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Global existence of periodic solutions in a tri-neuron network model with delays

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Abstract

We consider a delayed differential system that models a network of three neurons with memory. Using a global Hopf bifurcation theorem for FDE due to J. Wu and a Bendixson's criterion for high-dimensional ODE due to Li and Muldowney, we obtain a group of sufficient conditions for the system to have multiple periodic solutions when the sum of delays is sufficiently large.

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1. Introduction

Recently, there has been an increasing activity on and interest in the investigation of neuron systems, namely in the study of equations modeling neural networks and their artificial representations. These systems typically incorporate time delays since the transmission of information from one neuron to another is not instantaneous. There is an extensive research on the dynamics of delayed differential equations representing neural networks. Several papers are devoted to the existence and stability of periodic solutions of delayed neural network models with two neurons, see [1–10]. Several recent papers investigate the existence and properties of periodic solutions of delayed multi-neuron neural network models. Baldi and Atiya [11] investigated the effects of delays on the dynamics

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and, in particular on the oscillating properties of the simple neural network models

$$\frac{du_i}{dt} = -\frac{u_i}{T_i} + T_{ii-1} f_{i-1}(u_{i-1}(t - \tau_{ii-1})), \quad i = 1, \dots, n. \tag{1.1}$$

Wu [12] studied a delayed Hopfield-Cohen-Grossberg model of neural networks

$$\frac{du_i(t)}{dt} = -u_i(t) + \sum_{i=1}^{n} J_{ij} f(u_j(t - \tau)), \quad 1 \le i \le n, \tag{1.2}$$

where f is a sigmoidal function with $f(0) = 0$, and $J = (J_{ij})$ is a symmetric circulant matrix with all diagonal elements zero. Wu showed that system (1.2) exhibits rich dynamics and various types of oscillations for large delays. Campbell [13] generalized Baldi and Atiya's model (1.1) to a network that consists of a ring of neurons where the jth element receives two time-delayed inputs: one from itself and another from the previous element. Campbell obtained sufficient conditions for local stability and bifurcations. Campbell, Ruan and Wei [14] studied Campbell's model with $n = 4$. By analyzing the equivalent system of four scalar transcendental equations, they obtained sufficient conditions for the linear stability of the positive equilibrium, and proved that a Hopf bifurcation can occur when the positive equilibrium loses stability. Recently, Wei and Velarde [15] studied stability and other properties of Hopf bifurcation for the Baldi and Atiya's model (1.1) with $n = 3$,

$$\begin{aligned}\dot u_1(t) &= -a_1 u_1(t) + f_1(u_3(t - \tau_1)),\\ \dot u_2(t) &= -a_2 u_2(t) + f_2(u_1(t - \tau_2)),\\ \dot u_3(t) &= -a_3 u_3(t) + f_3(u_2(t - \tau_3)).\end{aligned} \tag{1.3}$$

By employing the results due to Ruan and Wei [16], they obtained sufficient conditions for the asymptotic stability of the equilibrium and for the existence of Hopf bifurcations. Applying the normal form theory and the center manifold theorem, they also obtained a formula that determines the direction and stability of the Hopf bifurcation.

The purpose of this paper is to investigate the global existence of multiple periodic solutions for (1.3). The method for showing the existence of non-constant periodic solutions is the S^1-equivariant degree (see [12,21]). More precisely, we shall use a global Hopf bifurcation result of Wu [12] for functional differential equations, which was established using a purely topological argument. Meanwhile, the Bendixson's criterion for higher dimensional ordinary differential equations due to Li and Muldowney [17] shall be used to rule out the existence of nonconstant periodic solution for zero delays. To the best of our knowledge, this paper is the first to deal with the global existence of non-constant periodic solutions of (1.1) for $n = 3$. The application of higher dimensional Bendixson's criterion in the study of global existence of periodic solutions to delay systems is also new. Other applications to infinite systems include a recent work of Beretta, Solimano and Takeuchi [18], in which Bendixson's criterion in higher dimensions, as developed in Li and Muldowney [17], was used to rule out periodic solutions for systems with infinite delays.

Our paper is organized as follows: In Section 2, we present the local Hopf bifurcation results of the model by Wei and Velarde [15], the global Hopf bifurcation theorem of Wu [12], and the higher dimensional Bendixson criterion of Li and Muldowney [17]. Section 3 deals with the nonexistence of periodic solutions when the delays are zero. The global existence of multiple periodic solutions is discussed in Section 4. As an example, the system

$$\begin{aligned}\dot u_1(t) &= -a u_1(t) + b_1 \tanh u_3(t - \tau_1),\\ \dot u_2(t) &= -a u_2(t) + b_2 \tanh u_1(t - \tau_2),\\ \dot u_3(t) &= -a u_3(t) + b_3 \tanh u_2(t - \tau_3),\end{aligned} \tag{1.4}$$

is analyzed and some numerical simulations are presented in Section 5.

2. Preliminary results

We present some preliminary results to be used in the subsequent sections to establish global existence of non-constant periodic solutions.

Let $x_1(t) = u_1(t - (\tau_2 + \tau_3))$, $x_2(t) = u_2(t - \tau_3)$, $x_3(t) = u_3(t)$ and $\tau = \tau_1 + \tau_2 + \tau_3$. Then Eq. (1.3) becomes the following equivalent system

$$\begin{aligned} \dot{x}_1(t) &= -a_1 x_1(t) + f_1(x_3(t-\tau)), \\ \dot{x}_2(t) &= -a_2 x_2(t) + f_2(x_1(t)), \\ \dot{x}_3(t) &= -a_3 x_3(t) + f_3(x_2(t)). \end{aligned} \tag{2.1}$$

We make the following assumption.

(H$_1$). For $i = 1, 2, 3$, constants $a_i > 0$, $f_i \in C^2$, $f_i(0) = 0$, and there exists $L > 0$ such that $|f_i(x)| \leq L$ for $x \in \mathbb{R}$. The origin (0, 0, 0) is the unique equilibrium of (2.1).

