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Global Stability for the SEIR Model in Epidemiology

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ABSTRACT

The SEIR model with nonlinear incidence rates in epidemiology is studied. Global stability of the endemic equilibrium is proved using a general criterion for the orbital stability of periodic orbits associated with higher-dimensional nonlinear autonomous systems as well as the theory of competitive systems of differential equations.

1. INTRODUCTION

The SEIR model in epidemiology for the spread of an infectious disease is described by the following system of differential equations:

$$\begin{aligned} S' &= -\lambda I^p S^q + \mu - \mu S \\ E' &= \lambda I^p S^q - (\epsilon + \mu) E \\ I' &= \epsilon E - (\gamma + \mu) I \\ R' &= \gamma I - \mu R, \end{aligned} \tag{1.1}$$

where $p, q, \gamma, \mu, \lambda$, and ϵ are positive parameters and S, E, I, and R denote the fractions of the population that are susceptible, exposed, infectious, and recovered, respectively. Some notable features of the model: the birth rate and the death rate are assumed to be equal (denoted by μ) and in consequence the total population is at an equilibrium; the incidence rate (the rate of new infections) is described by the nonlinear term $\lambda I^p S^q$ which includes the traditional bilinear case ($p = q = 1$); a latent period is introduced on the basis of the well-known SIS and SIR models. Individuals are susceptible, then exposed (in the latent period), then infectious, then recovered with permanent immunity.

Epidemiological models with nonlinear incidence rate have been under extensive studies in recent years. A good survey of results may be found in [4] and [8]. The existence and local stability of the nontrivial

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equilibrium as well as possible Hopf bifurcations for different values of the parameters are well understood (cf., [8]). The global stability of the nontrivial equilibrium, however, has long been conjectured and remained unsolved. It is the purpose of this paper to prove the global stability of the nontrivial equilibrium for the SEIR model (1.1).

Our main result, Theorem 2.1, is stated in the next section. In Section 3, we prove in Theorem 3.2 that any periodic orbit of the system (1.1), when it exists, is orbitally asymptotically stable based on a criterion for the asymptotic orbital stability of periodic orbits for general autonomous systems given in [11]. In Section 4, we first prove that (1.1) satisfies a strong Poincaré–Bendixson property using the theory of order preserving dynamical systems developed by M. W. Hirsch and H. L. Smith (cf., [5] or [13]). Then Theorem 2.1 is proved by ruling out periodic orbits using the result developed in the Section 3.

2. STATEMENT OF THE MAIN RESULT

We first outline some known results about the SEIR model (1.1). Their proofs and more detailed study can be found in [8].

Throughout this paper, we assume that $0 < p \leqslant 1$.

The feasible region for (1.1) is $\mathbf{R}^4_+$, the positive orthant of $\mathbf{R}^4$. Adding all the equations in (1.1) we have

$$(S+E+I+R)' = -\mu(S+E+I+R-1),$$

which has the following implications: the 3-dimensional simplex

$$\Gamma = \{(S,E,I,R) \in \mathbf{R}^4_+ : S+E+I+R = 1\}$$

is positively invariant; system (1.1) is dissipative and the global attractor is contained in Γ. Moreover, it suffices to study the dynamics of (1.1) on the simplex Γ.

On the simplex Γ,

$$R(t) = 1 - S(t) - E(t) - I(t).$$

Thus (1.1) reduces to the following 3-dimensional system:

$$\begin{aligned} S' &= -\lambda I^p S^q + \mu - \mu S \\ E' &= \lambda I^p S^q - (\epsilon + \mu)E \\ I' &= \epsilon E - (\gamma + \mu)I. \end{aligned} \tag{2.1}$$

The dynamical behavior of (1.1) on Γ is equivalent to that of (2.1). Therefore, in the rest of the paper we will study the system (2.1) in the region

$$T = \{(S,E,I) : 0 \leqslant S, E, I \leqslant 1, S+E+I \leqslant 1\},$$

and formulate our results accordingly.

