theorem is very similar to the main result of [VaVa1] where $\mathbf{H}_{t,c}$ is replaced by a double affine Hecke algebra and $LY_{\beta,\lambda}$ is replaced by the toroidal quantum algebra $\ddot{\mathbf{U}}_{q_1,q_2}$ (it is defined slightly differently in* op.cit*), under the assumption that the parameter q_1 is not a root of unity. However, theorem 5.2 is not an immediate consequence of the DAHA case in* op.cit. *since, in general, we don't have equivalences between categories of modules over $\mathbf{H}$ and $\mathbb{H}$ or over LY and $\ddot{\mathbf{U}}$. The first part of the proof of theorem 5.2 can be deduced from a proposition in [VaVa1]: this is explained in detail in the next section. However, the fact that the functor $\mathcal{F}$ is essentially surjective must be given an independent (but similar) proof for the aforementioned reason. The same is true for the rational case treated in section 8.*

6 Proof of theorem 5.2

The proof of theorem 5.2 consists of two parts. First, we show how to obtain an integrable $LY_{\beta,\lambda}$-module structure on $\mathcal{F}(M)$, and then we prove that any integrable representation of $LY_{\beta,\lambda}$ of level l is of the form $\mathcal{F}(M)$. If there is no confusion possible for the values of the parameters, we will write $\mathbb{H}, \mathbf{H}, \mathtt{H}, \mathsf{H}, LY$ instead of $\mathbb{H}_{q,\kappa}, \mathbf{H}_{t,c}, \mathtt{H}_c, \mathsf{H}_{t,c}, LY_{\beta,\lambda}$.

6.1 Proof of theorem 5.2, part 1

Fix $m \in M$, $\mathbf{v} = v_{i_1} \otimes \cdots \otimes v_{i_l} \in V^{\otimes l}$, where $\{v_1, \ldots, v_n\}$ is the standard basis of $\mathbb{C}^n$ and $1 \leq i_j \leq n$. The subalgebra $\mathfrak{sl}_n$ generated by the elements $X_i^{\pm}, H_i, 1 \leq i \leq n-1$, acts on $V^{\otimes l}$ as usual. The element $z \otimes u^{\pm} \in L\mathfrak{sl}_n$ acts on $\mathcal{F}(M)$ in the following way:

$$(z \otimes u^{\pm k})(m \otimes \mathbf{v}) = \sum_{j=1}^{l} m x_j^{\pm k} \otimes v_{i_1} \otimes \cdots \otimes (z v_{i_j}) \otimes \cdots \otimes v_{i_l}.$$

For $z \in \mathfrak{sl}_n$, we will write $z^j(\mathbf{v})$ for $v_{i_1} \otimes \cdots \otimes (z v_{i_j}) \otimes \cdot \otimes v_{i_l}$. The elements $J(X_i^{\pm}), J(H_i)$ and $X_{i,1}^{\pm}, H_{i,1}, 1 \leq i \leq n-1$, act on $\mathcal{F}(M)$ in the following way (see [Dr1],[ChPr2]):

$$J(X_i^{\pm})(m \otimes \mathbf{v}) = \sum_{j=1}^{l} m\mathcal{Y}_j \otimes X_i^{\pm,j}(\mathbf{v}), \quad X_{i,1}^{\pm}(m \otimes \mathbf{v}) = J(X_i^{\pm})(m \otimes \mathbf{v}) - \lambda\omega_i^{\pm}(m \otimes \mathbf{v}), \qquad (20)$$

$$J(H_i)(m \otimes \mathbf{v}) = \sum_{j=1}^{l} m\mathcal{Y}_j \otimes H_i^j(\mathbf{v}), \quad H_{i,1}(m \otimes \mathbf{v}) = J(H_i)(m \otimes \mathbf{v}) - \lambda\nu_i(m \otimes \mathbf{v}). \qquad (21)$$

The following observation will be very useful: the action of s_{jk} on $V^{\otimes l}$ if given in terms of matrices by: $s_{jk} = \sum_{r,s=1}^{n} E_{rs}^{j} E_{sr}^{k}$. It is possible to give another, somewhat simpler formula for the action of $X_{k,1}^{\pm}$ and $H_{k,1}$ if we assume that $i_1 \leq i_2 \leq \ldots \leq i_l$. We will denote by j_k (resp. $\tilde{\jmath}_k$) the first (resp. last) value of j such that $i_j = k$ and we set $l_k = \tilde{\jmath}_k - j_k + 1$. We will adopt the following notation: $\mathbf{v}_{j_{k+1}^-} = v_{i_1} \otimes \cdots \otimes v_{i_{j_{k+1}-1}} \otimes v_k \otimes v_{i_{j_{k+1}+1}} \otimes \cdots v_{i_l}$ or $\mathbf{v}_{j_{k+1}^-} = 0$ if $i_j \neq k+1$ for any $1 \leq j \leq l$; $\mathbf{v}_{j_k^+} = v_{i_1} \otimes \cdots \otimes v_{i_{\tilde{\jmath}_k - 1}} \otimes v_{k+1} \otimes v_{i_{\tilde{\jmath}_k+1}} \otimes \cdots \otimes v_{i_l}$ or $\mathbf{v}_{j_k^+} = 0$ if $i_j \neq k$ for any $1 \leq j \leq l$. For $1 \leq k \leq n-1$, we have:

$$X_{k,1}^{+}(m \otimes \mathbf{v}) = m\Big(\sum_{d=j_{k+1}}^{\tilde{\jmath}_{k+1}} s_{d,j_{k+1}}\Big)\Big(\mathcal{U}_{j_{k+1}} - \frac{\lambda}{4}(n-2k)\Big) \otimes \mathbf{v}_{j_{k+1}^-} \qquad (22)$$

$$X_{k,1}^{-}(m \otimes \mathbf{v}) = m\Big(\sum_{d=j_k}^{\tilde{\jmath}_k} s_{d,\tilde{\jmath}_k}\Big)\Big(\mathcal{U}_{\tilde{\jmath}_k} - \frac{\lambda}{4}(n-2k)\Big) \otimes \mathbf{v}_{j_k^+}$$

$$\begin{aligned} H_{k,1}(m\otimes \mathbf{v}) &= (\sum_{d=j_k}^{\tilde{j}_k} s_{d,j_k})\mathcal{U}_{j_k}(m\otimes \mathbf{v}) - (\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}} s_{d,j_{k+1}})\mathcal{U}_{j_{k+1}}(m\otimes \mathbf{v}) \\ & -\lambda\left(\frac{n-2k}{4}\right)(l_k-l_{k+1})(m\otimes\mathbf{v}) + \lambda l_k(l_k-l_{k+1}-1)(m\otimes \mathbf{v}) \end{aligned}$$

We prove only the identity for $X^+_{k,1}$, the other cases being similar. Suppose that $i_1 \le i_2 \le \cdots \le i_l$. We compute:

$$\begin{aligned} \omega^+_{k,1}(\mathbf{v}) &= \frac{(n-2k)}{4}\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}} E^d_{k,k+1}(\mathbf{v}) + \frac{1}{2}\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}} \sum_{\substack{a=1 \\ a\neq j_k,\dots,\tilde{j}_{k+1}}}^{l} \sum_{j=1}^{n} \mathrm{sign}(j-k)E^a_{kj}E^d_{j,k+1}(\mathbf{v}) \\ &= \frac{(n-2k)}{4}\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}} s_{dj_{k+1}}(\mathbf{v}_{j^-_{k+1}}) + \frac{1}{2}\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}} \sum_{\substack{a=1 \\ a\neq j_k,\dots,\tilde{j}_{k+1}}}^{l} \mathrm{sign}(a-j_{k+1})s_{ad}E^d_{k,k+1}(\mathbf{v}) \\ &= \left(\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}} s_{dj_{k+1}}\right)\left(\frac{n-2k}{4} + \frac{1}{2}\sum_{\substack{a=1 \\ a\neq j_k,\dots,\tilde{j}_{k+1}}}^{l} \mathrm{sign}(a-j_{k+1})s_{aj_{k+1}}\right)(\mathbf{v}_{j^-_{k+1}}) \end{aligned} \tag{23}$$

By s_{dd}, we mean simply the identity element in S_l.

$$\omega^+_{k,2}(\mathbf{v}) = \frac{1}{2}\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}} (l_k - l_{k+1}+1)E^d_{k,k+1}(\mathbf{v}) \tag{24}$$

Notice that

$$\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}} s_{dj_{k+1}} \sum_{a=j_k}^{\tilde{j}_k} s_{aj_{k+1}}(\mathbf{v}_{j^-_{k+1}}) = l_k \sum_{d=j_{k+1}}^{\tilde{j}_{k+1}} E^d_{k,k+1}(\mathbf{v})$$

and that

$$\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}} s_{dj_{k+1}} \sum_{a=j_{k+1}+1}^{\tilde{j}_{k+1}} s_{aj_{k+1}}(\mathbf{v}_{j^-_{k+1}}) = (l_{k+1}-1) \sum_{d=j_{k+1}}^{\tilde{j}_{k+1}} E^d_{k,k+1}(\mathbf{v})$$

Putting equalities (20), (23) and (24) together gives us:

$$X^+_{k,1}(m\otimes\mathbf{v}) = m\left(\sum_{d=j_{k+1}}^{\tilde{j}_{k+1}} s_{d,j_{k+1}}\right)\left(\mathcal{Y}_{j_{k+1}} - \frac{\lambda}{2}\left(\frac{n}{2} - k + \sum_{\substack{a=1 \\ a\neq j_{k+1}}}^{l} \mathrm{sign}(a-j_{k+1})s_{aj_{k+1}}\right)\right)\otimes \mathbf{v}_{j^-_{k+1}}$$

This is the formula (22) for the action of $X^+_{k,1}$ on $m\otimes\mathbf{v}$.

Remark 6.1. *For $j_1 \ge j_2$, we define elements in S_l by $\widetilde{s}_{j_1,j_2} = s_{j_1,j_1-1}s_{j_1-1,j_1-2}\cdots s_{j_2+1,j_2}$ and $\widetilde{s}_{j_2,j_1} = s_{j_2,j_2+1}s_{j_2+1,j_2+2}\cdots s_{j_1-1,j_1}$. Then, in formula (22), we can replace $s_{d,j_{k+1}}$ by $\widetilde{s}_{d,j_{k+1}}$ in the $^+$-case and by $\widetilde{s}_{d,\tilde{j}_k}$ in the $^-$ case: one has to notice that we can make this substitution in our computations above.*

Following one of the main ideas in [VaVa1], we define a linear automorphism T of $M \otimes_{\mathbb{C}[S_l]} V^{\otimes l}$ in the following way:

$$T(m \otimes v_{i_1} \otimes \cdots \otimes v_{i_l}) = (m x_1^{-\delta_{i_1,n}} \cdots x_l^{-\delta_{i_l,n}}) \otimes v_{i_1+1} \otimes \cdots \otimes v_{i_l+1},$$

with the convention that $v_{n+1} = v_1$. (Here, $\delta_{i,j}$ is the usual delta function.) We set $\mathbf{v}_{+1} = v_{i_1+1} \otimes \cdots \otimes v_{i_l+1}$. One can check that $T \circ \varphi(X_{i-1}^{\pm}) = \varphi(X_i^{\pm}) \circ T$ and $T \circ \varphi(H_{i-1}) = \varphi(H_i) \circ T$ for any $0 \leq i \leq n-1$, where $\varphi : Y \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathcal{F}(M))$ is the algebra map coming from the Y-module structure on $\mathcal{F}(M)$.

Recall the automorphism ρ from section 3. The following lemma will be crucial.

Lemma 6.1. *Let M be a module over $\mathbf{H}$. Suppose that $\lambda = c$ and $\beta = \frac{t}{2} - \frac{nc}{4} + \frac{c}{2}$. For any $2 \leq i \leq n-1$ and any $r \geq 0$, the following identities between operators on $\mathcal{F}(M)$ hold:*

$$\varphi\big(\rho(X_{i,r}^{\pm})\big) = T^{-1} \circ \varphi(X_{i,r}^{\pm}) \circ T \quad \varphi\big(\rho(H_{i,r})\big) = T^{-1} \circ \varphi(H_{i,r}) \circ T \tag{25}$$

$$\varphi\big(\rho^2(X_{1,r}^{\pm})\big) = T^{-2} \circ \varphi(X_{1,r}^{\pm}) \circ T^2 \quad \varphi\big(\rho^2(H_{1,r})\big) = T^{-2} \circ \varphi(H_{1,r}) \circ T^2 \tag{26}$$

There are two ways to prove this lemma. One is to deduce it from proposition 3.4 in [VaVa1] using the fact that the trigonometric Cherednik algebra is a limit version of the double affine Hecke algebra. The second one is by direct computations. We will start with the first approach and afterwards we will give a sketch of the relevant computations.

Proof. We can restrict ourselves to proving lemma 6.1 when $M = \mathbf{H}$. Since the elements $X_{i,r}^{\pm}, H_{i,r}$ with $r = 0, 1$ generate LY (see equation (12)), it is enough to prove the lemma for $r = 0, 1$. First, we prove relation (25) for $X_{i,1}^{+}$ with $2 \leq i \leq n-1$. The proof for $X_{i,1}^{-}$ is exactly the same and we omit it, and the proof for $H_{i,1}$ follows from either of these two cases using identity (3).