The linearization of (2.1) at (0, 0, 0) is

$$\begin{aligned} \dot{x}_1(t) &= -a_1 x_1(t) + f_1'(0) x_3(t-\tau), \\ \dot{x}_2(t) &= -a_2 x_2(t) + f_2'(0) x_1(t), \\ \dot{x}_3(t) &= -a_3 x_3(t) + f_3'(0) x_2(t), \end{aligned} \tag{2.2}$$

whose characteristic equation is

$$\lambda^3 + (a_1 + a_2 + a_3)\lambda^2 + (a_1 a_2 + a_1 a_3 + a_2 a_3)\lambda + a_1 a_2 a_3 - \prod_{j=1}^{3} f_j'(0) \mathrm{e}^{-\lambda\tau} = 0. \tag{2.3}$$

Set

$$\begin{aligned} c_1 &= a_1^2 + a_2^2 + a_3^2, & c_2 &= a_1^2 a_2^2 + a_1^2 a_3^2 + a_2^2 a_3^2, \\ c_3 &= a_1^2 a_2^2 a_3^2 - \left(\prod_{j=1}^{3} f_j'(0)\right)^2, & c_4 &= a_1 a_2 + a_1 a_3 + a_2 a_3, \end{aligned}$$

and

$$d = \prod_{j=1}^{3} f_j'(0).$$

Let $\omega_0 > 0$ be the unique positive root of

$$\omega^6 + c_1 \omega^4 + c_2 \omega^2 + c_3 = 0.$$

Denote

$$\bar{\tau}_j = \frac{1}{\omega_0} \left[\mathrm{Arc\,sin} \left(\frac{\omega_0^3 - c_4 \omega_0}{d} \right) + 2j\pi \right], \quad j = 0, 1, 2, \ldots. \tag{2.4}$$

The following local result is proved in Wei and Velarde [15, Theorem 3.1].

Proposition 2.1. *Suppose*

(H$_2$). $a_1a_2a_3 + \prod_{j=1}^{3} f_j'(0) < 0$ *and*

$$(a_2 + a_3)(a_1^2 + a_1a_2 + a_1a_3 + a_2a_3) + \prod_{j=1}^{3} f_j'(0) > 0.$$

Then the equilibrium $(0, 0, 0)$ *of* (2.1) *is asymptotically stable when* $\tau \in [0, \bar{\tau}_0)$, *and unstable when* $\tau > \bar{\tau}_0$. *Moreover, at* $\tau = \bar{\tau}_j$, $j = 0, 1, 2, \ldots$, $\pm \mathrm{i}\omega_0$ *is a pair of simple imaginary roots of* (2.3), *and* (2.1) *undergoes Hopf bifurcation near* $(0, 0, 0)$.

Let $\lambda(\tau) = \alpha_j(\tau) + \mathrm{i}\omega_j(\tau)$ be the root of (2.3) satisfying

$$\alpha_j(\bar{\tau}_j) = 0, \qquad \omega_j(\bar{\tau}_j) = \omega_0.$$

Then, from Lemma 3.3 of Wei and Velarde [15], we have the following transversality condition.

Proposition 2.2. *If* (H$_2$) *is satisfied, then*

$$\left.\frac{\mathrm{d}\alpha_j(\tau)}{\mathrm{d}\tau}\right|_{\tau=\bar{\tau}_j} > 0. \tag{2.5}$$

To extend the local Hopf branches described in Proposition 2.1 for large delay values, we apply a global Hopf bifurcation result of Wu [12], which we briefly explain in the following. Let X be the Banach space of bounded continuous mappings $x : \mathbb{R} \to \mathbb{R}^n$ with the supreme norm. For $x \in X$, $t \in \mathbb{R}$, define $x^t \in X$ as $x^t(s) = x(t + s)$ for $s \in \mathbb{R}$. Consider a functional differential equation

$$x'(t) = F(x^t, \alpha, T) \tag{2.6}$$

parametrized by two real parameters $(\alpha, T) \in \mathbb{R} \times \mathbb{R}_+$, where $F : X \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^n$ is completely continuous. Restrict F to the subspace of constant functions x, which is identified with $\mathbb{R}^n$, we obtain a mapping $\widehat{F} = F|_{\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_+} : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^n$. Assume

(A1). $\widehat{F}$ is C^2.

Let $\hat{x}_0 \in X$ be the constant mapping with value $x_0 \in \mathbb{R}^n$. The point $(\hat{x}_0, \alpha_0, T_0)$ is called a *stationary solution* of (2.6) if $\widehat{F}(\hat{x}_0, \alpha_0, T_0) = 0$. Assume

(A2). $D_x\widehat{F}(x, \alpha, T)|_{(\hat{x}_0, \alpha_0, T_0)}$ is an isomorphism at each stationary solution $(\hat{x}_0, \alpha_0, T_0)$.

Under Assumptions (A1) and (A2) and by the implicit function theorem, for each stationary solution $(\hat{x}_0, \alpha_0, T_0)$, there exists $\epsilon_0 > 0$ and a C^1 mapping $y : B_{\epsilon_0}(\alpha_0, T_0) \to \mathbb{R}^n$ such that $\widehat{F}(y(\alpha, T), \alpha, T) = 0$, for $(\alpha, T) \in B_{\epsilon_0}(\alpha_0, T_0) = (\alpha_0 - \epsilon_0, \alpha_0 + \epsilon_0) \times (T_0 - \epsilon_0, T_0 + \epsilon_0)$. Define the *characteristic matrix* at a stationary solution $(\hat{x}_0, \alpha_0, T_0)$ of (2.6), as

$$\Delta_{(\hat{x}_0, \alpha_0, T_0)}(\lambda) = \lambda\, \mathrm{Id} - D_\varphi F(\hat{x}_0, \alpha_0, T_0)(e^{\lambda \cdot}\mathrm{Id}),$$

where $D_\varphi F(\hat{x}_0, \alpha_0, T_0)$ is complexification of the derivative of $F(\varphi, \alpha, T)$ with respect to φ at $(\hat{x}_0, \alpha_0, T_0)$ (see Assumption (A3) below). The zeros of $\det \Delta_{(\hat{x}_0, \alpha_0, T_0)}(\lambda) = 0$ are the *characteristic roots*. Note that (A2) is equivalent to assuming that $\lambda = 0$ is not a characteristic root of any stationary solution. Assume

(A3). $F(\varphi, \alpha, T)$ is differentiable with respect to φ. The characteristic matrix $\Delta_{(\hat{y}(\alpha, T), \alpha, T)}(\lambda)$ is continuous in $(\alpha, T, \lambda) \in B_{\epsilon_0}(\alpha_0, T_0) \times \mathbb{C}$.