There are two possible equilibria to (2.1): the disease-free equilibrium $P_0 = (1,0,0)$ and the endemic equilibrium P^*. When $0 < p < 1$, P_0 is unstable and all solutions starting near P_0 except those from the S-axis move away from P_0; P^* is in the interior of T and is locally asymptotically stable. When $p = 1$, the contact number $\sigma = \lambda\epsilon/(\epsilon + \mu)(\gamma + \mu)$ satisfies a threshold condition: if $\sigma \leq 1$, P_0 is the only equilibrium in $\bar{T}$ and is globally asymptotically stable; if $\sigma > 1$, P_0 becomes an unstable saddle with one of the two unstable eigenvectors pointing to the inside of T while P^* emerges as a locally asymptotically stable equilibrium in the interior of T. It has been conjectured that P^* is globally asymptotically stable whenever it belongs to the interior of T [8].

The main aim of the present paper is to prove the following result:

THEOREM 2.1

If $0 < p < 1$ *or* $p = 1$ *and* $\sigma > 1$, *the endemic equilibrium* P^* *is globally asymptotically stable in the interior of* T.

Remark. Once Theorem 2.1 is proved, the global dynamical behavior of (1.1) is completely determined when $0 < p \leqslant 1$.

The proof of Theorem 2.1 will be given in Section 4. It depends on an orbital stability result for the periodic solutions to (2.1) which we will discuss in the next section.

The following are some properties of the system (2.1) that will be used in later sections.

PROPOSITION 2.2

The disease-free equilibrium P_0 *is the only omega limit point of* (2.1) *on the boundary of* T.

Proof. It is easy to see that the vector field of (2.1) is transversal to the boundary of T on all its faces except the S-axis which is invariant with respect to (2.1). On the S-axis the equation S satisfies is $S' = \mu - \mu S$, which implies that $S(t) \to 1$ as $t \to \infty$. Therefore, P_0 is the only omega limit point on the boundary of T. ■

PROPOSITION 2.3

Suppose $0 < p < 1$ *or* $p = 1$ *and* $\sigma > 1$. *Then* P_0 *cannot be the omega limit point of any orbit starting in the interior of* T.

Proof. Consider the function

$$L = E + \frac{\epsilon + \mu}{\epsilon} I.$$

Its derivative along the solutions of (2.1) is

$$L' = \lambda I^p \left(S^q - \frac{1}{\sigma} I^{1-p} \right).$$

Suppose now $p < 1$ or $p = 1$ and $\sigma > 1$. In the feasible region close enough to P_0, we always have $L' > 0$ as long as $I > 0$. Therefore, P_0 can only be the omega limit point of orbits on the invariant S-axis. Thus the lemma is proved. ∎

Remark. From Proposition 2.2 and Proposition 2.3 we know that when $0 < p < 1$ or $p = 1$ and $\sigma > 1$, the system (2.1) is persistent in the sense described in [1].

3. ORBITAL STABILITY OF PERIODIC ORBITS

Let $D \subset \mathbf{R}^n$ be an open set, and $x \mapsto f(x) \in \mathbf{R}^n$ be a C^1 function defined in D. We consider the autonomous system in $\mathbf{R}^n$

$$x' = f(x). \tag{3.1}$$

Let $x(t, x_0)$ denote the solution of (3.1) such that $x(t, 0) = x_0$. The linear variational equation of (3.1) with respect to $x(t, x_0)$ is given by

$$y'(t) = \frac{\partial f}{\partial x}(x(t, x_0))y(t), \tag{3.2}$$

where $\partial f / \partial x$ is the Jacobian matrix of f.

Our main result in this section concerns the orbital stability of a periodic orbit of the system (2.1). We first recall the basic definitions (cf., [3]). Suppose (3.1) has a periodic solution $x = p(t)$ with least period $\omega > 0$ and orbit $\gamma = \{p(t) : 0 \leqslant t \leqslant \omega\}$. This orbit is *orbitally stable* if for each $\epsilon > 0$, there exists a $\delta > 0$ such that any solution $x(t)$, for which the distance of $x(0)$ from γ is less δ, remains at a distance less than ϵ from γ for all $t \geqslant 0$. It is *asymptotically orbitally stable* if the distance of $x(t)$ from γ also tends to zero as $t \to \infty$. This orbit γ is *asymptotically orbitally stable with asymptotic phase* if it is asymptotically orbitally stable and there is a $b > 0$ such that, any solution $x(t)$, for which the distance of $x(0)$ from γ is less than b, satisfies $|x(t) - p(t - \tau)| \to 0$ as $t \to \infty$ for some τ which may depend on $x(0)$.