We start with proposition 3.4 in [VaVa1] in the case $M = \mathbb{H}_{q,\kappa}$. We choose a $\mathbf{v}$ as before and assume that $i_1 \leq i_2 \leq \cdots \leq i_l$. The aforementioned proposition, along with theorem 3.3 in *loc. cit.*, says that we have the following identities in $\mathrm{End}_{\mathbb{C}}(\mathbb{H} \otimes_{\mathcal{H}} V^{\otimes l})$ for $k \in \{2, \ldots, n-1\}$ concerning the action of $q^{\frac{n}{2}} e_{k,1} - e_{k,0}$ and $q^{\frac{n}{2}} e_{k-1,1} - e_{k-1,0}$ on $1 \otimes \mathbf{v}$ for $2 \leq k \leq n-1$:

$$q^{1-l_k} X_{j_n}^{-1} \cdots X_l^{-1} \left(1 + \sum_{d=j_k}^{\tilde{j}_k - 1} T_{d,j_k}\right) \big(q^{\frac{n}{2}} (q^{n-k} Y_{j_k})^{-1} - 1\big) \otimes \mathbf{v}_{j_k^- + 1} = \tag{27}$$

$$q^{1-l_k} \left(1 + \sum_{d=j_k}^{\tilde{j}_k - 1} T_{d,j_k}\right) \big(q^{\frac{n}{2}} (q^{n-(k-1)} q^{-1} Y_{j_k})^{-1} - 1\big) X_{j_n}^{-1} \cdots X_l^{-1} \otimes \mathbf{v}_{j_k^- + 1}$$

Here, for $d \geq j_k$, $T_{d,j_k} = T_d T_{d-1} \cdots T_{j_k}$.

Now we extend the base ring by replacing $\mathbb{C}$ by $\mathbb{C}[[h]]$ and set $q = e^{\frac{ch}{2}}, \kappa = e^{th}$. We view both sides of identity (27) as elements of $\mathbb{H}[[h]] \otimes_{\mathcal{H}[[h]]} V^{\otimes l}[[h]]$. Let us denote by $\mathfrak{a} : \mathbb{H}[[h]] \xrightarrow{\sim} \mathbf{H}[[h]]$ an isomorphism between these two completed algebras as described in [Ch3]. (Such an isomorphism can be obtained from a study of intertwiners.) In particular, we have $\mathfrak{a}(Y_i) = e^{-h\mathcal{U}_i}$ and, in the quotient $\mathbf{H} = \mathbf{H}[[h]]/h\mathbf{H}[[h]]$, $\mathfrak{a}(T_i) \mapsto s_{i,i+1}, \mathfrak{a}(X_i) \mapsto x_i$. Using $\mathfrak{a}$, we can identify $\mathbb{H}[[h]] \otimes_{\mathcal{H}[[h]]} V^{\otimes l}[[h]]$ with $\mathbf{H}[[h]] \otimes_{\mathbb{C}[S_l][[h]]} V^{\otimes l}[[h]]$.

After cancelling q^{1-l_k} in (27), we apply the isomorphism $\mathfrak{a}$ and equate the coefficients of h on both sides. We then obtain

$$x_{j_n}^{-1}\cdots x_l^{-1}\left(1+\sum_{d=j_k+1}^{\tilde{j}_k}\widetilde{s}_{d,j_k}\right)(\mathcal{U}_{j_k}-\frac{c}{4}(n-2k))\otimes \mathbf{v}_{j_k^-+1} =$$

$$\left(1+\sum_{d=j_k+1}^{\tilde{j}_k}\widetilde{s}_{d,j_k}\right)(\mathcal{U}_{j_k}-\frac{c}{4}(n-2k))x_{j_n}^{-1}\cdots x_l^{-1}\otimes \mathbf{v}_{j_k^-+1} \tag{28}$$

The equation (25), which we want to prove, says that for the case of $X_{k,1}^+, 2\le k\le n-1$, and $M=\mathbf{H}, m=1$:

$$\left(\sum_{d=j_k}^{\tilde{j}_k}\widetilde{s}_{d,j_k}\right)(\mathcal{U}_{j_k}-\frac{\lambda}{4}(n-2k+2))x_{j_n}^{-1}\cdots x_l^{-1}\otimes \mathbf{v}_{j_k^-+1}+\frac{\lambda}{2}\left(\sum_{d=j_k}^{\tilde{j}_k}\widetilde{s}_{d,j_k}\right)x_{j_n}^{-1}\cdots x_l^{-1}\otimes \mathbf{v}_{j_k^-+1} =$$

$$x_{j_n}^{-1}\cdots x_l^{-1}\left(\sum_{d=j_k}^{\tilde{j}_k}\widetilde{s}_{d,j_k}\right)\big(\mathcal{U}_{j_k}-\frac{\lambda}{4}(n-2k)\big)\otimes \mathbf{v}_{j_k^-+1}$$

This is equation (28) since $c=\lambda$.

As for equation (26) in the case of $X_{1,1}^+$ and $M=\mathbf{H}, m=1$, it says that

$$\left(\sum_{d=j_n}^{l}\widetilde{s}_{d,j_n}\right)\big(\mathcal{U}_{j_n}-\frac{\lambda}{4}(n-2n+2)\big)x_{j_{n-1}}^{-1}\cdots x_l^{-1}\otimes \mathbf{v}_{j_n^-+2} =$$

$$-2\beta\left(\sum_{d=j_n}^{l}\widetilde{s}_{d,j_n}\right)x_{j_{n-1}}^{-1}\cdots x_l^{-1}\otimes \mathbf{v}_{j_n^-+2}+x_{j_{n-1}}^{-1}\cdots x_l^{-1}\left(\sum_{d=j_n}^{l}\widetilde{s}_{d,j_n}\right)\big(\mathcal{U}_{j_n}-\frac{\lambda}{4}(n-2)\big)\otimes \mathbf{v}_{j_n^-+2}.$$

Since $\beta=\frac{t}{2}-\frac{n\lambda}{4}+\frac{\lambda}{2}$, this is equivalent to

$$\left(\sum_{d=j_n}^{l}\widetilde{s}_{d,j_n}\right)(\mathcal{U}_{j_n}+t)x_{j_{n-1}}^{-1}\cdots x_l^{-1}\otimes \mathbf{v}_{j_n^-+2} = x_{j_{n-1}}^{-1}\cdots x_l^{-1}\left(\sum_{d=j_n}^{l}\widetilde{s}_{d,j_n}\right)\mathcal{U}_{j_n}\otimes \mathbf{v}_{j_n^-+2}. \tag{29}$$

This can also be deduced from proposition 3.4 in [VaVa1] with $M=\mathbf{H}$. Indeed, this proposition says that we have the following relation concerning the action of $e_{1,1}-e_{1,0}$ and of $e_{n-1,1}-e_{n-1,0}$ on $1\otimes \mathbf{v}$:

$$q^{1-l+j_n}X_{j_{n-1}}^{-1}\cdots X_l^{-1}\left(1+\sum_{d=j_n}^{l-1}T_{d,j_n}\right)\big((q^{n-1}\kappa Y_{j_n})^{-1}-1\big)\otimes \mathbf{v}_{j_n^-+2} =$$

$$q^{1-l+j_n}\left(1+\sum_{d=j_n}^{l-1}T_{d,j_n}\right)\left((q^{n-(n-1)}q^{n-2}Y_{j_n})^{-1}-1\right)X_{j_{n-1}}^{-1}\cdots X_l^{-1}\otimes\mathbf{v}_{j_n^-+2}.$$

We proceed as previously: we cancel q^{1-l+j_n} on both sides, extend the base ring to $\mathbb{C}[[h]]$, set $q=e^{\frac{ch}{2}},\kappa=e^{th}$. We then apply the isomorphism $\mathfrak{a}$ and equate the coefficients of h; this yield

$$x_{j_{n-1}}^{-1}\cdots x_l^{-1}\left(\sum_{d=j_n}^{l}\widetilde{s}_{d,j_n}\right)(\mathcal{U}_{j_n}-t-\frac{c(n-1)}{2})\otimes\mathbf{v}_{j_n^-+2}=$$

$$\left(\sum_{d=j_n}^{l}\widetilde{s}_{d,j_n}\right)(\mathcal{U}_{j_n}-\frac{c(n-1)}{2})x_{j_{n-1}}^{-1}\cdots x_l^{-1}\otimes\mathbf{v}_{j_n^-+2}.$$

After a simple simplification, we obtain equation (29). □

Proof of lemma 6.1 - sketch of alternative approach. To simplify the notation, will not use φ in the proof. (We used it only to state the lemma in a convenient way.) We have to check the equality

$$(J(X_i^+)-\lambda\omega_i^+)\big(T(m\otimes\mathbf{v})\big)=T\big((J(X_{i-1}^+)-\lambda\omega_{i-1}^++\frac{\lambda}{2}X_{i-1}^+)(m\otimes\mathbf{v})\big). \tag{30}$$

With $\mathbf{v}$ as before (but without assuming that $i_1\le i_2\le\cdots\le i_l$), suppose that $j_1<\cdots<j_p$ are exactly the values of j for which $i_j=n$. Then $T(m\otimes\mathbf{v})=mx_{j_1}^{-1}\cdots x_{j_p}^{-1}\otimes\mathbf{v}_{+1}$. Set $x_{j_1,\dots,j_p}^{-1}=x_{j_1}^{-1}\cdots x_{j_p}^{-1}$.

$$\begin{aligned}J(X_i^+)\big(T(m\otimes\mathbf{v})\big) &= \sum_{k=1}^{l}mx_{j_1,\dots,j_p}^{-1}\mathcal{Y}_k\otimes E_{i,i+1}^k(\mathbf{v}_{+1})\\ &= \sum_{r=1}^{p}\sum_{\substack{k=1\\ k\neq j_s\forall s}}^{l}mx_{j_1}^{-1}\cdots x_{j_{r-1}}^{-1}[x_{j_r}^{-1},\mathcal{Y}_k]x_{j_{r+1}}^{-1}\cdots x_{j_p}^{-1}\otimes E_{i,i+1}^k(\mathbf{v}_{+1}) \qquad (31)\\ &\quad+\sum_{k=1}^{l}m\mathcal{Y}_kx_{j_1,\dots,j_p}^{-1}\otimes E_{i,i+1}^k(\mathbf{v}_{+1}) \qquad (32)\end{aligned}$$

and

$$\sum_{k=1}^{l}m\mathcal{Y}_kx_{j_1,\dots,j_p}^{-1}\otimes E_{i,i+1}^k(\mathbf{v}_{+1})=T\big(J(X_{i-1}^+)(m\otimes\mathbf{v})\big) \tag{33}$$

Therefore, we must prove that

$$(31)-\lambda\omega_i^+\big(T(m\otimes\mathbf{v})\big)=-\lambda T\big(\omega_{i-1}^+(m\otimes\mathbf{v})\big)+\frac{\lambda}{2}T\big(X_{i-1}^+(m\otimes\mathbf{v})\big) \tag{34}$$

To compute the action of ω_i^+ on $\mathbf{v}_{+1}$, we distinguish two cases: when $E_{j,i+1}$ and E_{ij} act on the same tensorand, and when they act on different ones.

$$\begin{aligned}\omega_i^+\big(T(m\otimes\mathbf{v})\big) &= \frac{n-2i}{4}\sum_{k=1}^{l} m x_{j_1,\ldots,j_p}^{-1}\otimes E_{i,i+1}^k(\mathbf{v}_{+1}) = \frac{(n-2i)}{4}T\left(\sum_{k=1}^{l} m\otimes E_{i-1,i}^k(\mathbf{v})\right) && (35)\\ &+\frac{1}{2}\sum_{\substack{j=1\\ j\neq i,i+1}}^{n}\sum_{\substack{k=1\\ i_k=j-1}}^{l}\sum_{\substack{d=1\\ i_d=i}}^{l}\operatorname{sign}(j-i) m x_{j_1,\ldots,j_p}^{-1}\otimes E_{i,i+1}^k\big(s_{kd}(\mathbf{v}_{+1})\big) && (36)\\ &-\frac{1}{4} m x_{j_1,\ldots,j_p}^{-1}\otimes\big(E_{i,i+1}H_i+H_iE_{i,i+1})(\mathbf{v}_{+1}\big) && (37)\end{aligned}$$

Doing the same for ω_{i-1}^+, we obtain that $\omega_i^+(T(m\otimes\mathbf{v}))-T(\omega_{i-1}^+(m\otimes\mathbf{v}))+\frac{1}{2}T(X_{i-1}^+(m\otimes\mathbf{v}))$ equals

$$\frac{1}{2}\sum_{\substack{j=1\\ j\neq i,i+1}}^{l}\sum_{\substack{k=1\\ i_k=j-1}}^{l}\sum_{\substack{d=1\\ i_d=i}}^{l}\operatorname{sign}(j-i) m x_{j_1,\ldots,j_p}^{-1}\otimes E_{i,i+1}^k\big(s_{kd}(\mathbf{v}_{+1})\big) \quad (38)$$

$$-T\left(\frac{1}{2}\sum_{\substack{j=1\\ j\neq i-1,i}}^{n}\sum_{\substack{k=1\\ i_k=j}}^{l}\sum_{\substack{d=1\\ i_d=i}}^{l}\operatorname{sign}(j-i+1) m\otimes E_{i-1,i}^k\big(s_{kd}(\mathbf{v})\big)\right) \quad (39)$$

Therefore, the equality (34) that we have to prove simplifies to (31) = (38) – (39).