A stationary solution $(\hat{x}_0, \alpha_0, T_0)$ is said to be a *center*, if it has purely imaginary characteristic values of the form $\mathrm{i}m2\pi/T_0$ for some positive integer *m*. A center $(\hat{x}_0, \alpha_0, T_0)$ is *isolated* if (i) it is the only center in a neighborhood of $(\hat{x}_0, \alpha_0, T_0)$, and (ii) it has only finitely many purely imaginary characteristic values of the form $\mathrm{i}m2\pi/T_0$ for some integer *m*. Let $J(\hat{x}_0, \alpha_0, T_0)$ be the set of all such positive integers *m* at an isolated center $(\hat{x}_0, \alpha_0, T_0)$. Assume that there exists $m \in J(\hat{x}_0, \alpha_0, T_0)$ such that the following holds.

(A4). There exist $\epsilon, \delta \in (0, \epsilon_0)$ such that on $[\alpha_0 - \delta, \alpha_0 + \delta] \times \partial\Omega_{\epsilon,T_0}$, $\det \Delta_{(\hat{y}(\alpha,T),\alpha,T)}(u + \mathrm{i}m2\pi/T) = 0$ if and only if $\alpha = \alpha_0$, $u = 0$, $T = T_0$, where

$$\Omega_{\epsilon,T_0} = \{(u, T) \,:\, 0 < u < \epsilon,\ |T - T_0| < \epsilon\}.$$

Define

$$H_m^{\pm}(\hat{x}_0, \alpha_0, T_0)(u, T) = \det \Delta_{(\hat{y}(\alpha_0 \pm \delta, T), \alpha_0 \pm \delta, T)}\left(u + \mathrm{i}m\frac{2\pi}{T}\right). \quad (2.7)$$

Then Assumption (A4) implies that $H_m^{\pm}(\hat{x}_0, \alpha_0, T_0) \neq 0$ on $\partial\Omega_{\epsilon,T_0}$. Thus, the *mth crossing number* $\gamma_m(\hat{x}_0, \alpha_0, T_0)$ of $(\hat{x}_0, \alpha_0, T_0)$ can be defined using topological degree of mappings $H_m^{\pm}$, as

$$\gamma_m(\hat{x}_0, \alpha_0, T_0) = \deg_{\mathrm{B}}(H_m^{-}(\hat{x}_0, \alpha_0, T_0), \Omega_{\epsilon,T_0}) - \deg_{\mathrm{B}}(H_m^{+}(\hat{x}_0, \alpha_0, T_0), \Omega_{\epsilon,T_0}). \quad (2.8)$$

It is shown in Wu [12, Theorem 3.2] that $\gamma_m(\hat{x}_0, \alpha_0, T_0) \neq 0$ implies the existence of a local bifurcation of periodic solutions with periods near T_0/m. To extend globally the local bifurcating branch, assume

(A5). All centers of (2.6) are isolated and (A4) holds for each center $(\hat{x}_0, \alpha_0, T_0)$ and each $m \in J(\hat{x}_0, \alpha_0, T_0)$.

(A6). For each bounded set $W \subseteq X \times \mathbb{R} \times \mathbb{R}_+$ there exists constant $L > 0$ such that, $|F(\varphi, \alpha, T) - F(\psi, \alpha, T)| \leq L \sup_{s\in\mathbb{R}} |\varphi(s) - \psi(s)|$, for $(\varphi, \alpha, T), (\psi, \alpha, T) \in W$.

The following is a global Hopf bifurcation result in Wu [12, Theorem 3.3].

Proposition 2.3. *Assume that (A1)–(A6) hold. Let*

$$\begin{aligned}&\textstyle\sum(F) = \mathrm{Cl}\{(x, \alpha, T),\ x \text{ is a } T\text{-periodic solutions of } (2.6)\} \subset X \times \mathbb{R} \times \mathbb{R}_+,\\ &N(F) = \{(\hat{x}, \alpha, T),\ F(\hat{x}, \alpha, T) = 0\}.\end{aligned}$$

Let $C(\hat{x}_0, \alpha_0, T_0)$ *be the connected component in* $\sum(F)$ *of an isolated center* $(\hat{x}_0, \alpha_0, T_0)$*. Then either*

(i) $C(\hat{x}_0, \alpha_0, T_0)$ *is unbounded, or*
(ii) $C(\hat{x}_0, \alpha_0, T_0)$ *is bounded,* $C(\hat{x}_0, \alpha_0, T_0) \cap N(F)$ *is finite, and*

$$\sum_{(\hat{x},\alpha,T)\in C(\hat{x}_0,\alpha_0,T_0)\cap N(F)} \gamma_m(\hat{x}, \alpha, T) = 0 \quad (2.9)$$

for all $m = 1, 2, \ldots$, *where* $\gamma_m(\hat{x}, \alpha, T)$ *is the mth crossing number of* $(\hat{x}, \alpha, T)$ *if* $m \in J(\hat{x}, \alpha, T)$, *or it is zero otherwise.*

By Proposition 2.3, to show $C(\hat{x}_0, \alpha_0, T_0)$ is unbounded, one can show that the sum in (2.9) is nonzero, for a particular integer *m*. This will be done for system (2.1) in Section 4.

Another technical issue when applying Proposition 2.3 is to prove that (2.1) with $\tau = 0$ has no non-constant periodic solutions. This will be done by applying a high-dimensional Bendixson's criterion of Li and Muldowney [17], which we briefly summarize in the rest of the section.

