ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 25, Number 1, Winter 1995

BENDIXSON'S CRITERION FOR AUTONOMOUS SYSTEMS WITH AN INVARIANT LINEAR SUBSPACE

MICHAEL Y. LI

Dedicated to the memory of Professor G.J. Butler

ABSTRACT. A class of nonlinear autonomous systems ordinary differential equations in $\mathbf{R}^n$ with an invariant linear subspace which includes as examples a wide range of biological and chemical systems is defined and studied. Among other things, criteria precluding the existence of periodic solutions are obtained for such systems using a general method developed in [**4**].

1. Introduction. Let $D \subset \mathbf{R}^n$ be a convex open set and $x \mapsto f(x) \in \mathbf{R}^n$ a C^1 function defined in D. We consider the autonomous system in $\mathbf{R}^n$

$$x' = f(x) \tag{1.1}$$

under the following assumptions:

(H1) The Jacobian matrix $\partial f/\partial x$ of the vector field f of (1.1) can be written as

$$\frac{\partial f}{\partial x}(x) = -\nu I + A(x) \quad \text{for all } x \text{ in } D, \tag{1.2}$$

where ν is a constant and $x \mapsto A(x)$ is an $n \times n$ matrix-valued function.

(H2) There exists a constant matrix B with $\operatorname{rank} B = r$ such that

$$BA(x) = 0 \quad \text{for all } x \text{ in } D. \tag{1.3}$$

We will call a nonlinear system (1.1) satisfying (H1) and (H2) an autonomous system with an invariant linear subspace. Examples of

Received by the editors on September 10, 1992, and in revised form on January 19, 1993.
Research is partially supported by the Natural Science and Engineering Research Council of Canada under Grant NSERC A7197 and by the Dissertation Fellowship at the University of Alberta.

such systems include many biological models such as chemostat and chemostat-like models in population biology [**3, 13**]; SEIRS models in epidemiology [**5**]; some HIV transmission models in AIDS epidemiology [**12**]. Examples can also be found among chemical kinetics models [**10**]. It has been known that all these concrete systems have as the global center manifold an invariant lower dimensional simplex in their phase spaces. We will shown in Theorem 3.1 that this property is preserved in our general setting: under the assumptions (H1) and (H2), system (1.1) has in D a global invariant $(n-r)$-dimensional affine manifold Γ given by

$$\Gamma = \{x \in D \mid B(x-\bar{x}) = 0\}, \quad \text{for some } \bar{x} \in \mathbf{R}^n. \tag{1.4}$$

It is also proved in Theorem 3.1 that Γ is either the global center manifold in D or a level surface defined by a set of linear first integrals. In any case, it is sufficient to study the dynamics on Γ if only the asymptotic behavior of the solutions is concerned. One way to do this is to employ the equation $B(x-\bar{x}) = 0$ defining Γ to reduce the number of variables by r and to study the resulting $n-r$ system. This is the method used by many authors [**3, 5, 13**]. Our approach in this paper is to exploit the property that the linear subspace $\ker B$ of $\mathbf{R}^n$ is invariant with respect to the linear variational equation

$$y'(t) = \frac{\partial f}{\partial x}(x(t, x_0))y(t) \tag{1.5}$$

for any solution $x = x(t, x_0)$ of (1.1), in the sense that $y(t) \in \ker B$ for all $t \in \mathbf{R}$ if $y(0) \in \ker B$. Since there is no need to break up the original system, more balanced results may be obtained. We also hope this can partially justify the name given to such systems.

In Section 2 we study linear systems in $\mathbf{R}^n$ with an invariant subspace in order to understand the implications of (H1) and (H2) on the linear variational equations in (1.5); in Section 3 we give in Theorem 3.3 a general Bendixson criterion for a nonlinear autonomous system (1.1) satisfying (H1) and (H2). It is based on a general method of proving the nonexistence of periodic solutions for general autonomous systems in $\mathbf{R}^n$ developed in [**4**]. Finally, in Section 4, as an illustration, the SEIS models in epidemiology considered by Liu et al. [**5**] is revisited, the global asymptotic stability of the endemic equilibrium whenever it exists can also be proved easily using our results.