By considering the two different cases: $j\neq 1$ and $j=1$ in (38) (and $j\neq n, j=n$ in (39)), we find the following expression for (38) – (39), which equals (31) using $[x_{j_r}^{-1},\mathcal{Y}_k]=-\frac{c}{2}(x_{j_r}^{-1}+x_k^{-1})s_{kj_r}$:

$$(38)-(39) = -\frac{1}{2}\sum_{\substack{k=1\\ i_k=n}}^{l}\sum_{\substack{d=1\\ i_d=i}}^{l} m x_{j_1,\ldots,j_p}^{-1}\otimes E_{i,i+1}^k\big(s_{kd}(\mathbf{v}_{+1})\big)-\frac{1}{2}T\left(\sum_{\substack{k=1\\ i_k=n}}^{l}\sum_{\substack{d=1\\ i_d=i}}^{l} m\otimes E_{i-1,i}^k\big(s_{kd}(\mathbf{v})\big)\right)$$

We now prove the case $i=1$ in lemma 6.1. Suppose that $j_1,\ldots,j_p$ (resp. $\gamma_1,\ldots,\gamma_e$) are exactly the values of j (resp. of γ) such that $i_j=n$ (resp. $i_\gamma=n-1$).

$$\begin{aligned}J(X_{1,0}^+)\big(T^2(m\otimes\mathbf{v})\big) &= \sum_{s=1}^{p}\sum_{u=1}^{e} m x_{j_1,\ldots,j_p}^{-1}x_{\gamma_1}^{-1}\cdots x_{\gamma_{u-1}}^{-1}[x_{\gamma_u}^{-1},\mathcal{Y}_{j_s}]x_{\gamma_{u+1}}^{-1}\cdots x_{\gamma_e}^{-1}\otimes E_{12}^{j_s}(\mathbf{v}_{+2}) && (40)\\ &+\sum_{s=1}^{p}\sum_{r=1}^{p} m x_{j_1}^{-1}\cdots x_{j_{r-1}}^{-1}[x_{j_r}^{-1},\mathcal{Y}_{j_s}]x_{j_{r+1}}^{-1}\cdots x_{j_p}^{-1}x_{\gamma_1,\ldots,\gamma_e}^{-1}\otimes E_{12}^{j_s}(\mathbf{v}_{+2}) && (41)\\ &+\sum_{k=1}^{l} m\mathcal{Y}_k x_{j_1,\ldots,j_p}^{-1}x_{\gamma_1,\ldots,\gamma_e}^{-1}\otimes E_{12}^k(\mathbf{v}_{+2})\end{aligned}$$

The last term is equal to $T^2\big(J(X_{n-1}^+)(m\otimes\mathbf{v})\big)$. We can decompose (40) and (41) into several sums using the relations $[x_{\gamma_u}^{-1},\mathcal{Y}_{j_s}]=-\frac{c}{2}(x_{\gamma_u}^{-1}+x_{j_s}^{-1})s_{\gamma_u,j_s}$ and $[x_{j_r}^{-1},\mathcal{Y}_{j_r}]=tx_{j_r}^{-1}+\frac{c}{2}\sum_{\substack{k=1\\ k\neq j_r}}^{l}(x_{j_r}^{-1}+x_k^{-1})s_{kj_r}$.

After some long computations, we obtain:

$$(40)+(41) = t\sum_{a=1}^{p} m x^{-1}_{j_1,\ldots,j_p} x^{-1}_{\gamma_1,\ldots,\gamma_e} \otimes E^{j_a}_{12}(\mathbf{v}_{+2}) \tag{42}$$

$$+\frac{c}{2}\sum_{a=1}^{p}\sum_{\substack{q=1\\ q\neq j_d,\gamma_h}}^{l} m x^{-1}_{j_1,\ldots,j_{a-1}} x^{-1}_{j_a} x^{-1}_{j_{a+1},\ldots,j_p} x^{-1}_{\gamma_1,\ldots,\gamma_e} \otimes E^{q}_{12}\big(s_{j_a,q}(\mathbf{v}_{+2})\big) \tag{43}$$

$$+\frac{c}{2}\sum_{a=1}^{p}\sum_{\substack{q=1\\ q\neq j_d,\gamma_h}}^{l} m x^{-1}_{j_1,\ldots,j_{a-1}} x^{-1}_{q} x^{-1}_{j_{a+1},\ldots,j_p} x^{-1}_{\gamma_1,\ldots,\gamma_e} \otimes E^{q}_{12}\big(s_{j_a,q}(\mathbf{v}_{+2})\big) \tag{44}$$

We now focus on $\omega_1^+\big(T^2(m\otimes\mathbf{v})\big)$ and $T^2\big(\omega_{n-1}^+(m\otimes\mathbf{v})\big)$. By considering the cases when E_{1j}, E_{j2} (and $E_{n-1,j}, E_{jn}$) act on the same tensorand and on different ones, we can write:

$$\omega_1^+\big(T^2(m\otimes\mathbf{v})\big) - T^2\big(\omega_{n-1}^+(m\otimes\mathbf{v})\big) = \frac{1}{2}\sum_{j=3}^{n}\sum_{\substack{q=1\\ i_q+2=j}}^{l}\sum_{b=1}^{p} m x^{-1}_{\gamma_1,\ldots,\gamma_e}\otimes E^{q}_{12}\big(s_{q,j_b}(\mathbf{v}_{+2})\big) \tag{45}$$

$$+\frac{1}{2}T^2\left(\sum_{j=1}^{n-2}\sum_{\substack{q=1\\ i_q=j}}^{l}\sum_{b=1}^{p} m\otimes E^{q}_{n-1,n}\big(s_{q,j_b}(\mathbf{v})\big)\right) \tag{46}$$

$$+\left(\frac{n-2}{2}\right)T^2\big(E_{n-1,n}(m\otimes\mathbf{v})\big) \tag{47}$$

One can check that $(43)=(45)$ and $(46)=-(44)$. Finally, we get

$$\begin{aligned} X^+_{1,1}\big(T^2(m\otimes\mathbf{v})\big) - T^2\big(X^+_{n-1,1}(m\otimes\mathbf{v})\big) &= (40)+(41)+(42)-\lambda(45)+\lambda(46)-\lambda(47) \\ &= 2\beta T^2\big(E_{n-1,n}(m\otimes\mathbf{v})\big). \end{aligned}$$

□

Using the lemma, we can now define the action of $X^{\pm}_{0,1}$ and of $H_{0,1}$ on $\mathcal{F}(M)$ by setting

$$X^{\pm}_{0,1}(m\otimes\mathbf{v}) = T^{-1}\Big(X^{\pm}_{1,1}\big(T(m\otimes\mathbf{v})\big)\Big) - \beta X^{\pm}_0(m\otimes\mathbf{v})$$

and

$$H_{0,1}(m\otimes\mathbf{v}) = T^{-1}\Big(H_{1,1}\big(T(m\otimes\mathbf{v})\big)\Big) - \beta H_0(m\otimes\mathbf{v}).$$

Note that lemma 6.1 implies that $X^{\pm}_{0,1}\big(T(m\otimes\mathbf{v})\big) = T\big(X^{\pm}_{n-1,1}(m\otimes\mathbf{v}) + \beta X^{\pm}_{n-1}(m\otimes\mathbf{v})\big)$ and similarly for $H_{0,1}$. In other words, and more generally, we set

$$\varphi(X^{\pm}_{0,r}) = T\circ\varphi\big(\rho(X^{\pm}_{0,r})\big)\circ T^{-1},\ \ \varphi(H_{0,r}) = T\circ\varphi\big(\rho(H_{0,r})\big)\circ T^{-1}\ \forall r\geq 0.$$

We now have to check that this indeed gives $\mathcal{F}(M)$ a structure of integrable module over LY. Choose $i,j,k\in\{0,1,\ldots,n-1\}$ with $k\neq i, k\neq j$. We have to verify that $\varphi(X^{\pm}_{i,r}), \varphi(H_{i,r}), \varphi(X^{\pm}_{j,s})$

and $\varphi(H_{j,s})$ satisfy the defining relations of LY. This is true when $k=0$ from theorem 1 of [Dr1]. Using lemma 6.1, we conclude that it is also true for $k \neq 0$. This means that we have a well-defined algebra homomorphism φ from LY to $\mathrm{End}_{\mathbb{C}}(\mathcal{F}(M))$. That $\mathcal{F}(M)$ is integrable follows from the fact that $V^{\otimes l}$ is an integrable $\mathfrak{sl}_n$-module, and that it is of level l follows from theorem 5.1 in the case of $\mathbb{C}[S_l]$ and $\mathfrak{Usl}_n$.

6.2 Proof of theorem 5.2, part 2

For the rest of this section, we assume that $l+2<n$ (so, in particular, $n \geq 4$). In the second step of the proof, we have to show that, given an integrable module $\widehat{M}$ of level l over LY, we can find a module M over $\mathbf{H}$ such that $\mathcal{F}(M)=\widehat{M}$. Such an $\widehat{M}$ cannot, in general, be lifted to a module over $\ddot{U}_{q_1,q_2}$, so this second step is not an immediate consequence of [VaVa1], although the approach is similar. Integrable $\mathfrak{Usl}_n$-modules are direct sums of finite dimensional ones, so, by the results of Drinfeld [Dr1] and Chari-Pressley [ChPr1], we know that there exists modules M^1 and M^2 over, respectively, $\mathtt{H}$ and $\mathbb{C}[\widetilde{S}_l]$, such that $\widehat{M}=\mathcal{F}(M^1)$ as Y-module and $\widehat{M}=\mathcal{F}(M^2)$ as $L\mathfrak{sl}_n$-module. Since $\mathbb{C}[S_l] \subset \mathtt{H}$ and $\mathbb{C}[S_l] \subset \mathbb{C}[\widetilde{S}_l]$, we have an isomorphism $M^1 \cong M^2$ of S_l-modules, so we can denote them simply by M. We have to show that M is an $\mathbf{H}$-module. The following will be useful.

Lemma 6.2. *If $\mathbf{v}=v_{i_1}\otimes\cdots\otimes v_{i_l}$ is a generator of $V^{\otimes l}$ as a module over $\mathfrak{Usl}_n$ (that is, if $i_j \neq i_k$ for any $j \neq k$), then $m\otimes\mathbf{v}=0 \Longrightarrow m=0$.*

Fix $1 \leq j,k \leq l, j \neq k$. We choose $\mathbf{v}$ to be the following generator of $V^{\otimes l}$ as $\mathfrak{Usl}_n$-module: $\mathbf{v}=v_{i_1}\otimes v_{i_2}\otimes\cdots\otimes v_{i_l}$ where $i_d=d+3$ if $d<j, d\neq k$, $i_d=d+2$ if $d>j, d\neq k$, $i_j=2$ and $i_k=1$. We can express ω_2^- as an operator on $V^{\otimes l}$ in the following way:

$$\omega_2^-|_{V^{\otimes l}} = -\frac{1}{2}\sum_{\substack{d=1\\d\neq 2,3}}^{n}\sum_{r=1}^{l}\sum_{\substack{s=1\\s\neq r}}^{l}\mathrm{sign}(2-d)(E_{3d}^{r}E_{d2}^{s}) + \left(\frac{n-4}{4}\right)\sum_{r=1}^{l}E_{32}^{r} - \frac{1}{2}\sum_{r=1}^{l}\sum_{\substack{s=1\\s\neq r}}^{l}E_{32}^{r}H_2^{s}$$

Therefore, $[E_{n1}^a,\omega_2^-] = -\frac{1}{2}\sum_{\substack{r=1\\r\neq a}}^{l}E_{31}^{r}E_{n2}^{a} - \frac{1}{2}\sum_{\substack{s=1\\s\neq a}}^{l}E_{31}^{a}E_{n2}^{s}$ and applying this to $m\otimes\mathbf{v}$ with $a=j,k$ gives

$$[E_{n1}^{j},\omega_2^-](m\otimes\mathbf{v}) = -\frac{1}{2}E_{31}^{k}E_{n2}^{j}(m\otimes\mathbf{v}) \quad\text{and}\quad [E_{n1}^{k},\omega_2^-](m\otimes\mathbf{v}) = -\frac{1}{2}E_{31}^{k}E_{n2}^{j}(m\otimes\mathbf{v}).$$

$$\begin{aligned}
(X_{2,1}^-X_0^+ - X_0^+X_{2,1}^-)(m\otimes\mathbf{v}) &= \sum_{r=1}^{l}\sum_{s=1}^{l}\big(mx_r\mathcal{Y}_s\otimes X_{2,0}^{-,s}E_{n1}^{r}(\mathbf{v}) - m\mathcal{Y}_s x_r\otimes E_{n1}^{r}X_{2,0}^{-,s}(\mathbf{v})\big)\\
&\quad -\lambda[\omega_2^-,X_0^+](m\otimes\mathbf{v})\\
&= \sum_{r=1}^{l}\sum_{s=1}^{l}m[x_r,\mathcal{Y}_s]\otimes E_{32}^{s}E_{n1}^{r}(\mathbf{v}) + \lambda\sum_{a=1}^{s}mx_a\otimes[E_{n1}^{a},\omega_2^-](\mathbf{v})\\
&= m[x_k,\mathcal{Y}_j]\otimes E_{32}^{j}E_{n1}^{k}(\mathbf{v}) - \frac{\lambda}{2}mx_j\otimes E_{31}^{k}E_{n2}^{j}(\mathbf{v})\\
&\quad -\frac{\lambda}{2}mx_k\otimes E_{31}^{k}E_{n2}^{j}(\mathbf{v})\\
&= m\big([x_k,\mathcal{Y}_j] - \frac{\lambda}{2}(x_j+x_k)s_{jk}\big)\otimes\widetilde{\mathbf{v}}
\end{aligned}$$

where $\widetilde{\mathbf{v}} = E_{32}^{j}E_{n1}^{k}(\mathbf{v})$. We know from relation (3) that $[X_{2,1}^{-}, X_0^{+}] = 0$, so the last expression is equal to 0. Since $\widetilde{\mathbf{v}}$ is a generator of $V^{\otimes l}$ as a $\mathfrak{U}\mathfrak{sl}_n$-module, it follows, from lemma 6.2 and our assumption that $\lambda = c$, that $m\big([x_k, \mathcal{Y}_j] - \frac{c}{2}(x_j + x_k)s_{jk}\big) = 0$.