Consider a system of ordinary differential equations

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \; f \in C^1 \tag{2.10}$$

for any finite n. As shown in [17], to derive a high-dimensional Bendixson criterion, it is sufficient to show that the second compound equation

$$z'(t) = \frac{\partial f}{\partial x}^{[2]}(x(t, x_0))z(t), \tag{2.11}$$

with respect to a solution $x(t, x_0) \in D$ to (2.10) is equi-uniformly asymptotically stable, namely, for each $x_0 \in D$, system (2.11) is uniformly asymptotically stable, and the exponential decay rate is uniform for x_0 in each compact subset of D, where $D \subset \mathbb{R}^n$ is an open connected set. Here $\partial f/\partial x^{[2]}$ is the second additive compound matrix of the Jacobian matrix $\partial f/\partial x$. It is an $\binom{n}{2} \times \binom{n}{2}$ matrix, and thus (2.11) is a linear system of dimension $\binom{n}{2}$ (see Fiedler [19] and Múldowney [20]). For a 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

its second additive compound matrix $A^{[2]}$ is

$$A^{[2]} = \begin{pmatrix} a_{11} + a_{22} & a_{23} & -a_{13} \\ a_{32} & a_{11} + a_{33} & a_{12} \\ -a_{31} & a_{21} & a_{22} + a_{33} \end{pmatrix}. \tag{2.12}$$

The equi-uniform asymptotic stability of (2.11) implies the exponential decay of the surface area of any compact two-dimensional surface in D. If D is simply connected, this precludes the existence of any invariant simple closed rectifiable curve in D, including periodic orbits. In particular, the following result is proved in Li and Muldowney [17].

Proposition 2.4. *Let $D \subset \mathbb{R}^n$ be a simply connected region. Assume that the family of linear systems*

$$z'(t) = \frac{\partial f}{\partial x}^{[2]}(x(t, x_0))z(t), \quad x_0 \in D$$

is equi-uniformly asymptotically stable. Then

(a) *D contains no simple closed invariant curves including periodic orbits, homoclinic orbits, heteroclinic cycles;*
(b) *each semi-orbit in D converges to a single equilibrium.*

In particular, if D is positively invariant and contains an unique equilibrium $\bar{x}$, then $\bar{x}$ is globally asymptotically stable in D.

The required uniform asymptotic stability of the family of linear systems (2.11) can be proved by constructing a suitable Lyapunov function. For instance, (2.11) is equi-uniformly asymptotically stable if there exists a positive definite function $V(z)$, such that, $\mathrm{d}V(z)/\mathrm{d}t\,|_{(2.11)}$ is negative definite, and V and $\mathrm{d}V/\mathrm{d}t\,|_{(2.11)}$ are both independent of x_0.

3. Nonexistence of nonconstant periodic solution when $\tau = 0$

Consider system (2.1) with $\tau = 0$,

$$\begin{aligned} \dot{x}_1 &= -a_1x_1 + f_1(x_3),\\ \dot{x}_2 &= -a_2x_2 + f_2(x_1),\\ \dot{x}_3 &= -a_3x_3 + f_3(x_2). \end{aligned} \tag{3.1}$$

We make the following assumption.

(H$_3$). There exist $\alpha, \beta > 0$ such that

$$\sup_{x\in\mathbb{R}} \left\{ -(a_1+a_2)+\alpha|f_1'(x)|,\ -(a_1+a_3)+\frac{\beta}{\alpha}|f_3'(x)|, -(a_2+a_3)+\frac{1}{\beta}|f_2'(x)| \right\} < 0. \tag{3.2}$$

Proposition 3.1. *If the Hypotheses* (H$_1$) *and* (H$_3$) *are satisfied, then the system* (3.1) *has no non-constant periodic solutions. Furthermore, the unique equilibrium* (0, 0, 0) *is globally asymptotically stable in* $\mathbb{R}^3$.

Proof. We first prove that solutions of (3.1) are bounded. Let

$$V(x_1, x_2, x_3) = \tfrac{1}{2}[x_1^2 + x_2^2 + x_3^2].$$

Then the derivative of V along a solution of (3.1) is

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(3.1)} = -a_1x_1^2 - a_2x_2^2 - a_3x_3^2 + x_1f_1(x_3) + x_2f_2(x_1) + x_3f_3(x_2).$$

Using Assumption (H$_1$) we get

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2.9)} \le -a_1x_1^2 - a_2x_2^2 - a_3x_3^2 + L|x_1| + L|x_2| + L|x_3|. \tag{3.3}$$

There exists $M > 1$ such that $\sqrt{x_1^2+x_2^2+x_3^2} \ge M$ implies

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(3.1)} < 0.$$

This shows that solutions of (3.1) are uniformly ultimately bounded.

Denote $x = (x_1, x_2, x_3)^{\mathrm{T}}$ and

$$f(x_1, x_2, x_3) = (-a_1x_1 + f_1(x_3),\ -a_2x_2 + f_2(x_1),\ -a_3x_3 + f_3(x_2))^{\mathrm{T}}.$$

We have

$$\frac{\partial f}{\partial x} = \begin{pmatrix} -a_1 & 0 & f_1'(x_3) \\ f_2'(x_1) & -a_2 & 0 \\ 0 & f_3'(x_2) & -a_3 \end{pmatrix},$$

and, by (2.12)

$$\frac{\partial f^{[2]}}{\partial x} = \begin{pmatrix} -(a_1+a_2) & 0 & -f_1'(x_3) \\ f_3'(x_2) & -(a_1+a_3) & 0 \\ 0 & f_2'(x_1) & -(a_2+a_3) \end{pmatrix}.$$

The second compound system

$$\begin{pmatrix} \dot z_1 \\ \dot z_2 \\ \dot z_3 \end{pmatrix} = \frac{\partial f^{[2]}}{\partial x} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix},$$

is

$$\begin{aligned} \dot z_1 &= -(a_1+a_2)z_1 - f_1'(x_3(t))z_3, \\ \dot z_2 &= -(a_1+a_3)z_2 + f_3'(x_2(t))z_1, \\ \dot z_3 &= -(a_2+a_3)z_3 + f_2'(x_1(t))z_2, \end{aligned} \tag{3.4}$$

where $x(t) = (x_1(t), x_2(t), x_3(t))^{\mathrm{T}}$ is a solution of the system (3.1) with $x(0) = x_0 \in \mathbb{R}^3$. Set

$$W(z) = \max\{\alpha|z_1|, \beta|z_2|, |z_3|\}, \tag{3.5}$$

where $\alpha, \beta > 0$ are constants. Then direct calculation leads to the following inequalities

$$\begin{aligned} \frac{\mathrm{d}^+}{\mathrm{d}t}\,\alpha|z_1(t)| &\le -(a_1+a_2)\alpha|z_1| + \alpha|f_1'(x_3(t))|\,|z_3|, \\ \frac{\mathrm{d}^+}{\mathrm{d}t}\,\beta|z_2(t)| &\le -(a_1+a_3)\beta|z_2| + \frac{\beta}{\alpha}|f_3'(x_2(t))| \cdot \alpha\,|z_1|, \\ \frac{\mathrm{d}^+}{\mathrm{d}t}\,|z_3(t)| &\le -(a_2+a_3)|z_3(t)| + \frac{1}{\beta}|f_2'(x_1(t))| \cdot \beta|z_2(t)|, \end{aligned}$$

where $\mathrm{d}^+/\mathrm{d}t$ denotes the right-hand derivative. Therefore,

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\,W(z(t)) \le \mu(t)\,W(z(t)),$$

with

$$\mu(t) = \max\left\{-(a_1+a_2) + \alpha|f_1'(x_3(t))|,\ -(a_1+a_3) + \frac{\beta}{\alpha}\,|f_3'(x_2(t))|, -(a_2+a_3) + \frac{1}{\beta}|f_2'(x_1(t))|\right\}.$$