2. Linear systems with an invariant subspace. Let $\langle\ ,\ \rangle$ and $||\cdot||$ denote the Euclidean inner product and norm, respectively, and $t \mapsto A(t)$ an $n \times n$ matrix-valued function continuous in $\mathbf{R}$. We consider the linear system of n differential equations

$$x'(t) = A(t)x(t) \tag{2.1}$$

subject to the condition that there exists a constant matrix B such that

$$BA(t) = 0, \quad \text{for all } t \in \mathbf{R}. \tag{2.2}$$

We denote the kernel of B by V_0 and its orthogonal complement in $\mathbf{R}^n$ by $V_0^{\perp}$. Then $\mathbf{R}^n = V_0 \oplus V_0^{\perp}$ and $V_0^{\perp} \cong \operatorname{Im} B^*$ (cf. [**9**, Theorem 12.10]), where the asterisk denotes the transposition. Moreover, if $\operatorname{rank} B = r$, then $\dim V_0 = n - r$, $\dim V_0^{\perp} = r$.

Let $\mathcal{X}$ be the solution space of (2.1) and $\mathcal{X}_0$ be the subspace of $\mathcal{X}$ consisting of those solutions $x = x(t)$ of (2.1) with $x(t_0) \in V_0$, for some $t_0 \in \mathbf{R}$. The subspace V_0 of $\mathbf{R}^n$ is called *invariant* with respect to (2.1) if $x(t) \in V_0$ for all $t \in \mathbf{R}$ when $x = x(t)$ is a solution in $\mathcal{X}_0$.

Theorem 2.1. *Suppose* (2.2) *is satisfied. Then* V_0 *is invariant with respect to* (2.1).

Proof. From (2.1) and (2.2) we have $(Bx(t))' = BA(t)x(t) \equiv 0$. Hence, for every solution $x = x(t)$ of (2.1), $Bx(t) = Bx(t_0)$ for all $t, t_0 \in \mathbf{R}$, which leads to the claim of the theorem. $\square$

Remark. In the rest of this section, our interest will be primarily in the behavior of solutions in $\mathcal{X}_0$ or, equivalently speaking, we will study (2.1) restricted to the invariant subspace V_0. We will see later that the need for this consideration arises in Section 3.

For vectors $u_1, \dots, u_k \in \mathbf{R}^n$, $u_1 \wedge \cdots \wedge u_k$ denotes their *exterior product*, which is a vector in $\wedge^k \mathbf{R}^n \cong \mathbf{R}^N$, $N = \binom{n}{k}$. An inner product and the corresponding norm can be defined canonically in $\wedge^k \mathbf{R}^n$ from those of $\mathbf{R}^n$ (cf. [**11**, Chapter 5]). We will also denote them by $\langle\ ,\ \rangle$ and $||\cdot||$, respectively, for simplicity of notation. We need the following property of this canonical inner product in $\wedge^k \mathbf{R}^n$.

Lemma 2.2. *Suppose that* $u_1, \dots, u_k, v_1, \dots, v_m \in \mathbf{R}^n$ *and* $\langle u_i, v_j \rangle = 0$, $1 \le i \le k$, $1 \le j \le m$. *Let* $\Delta = u_1 \wedge \cdots \wedge u_k$, $\Lambda = v_1 \wedge \cdots \wedge v_m$. *Then* $\langle \Delta \wedge y, x \wedge \Lambda \rangle = \langle \Delta, x \rangle \langle y, \Lambda \rangle$ *for all* $x \in \wedge^k \mathbf{R}^n$ *and* $y \in \Lambda^m \mathbf{R}^n$.

Proof. Since each element in $\wedge^k \mathbf{R}^n$ is a linear combination of terms like $e_1 \wedge \cdots \wedge e_k$, we may assume that $x = u_1' \wedge \cdots \wedge u_k'$ and $y = v_1' \wedge \cdots \wedge v_m'$, the general case can be proved using the bilinearity of $\langle\ ,\ \rangle$. By definition

$$\langle \Delta \wedge y, x \wedge \Lambda \rangle = \det \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}_{(k+m)\times(k+m)} \tag{2.3}$$

where D_{ij} are blocks given by

$$\begin{aligned} D_{11} &= (\langle u_i, u_j' \rangle)_{k\times k}, \qquad & D_{12} &= (\langle u_i, v_j \rangle)_{k\times m}, \\ D_{21} &= (\langle v_i', u_j' \rangle)_{m\times k}, \qquad & D_{22} &= (\langle v_i', v_j \rangle)_{m\times m}. \end{aligned}$$

Observe that $\det D_{11} = \langle \Delta, x \rangle$, $\det D_{22} = \langle y, \Lambda \rangle$, by definition, and $D_{12} = 0$ from assumptions, the lemma is proved by expanding the determinant in (2.3). □

Let $X(t)$ be the fundamental solution of (2.1) with $X(t_0) = I_{n\times n}$ for some $t_0 \in \mathbf{R}$, where $I_{n\times n}$ is the $n \times n$ identity matrix. The following result is essential to all developments in this section.