The following is a criterion given in [11] for the asymptotic orbital stability of a periodic orbit to the general autonomous system (3.1).

THEOREM 3.1

A sufficient condition for a periodic orbit $\gamma = \{p(t) : 0 \leqslant t \leqslant \omega\}$ *of* (3.1) *to be asymptotically orbitally stable with asymptotic phase is that the linear system*

$$z'(t) = \left(\frac{\partial f^{[2]}}{\partial x} (p(t)) \right) z(t) \tag{3.3}$$

be asymptotically stable.

Equation (3.3) is called the *second compound equation* of (3.2) and $\partial f^{[2]} / \partial x$ is the *second compound matrix* of the Jacobian matrix $\partial f / \partial x$ of f. Generally speaking, for a $n \times n$ matrix A and integer $1 \leqslant k \leqslant n$, the *k-th additive compound matrix* of A is denoted by $A^{[k]}$. This is a $N \times N$ matrix, $N = \binom{n}{k}$, defined by

$$A^{[k]} = D_{+} (I + hA)^{(k)}|_{h=0},$$

where $B^{(k)}$ is the kth exterior power of a $n \times n$ matrix B and D_{+} denotes the right-hand derivative. A survey on the definition and properties of additive compound matrices together with their connections to differential equations may be found in [9, 11]. Further applications may be found in [6, 7]. Examples of various compound matrices when $n = 3$ are provided in the Appendix of this paper. We summarize some properties of the additive compound matrices in the following; readers are referred to [11] for their proofs.

The term *additive* comes from the property $(A + B)^{[k]} = A^{[k]} + B^{[k]}$; if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A, then all the possible sums of the form $\lambda_{i_1} + \cdots + \lambda_{i_k}$, $1 \leqslant i_1 < \cdots < i_k \leqslant n$, are the eigenvalues of $A^{[k]}$; in the two extreme cases when $k = 1$ and n, we have

$$A^{[1]} = A \qquad \text{and} \qquad A^{[n]} = \operatorname{tr}(A),$$

respectively.

Remark. It is also demonstrated in [11] that Theorem 3.1 generalizes a classic criterion of Poincaré for the orbital stability of periodic solutions to planar autonomous systems.

Using Theorem 3.1 we can prove the following result.

THEOREM 3.2

The trajectory of any nonconstant periodic solution to (2.1), *if it exists, is asymptotically orbitally stable with asymptotic phase.*

Proof. The Jacobian matrix $J(S,E,I)$ of (2.1) is given by

$$J(S,E,I) = \begin{bmatrix} -\lambda q I^p S^{q-1} - \mu & 0 & -\lambda p I^{p-1} S^q \\ \lambda q I^p S^{q-1} & -\epsilon - \mu & \lambda p I^{p-1} S^q \\ 0 & \epsilon & -\gamma - \mu \end{bmatrix}.$$

Using the examples of compound matrices given in the Appendix, we can write the linear system (3.3) with respect to a solution ($S(t)$, $E(t)$, $I(t)$) of (2.1) as the following 3×3 system:

$$\begin{aligned} X' &= -(\lambda q I^p S^{q-1} + \epsilon + 2\mu)X + \lambda p I^{p-1} S^q (Y+Z) \\ Y' &= \epsilon X - (\lambda q I^p S^{q-1} + \gamma + 2\mu) Y \\ Z' &= \lambda q I^p S^{q-1} Y - (\epsilon + \gamma + 2\mu) Z. \end{aligned} \tag{3.4}$$