Thus, under Hypothesis (H_3), and by the boundedness of solution to (3.1), there exists a $\delta > 0$ such that $\mu(t) \le -\delta < 0$, and thus

$$W(z(t)) \le W(z(s))\,\mathrm{e}^{-\delta(t-s)}, \quad t \ge s > 0.$$

This establishes the equi-uniform asymptotic stability of the second compound system (3.4), and hence the conclusion of Proposition 3.1 follows from Proposition 2.4. □

Corollary 3.2. *The conclusions of Proposition 3.1 hold if any of the following conditions holds,*

$$\sup_{x\in\mathbb{R}}\{-(a_1+a_2)+|f_1'(x)|, -(a_1+a_3)+|f_3'(x)|, -(a_2+a_3)+|f_2'(x)|\} < 0, \tag{3.6}$$

$$\sup_{x\in\mathbb{R}}\{-(a_1+a_2)+|f_3'(x)|, -(a_1+a_3)+|f_2'(x)|, -(a_2+a_3)+|f_1'(x)|\} < 0, \tag{3.7}$$

$$\sup_{x\in\mathbb{R}}\{-(a_1+a_2)+|f_2'(x)|, -(a_1+a_3)+|f_1'(x)|, -(a_2+a_3)+|f_3'(x)|\} < 0, \tag{3.8}$$

$$\sup_{x\in\mathbb{R}}\Big\{-\sqrt{(a_1+a_2)(a_2+a_3)}+|f_1'(x)|, -\sqrt{(a_1+a_3)(a_2+a_3)}+|f_2'(x)|, -\sqrt{(a_1+a_3)(a_1+a_2)}+|f_3'(x)|\Big\} < 0, \tag{3.9}$$

$$\sup_{x\in\mathbb{R}}\Big\{-\sqrt{(a_1+a_2)(a_1+a_3)}+|f_1'(x)|, -\sqrt{(a_1+a_2)(a_2+a_3)}+|f_2'(x)|, -\sqrt{(a_1+a_3)(a_2+a_3)}+|f_3'(x)|\Big\} < 0, \tag{3.10}$$

$$\sup_{x\in\mathbb{R}}\Big\{-\sqrt{(a_2+a_3)(a_1+a_3)}+|f_1'(x)|, -\sqrt{(a_1+a_3)(a_1+a_2)}+|f_2'(x)|, -\sqrt{(a_1+a_2)(a_2+a_3)}+|f_3'(x)|\Big\} < 0. \tag{3.11}$$

Proof. In the inequality (3.2) of (H_3), choosing $\alpha=\beta=1$, $\alpha=(a_1+a_2)/(a_2+a_3)$, $\beta=(a_1+a_3)/(a_2+a_3)$, and $\alpha=(a_1+a_2)/(a_1+a_3)$, $\beta=(a_1+a_2)/(a_2+a_3)$, we arrive at the first three inequalities, respectively. The last three inequalities can be derived from (3.2) by choosing, respectively,

$$\alpha=\sqrt{\frac{a_1+a_2}{a_2+a_3}}, \qquad \beta=\sqrt{\frac{a_1+a_3}{a_2+a_3}},$$

$$\alpha=\sqrt{\frac{a_1+a_2}{a_1+a_3}}, \qquad \beta=\sqrt{\frac{a_1+a_2}{a_2+a_3}}, \text{ and}$$

$$\alpha=\frac{a_1+a_2}{\sqrt{(a_2+a_3)(a_1+a_3)}}, \quad \beta=\frac{\sqrt{(a_1+a_3)(a_1+a_2)}}{a_2+a_3}.$$

□

4. Global existence of periodic solutions

In this section, we prove that the local Hopf branches of (2.1) obtained in Proposition 2.1 can be extended for large values of the total delay $\tau=\sum_{j=1}^{3}\tau_j$, by applying Proposition 2.3. Our main result is the following.

Theorem 4.1. *Suppose that the Hypotheses (H_1)–(H_3), are satisfied. Then system* (2.1) *has at least k non-constant periodic solutions when $\tau>\overline{\tau}_k$, $k\geq 1$, where $\overline{\tau}_k$ is defined by* (2.4).

Consider the Fuller space

$$\sum = \{(\hat{x}, \tau, T) \,:\, x \text{ is a } T\text{-periodic solution of (2.1)}\} \subset X \times \mathbb{R} \times \mathbb{R}_+.$$

Note that (2.1) does not depend explicitly on T. We want to verify the Assumptions (A1)–(A6) of Proposition 2.3, for (2.1). Smoothness conditions in (A1) and (A6) are ensured by our Assumption (H_1). Also by (H_1), we know that $(0, 0, 0)$ is the only equilibrium of (2.1), and thus all stationary solution of (2.1) are of the form $(\hat{0}, \tau, T)$. By Proposition 2.1, $\lambda = 0$ is not a characteristic root of he equilibrium $(0, 0, 0)$, and thus (A2) is satisfied. The characteristic function of (2.1) is

$$q(\lambda) = \Lambda^3 + \left(\sum_{j=1}^{3} a_j\right)\lambda^2 + \left(\sum_{i\neq j}^{3} a_{ij}\right)\lambda + \prod_{j=1}^{3} a_j - \left(\prod_{j=1}^{3} f_j'(0)\right)\mathrm{e}^{-\lambda\tau}, \tag{4.1}$$

and $q(\lambda)$ is continuous in $(\tau, T, \lambda) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{C}$. This verifies (A3). A stationary solution $(\hat{0}, \tau, T)$ is a center if $(0, 0, 0)$ has purely imaginary eigenvalues of form $im2\pi/T$. By Proposition 2.1 we know that this occurs if and only if $m = 1$, $\tau = \bar{\tau}_k$ and $T = 2\pi/\omega_0$. Therefore, the set of centers is given by

$$\left\{\left(\hat{0}, \bar{\tau}_k, \frac{2\pi}{\omega_0}\right);\, k = 0, 1, 2, \dots\right\},$$

where $\bar{\tau}_k$ is defined by (2.4), and all centers are isolated.