We consider now the relation between x_k and $\mathcal{Y}_k$. From the definition of ν_1:

$$\nu_1 = \frac{1}{4}\sum_{d=3}^{n}(E_{1d}E_{d1} + E_{d1}E_{1d}) + \frac{1}{2}(E_{12}E_{21} + E_{21}E_{12}) - \frac{1}{4}\sum_{d=3}^{n}(E_{2d}E_{d2} + E_{d2}E_{2d}) - \frac{1}{2}H_1^2$$

whence, as an operator on $V^{\otimes l}$, it is equal to

$$\nu_1|_{V^{\otimes l}} = \frac{1}{2}\sum_{d=3}^{n}\sum_{j=1}^{l}\sum_{\substack{s=1\\s\neq j}}^{l}(E_{1d}^{j}E_{d1}^{s} - E_{2d}^{j}E_{d2}^{s}) + \sum_{j=1}^{l}\sum_{\substack{s=1\\s\neq j}}^{l}E_{12}^{j}E_{21}^{s} - \frac{1}{2}\sum_{j=1}^{l}\sum_{\substack{s=1\\s\neq j}}^{l}H_1^{j}H_1^{s} + \left(\frac{n-2}{4}\right)\sum_{j=1}^{l}H_1^{j}.$$

Therefore,

$$[E_{n1}^{r}, \nu_1] = \frac{1}{2}\sum_{d=3}^{n-1}\sum_{\substack{s=1\\s\neq r}}^{l}E_{nd}^{r}E_{d1}^{s} + \frac{1}{2}\sum_{\substack{s=1\\s\neq r}}^{l}(H_0^{r}E_{n1}^{s} + E_{21}^{r}E_{n2}^{s}) + \sum_{\substack{s=1\\s\neq r}}^{l}E_{n2}^{r}E_{21}^{s} - \sum_{\substack{s=1\\s\neq r}}^{l}E_{n1}^{r}H_1^{s} + \left(\frac{n-2}{4}\right)E_{n1}^{r}.$$

Fix k, $1 \le k \le l$. We now choose $\mathbf{v}$ to be equal to $\mathbf{v} = v_{i_1} \otimes \cdots \otimes v_{i_l}$ with $i_d = d+2$ if $d < k$, $i_d = d+1$ if $d > k$ and $i_k = 1$. Note that $i_d \neq 2, n, n-1\,\forall d$ since $l+1 < n-1$ by assumption. Applying the previous expression for $[E_{n1}^{r}, \nu_1]$ to $\mathbf{v}$, we obtain the following:

$$[E_{n1}^{r}, \nu_1](\mathbf{v}) = \frac{1}{2}\sum_{d=3}^{n}E_{nd}^{r}E_{d1}^{k}(\mathbf{v}) = \frac{1}{2}s_{kr}E_{n1}^{k}(\mathbf{v}) \text{ if } r \neq k \quad [E_{n1}^{k}, \nu_1](\mathbf{v}) = \left(\frac{n-2}{4}\right)E_{n1}^{k}(\mathbf{v}). \tag{48}$$

We need (48) to obtain equation (49) below. Note that $H_{1,0}^{s}(\mathbf{v}) = 0$ if $s \neq k$.

$$\begin{aligned}
(H_{1,1}X_0^{+} - X_0^{+}H_{1,1})(m \otimes \mathbf{v}) &= \sum_{r=1}^{l}\sum_{s=1}^{l} mx_r\mathcal{Y}_s \otimes H_{1,0}^{s}E_{n1}^{r}(\mathbf{v}) \\
&\quad - \sum_{s=1}^{l}\sum_{r=1}^{l} m\mathcal{Y}_s x_r \otimes E_{n1}^{r}H_{1,0}^{s}(\mathbf{v}) - \lambda[\nu_1, X_0^{+}](m \otimes \mathbf{v}) \\
&= -m\mathcal{Y}_k x_k \otimes E_{n1}^{k}H_{1,0}^{k}(\mathbf{v}) + \lambda\sum_{r=1}^{l} mx_r \otimes [E_{n1}^{r}, \nu_1](\mathbf{v}) \\
&= -m\mathcal{Y}_k x_k \otimes E_{n1}^{k}(\mathbf{v}) + \frac{\lambda}{2}\sum_{\substack{r=1\\r\neq k}}^{l} mx_r \otimes s_{kr}E_{n1}^{k}(\mathbf{v}) \\
&\quad + \lambda\left(\frac{n-2}{4}\right) m \otimes E_{n1}^{k}(\mathbf{v}) \\
&= -m\mathcal{Y}_k x_k \otimes \widetilde{\mathbf{v}} + \frac{\lambda}{2}\sum_{\substack{r=1\\r\neq k}}^{l} mx_r s_{kr} \otimes \widetilde{\mathbf{v}} + \lambda\left(\frac{n-2}{4}\right) m \otimes \widetilde{\mathbf{v}} \quad (49)
\end{aligned}$$

where $\widetilde{\mathbf{v}} = E_{n1}^k(\mathbf{v})$. We want to obtain a similar relation with $H_{1,1}$ replaced by $H_{n-1,1}$.

From the definition of ν_{n-1},

$$\begin{aligned}\nu_{n-1} &= \frac{1}{4}\sum_{d=1}^{n-2}(E_{dn}E_{nd}+E_{nd}E_{dn}) + \frac{1}{2}(E_{n-1,n}E_{n,n-1}+E_{n,n-1}E_{n-1,n}) \\ &\quad -\frac{1}{4}\sum_{d=1}^{n-2}(E_{d,n-1}E_{n-1,d}+E_{n-1,d}E_{d,n-1}) - \frac{1}{2}H_{n-1}^2\end{aligned}$$

whence, as an operator on $V^{\otimes l}$, it is equal to

$$\begin{aligned}\nu_{n-1}|_{V^{\otimes l}} &= \frac{1}{2}\sum_{d=1}^{n-2}\sum_{j=1}^{l}\sum_{\substack{s=1\\ s\neq j}}^{l}(E_{dn}^jE_{nd}^s - E_{d,n-1}^jE_{n-1,d}^s) + \sum_{j=1}^{l}\sum_{\substack{s=1\\ s\neq j}}^{l}(E_{n-1,n}^jE_{n,n-1}^s) \\ &\quad -\frac{1}{2}\sum_{j=1}^{l}\sum_{\substack{s=1\\ s\neq j}}^{l}H_{n-1}^jH_{n-1}^s - \left(\frac{n-2}{4}\right)\sum_{j=1}^{l}H_{n-1}^j.\end{aligned}$$

Therefore,

$$\begin{aligned}[E_{n1}^r,\nu_{n-1}] &= -\frac{1}{2}\sum_{d=2}^{n-2}\sum_{\substack{s=1\\ s\neq r}}^{l}E_{d1}^rE_{nd}^s + \frac{1}{2}\sum_{\substack{s=1\\ s\neq r}}^{l}(H_0^rE_{n1}^s - E_{n,n-1}^rE_{n-1,1}^s) \\ &\quad -\sum_{\substack{s=1\\ s\neq r}}^{l}E_{n-1,1}^rE_{n,n-1}^s - \sum_{\substack{s=1\\ s\neq r}}^{l}E_{n1}^rH_{n-1}^s - \left(\frac{n-2}{4}\right)E_{n1}^r.\end{aligned}$$

Applying the previous expression for $[E_{n1}^r,\nu_{n-1}]$ to $\mathbf{v}$, we conclude that $[E_{n1}^r,\nu_{n-1}](\mathbf{v}) = 0$ if $r\neq k$ and

$$[E_{n1}^k,\nu_{n-1}](\mathbf{v}) = -\frac{1}{2}\sum_{d=2}^{n-2}\sum_{\substack{s=1\\ s\neq k}}^{l}E_{d1}^kE_{nd}^s(\mathbf{v}) - \left(\frac{n-2}{4}\right)E_{n1}^k(\mathbf{v}) = -\frac{1}{2}\sum_{\substack{s=1\\ s\neq k}}^{l}s_{ks}E_{n1}^k(\mathbf{v}) - \left(\frac{n-2}{4}\right)E_{n1}^k(\mathbf{v})$$

This equation allows us to compute $[H_{n-1,1},X_0^+](m\otimes\mathbf{v})$:

$$\begin{aligned}(H_{n-1,1}X_0^+ - X_0^+H_{n-1,1})(m\otimes\mathbf{v}) &= \sum_{r,s=1}^{l}\left(mx_r\mathcal{Y}_s\otimes H_{n-1,0}^sE_{n1}^r(\mathbf{v}) - m\mathcal{Y}_sx_r\otimes E_{n1}^rH_{n-1,0}^s(\mathbf{v})\right) \\ &\quad -\lambda[\nu_{n-1},X_0](m\otimes\mathbf{v}) \\ &= mx_k\mathcal{Y}_k\otimes H_{n-1,0}^kE_{n1}^k(\mathbf{v}) + \lambda\sum_{r=1}^{l}mx_r\otimes[E_{n1}^r,\nu_{n-1}](\mathbf{v}) \\ &= -mx_k\mathcal{Y}_k\otimes\widetilde{\mathbf{v}} - \frac{\lambda}{2}\left(\sum_{\substack{s=1\\ s\neq k}}^{l}mx_ks_{ks} + \frac{n-2}{2}\right)\otimes\widetilde{\mathbf{v}} \qquad (50)\end{aligned}$$

From the relations (1), (6) and (9) in LY, we know that

$$-X^+_{0,1} = [H_{1,1}, X^+_0] + ((\lambda-\beta)H_1X^+_0 + \beta X^+_0 H_1) \quad (51)$$
$$= [H_{n-1,1}, X^+_0] + (\beta H_{n-1}X^+_0 + (\lambda-\beta)X^+_0 H_{n-1}) \quad (52)$$

Applying these two expressions for $-X^+_{0,1}$ to $m\otimes \mathbf{v}$, using equalities (49),(50) and the fact that $H_1X^+_0(\mathbf{v})=0$ and $X^+_0H_{n-1}(\mathbf{v})=0$ because of our choice of $\mathbf{v}$, we obtain:

$$-m\mathcal{Y}_k x_k \otimes \widetilde{\mathbf{v}} + \frac{\lambda}{2}\sum_{\substack{r=1\\ r\neq k}}^{l} m x_r s_{kr}\otimes\widetilde{\mathbf{v}} + \lambda\left(\frac{n-2}{4}\right) m x_k\otimes\widetilde{\mathbf{v}} + \beta X^+_0 H_1(m\otimes\mathbf{v}) =$$

$$-m x_k\mathcal{Y}_k \otimes \widetilde{\mathbf{v}} - \frac{\lambda}{2}\sum_{\substack{s=1\\ s\neq k}}^{l} m x_k s_{ks}\otimes\widetilde{\mathbf{v}} - \lambda\left(\frac{n-2}{4}\right) m x_k\otimes\widetilde{\mathbf{v}} + \beta H_{n-1} X^+_0 (m\otimes\mathbf{v})$$

$$\Longrightarrow m[x_k,\mathcal{Y}_k]\otimes\widetilde{\mathbf{v}} + \frac{\lambda}{2}\sum_{\substack{r=1\\ r\neq k}}^{l} m(x_r+x_k)s_{kr}\otimes\widetilde{\mathbf{v}} + \lambda\left(\frac{n-2}{2}\right) m x_k\otimes\widetilde{\mathbf{v}} + 2\beta m x_k\otimes\widetilde{\mathbf{v}} = 0$$

Since $\widetilde{\mathbf{v}}$ is a generator of $V^{\otimes l}$ as a $\mathfrak{U}\mathfrak{sl}_n$-module, it follows from lemma 6.2 and our assumptions that $2\beta + \frac{\lambda(n-2)}{2} = t, \lambda = c$ that

$$m\big([x_k,\mathcal{Y}_k] + \frac{c}{2}\sum_{\substack{r=1\\ r\neq k}}^{l}(x_r+x_k)s_{kr} + t x_k\big) = 0$$

We proved above that $m([x_k,\mathcal{Y}_j] - \frac{c}{2}(x_j+x_k)s_{jk}) = 0$ if $j\neq k$. These last two equalities imply that M is a right module over $\mathbf{H}$.

Therefore, we have shown that the $\mathtt{H}$- and the $\mathbb{C}[\widetilde{S}_l]$-module structure on M can be glued to yield a module over $\mathbf{H}$. To prove that $\mathcal{F}$ is an equivalence, we are left to show that it is fully faithful. That $\mathcal{F}$ is injective on morphisms is true because this is true for the Schur-Weyl duality functor between $\mathbb{C}[\widetilde{S}_l]$ and $\mathfrak{U}(L\mathfrak{sl}_n)$, so suppose that $f:\mathcal{F}(M_1)\longrightarrow\mathcal{F}(M_2)$ is a LY-homomorphism. From the main results of [ChPr1] and [Dr1], f is of the form $f(m\otimes\mathbf{v}) = g(m)\otimes\mathbf{v}, \forall m\in M_1$, where $g\in \mathrm{Hom}_{\mathbb{C}}(M_1,M_2)$ is a linear map which is also a homomorphism of right $\mathbb{C}[\widetilde{S}_l]$- and $\mathtt{H}$-modules. Since $\mathbf{H}$ is generated by its two subalgebras $\mathbb{C}[\widetilde{S}_l]$ and $\mathtt{H}$, g is even a homomorphism of $\mathbf{H}$-modules. Therefore, $f=\mathcal{F}(g)$ and this completes the proof of theorem 5.2. □

7 Action of the elements $X^{\pm}_{0,1}, H_{0,1}$

Now that we know that $\mathcal{F}(M)$ is a module over LY, it may be interesting to see explicitly how the elements $X^{\pm}_{0,1}$ and $H_{0,1}$ act on it. What we will discover will be useful in the next section. We will assume throughout this section that $\lambda = c$, $\beta = \frac{t}{2} - \frac{nc}{4} + \frac{c}{2}$ and $n\geq 3$.