To show the asymptotic stability of the system (3.4) we consider the following function:

$$V(X,Y,Z;S,E,I) = |P(S,E,I)\cdot(X,Y,Z)^*|$$

where the matrix $P = \operatorname{diag}(1, E/I, E/I)$ and $|\cdot|$ is the norm in $\mathbf{R}^3$ defined by

$$|(X,Y,Z)| = \sup\{|X|, |Y| + |Z|\}. \tag{3.5}$$

Suppose that the solution $(S(t), E(t), I(t))$ is periodic of least period $\omega > 0$. Then Proposition 2.2 implies that its orbit γ remains at a positive distance from the boundary of T. The matrix P and its inverse are thus well defined and smooth along γ. There exists constant $c > 0$ such that

$$V(X,Y,Z;S,E,I) \geqslant c|(X,Y,Z)| \tag{3.6}$$

for all $(X,Y,Z) \in \mathbf{R}^3$ and $(S,E,I) \in \gamma$. Let $(X(t), Y(t), Z(t))$ be a solution to (3.4) and

$$\begin{aligned} V(t) &= V(X(t),Y(t),Z(t);S(t),E(t),I(t)) \\ &= \sup\left\{|X(t)|, \frac{E(t)}{I(t)}(|Y(t)| + |Z(t)|)\right\}. \end{aligned}$$

The right-hand derivative of $V(t)$ exists and its calculation is described in [10] and [12]; see also [2]. In fact, direct calculation yields

$$\begin{aligned} &D_+|X(t)| \\ &\quad \leqslant -(\lambda q I^p S^{q-1} + \epsilon + 2\mu)|X(t)| + \lambda p I^{p-1} S^q (|Y(t)| + |Z(t)|) \\ &\quad \leqslant -(\lambda q I^p S^{q-1} + \epsilon + 2\mu)|X(t)| + \frac{\lambda I^p S^q}{E}\left\{\frac{E}{I}(|Y(t)| + |Z(t)|)\right\}, \end{aligned} \tag{3.7}$$

and

$$D_+|Y(t)| \leqslant \epsilon|X(t)| - (\lambda q I^p S^{q-1} + \gamma + 2\mu)|Y(t)|,$$
$$D_+|Z(t)| \leqslant \lambda q I^p S^{q-1}|Y(t)| - (\epsilon + \gamma + 2\mu)|Z(t)|,$$

and thus

$$\begin{aligned} D_+ \frac{E}{I}(|Y(t)| + |Z(t)|) &= \left(\frac{E'}{E} - \frac{I'}{I}\right)\frac{E}{I}(|Y(t)| + |Z(t)|) + \frac{E}{I}D_+(|Y(t)| + |Z(t)|) \\ &\leqslant \frac{\epsilon E}{I}|X(t)| + \left(\frac{E'}{E} - \frac{I'}{I} - \gamma - 2\mu\right)\frac{E}{I}(|Y(t)| + |Z(t)|). \quad (3.8) \end{aligned}$$

We claim that (3.7) and (3.8) lead to

$$D_+V(t) \leqslant \sup\{g_1(t), g_2(t)\} \cdot V(t), \quad (3.9)$$

where

$$g_1(t) = -\lambda q I^p S^{q-1} - \epsilon - 2\mu + \frac{\lambda I^p S^q}{E},$$
$$g_2(t) = \frac{\epsilon E}{I} + \frac{E'}{E} - \frac{I'}{I} - \gamma - 2\mu.$$

To see this we consider three cases near $t > 0$: (a) $V(t) = |X(t)|$; (b) $V(t) = E/I(|Y(t)| + |Z(t)|)$; (c) neither (a) nor (b) holds. Clearly (3.9) follows from (3.7) in case (a) and from (3.8) in case (b). In case (c), we necessarily have $|X(t)| = E/I(|Y(t)| + |Z(t)|)$ and $D_+|X(t)| = D_+(E/I)(|Y(t)| + |Z(t)|)$ at t, and thus (3.9) follows from either (3.7) or (3.8). Using (2.1)

$$\frac{\lambda I^p S^q}{E} = \frac{E'}{E} + \epsilon + \mu,$$
$$\frac{\epsilon E}{I} = \frac{I'}{I} + \gamma + \mu,$$

we find

$$\sup\{g_1(t), g_2(t)\} \leqslant \frac{E'}{E} - \mu,$$

and thus

$$\int_0^\omega \sup\{g_1(t), g_2(t)\}\,dt \leqslant \log E(t)|_0^\omega - \mu\omega = -\mu\omega < 0,$$

which, together with (3.9), implies that $V(t) \to 0$ as $t \to \infty$, and in turn that $(X(t), Y(t), Z(t)) \to 0$ as $t \to \infty$ by (3.6). As a result, the linear system (3.4) is asymptotically stable and the periodic solution $(S(t), E(t), I(t))$ is asymptotically orbitally stable with asymptotic phase by Theorem 3.1. ■

4. PROOF OF THEOREM 2.1

Before presenting the proof of Theorem 2.1, we will study further dynamical behavior of the SEIR model (2.1).