Consider $q(\lambda)$ with $m = 1$. By Propositions 2.1 and 2.2, for fixed k, there exist $\varepsilon > 0$, $\delta > 0$ and a smooth curve $\lambda : (\bar{\tau}_k - \delta, \bar{\tau}_k + \delta) \to \mathbb{C}$ such that $q(\lambda(\tau)) = 0$, $|\lambda(\tau) - \mathrm{i}\omega_0| < \varepsilon$ for all $\tau \in (\bar{\tau}_k - \delta, \bar{\tau}_k + \delta)$, and

$$\lambda(\bar{\tau}_k) = \mathrm{i}\omega_0, \qquad \frac{\mathrm{d}}{\mathrm{d}\tau}\operatorname{Re}\lambda(\tau)\bigg|_{\tau=\bar{\tau}_k} > 0.$$

Let

$$\Omega_\varepsilon = \left\{(v, p) : 0 < v < \varepsilon,\ \left|T - \frac{2\pi}{\omega_0}\right| < \varepsilon\right\}. \tag{4.2}$$

Clearly, if $|\tau - \bar{\tau}_k| < \delta$ and $(v, T) \in \partial\Omega_\varepsilon$ such that $q(v + \mathrm{i}\,2\pi/T) = 0$, then $\tau = \bar{\tau}_k$, $v = 0$ and $T = 2\pi/\omega_0$. This verifies (A4) for $m = 1$. Moreover, if we put

$$H_m^{\pm}\left(\hat{0}, \bar{\tau}_k, \frac{2\pi}{\omega_0}\right)(v, T) = \Delta_{(\hat{0}, \bar{\tau}_k \pm \delta, T)}\left(v + \mathrm{i}m\,\frac{2\pi}{T}\right),$$

then, at $m = 1$, we have

$$\gamma_m\left(\hat{0}, \bar{\tau}_k, \frac{2\pi}{\omega_0}\right) = \deg_{\mathrm{B}}\left(H_m^{-}\left(\hat{0}, \bar{\tau}_k, \frac{2\pi}{\omega_0}\right), \Omega_\varepsilon\right) - \deg_{\mathrm{B}}\left(H_m^{+}\left(\hat{0}, \bar{\tau}_k, \frac{2\pi}{\omega_0}\right), \Omega_\varepsilon\right) = -1.$$

Thus the connected component $C(\hat{0}, \bar{\tau}_k, 2\pi/\omega_0)$ through $(\hat{0}, \bar{\tau}_k, 2\pi/\omega_0)$ in $\sum$ is nonempty. Since the first crossing number of each center is always -1, by Proposition 2.3, we conclude that $C(\hat{0}, \bar{\tau}_k, 2\pi/\omega_0)$ is unbounded. We thus have proved the following lemma.

Lemma 4.2. *$C(\hat{0}, \bar{\tau}_k, 2\pi/\omega_0)$ is unbounded for each center $C(\hat{0}, \bar{\tau}_k, \frac{2\pi}{\omega_0})$.*

Next, we prove the following results.

Lemma 4.3. *Periodic solutions of (2.1) are uniformly bounded.*

Proof. Let $a = \min\{a_1, a_2, a_3\}$, $M \geq \max\{1,\ L(a_1 + a_2 + a_3)/a\}$ and $r(t) = \sqrt{x_1^2(t) + x_2^2(t) + x_3^2(t)}$. Differentiating $r(t)$ along a solution of (2.1) we have

$$\begin{aligned}\dot r(t) &= \frac{1}{r(t)}[x_1(t)\dot x_1(t) + x_2(t)\dot x_2(t) + x_3(t)\dot x_3(t)]\\ &= \frac{1}{r(t)}[-(a_1x_1^2(t) + a_2x_2^2(t) + a_3x_3^2(t)) + a_1x_1(t)f_1(x_3(t-\tau)) + a_2x_2(t)f_2(x_1(t)) + a_3x_3(t)f_3(x_2(t))]\\ &\leq \frac{1}{r(t)}[-a(x_1^2(t) + x_2^2(t) + x_3^2(t)) + L(a_1|x_1(t)| + a_2|x_2(t)| + a_3|x_3(t)|)].\end{aligned}$$

If there exists $t_0 > 0$ such that $r(t_0) = A \geq M$, then we have

$$\dot r(t_0) \leq \frac{1}{A}[-aA^2 + AL(a_1 + a_2 + a_3)] = -aA + L(a_1 + a_2 + a_3) < 0.$$

It follows that if $x(t) = (x_1(t), x_2(t), x_3(t))^{\mathrm{T}}$ is a periodic solution of (2.1), then $r(t) < M$ for all t. This shows that the periodic solutions of (2.1) are uniformly bounded. □

Lemma 4.4. *The periods of periodic solution of (2.1) are uniformly bounded.*

Proof. Note that if $x(t) = (x_1(t), x_2(t), x_3(t))^{\mathrm{T}}$ is a τ-periodic solution of system (2.1), then $x(t)$ is τ-periodic solution of the ordinary differential Eq. (3.1). Applying Proposition 2.4 we know that, under Hypothesis (H_3), system (3.1) has no non-constant periodic solutions. Therefore, system (2.1) has no non-constant τ-periodic solutions.