7.1 Action of $X_{0,1}^+$

Equations (51) and (52) yield

$$X_{0,1}^+ = -\frac{1}{2}[H_{1,1} + H_{n-1,1}, X_0^+] - \frac{1}{2}\big(((\lambda-\beta)H_1 + \beta H_{n-1})X_0^+ + X_0^+(\beta H_1 + (\lambda-\beta)H_{n-1})\big).$$

We will use the notation $K_r(z)$ to denote the element $z \otimes u^r \in L\mathfrak{sl}_n$ for $z \in \mathfrak{sl}_n$; in particular, $K_1(E_{n1}) = X_0^+$ and $K_{-1}(E_{1n}) = X_0^-$. The element $K_r(z)$ maps to the operator in $\mathrm{End}_{\mathbb{C}}(\mathcal{F}(M))$ given by $K_r(z)(m \otimes \mathbf{v}) = \sum_{k=1}^{l} m x_k^r \otimes z^k(\mathbf{v})$. Writing $H_{1,1}$ as $H_{1,1} = J(H_1) - \lambda\nu_1$, and similarly for $H_{n-1,1}$, we can express $X_{0,1}^+$ in the following way. (We will use that $[H_{n-1} - H_1, X_0^+] = 0$.)

$$\begin{aligned}
X_{0,1}^+ &= -\frac{1}{2}[J(H_1 + H_{n-1}), X_0^+] - \frac{\lambda}{8}\sum_{d=3}^{n-1}\big(K_1(E_{nd})E_{d1} + E_{d1}K_1(E_{nd})\big) \\
&\quad + \frac{\lambda}{8}\Big(\big(K_1(E_{11}) - K_1(E_{nn})\big)E_{n1} + E_{n1}\big(K_1(E_{11}) - K_1(E_{nn})\big)\Big) + \frac{\lambda}{4}(H_1X_0^+ + X_0^+H_1) \\
&\quad - \frac{\lambda}{4}\big(K_1(E_{n2})E_{21} + E_{21}K_1(E_{n2})\big) - \frac{\lambda}{8}\big(K_1(E_{21})E_{n2} + E_{n2}K_1(E_{21})\big) \\
&\quad + \frac{\lambda}{8}\sum_{d=2}^{n-2}\big(K_1(E_{d1})E_{nd} + E_{nd}K_1(E_{d1})\big) + \frac{\lambda}{8}\Big(\big(K_1(E_{11}) - K_1(E_{nn})\big)E_{n1} \\
&\quad + E_{n1}\big(K_1(E_{11}) - K_1(E_{nn})\big)\Big) + \frac{\lambda}{4}\big(K_1(E_{n-1,1})E_{n,n-1} + E_{n,n-1}K_1(E_{n-1,1})\big) \\
&\quad + \frac{\lambda}{8}\big(K_1(E_{n,n-1})E_{n-1,1} + E_{n-1,1}K_1(E_{n,n-1})\big) \\
&\quad + \frac{\lambda}{4}(X_0^+H_{n-1} + H_{n-1}X_0^+) - \frac{1}{2}\big(((\lambda-\beta)H_1 + \beta H_{n-1})X_0^+ + X_0^+(\beta H_1 + (\lambda-\beta)H_{n-1})\big) \\
&= -\frac{1}{2}[J(H_1 + H_{n-1}), X_0^+] - \frac{\lambda}{8}\sum_{d=3}^{n-2}\big(K_1(E_{nd})E_{d1} + E_{d1}K_1(E_{nd})\big) \\
&\quad - \frac{\lambda}{4}\big(K_1(E_{n2})E_{21} + E_{21}K_1(E_{n2})\big) + \frac{\lambda}{8}\sum_{d=3}^{n-2}\big(K_1(E_{d1})E_{nd} + E_{nd}K_1(E_{d1})\big) \\
&\quad + \frac{\lambda}{4}\Big(\big(K_1(E_{11}) - K_1(E_{nn})\big)E_{n1} + E_{n1}\big(K_1(E_{11}) - K_1(E_{nn})\big)\Big) \\
&\quad + \frac{\lambda}{4}\big(K_1(E_{n-1,1})E_{n,n-1} + E_{n,n-1}K_1(E_{n-1,1})\big) \\
&= -\frac{1}{2}[J(E_{11} - E_{nn}), X_0^+] + \frac{1}{2}[H_{2,1} + \cdots H_{n-2,1}, X_0^+] + \frac{\lambda}{2}[\nu_2 + \cdots + \nu_{n-2}, X_0^+] \qquad (53) \\
&\quad - \frac{\lambda}{8}\sum_{d=2}^{n-2}\big(K_1(E_{nd})E_{d1} + E_{d1}K_1(E_{nd})\big) + \frac{\lambda}{8}\sum_{d=3}^{n-1}\big(K_1(E_{d1})E_{nd} + E_{nd}K_1(E_{d1})\big) \\
&\quad - \frac{\lambda}{8}\big(K_1(E_{n2})E_{21} + E_{21}K_1(E_{n2})\big) - \frac{\lambda}{4}\big(K_1(H_0)E_{n1} + E_{n1}K_1(H_0)\big) \\
&\quad + \frac{\lambda}{8}\big(K_1(E_{n-1,1})E_{n,n-1} + E_{n,n-1}K_1(E_{n-1,1})\big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}[J(E_{nn}-E_{11}),X_0^+] - \frac{\lambda}{8}\sum_{d=2}^{n-1}\big(K_1(E_{nd})E_{d1}+E_{d1}K_1(E_{nd})\big)\\
&\quad +\frac{\lambda}{8}\sum_{d=2}^{n-1}\big(K_1(E_{d1})E_{nd}+E_{nd}K_1(E_{d1})\big) - \frac{\lambda}{4}\big(K_1(H_0)E_{n1}+E_{n1}K_1(H_0)\big)\\
&= \widetilde{J}(X_0^+) - \frac{\lambda}{8}\sum_{\epsilon\in\Delta^+}\big([X_0^+,X_\epsilon^+]X_\epsilon^- + X_\epsilon^-[X_0^+,X_\epsilon^+]\big) - \frac{\lambda}{8}\big(K_1(H_0)E_{n1}+E_{n1}K_1(H_0)\big).
\end{aligned}
$$

We define $\widetilde{J}(X_0^+)$ to be $\frac{1}{2}[J(H_0),X_0^+]$. Set $\widetilde{\mathcal{Y}}_j = \frac{1}{2}(x_j\mathcal{Y}_j+\mathcal{Y}_jx_j)$.

$$
\begin{aligned}
\widetilde{J}(X_0^+)(m\otimes \mathbf{v}) &= \frac{1}{2}\sum_{j=1}^{l}\sum_{\substack{k=1\\k\neq j}}^{l} m[x_k,\mathcal{Y}_j]\otimes H_0^jE_{n1}^k(\mathbf{v}) + \frac{1}{2}\sum_{j=1}^{l} m(x_j\mathcal{Y}_j+\mathcal{Y}_jx_j)\otimes E_{n1}^j(\mathbf{v})\\
&= \frac{\lambda}{4}\sum_{j=1}^{l}\sum_{\substack{k=1\\k\neq j}}^{l} m(x_k+x_j)s_{jk}\otimes H_0^jE_{n1}^k(\mathbf{v}) + \sum_{j=1}^{l} m\widetilde{\mathcal{Y}}_j\otimes E_{n1}^j(\mathbf{v})\\
&= \left(\sum_{j=1}^{l}\widetilde{\mathcal{Y}}_j\otimes E_{n1}^j + \frac{\lambda}{8}\big(X_0^+H_0+H_0X_0^+ + E_{n1}K_1(H_0)+K_1(H_0)E_{n1}\big)\right)(m\otimes\mathbf{v})
\end{aligned}
$$

Set $J(X_0^+) = \widetilde{J}(X_0^+) - \frac{\lambda}{8}(X_0^+H_0+H_0X_0^+ + E_{n1}K_1(H_0)+K_1(H_0)E_{n1})$, so

$$
X_{0,1}^+ = J(X_0^+) - \frac{\lambda}{8}\sum_{\epsilon\in\Delta^+}\big([X_0^+,X_\epsilon^+]X_\epsilon^- + X_\epsilon^-[X_0^+,X_\epsilon^+]\big) + \frac{\lambda}{8}(X_0^+H_0+H_0X_0^+)
$$

7.2 Action of $X_{0,1}^-$

The action of $X_{0,1}^-$ on $\mathcal{F}(M)$ can be expressed in a simple way. Proceeding exactly as for $X_{0,1}^+$, we can write

$$
X_{0,1}^- = J(X_0^-) + \frac{\lambda}{8}\sum_{\epsilon\in\Delta^+}\big([X_0^-,X_\epsilon^-]X_\epsilon^+ + X_\epsilon^+[X_0^-,X_\epsilon^-]\big) + \frac{\lambda}{8}(X_0^-H_0+H_0X_0^-) \tag{54}
$$

where $J(X_0^-)$ acts on $m\otimes\mathbf{v}$ by $J(X_0^-)(m\otimes\mathbf{v}) = \frac{1}{2}\sum_{j=1}^{l} m(x_j^{-1}\mathcal{Y}_j+\mathcal{Y}_jx_j^{-1})\otimes E_{1n}^j(\mathbf{v})$. This can be written in the following form:

$$
\begin{aligned}
\frac{1}{2}\sum_{j=1}^{l} m(x_j^{-1}\mathcal{Y}_j+\mathcal{Y}_jx_j^{-1})\otimes E_{1n}^j(\mathbf{v}) &= \frac{1}{2}\sum_{j=1}^{l} m\big(y_j+\frac{1}{2}(x_j^{-1}y_jx_j+x_jy_jx_j^{-1})\big)\otimes E_{1n}^j(\mathbf{v})\\
&= \sum_{j=1}^{l} m\big(y_j+\frac{1}{4}([x_j^{-1},y_j]x_j+x_j[y_j,x_j^{-1}])\big)\otimes E_{1n}^j(\mathbf{v})
\end{aligned}
$$

$$\frac{1}{2}\sum_{j=1}^{l} m(x_j^{-1}\mathcal{Y}_j + \mathcal{Y}_j x_j^{-1}) \otimes E_{1n}^j(\mathbf{v}) = \sum_{j=1}^{l} m\big(y_j + \frac{c}{4}(\sum_{k\neq j} x_j^{-1}s_{jk} - \sum_{k\neq j} x_k^{-1}s_{jk})\big) \otimes E_{1n}^j(\mathbf{v})$$
$$= \sum_{j=1}^{l} m\big(y_j + \frac{c}{4}\sum_{k\neq j}(x_j^{-1} - x_k^{-1})s_{jk}\big) \otimes E_{1n}^j(\mathbf{v}) \qquad (55)$$

As for the sum $\sum_{\epsilon\in\Delta^+}\big([X_0^-, X_\epsilon^-]X_\epsilon^+ + X_\epsilon^+[X_0^-, X_\epsilon^-]\big)$, it equals

$$\sum_{d=2}^{n-1}\big((K_{-1}(E_{1d})E_{dn}+E_{dn}K_{-1}(E_{1d})-(K_{-1}(E_{dn})E_{1d}+E_{1d}K_{-1}(E_{dn}))\big)-K_{-1}(H_0)E_{1n}+E_{1n}K_{-1}(H_0),$$

so it acts on $m\otimes\mathbf{v}$ in the following way:

$$\sum_{\epsilon\in\Delta^+}\big([X_0^-, X_\epsilon^-]X_\epsilon^+ + X_\epsilon^+[X_0^-, X_\epsilon^-]\big)(m\otimes\mathbf{v}) = 2\sum_{\substack{j,k=1\\k\neq j}}^{l}\sum_{d=2}^{n-1} mx_k^{-1}\otimes(E_{1d}^kE_{dn}^j - E_{dn}^kE_{1d}^j)(\mathbf{v})$$
$$-2\sum_{\substack{j,k=1\\k\neq j}}^{l} mx_k^{-1}\otimes H_0^kE_{1n}^j(\mathbf{v})$$
$$= 2\sum_{\substack{j,k=1\\k\neq j}}^{l}\sum_{d=1}^{n} m(x_k^{-1} - x_j^{-1})\otimes E_{1d}^kE_{dn}^j(\mathbf{v})$$
$$+2\sum_{\substack{j,k=1\\k\neq j}}^{l} mx_k^{-1}\otimes(E_{1n}^kE_{11}^j - E_{1n}^kE_{nn}^j)(m\otimes\mathbf{v})$$
$$= -2\sum_{\substack{j,k=1\\k\neq j}}^{l} m(x_j^{-1} - x_k^{-1})s_{jk}\otimes E_{1n}^j(\mathbf{v})$$
$$-\big(X_0^-H_0 + H_0X_0^-\big)(m\otimes\mathbf{v}) \qquad (56)$$

Combining equations (54),(55) and (56), we conclude that $X_{0,1}^-(m\otimes\mathbf{v}) = \sum_{j=1}^{l} my_j\otimes E_{1n}^j(\mathbf{v})$. The element $X_{0,1}^-$ will become important in the next section. We will sometimes denote it by Y_0^+.