The autonomous system (3.1) is said to be *competitive in D* if, for some diagonal matrix $H = \operatorname{diag}(\epsilon_1, \ldots, \epsilon_n)$ where each ϵ_i is either 1 or -1, $H \partial f / \partial x H$ has nonpositive off diagonal elements for all $x \in D$. It is shown in [13] that, if D is convex, the flow of such a system preserves, for $t < 0$, the partial ordering in $\mathbf{R}^n$ defined by the orthant $K = \{(x_1, \ldots, x_n) \in \mathbf{R}^n : \epsilon_i x_i \geqslant 0, i = 1, \ldots, n\}$. We also want to remark that the concept of competitiveness defined above is more general than that in [5] in that the partial ordering is not necessarily that defined by the standard orthant of $\mathbf{R}^n$. However, as is pointed out in [13], by a change of variables $y = Hx$, a competitive system defined above can be transformed into a system that is "competitive" in the sense of [5].

By looking at its Jacobian matrix and choosing the matrix H as $H = \operatorname{diag}(-1, 1, -1)$, we can see that the system (2.1) is competitive in the convex region T, with respect to the partial ordering defined by the orthant $\{(S, E, I) \in \mathbf{R}^3 : S \leqslant 0, E \geqslant 0, I \leqslant 0\}$. It is known that 3-dimensional competitive systems have the Poincaré–Bendixson property [5, 13].

THEOREM 4.1

Assume that $n = 3$ and D is convex. Suppose (3.1) *is competitive in D and L is a nonempty compact omega limit set of* (3.1). *If L contains no equilibria, then L is a closed orbit* (*cf.* [5, *Theorem* 1]).

Using Theorem 4.1 we can show that the system (2.1) possesses the following strong Poincaré–Bendixson property:

THEOREM 4.2

Any compact omega limit set of (2.1) *in the interior of T is either a closed orbit or the endemic equilibrium P^*.*

Proof. Suppose that Ω is an omega limit set of (2.1) in the interior of T. If Ω does not contain P^*, then it contains no equilibria since P^* is the only interior equilibrium. Theorem 4.1 will then imply that Ω is a closed orbit. Suppose Ω contains P^*. Since P^* is asymptotically stable whenever it exists in the interior of T, any orbit that gets arbitrarily close to P^* must converge to P^*. Thus $\Omega = \{P^*\}$. ■

Proof of Theorem 2.1

The basin of attraction U of the endemic equilibrium P^* is a relatively open subset of T since P^* is locally asymptotically stable. The theorem is proved if we can show that U contains the interior of T. Assuming that the contrary is true, then the boundary ∂U of U has nonempty intersection with the interior of T (we denote this intersection by Σ). Now Σ is invariant and thus $\bar{\Sigma}$ contains a nonempty compact omega limit set γ which is in the interior of T by Proposition 2.2 and Proposition 2.3. Moreover, γ does not contain P^* and thus contains no equilibria. We can then conclude from Theorem 4.2 and Theorem 3.2 that γ is a nontrivial periodic orbit which is asymptotically orbitally stable. But this contradicts the fact that Σ, hence γ, is contained in the alpha limit set of P^*. This contradiction completes the proof of the theorem. ■

APPENDIX

The matrices $A^{[k]}$ in the case $n = 3$ are as follows:

$$A^{[1]} = \begin{bmatrix} a_1^1 & a_1^2 & a_1^3 \\ a_2^1 & a_2^2 & a_2^3 \\ a_3^1 & a_3^2 & a_3^3 \end{bmatrix} = A$$

$$A^{[2]} = \begin{bmatrix} a_1^1 + a_2^2 & a_2^3 & -a_1^3 \\ a_3^2 & a_1^1 + a_3^3 & a_1^2 \\ -a_3^1 & a_2^1 & a_2^2 + a_3^3 \end{bmatrix}$$

$$A^{[3]} = a_1^1 + a_2^2 + a_3^3 = \operatorname{Tr} A.$$

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