By the definition of $\bar\tau_k$ in (2.4), we know that

$$\omega_0\bar\tau_k > 2\pi, \quad k = 1, 2, \ldots$$

and hence

$$\frac{2\pi}{\omega_0} < \bar\tau_k, \quad k = 1, 2, \ldots$$

From Proposition 2.1, we know that $\bar\tau_0 > 0$. Hence for $\tau > \bar\tau_k$, there exists an integer m such that $\tau/m < 2\pi/\omega_0 < \tau$. Since system (2.1) has no τ-periodic solution, it has no τ/n-periodic solution for any integer n. This implies that the period T of a periodic solution on the connected component $C(\hat 0, \bar\tau_k, 2\pi/\omega_0)$ satisfies $\tau/m < T < \tau$. So we know that the periods of the periodic solutions of the system (2.1) on $C(\hat 0, \bar\tau_k, 2\pi/\omega_0)$ are uniformly bounded. □

Proof of Theorem 4.1. By Lemma 4.2, we know that $C(\hat 0, \bar\tau_k, 2\pi/\omega_0)$ is nonempty and unbounded. By Lemmas 4.3 and 4.4, the projections of $C(\hat 0, \bar\tau_k, 2\pi/\omega_0)$ onto the x- and T-space are bounded. This implies that the projection of $C(\hat 0, \bar\tau_k, 2\pi/\omega_0)$ onto the τ-space must be unbounded. □

Applying Proposition 2.4 again, we know that system (2.1) has no non-constant periodic solutions when $\tau = 0$. Thus, the projection of $C(\hat 0, \bar\tau_k, 2\pi/\omega_0)$ onto the τ-space must be an interval $[\bar\tau, \infty)$ with $0 < \bar\tau \leq \bar\tau_k$. This shows that, for each $\tau > \bar\tau_k$ ($k \geq 1$), system (2.1) has a non-constant periodic solution on $C(\hat 0, \bar\tau_k, 2\pi/\omega_0)$. Therefore, if $\tau > \bar\tau_k \geq \bar\tau_1$, system (2.1) has at least k periodic solutions. This completes the proof of Theorem 4.1.

The following corollary follows from the Theorem 4.1 and Corollary 3.2.

Corollary 4.5. *The conclusions of Theorem 4.1 hold if* (H_1), (H_2) *and any of the inequalities (3.6)–(3.11) are satisfied.*

From Proposition 2.4 we know that, under the Hypotheses (H_1) and (H_3), the unique equilibrium $(0, 0, 0)$ of system (2.1) with $\tau = 0$ is globally asymptotically stable in $\mathbb{R}^3$. However, under the Hypotheses (H_1)–(H_3), system (2.1) has at least k non-constant periodic solutions when $\tau > \overline{\tau}_k (k \geq 1)$. This demonstrates how time delays influence the dynamics of system (2.1).

5. An example

Consider a tri-neuron network model without self-connection

$$\begin{aligned}\dot{u}_1(t) &= -u_1(t) + b_1 f(u_3(t - \tau_1)),\\ \dot{u}_2(t) &= -u_2(t) + b_2 f(u_1(t - \tau_2)),\\ \dot{u}_3(t) &= -u_3(t) + b_3 f(u_2(t - \tau_3)).\end{aligned} \tag{5.1}$$

To the best of our knowledge, there are no previous works on the global existence of non-constant periodic solutions of the system (5.1). Applying the results of Sections 2 and 3, we establish the global existence of periodic solutions for the system (5.1). We make the following assumptions on $f(x)$.

(P_1). $f \in C^2$, $xf(x) > 0$ for $x \neq 0$, and $(0, 0, 0)$ is the unique equilibrium of (5.1).

(P_2). There exists $L > 0$ such that $|f(x)| \leq L$ for $x \in \mathbb{R}$, and

$$-2 < \left(\prod_{j=1}^{3} b_j\right)^{1/3} f'(0) < -1$$

(P_3). $|b_j f'(x)| < 2$ for $x \in \mathbb{R}$, $j = 1, 2, 3$.

Let $\omega_0 > 0$ be the positive root of the equation

$$\omega^6 + 3\omega^4 + 3\omega^2 + \left[1 - \left(\prod_{j=1}^{3} b_j\right)^2 f'^6(0)\right] = 0$$

and

$$\tilde{\tau}_k = \frac{1}{\omega_0}\left[\text{Arc sin}\left(\frac{\omega_0^3 - 3\omega_0}{(\prod_{j=1}^{3} b_j) f'^3(0)}\right) + 2k\pi\right], \quad k = 0, 1, 2, \ldots. \tag{5.2}$$

Set $\tau = \sum_{j=1}^{3} \tau_j$. By Corollary 4.5, we have the following result.

Theorem 5.1. *Suppose that (P_1)–(P_3) are satisfied. Then system* (5.1) *has at least k non-constant periodic solutions when* $\tau > \tilde{\tau}_k$, $k \geq 1$.

Applying Theorem 5.1 to $f(x) = \tanh(x)$ we have the following corollary.

Corollary 5.2. *For the neural network model with three neurons*

$$\begin{aligned}\dot{u}_1(t) &= -u_1(t) + b_1 \tanh[u_3(t - \tau_1)]\\ \dot{u}_2(t) &= -u_2(t) + b_2 \tanh[u_1(t - \tau_2)]\\ \dot{u}_3(t) &= -u_3(t) + b_3 \tanh[u_2(t - \tau_3)],\end{aligned} \tag{5.3}$$

if

$$-2 < \left(\sum_{j=1}^{3} b_j\right)^{1/3} < -1$$

is satisfied, then system (5.3) has at least k non-constant periodic solutions when $\tau > \tilde{\tau}_k$ *and* $k \geq 1$, *where* $\tau = \sum_{j=1}^{3} \tau_j$, *and* $\tilde{\tau}_k$ *is defined in (5.2) with* $f'(0) = 1$.

5.1. Numerical experiments

To demonstrate the global Hopf bifurcation results in Theorem 4.1, we carry out numerical simulations on system (5.3). The simulations are done using Mathematica with different values of b_i and τ_i, and different initial values for u_i. The simulations consistently show global existence of periodic solution: existence of large amplitude periodic solutions for values of $\tau = \tau_1 + \tau_2 + \tau_3$ far away from $\tilde{\tau}_k$. In Fig. 1, we show one of the simulations using $b = b_1 = b_2 = b_3 = -2$ such that b_i satisfy the condition in Corollary 5.2. In this case it can be calculated that $\omega_0 = 1.517$ and, for $k = 0, 1, 2, \ldots$, $\tilde{\tau}_k = 0.12, 4.26, 8.40, 12.54, 16.68, 20.82, \ldots$. The delays are chosen