7.3 Action of $H_{0,1}$

We use the equality $H_{0,1} = [X_{0,0}^+, X_{0,1}^-]$.

$$
\begin{aligned}
H_{0,1}(m\otimes \mathbf{v}) &= \sum_{j=1}^{l}\sum_{\substack{k=1\\k\neq j}}^{l} m[y_j,x_k]\otimes E_{n1}^{k}E_{1n}^{j}(\mathbf{v}) + \sum_{j=1}^{l} my_jx_j\otimes E_{nn}^{j}(\mathbf{v}) - \sum_{j=1}^{l} mx_jy_j\otimes E_{11}^{j}(\mathbf{v})\\
&= -c\sum_{j=1}^{l}\sum_{\substack{k=1\\k\neq j}}^{l} ms_{jk}\otimes E_{n1}^{k}E_{1n}^{j}(\mathbf{v}) + \sum_{j=1}^{l} my_jx_j\otimes E_{nn}^{j}(\mathbf{v}) - \sum_{j=1}^{l} mx_jy_j\otimes E_{11}^{j}(\mathbf{v})\\
&= -c\sum_{j=1}^{l}\sum_{\substack{k=1\\k\neq j}}^{l} m\otimes E_{11}^{k}E_{nn}^{j}(\mathbf{v}) + \sum_{j=1}^{l} m\mathcal{Y}_j\otimes H_0^{j}(\mathbf{v})\\
&\quad + \sum_{j=1}^{l}\left(\frac{t}{2}+\frac{c}{2}\sum_{\substack{i=1\\i\neq j}}^{l}s_{ij}\right)\otimes E_{nn}^{j}(\mathbf{v}) + \sum_{j=1}^{l}\left(\frac{t}{2}+\frac{c}{2}\sum_{\substack{i=1\\i\neq j}}^{l}s_{ij}\right)\otimes E_{11}^{j}(\mathbf{v})\\
&= -c\sum_{j=1}^{l}\sum_{k=1}^{l} m\otimes E_{11}^{k}E_{nn}^{j}(\mathbf{v}) + J(H_0)(m\otimes\mathbf{v}) + \left(\frac{t}{2}-\frac{cn}{4}\right)(E_{11}+E_{nn})(m\otimes\mathbf{v})\\
&\quad + \frac{c}{4}\sum_{d=1}^{n}(E_{nd}E_{dn}+E_{dn}E_{nd}+E_{1d}E_{d1}+E_{d1}E_{1d})(m\otimes\mathbf{v}) - \frac{c}{2}\left(\sum_{d=1}^{n}E_{dd}\right)(m\otimes\mathbf{v})\\
&= \Big(J(H_0) + \frac{\lambda}{4}\sum_{\epsilon\in\Delta^+}(\epsilon,\theta)(E_\epsilon^+E_\epsilon^- + E_\epsilon^-E_\epsilon^+) + \frac{\lambda}{2}H_0^2 - \frac{\lambda l}{2}\\
&\quad + \left(\beta-\frac{\lambda}{2}\right)(E_{11}+E_{nn})\Big)(m\otimes\mathbf{v})
\end{aligned}
$$

It can be proved that the subalgebra of LY generated by Y_0^+ and $\mathfrak{sl}_n$ is isomorphic to $\mathfrak{U}(\mathfrak{sl}_n\otimes_{\mathbb{C}}\mathbb{C}[v])$. (See also proposition 8.1 below.) We introduce the notation $Q_r(z), r\in\mathbb{Z}_{\geq 0}$, to denote $z\otimes v^r$ as an element of this subalgebra; in particular, $Q_1(E_{1n}) = Y_0^+$. There are three types of operators in $\mathrm{End}_{\mathbb{C}}(M\otimes_{\mathbb{C}[S_l]} V^{\otimes l})$ which are of particular interest to us: those coming from the action of $J(z), K_r(z)$ and of $Q_r(z)$. They are related to each other in the following way.

Proposition 7.1 (See also [BHW]). *Suppose that $a\neq b$ and $c\neq d$. Then we have the equality* $[Q_1(E_{ab}),K_1(E_{cd})] + [K_1(E_{ab}),Q_1(E_{cd})] = 2\big(\delta_{bc}J(E_{ad}) - \delta_{da}J(E_{cb})\big)$.

Proof. First, we will prove the equality

$$[Q_1(E_{1n}),K_1(H_0)] + [K_1(E_{1n}),Q_1(H_0)] = 4J(E_{1n}) \tag{57}$$

$$
\begin{aligned}
[Q_1(E_{1n}),K_1(H_0)] + [K_1(E_{1n}),Q_1(H_0)] &= \big[X_{0,1}^-,[X_0^+,E_{1n}]\big] + \frac{1}{2}\Big[\big[E_{1n},[X_0^+,E_{1n}]\big],[E_{n1},X_{0,1}^-]\Big]\\
&= -[H_{0,1},E_{1n}] - \frac{1}{2}\Big[\big[H_0,[X_0^+,E_{1n}]\big],X_{0,1}^-\Big] - \frac{1}{2}\Big[\big[E_{1n},[X_0^+,H_0]\big],X_{0,1}^-\Big]\\
&\quad + \frac{1}{2}\Big[E_{n1},\big[E_{1n},[[X_0^+,X_{0,1}^-],E_{1n}]\big]\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= -[H_{0,1}, E_{1n}] + \big[[E_{1n}, X_0^+], X_{0,1}^-\big] + \frac{1}{2}\Big[E_{n1}, \big[E_{1n}, [H_{0,1}, E_{1n}]\big]\Big] \\
&= 2[E_{1n}, H_{0,1}] + \frac{1}{2}\Big[E_{n1}, \big[E_{1n}, [H_{0,1}, E_{1n}]\big]\Big] \qquad (58)
\end{aligned}
$$

$$
\begin{aligned}
[H_{0,1}, E_{1n}] &= \big[[H_{0,1}, E_{12}], E_{2n}\big] + \Big[E_{12}, \big[E_{23}, [\cdots [E_{n-2,n-1}, [H_{0,1}, E_{n-1,n}] \cdots \Big] \\
&= [-X_{1,1}^+ - (\beta H_0 X_1^+ + (\lambda - \beta) X_1^+ H_0), E_{2n}] \\
&\quad + \Big[E_{12}, \big[E_{23}, [\cdots [E_{n-2,n-1}, -X_{n-1,1}^+ - ((\lambda - \beta) H_0 X_{n-1}^+ + \beta X_{n-1}^+ H_0)\big] \cdots \Big] \\
&= [-J(X_1^+) + \lambda \omega_1^+ - (\beta H_0 X_1^+ + (\lambda - \beta) X_1^+ H_0), E_{2n}] + \\
&\quad \Big[E_{12}, \big[E_{23}, [\cdots [E_{n-2,n-1}, -J(X_{n-1}^+) + \lambda \omega_{n-1}^+ - ((\lambda - \beta) H_0 X_{n-1}^+ + \beta X_{n-1}^+ H_0)\big] \cdot \Big] \\
&= -J(E_{1n}) - (\beta H_0 E_{1n} + (\lambda - \beta) E_{1n} H_0) + (\beta E_{2n} E_{12} + (\lambda - \beta) E_{12} E_{2n}) \\
&\quad + \lambda [\omega_1^+, E_{2n}] - J(E_{1n}) + \lambda [E_{1,n-1}, \omega_{n-1}^+] \\
&\quad - \Big[E_{12}, \big[E_{23}, [\cdots [E_{n-2,n-1}, ((\lambda - \beta) H_0 X_{n-1}^+ + \beta X_{n-1}^+ H_0)\big] \cdots \Big] \qquad (59)
\end{aligned}
$$

The expression $\Big[E_{12}, \big[E_{23}, [\cdots [E_{n-2,n-1}, ((\lambda - \beta) H_0 X_{n-1}^+ + \beta X_{n-1}^+ H_0)\big] \cdots \Big]$ is equal to

$$
[E_{12}, (\lambda - \beta) H_0 E_{2n} + \beta E_{2n} H_0] = ((\lambda - \beta) E_{12} E_{2n} + \beta E_{2n} E_{12}) + ((\lambda - \beta) H_0 E_{1n} + \beta E_{1n} H_0) \qquad (60)
$$

$$
\begin{aligned}
[E_{1,n-1}, \omega_{n-1}^+] &= -\frac{1}{4} \sum_{j=2}^{n-2} (E_{jn} E_{1j} + E_{1j} E_{jn}) - \frac{1}{4} \big(E_{1n}(E_{11} - E_{n-1,n-1}) + (E_{11} - E_{n-1,n-1}) E_{1n}\big) \\
&\quad - \frac{1}{4} (E_{1n} H_{n-1} + H_{n-1} E_{1n}) - \frac{1}{4} (E_{n-1,n} E_{1,n-1} + E_{1,n-1} E_{n-1,n}) \\
&= -\frac{1}{4} \sum_{j=2}^{n-1} (E_{jn} E_{1j} + E_{1j} E_{jn}) + \frac{1}{4} (E_{1n} H_0 + H_0 E_{1n}) \qquad (61)
\end{aligned}
$$

$$
\begin{aligned}
[\omega_1^+, E_{2n}] &= \frac{1}{4} \sum_{j=3}^{n-1} (E_{jn} E_{1j} + E_{1j} E_{jn}) - \frac{1}{4} \big((E_{22} - E_{nn}) E_{1n} + E_{1n}(E_{22} - E_{nn})\big) \\
&\quad - \frac{1}{4} (E_{1n} H_1 + H_1 E_{1n}) + \frac{1}{4} (E_{12} E_{2n} + E_{2n} E_{12}) \\
&= \frac{1}{4} \sum_{j=2}^{n-1} (E_{jn} E_{1j} + E_{1j} E_{jn}) + \frac{1}{4} (E_{1n} H_0 + H_0 E_{1n}) \qquad (62)
\end{aligned}
$$

Therefore, combining equations (59),(61),(62) and (60), we obtain the following simple expression for $[H_{0,1}, E_{1n}]$:

$$[H_{0,1}, E_{1n}] = -2J(E_{1n}) - \frac{\lambda}{2}(H_0 E_{1n} + E_{1n} H_0) \tag{63}$$

Putting together equations (58) and (63) yields equality (57):

$$\begin{aligned}
[Q(E_{1n}), K_1(H_0)] + [K_1(E_{1n}), Q(H_0)] &= 4J(E_{1n}) + \lambda(H_0 E_{1n} + E_{1n} H_0) \\
&\quad -\frac{1}{2}[E_{n1}, [E_{1n}, 2J(E_{1n}) + \frac{\lambda}{2}(H_0 E_{1n} + E_{1n} H_0)]] \\
&= 4J(E_{1n}) + \lambda(H_0 E_{1n} + E_{1n} H_0) - \lambda[E_{n1}, E_{1n}^2] \\
&= 4J(E_{1n})
\end{aligned}$$

The bracket of E_{n1} with both sides of equation (57) yields

$$[K_1(E_{n1}), Q_1(E_{1n})] + [Q_1(E_{n1}), K_1(E_{1n})] = 2J(H_0) \tag{64}$$

This proves proposition 7.1 when $a = n, b = 1, c = 1, d = n$.

Assuming that $a \neq 1, n$, we apply $[E_{an}, \cdot]$ to (64) to get $[K_1(E_{a1}), Q_1(E_{1n})] + [Q_1(E_{a1}), K_1(E_{1n})] = 2J(E_{an})$. If $b \neq 1, a$, we apply $[\cdot, E_{1b}]$ to the previous equation: this yields $[K_1(E_{ab}), Q_1(E_{1n})] + [Q_1(E_{ab}), K_1(E_{1n})] = 0$. If $c \neq 1, n$, we use $[E_{c1}, \cdot]$ to get

$$\begin{aligned}
[K_1(E_{ab}), Q_1(E_{cn})] + [Q_1(E_{ab}), K_1(E_{cn})] &= 2\delta_{bc}\big([K_1(E_{a1}), Q_1(E_{1n})] + [Q_1(E_{a1}), K_1(E_{1n})]\big) \\
&= 2\delta_{bc} J(E_{an})
\end{aligned}$$

We now apply $[\cdot, E_{nd}]$ if $b, d \neq n$ and obtain

$$[K_1(E_{ab}), Q_1(E_{cd})] + [Q_1(E_{ab}), K_1(E_{cd})] - 2\delta_{ad}\big([K_1(E_{nb}), Q_1(E_{cn})] + [Q_1(E_{nb}), K_1(E_{cn})]\big)$$

$$= 2\delta_{bc} J(E_{ad}) - 2\delta_{bc}\delta_{ad} J(E_{nn}).$$

Note that, although $J(E_{nn})$ is not defined, if $b = c$ and $a = d$, then the right-hand side becomes $2J(E_{aa} - E_{nn})$. It is enough to show that $[K_1(E_{nb}), Q_1(E_{cn})] + [Q_1(E_{nb}), K_1(E_{cn})] = -2J(E_{cb}) + 2\delta_{bc} J(E_{nn})$. Starting with (64) and assuming that $b, c \neq 1, n$, we apply $[\cdot, E_{1b}]$ and $[E_{c1}, \cdot]$ to get this last equation. The remaining cases can be handled in a similar manner. □

8 Schur-Weyl dual of the rational Cherednik algebra

Our goal in this section is to establish an equivalence of categories for the rational Cherednik algebra similar to the one given in theorem 5.2 and to identify the Schur-Weyl dual of H with a subalgebra of LY.

8.1 Case of type $\mathfrak{gl}_l$

Definition 8.1. *The subalgebra of LY generated by* $X_i^{\pm}, 1 \le i \le n-1, X_0^+$ *and* Y_0^+ *is denoted by* $\mathbb{L}_{\beta,\lambda}$ *and called a Yangian deformed double-loop algebra, as suggested in [BHW]. The polynomial loop algebra generated by* $X_i^{\pm}, 1 \le i \le n-1$ *and* X_0^+ *(resp.* Y_0^+*) is denoted* L_X *(resp.* L_Y*).*

Remark 8.1. *The algebra* $LY_{\beta,\lambda}$ *is the same as the subalgebra generated by* $z, K_1(z), Q_1(z), \forall z \in \mathfrak{sl}_n$*. Furthermore, proposition 7.1 implies that* $\mathbb{L}_{\beta,\lambda}$ *contains all the elements* $X_{i,r}^{\pm}, H_{i,r}$ *for* $1 \le i \le n, r \ge 0$ *and relation (12) shows that it also contains* $X_{0,r}^+, \forall r \ge 0$ *and* $X_{0,r}^-, \forall r \ge 1$*. We will abbreviate* $\mathbb{L}_{\beta,\lambda}$ *by* $\mathbb{L}$*.*

The computations for the action of $X_{0,1}^-$ on $M \otimes_{\mathbb{C}[S_l]} V^l$ and the anti-symmetric role of $\mathfrak{h}$ and $\mathfrak{h}^*$ in the definition of H, along with the last proposition of the previous section, suggest that the following result is true.

Proposition 8.1. *There exists an anti-involution* ι *of* $\mathbb{L}$ *which interchanges* L_X *and* L_Y *and which is given on the generators by the formulas*

$$\iota(X_{i,r}^{\pm}) = X_{i,r}^{\mp} \text{ if } i \neq 0, \ \ \iota(H_{i,r}) = H_{i,r}$$

$$\iota(X_{0,r}^{+}) = X_{0,r+1}^{-} \text{ for } r \ge 0, \ \ \iota(X_{0,r}^{-}) = X_{0,r-1}^{+} \text{ for } r \ge 1$$

Proof. This can be checked using the relations given in definition 3.2. □

Theorem 8.1. *Suppose that* $l \ge 1, n \ge 3$*. Set* $\lambda = c$ *and* $\beta = \frac{t}{2} - \frac{cn}{4} + \frac{c}{2}$*. The functor* $M \mapsto M \otimes_{\mathbb{C}[S_l]} V^{\otimes l}$ *sends a right* H*-module to an integrable left* $\mathbb{L}$*-module of level* l*. Furthermore, if* $l+2 < n$*, this functor is an equivalence.*

Proof. As for theorem 5.2, the proof is in two parts. First, it is enough to take $M = \mathsf{H}$ and show that $\mathcal{F}(M)$ is a module over $\mathbb{L}$. We can view $\mathsf{H} \otimes_{\mathbb{C}[S_l]} \mathbb{C}^{\otimes l}$ as a subspace of $\mathbf{H} \otimes_{\mathbb{C}[S_l]} \mathbb{C}^{\otimes l}$; the later is a module over $\mathbb{L}$ since it is even a module over LY. The subspace $\mathcal{F}(\mathsf{H})$ is stable under the action of the subalgebras L_X and L_Y, so it is a module over the subalgebra of LY generated by L_X and L_Y, which is exactly $\mathbb{L}$. The fact that $\mathcal{F}(M)$ is integrable of level l follows from the same argument as in the proof of theorem 5.2.

Now let N be an integrable module of level l over $\mathbb{L}$ and suppose that $l+2 < n$. We have to show that there exists a module M over H such that $\mathcal{F}(M) = N$. We can argue as for the trigonometric case to conclude that there exists an S_l-module M, which is also a $\mathbb{C}[\mathfrak{h}] \rtimes W$- and a $\mathbb{C}[\mathfrak{h}^*] \rtimes W$-module, such that $\mathcal{F}(M) \cong N$. As before, we must show that M is actually a module over H.

Fix $1 \le j, k \le l, j \neq k$. Choose $\mathbf{v} = v_{i_1} \otimes \cdots \otimes v_{i_l}$ such that $i_k = 2, i_j = n-1$, $i_r = r+2$ if $r < j, r \neq k$, $i_r = r+1$ if $r > j, r \neq k$. Set $\widetilde{\mathbf{v}} = E_{n2}^k E_{1,n-1}^j(\mathbf{v})$.

On one hand,

$$\big(Q_1(E_{1,n-1})K_1(E_{n2}) - K_1(E_{n2})Q_1(E_{1,n-1})\big)(m \otimes \mathbf{v}) =$$

$$\sum_{s=1}^{l}\sum_{r=1}^{l} m x_r y_s \otimes E_{1,n-1}^s E_{n2}^r(\mathbf{v}) - \sum_{s=1}^{l}\sum_{r=1}^{l} m y_s x_r \otimes E_{n2}^r E_{1,n-1}^s(\mathbf{v}) \ = \ m(x_k y_j - y_j x_k) \otimes \widetilde{\mathbf{v}} \quad (65)$$

On the other hand, $Q_1(E_{1,n-1}) = [Y_0^+, E_{n,n-1}]$ and $K_1(E_{n2}) = [X_0^+, E_{12}]$, so:

$$\begin{aligned}
[Q_1(E_{1,n-1}), K_1(E_{n2})] &= \big[[Y_0^+, E_{n,n-1}], [X_0^+, E_{12}]\big] = \big[[X_{0,1}^-, [X_0^+, E_{12}]], E_{n,n-1}\big] \\
&= \Big[\big[[X_{0,1}^-, X_0^+], X_1^+\big], E_{n,n-1}\Big] = -\big[[H_{0,1}, X_1^+], E_{n,n-1}\big] \\
&= -[-X_{1,1}^+ - (\beta H_0 X_1^+ + (\lambda-\beta) X_1^+ H_0), X_{n-1}^-] \\
&= [\beta H_0 X_1^+ + (\lambda-\beta) X_1^+ H_0, X_{n-1}^-] \\
&= \beta E_{n,n-1} E_{12} + (\lambda-\beta) E_{12} E_{n,n-1}
\end{aligned}$$

Therefore,

$$\begin{aligned}
[Q_1(E_{1,n-1}), K_1(E_{n2})](m \otimes \mathbf{v}) &= m \otimes (\beta E_{n,n-1} E_{12} + (\lambda-\beta) E_{12} E_{n,n-1})(\mathbf{v}) \\
&= \lambda m \otimes E_{12}^k E_{n,n-1}^j(\mathbf{v}) = \lambda m s_{jk} \otimes \widetilde{\mathbf{v}} \qquad (66)
\end{aligned}$$

Equations (65) and (66) imply that $m(x_k y_j - y_j x_k - \lambda s_{jk}) \otimes \widetilde{\mathbf{v}} = 0$. From lemma 6.2 and our assumption that $\lambda = c$, we conclude that

$$m(x_k y_j - y_j x_k - c s_{jk}) = 0. \qquad (67)$$

Now let $\mathbf{v}$ be determined by $i_k = n-1$, $i_j = j+1$ if $j \neq k$. Set $\widehat{\mathbf{v}} = E_{n,n-1}^k(\mathbf{v})$. On one hand,

$$[K_1(E_{n1}), Q_1(E_{1,n-1})](m \otimes \mathbf{v}) = m y_k x_k \otimes E_{n1}^k E_{1,n-1}^k(\mathbf{v}) = m y_k x_k \otimes \widehat{\mathbf{v}} \qquad (68)$$

On the other hand,

$$\begin{aligned}
[K_1(E_{n1}), Q_1(E_{1,n-1})] &= [X_0^+, [Y_0^+, E_{n,n-1}]] = [X_0^+, [X_{0,1}^-, X_{n-1}^-]] \\
&= [H_{0,1}, X_{n-1}^-] = X_{n-1,1}^- + (\beta H_0 X_{n-1}^- + (\lambda-\beta) X_{n-1}^- H_0) \\
&= J(X_{n-1}^-) - \lambda \omega_{n-1}^- + (\beta H_0 X_{n-1}^- + (\lambda-\beta) X_{n-1}^- H_0)
\end{aligned}$$

where

$$\omega_{n-1}^- = -\frac{1}{4}\sum_{d=1}^{n-2}(E_{nd}E_{d,n-1} + E_{d,n-1}E_{nd}) - \frac{1}{4}(X_{n-1}^- H_{n-1} + H_{n-1}X_{n-1}^-).$$

Therefore, we also have:

$$\begin{aligned}
[K_1(E_{n1}), Q_1(E_{1,n-1})](m \otimes \mathbf{v}) &= \big(J(X_{n-1}^-) - \lambda\omega_{n-1}^- + (\beta H_0 X_{n-1}^- + (\lambda-\beta) X_{n-1}^- H_0)\big)(m \otimes \mathbf{v}) \\
&= m\mathcal{Y}_k \otimes E_{n,n-1}^k(\mathbf{v}) - \lambda m \otimes (\omega_{n-1}^-(\mathbf{v})) + \beta m \otimes H_0^k E_{n,n-1}^k(\mathbf{v}) \\
&= m\mathcal{Y}_k \otimes \widehat{\mathbf{v}} + \frac{\lambda}{2}\sum_{d=1}^{n-2}\sum_{r=1}^{l}\sum_{\substack{s=1 \\ s\neq d}}^{l} m \otimes (E_{nd}^s E_{d,n-1}^r)(\mathbf{v}) \\
&\quad + \lambda\left(\frac{n-2}{4}\right) m \otimes E_{n,n-1}(\mathbf{v}) + \frac{\lambda}{4} m \otimes (X_{n-1}^- H_{n-1} \\
&\quad + H_{n-1}X_{n-1}^-)(\mathbf{v}) + \beta m \otimes H_0^k E_{n,n-1}^k(\mathbf{v})
\end{aligned}$$

$$\begin{aligned}[K_1(E_{n1}),Q_1(E_{1,n-1})](m\otimes \mathbf{v}) &= m\mathcal{Y}_k\otimes\widehat{\mathbf{v}}+\frac{\lambda}{2}\sum_{d=1}^{n-2}\sum_{\substack{j=1\\ j\neq k}}^{l} m\otimes(E^j_{nd}E^k_{d,n-1})(\mathbf{v})+\lambda\left(\frac{n-2}{4}\right)m\otimes\widehat{\mathbf{v}}\\
&\quad+\frac{\lambda}{4}m(E^k_{n,n-1}E^k_{n-1,n-1}-E^k_{nn}E^k_{n,n-1})(\mathbf{v})+\beta m\otimes E^k_{n,n-1}(\mathbf{v})\\
&= \frac{1}{2}m(x_ky_k+y_kx_k)\otimes\widehat{\mathbf{v}}+\frac{\lambda}{2}\sum_{\substack{j=1\\ j\neq k}}^{l} ms_{jk}\otimes\widehat{\mathbf{v}}\\
&\quad+(\frac{\lambda n}{4}-\frac{\lambda}{2}+\beta)m\otimes\widehat{\mathbf{v}} \qquad (69)\end{aligned}$$

From the equations (68) and (69) and our hypothesis that $\beta=\frac{t}{2}-\frac{\lambda n}{4}+\frac{\lambda}{2}$, we deduce the following equality:

$$my_kx_k\otimes\widehat{\mathbf{v}}=\frac{1}{2}m(x_ky_k+y_kx_k)\otimes\widehat{\mathbf{v}}+\frac{\lambda}{2}\sum_{\substack{j=1\\ j\neq k}}^{l} ms_{jk}\otimes\widehat{\mathbf{v}}+\frac{t}{2}m\otimes\widehat{\mathbf{v}}$$

which implies that $m\big(y_kx_k-x_ky_k-t-\lambda\sum_{\substack{j=1\\ j\neq k}}^{l}s_{jk}\big)\otimes\widehat{\mathbf{v}}=0$. Since $\lambda=c$ by assumption and $\widehat{\mathbf{v}}$ is a generator of $V^{\otimes l}$ as $\mathfrak{U}\mathfrak{sl}_n$-module, we conclude, using again lemma 6.2, that the equality

$$m\big(y_kx_k-x_ky_k-t-c\sum_{\substack{j=1\\ j\neq k}}^{l}s_{jk}\big)=0 \qquad (70)$$

must be satisfied. Equations (67) and (70) show that M is a right module over H. Finally, that $\mathcal{F}$ is bijective on the set of morphisms follows from an argument similar to the one used in the trigonometric case. □

8.2 Case of type A_{l-1}

So far, we have considered only Cherednik algebras of type $\mathfrak{gl}_l$. There is at least one major difference between these and the Cherednik algebras $\widetilde{\mathsf{H}}_{t,c}$ of type A_{l-1}: the latter admit finite dimensional representations for certain specific values of $t\neq 0$ and c (see [BEG2],[Go]), whereas the former don't have such representations if $t\neq 0$ because, in this case, they contain a copy of the first Weyl algebra (which is the subalgebra of $\mathsf{H}_{t,c}$ generated by $x_1+\ldots+x_n$ and $y_1+\ldots+y_n$).

We need to introduce two new algebras.

Definition 8.2. *The algebra $\mathbb{L}'_{\beta,\lambda,t,l}$ is obtained by adjoining to $\mathbb{L}$ two generators, ξ and δ, which satisfy the following relations:*

$$[\xi,z]=0,\quad [\xi,J(z)]=\frac{t}{l}K_1(z),\quad [\xi,Q_1(z)]=\frac{t}{l}z,\quad [\xi,K_1(z)]=0,\quad \forall z\in\mathfrak{sl}_n$$

$$[\delta,z]=0,\quad [\delta,J(z)]=-\frac{t}{l}Q_1(z),\quad [\delta,Q_1(z)]=0,\quad [\delta,K_1(z)]=-\frac{t}{l}z,\quad \forall z\in\mathfrak{sl}_n,\quad [\xi,\delta]=\frac{t}{l}$$

Definition 8.3. *We define* $\widetilde{\mathbb{L}}_{\beta,\lambda,t,l}$ *to be the subalgebra of* $\mathbb{L}'_{\beta,\lambda,t,l}$ *which is generated by* $\mathfrak{sl}_n$*,* $K_1(z) - z \cdot \xi$ *and by* $Q_1(z) - z \cdot \delta$ *for all* $z \in \mathfrak{sl}_n$*.*

We will write $\widetilde{\mathbb{L}}$ instead of $\widetilde{\mathbb{L}}_{\beta,\lambda,t,l}$ and $\mathbb{L}'$ instead of $\mathbb{L}'_{\beta,\lambda,t,l}$ in general. We will denote by $\widetilde{L}_X$ the subalgebra of $\widetilde{\mathbb{L}}$ (isomorphic to $\mathfrak{U}(\mathfrak{sl}_n \otimes_{\mathbb{C}} \mathbb{C}[u])$) generated by $\mathfrak{sl}_n$ and $\widetilde{K}_1(z) = K_1(z) - z \cdot \xi, \forall z \in \mathfrak{sl}_n$, and by $\widetilde{L}_Y$ the one (also isomorphic to the enveloping algebra of the polynomial loop algebra of $\mathfrak{sl}_n$) generated by $\mathfrak{sl}_n$ and $\widetilde{Q}_1(z) = Q_1(z) - z \cdot \delta, \forall z \in \mathfrak{sl}_n$.

Set $\bar{x} = \frac{1}{l}(x_1 + \ldots + x_l)$ and $\bar{y} = \frac{1}{l}(y_1 + \ldots + y_l)$. Note that $x_j - \bar{x} \in \mathfrak{h}_0^*$ and $y_j - \bar{y} \in \mathfrak{h}_0$, where $\mathfrak{h}_0^* = \text{span}\{x_i - x_j | 1 \le i \neq j \le l\} \subset \mathfrak{h}^*$ and $\mathfrak{h}_0 = \text{span}\{y_i - y_j | 1 \le i \neq j \le l\} \subset \mathfrak{h}$.