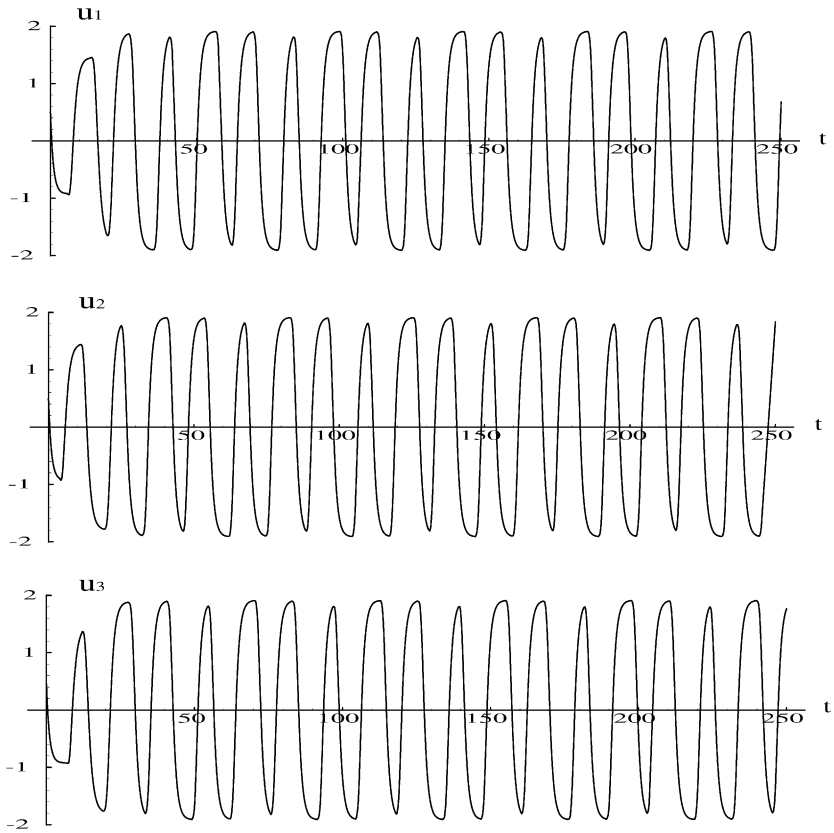

Fig. 1. Mathematica simulations of a periodic solution to system (5.3) with $\tau_1 = 6.56$, $\tau_2 = 4.56$, $\tau_3 = 7.56$ and $b = -2$. The total delay $\tau = \tau_1 + \tau_2 + \tau_3 = 18.68$ is between the two Hopf bifurcation values $\tilde{\tau}_4 = 16.68$ and $\tilde{\tau}_5 = 20.82$.

as $\tau_1 = 6.56$, $\tau_2 = 4.56$, $\tau_3 = 7.56$ so that $\tau = \tau_1 + \tau_2 + \tau_3 = 18.68$ is between the two Hopf bifurcation values $\tilde{\tau}_4 = 16.68$ and $\tilde{\tau}_5 = 20.82$. Periodic solutions of amplitude 2 are shown in Fig. 1.

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References

[1] Y. Chen, J. Wu, Minimal instability and unstable set of a phase-locked periodic orbit in a delayed neural network, Physica D 134 (1999) 185–199.
[2] Y. Chen, J. Wu, Existence and attraction of a phase-locked oscillation in a delayed network of two neurons, Differential Integral Equations 14 (2001) 1181–1236.
[3] Y. Chen, J. Wu, Slowly oscillating periodic solutions for a delayed frustrated network of two neurons, J. Math. Anal. Appl. 259 (2001) 188–208.
[4] T. Faria, On a planar system modelling a neuron network with memory, J. Differential Equations 168 (2000) 129–149.
[5] F. Giamnakopoulos, A. Zapp, Bifurcation in a planar system of differential delay equations modelling neural activity, Physica D 159 (2001) 215–232.
[6] L. Olien, J. Bélair, Bifurcation stability and monotonicity properties of a delayed neural network model, Physica D 102 (1997) 349–363.
[7] K. Gopalsamy, I. Leung, Delay induced periodicity in a neural network of excitation and inhibition, Physica D 89 (1996) 395–426.
[8] N.C. Majee, A.B. Roy, Temporal dynamics of a two-neuron continuous network model with time delay, Appl. Math. Modeling 21 (1997) 673–679.
[9] X. Liao, K. Wong, Z. Wu, Bifurcation analysis on a two-neuron system with distribution delays, Physica D 149 (2001) 123–141.
[10] J. Wei, S. Ruan, Stability and bifurcation in a neural network model with two delays, Physica D 130 (1999) 255–272.
[11] P. Baldi, A. Atiya, How delays affect neural dynamics and learning, IEEE. Tran. NN 5 (1994) 610–621.
[12] J. Wu, Symmetric functional differential equations and neural networks with memory, Trans. Am. Math. Soc. 350 (1998) 4799–4838.
[13] S.A. Campbell, Stability and bifurcation of a simple neural network with multiple time delays, Fields Inst. Commun. 21 (1999) 65–79.
[14] S.A. Campbell, S. Ruan, J. Wei, Qualitative analysis of a neural network model with multiple time delays, Int. J. Bifu. Chaos 9 (1999) 1585–1595.
[15] J. Wei, M. Velarde, Bifurcation analysis and existence of periodic solutions in a simple neural network with delays, Chaos (2004), in press.
[16] S. Ruan, J. Wei, On the zeros of a third degree exponential polynomial with applications to a delayed model for the control of testosterone secretions, IMA. J. Math. Appl. Medi. Biol. 18 (2001) 41–52.
[17] M.Y. Li, J. Muldowney, On Bendixson's criterion, J. Differential Equations 106 (1994) 27–39.
[18] E. Beretta, F. Solimano, Y. Tacheuchi, Negative criteria for the existence of periodic solutions in a class of delay- differential equations, Nonlinear Anal. 50 (2002) 941–966.
[19] M. Fiedler, Additive compound matrices and inequality for eigenvalues of stochastic matrices, Czech. Math. J. 99 (1974) 392–402.
[20] J.S. Muldowney, Compound matrices and ordinary differential equations, Rocky Mountain J. Math. 20 (1990) 857–871.
[21] L.H. Erbe, K. Geba, W. Krawcewicz, J. Wu, S^1-degree and global Hopf bifurcation theory of functional differential equations, J. Differential Equations 98 (1992) 277–298.