Given a module M over H, it is possible to make $\mathcal{F}(M)$ into a module over $\mathbb{L}'$ by letting ξ and δ act on $M \otimes_{\mathbb{C}[S_l]} V^l$ in the following way: $\xi(m \otimes \mathbf{v}) = m\bar{x} \otimes \mathbf{v}$ and $\delta(m \otimes \mathbf{v}) = m\bar{y} \otimes \mathbf{v}$. This follows directly from our choice of relations in definition 8.2. From this we can deduce that, given a module M over $\widetilde{\mathsf{H}}$, $\mathcal{F}(M)$ can be made into a module over $\widetilde{\mathbb{L}}$ by letting $\widetilde{K}_1(z)$ and $\widetilde{Q}_1(z)$ act on $M \otimes_{\mathbb{C}[S_l]} V^l$ in the following way:

$$\widetilde{K}_1(z)(m \otimes \mathbf{v}) = \sum_{j=1}^{l} m(x_j - \bar{x}) \otimes z^j(\mathbf{v}), \quad \widetilde{Q}_1(z)(m \otimes \mathbf{v}) = \sum_{j=1}^{l} m(y_j - \bar{y}) \otimes z^j(\mathbf{v}).$$

Note that this module structure has the following particularity if $l+1 \le n$: choose $\mathbf{v} = v_{i_1} \otimes \cdots \otimes v_{i_l}$ such that the i_k are pairwise distinct and choose $1 \le j \le n$ such that $j \neq i_k$ for any $k, 1 \le k \le l$. Set $z = E_{i_1 i_1} + \cdots + E_{i_l i_l} - lE_{jj} \in \mathfrak{sl}_n$. Then, since $\sum_{k=1}^{l}(x_k - \bar{x}) = 0$,

$$\widetilde{K}_1(z)(m \otimes \mathbf{v}) = \sum_{k=1}^{l} m(x_k - \bar{x}) \otimes z^k(\mathbf{v}) = \sum_{k=1}^{l} m(x_k - \bar{x}) \otimes E^k_{i_k i_k}(\mathbf{v}) = \sum_{k=1}^{l} m(x_k - \bar{x}) \otimes \mathbf{v} = 0$$

Let $\mu_1, \ldots, \mu_n$ be the fundamental weights of the usual Cartan subalgebra of $\mathfrak{gl}_n$. The vector $\mathbf{v} = v_{i_1} \otimes \cdots \otimes v_{i_l}$ has weight $\mu_{i_1} + \ldots + \mu_{i_l}$. The observation in this paragraph motivates the following definition.

Definition 8.4. *Suppose that* $l+1 \le n$*. An integrable module* N *of level* l *over* $\widetilde{\mathbb{L}}$ *is said to satisfy condition* Cond(l) *if the following vanishing condition is satisfied: if* $\eta \in N$ *is a weight vector of weight* $\mu_{i_1} + \ldots + \mu_{i_l}$ *for* $i_1, \ldots, i_l \in \{1, \ldots, n\}$*, then* $\widetilde{K}_1(z)(\eta) = 0$ *and* $\widetilde{Q}_1(z)(\eta) = 0$*, where* $z = E_{i_1 i_1} + \ldots + E_{i_l i_l} - lE_{jj} \in \mathfrak{sl}_n$ *for any choice of* $j \in \{1, \ldots, n\}, j \neq i_k \forall k$*.*

We can now establish a Schur-Weyl equivalence between $\widetilde{\mathsf{H}}$ and $\widetilde{\mathbb{L}}$.

Theorem 8.2. *Suppose that* $l \ge 2, n \ge 3$*. Set* $\lambda = c, \beta = \frac{t}{2} - \frac{cn}{4} + \frac{c}{2}$*. The functor* $\mathcal{F}$ *given by* $\mathcal{F}(M) = M \otimes_{\mathbb{C}[S_l]} V^l$ *sends a right module over* $\widetilde{\mathsf{H}}$ *to an integrable left module over* $\widetilde{\mathbb{L}}$ *of level* l*. Furthermore, if* $l+2 < n$*, this functor is an equivalence of categories if we restrict ourselves to left modules over* $\widetilde{\mathbb{L}}$ *satisfying condition* Cond(l)*.*

Proof. We have already established the first part of the theorem, so suppose that $l + 2 < n$. The proof of the equivalence follows exactly the same steps as the proof of theorem 8.1 with one major difference and a few minor ones. The major one is the following. Let $\widetilde{M}$ be an integrable left module over $\widetilde{\mathbb{L}}$ satisfying condition Cond(l). By the Schur-Weyl equivalence between $\widetilde{L}_X$ and the smash product $\mathrm{Sym}(\mathfrak{h}^*) \rtimes S_l$, we know that there exists a module M_1 over $\mathrm{Sym}(\mathfrak{h}^*) \rtimes S_l$ such that $\mathcal{F}(M_1) = \widetilde{M}$ as $\widetilde{L}_X$-modules and the action of $\widetilde{K}_1(z)$ is given by $\widetilde{K}_1(z)(m \otimes \mathbf{v}) = \sum_{k=1}^{l} mx_k \otimes z^k(\mathbf{v}), m \in M_1$. Set $z = E_{11} + \ldots + E_{ll} - lE_{nn}$ and $\mathbf{v} = v_1 \otimes \cdots \otimes v_l$. Since we are assuming that $\widetilde{M}$ satisfies condition Cond(l), we know that, in particular, $\widetilde{K}_1(z)(m \otimes \mathbf{v}) = 0$, so this means that $0 = \sum_{k=1}^{l} mx_k \otimes z^k(\mathbf{v}) = \sum_{k=1}^{l} mx_k \otimes \mathbf{v}$. Lemma 6.2 now implies that $m\left(\sum_{k=1}^{l} x_k\right) = 0$. This means that M_1 can be viewed as a module over the quotient of $\mathrm{Sym}(\mathfrak{h}^*) \rtimes S_l$ by the ideal generated by $\bar{x}$: this quotient is isomorphic to $\mathrm{Sym}(\mathfrak{h}_0^*) \rtimes S_l$, which is exactly what we needed. Similarly, we can argue that there exists a module M_2 over $\mathrm{Sym}(\mathfrak{h}_0) \rtimes S_l$ such that $\mathcal{F}(M_2) \cong M$ as modules over $\widetilde{L}_Y$. We can identify M_1 and M_2 as modules over $\mathfrak{sl}_n$.

As for the minor differences, one should use $\widetilde{x}_i = x_i - \bar{x}$ and $\widetilde{y}_i = y_i - \bar{y}$ as generators of $\widetilde{\mathsf{H}}$ and note that $[\widetilde{y}_i, \widetilde{x}_j] = [y_i, x_j] - \frac{t}{l}$. The following relations must also be used:

$$\begin{aligned}
[\widetilde{Q}_1(E_{1,n-1}), \widetilde{K}_1(E_{n2})] &= [Q_1(E_{1,n-1}), K_1(E_{n2})] - E_{n2}[Q_1(E_{1,n-1}), \xi] \\
&\quad - E_{1,n-1}[\delta, K_1(E_{n2})] + E_{1,n-1}E_{n2}[\delta, \xi] \\
&= [Q_1(E_{1,n-1}), K_1(E_{n2})] + \frac{t}{l}E_{n2}E_{1,n-1}
\end{aligned}$$

$$\begin{aligned}
[\widetilde{K}_1(E_{n1}), \widetilde{Q}_1(E_{1,n-1})] &= [K_1(E_{n1}), Q_1(E_{1,n-1})] - [E_{n1}\xi, Q_1(E_{1,n-1})] \\
&\quad - [K_1(E_{n1}), E_{1,n-1}\delta] + [E_{n1}\xi, E_{1,n-1}\delta] \\
&= [K_1(E_{n1}), Q_1(E_{1,n-1})] - Q_1(E_{n,n-1})\xi - \frac{t}{l}E_{n1}E_{1,n-1} \\
&\quad - K_1(E_{n,n-1})\delta - \frac{t}{l}E_{1,n-1}E_{n1} + E_{n,n-1}\xi\delta + \frac{t}{l}E_{1,n-1}E_{n1} \\
&= [K_1(E_{n1}), Q_1(E_{1,n-1})] - Q_1(E_{n,n-1})\xi - \frac{t}{l}E_{n1}E_{1,n-1} \\
&\quad - K_1(E_{n,n-1})\delta + E_{n,n-1}\xi\delta
\end{aligned}$$

$$\begin{aligned}
(\widetilde{x}_k\widetilde{y}_k + \widetilde{y}_k\widetilde{x}_k) &= (x_k y_k + y_k x_k) - (\bar{x}y_k + y_k\bar{x}) - (x_k\bar{y} + \bar{y}x_k) + (\bar{x}\bar{y} + \bar{y}\bar{x}) \\
&= (x_k y_k + y_k x_k) - 2\bar{x}y_k - [y_k, \bar{x}] - 2\bar{y}x_k - [x_k, \bar{y}] + 2\bar{y}\bar{x} + [\bar{x}, \bar{y}] \\
&= (x_k y_k + y_k x_k) - 2\bar{x}y_k - 2\bar{y}x_k + 2\bar{y}\bar{x} - \frac{t}{l}
\end{aligned}$$

□

8.3 Category $\mathcal{O}$

One important category of modules over $\mathsf{H}_{t,c}$ and $\widetilde{\mathsf{H}}_{t,c}$ (when $t \neq 0$) is the category $\mathcal{O}$ studied in [GGOR].

Definition 8.5. *We define $\mathcal{O}_{t,c}$ (resp. $\widetilde{\mathcal{O}}_{t,c}$) for $t \neq 0$ to be the category of right modules over $\mathsf{H}_{t,c}$ (resp. $\widetilde{\mathsf{H}}_{t,c}$) which are finitely generated over $\mathsf{H}_{t,c}$ (resp. $\widetilde{\mathsf{H}}_{t,c}$) and locally nilpotent over $\mathbb{C}[\mathfrak{h}^*]$ (resp. $\mathbb{C}[\mathfrak{h}_0^*]$). We set $\mathcal{O} = \mathcal{O}_{t,c}$ and $\widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}_{t,c}$.*

We see from the definition of the $\mathbb{L}$-module structure on $\mathcal{F}(M)$ that if $M \in \mathcal{O}$ then $\mathcal{F}(M)$ is locally nilpotent over the subalgebra A of $\mathbb{L}$ generated by $Q_r(z), \forall z \in \mathfrak{sl}_n, \forall r \geq 1$. A similar observation is true for $\widetilde{\mathsf{H}}$, the subalgebra $\widetilde{A}$ being the one generated by $\widetilde{Q}_1(z), \forall z \in \mathfrak{sl}_n$. This leads us to our last theorem.

Theorem 8.3. *Assume that $l+2 < n$, $\lambda = c$ and $\beta = \frac{t}{2} - \frac{cn}{4} + \frac{c}{2}$. The functor $\mathcal{F}$ establishes an equivalence between the category $\mathcal{O}$ (resp. $\widetilde{\mathcal{O}}$) and the category of finitely generated left modules over $\mathbb{L}$ (resp. $\widetilde{\mathbb{L}}$) which are locally nilpotent over the subalgebra A (resp. $\widetilde{A}$) and integrable of level l (resp. and satisfy condition* Cond(l)*).*

Proof. We prove this theorem for H, the proof being the same for $\widetilde{\mathsf{H}}$. If $m_1, \ldots, m_k$ are generators of M, then $\{m_i \otimes \mathbf{v}, 1 \leq i \leq k, \mathbf{v} = v_{i_1} \otimes \cdots \otimes v_{i_l}\}$ is a finite set of generators for $\mathcal{F}(M)$. To see this, we can assume that M is generated over $\mathbb{C}[\mathfrak{h}]$ by $m_1, \ldots, m_k$. Take an element $m \otimes \mathbf{v} \in \mathcal{F}(M)$ with $m = m_1 x_1^{a_1} \cdots x_l^{a_l}$. We suppose first that $\mathbf{v} = v_1 \otimes v_2 \otimes v_3 \otimes \cdots \otimes v_l$ and set $\mathbf{v}' = v_1 \otimes v_3 \otimes v_4 \otimes \cdots \otimes v_{l+1}$. Then

$$m \otimes \mathbf{v} = K_{a_l}(E_{l,l+1}) \cdots K_{a_2}(E_{23}) K_{a_1}(H_1)(m_1 \otimes \mathbf{v}').$$

Now we can apply elements of $\mathfrak{Usl}_n$ to $v_1 \otimes v_2 \otimes \cdots \otimes v_l$ to obtain any other element of V^l. The general case when $m = \sum_{j=1}^{k} m_j p_j(x_1, \ldots, x_l) \otimes \mathbf{v}_j$, $p_j(x_1, \ldots, x_l)$ being a polynomial, follows from this. Conversely, suppose that N is a finitely generated integrable module over $\mathbb{L}$ of level l and $N = \mathcal{F}(M)$. Let $\{n_1, \ldots, n_k\}$ be a set of generators of N and write $n_i = \sum_{j=1}^{k_i} m_{ij} \otimes \mathbf{v}_{ij}$ for some $m_{ij} \in M$ and some $\mathbf{v}_{ij} \in V^l$. Then $\{m_{ij} | 1 \leq i \leq k, 1 \leq j \leq k_i\}$ is a set of generators of M.

Now suppose that N is an integrable left module over $\mathbb{L}$ of level l which is locally nilpotent over A. By theorem 8.1, we know that $N = \mathcal{F}(M)$ for a right module M over H. Pick $m \in M$. It is enough to show that $m y_i^p = 0$ for some $p \in \mathbb{Z}_{>0}$. Set $\mathbf{v} = v_1 \otimes \cdots \otimes v_i \otimes v_{i+2} \otimes \cdots \otimes v_{l+1}$ and choose p so that $Q_1(H_i)^p(m \otimes \mathbf{v}) = 0$. Then $Q_1(H_i)(m \otimes \mathbf{v}) = m y_i \otimes \mathbf{v}$, so $m y_i^p \otimes \mathbf{v} = Q_1(H_i)^p(m \otimes \mathbf{v}) = 0$ and lemma 6.2 implies that $m y_i^p = 0$. □

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