Proposition 2.3. *Assume that the system* (2.1) *satisfies* (2.2). *If* $u \in \mathbf{R}^n$ *and* $w \in V_0^{\perp}$, *then*

$$\langle X(t)u, w \rangle = \langle u, w \rangle \quad \text{for all } t \in \mathbf{R}. \tag{2.4}$$

Proof. Since $V_0^{\perp} \cong \operatorname{Im} B^*$, we can assume that $w = B^* v$ for some v. Now

$$\begin{aligned} \frac{d}{dt} \langle X(t)u, w \rangle &= \langle A(t)X(t)u, w \rangle \\ &= \langle BA(t)X(t)u, v \rangle = 0 \end{aligned}$$

for all $t \in \mathbf{R}$, hence $\langle X(t)u, w \rangle = \langle X(t_0)u, w \rangle = \langle u, w \rangle$. □

In the following, for an $n \times n$ matrix A, and integer $1 \leq k \leq n$, we use $A^{[k]}$ to denote the k-th *additive compound matrix* of A. This is an $N \times N$ matrix, $N = \binom{n}{k}$. A good survey on the definition and properties of additive compound matrices, together with their connections to differential equations, may be found in [**8**]. Here we only mention a few properties that will be used in this paper. We refer the readers to [**8**] for their proof.

The term *additive* comes from the property $(A+B)^{[k]} = A^{[k]} + B^{[k]}$; if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A, then all the possible sums of form $\lambda_{i_1} + \cdots + \lambda_{i_k}$, $1 \leq i_1 < \cdots < i_k \leq n$, give the eigenvalues of $A^{[k]}$; in the two extreme cases when $k = 1$ and n, we have

$$A^{[1]} = A \quad \text{and} \quad A^{[n]} = \mathrm{tr}\,(A), \tag{2.5}$$

respectively.

The connection between additive compound matrices and linear systems of ordinary differential equations can be seen from the following proposition, whose proof can be found in [**6**] or [**8**].

Proposition 2.4. *If $x_1(t), \dots, x_k(t)$ are solutions of* (2.1), *then $y(t) = x_1(t) \wedge \cdots \wedge x_k(t)$ is a solution of the linear system*

$$y'(t) = A^{[k]}(t)y(t). \tag{2.6}$$

Equation (2.6) is called the k-th *compound equation* of (2.1). When $k = 1$ and n, as a result of (2.5), (2.6) becomes the original system (2.1) and the well-known Liouville equation (cf. [**2**, Chapter V]), respectively. To see the latter, recall that $u_1 \wedge \cdots \wedge u_n = \det\,(u_1, \dots, u_n)$ for any n vectors $u_1, \dots, u_n$ in $\mathbf{R}^n$, where $(u_1, \dots, u_n)$ denotes the $n \times n$ matrix with the i-th column given by the coordinate vector of u_i.

The following result is an attempt to study the stability of (2.1) when restricted to the invariant subspace V_0.

Theorem 2.5. *Assume that the system* (2.1) *satisfies* (2.2), *and* $\mathrm{rank}\, B = r$. *Then, for any $u_1, \dots, u_k \in V_0$, $\lim_{t\to\infty} X(t)u_1 \wedge \cdots \wedge X(t)u_k = 0$ if the linear system*

$$z'(t) = A^{[k+r]}(t)z(t) \tag{2.7}$$

is asymptotically stable.

Proof. If $u_1, \dots, u_k$ are linearly dependent, so are $X(t)u_1, \dots, X(t)u_k$, from the uniqueness of solutions of the linear system (2.1). This leads to $\Omega(t) := X(t)u_1 \wedge \cdots \wedge X(t)u_k \equiv 0$ and the proposition holds trivially. Now assume that $u_1, \dots, u_k$ are linearly independent; then $\Omega(t) \neq 0$ for all $t \in \mathbf{R}$. Choose an orthonormal basis $\{w_1, \dots, w_r\}$ of $V_0^\perp$. We claim the following: for all $t \in \mathbf{R}$,

(1) $\langle X(t)w_1 \wedge \cdots \wedge X(t)w_r, w_1 \wedge \cdots \wedge w_r \rangle = 1$,

(2) $\langle X(t)w_1 \wedge \cdots \wedge X(t)w_r \wedge \Omega(t), w_1 \wedge \cdots \wedge w_r \Omega(t) \rangle = ||\Omega(t)||^2$,

(3) $||w_1 \wedge \cdots \wedge w_r \wedge \Omega(t)|| = ||\Omega(t)||$,

(4) $||\Omega(t)|| \leq ||X(t)w_1 \wedge \cdots \wedge X(t)w_r \wedge \Omega(t)||$.

Observe that $\langle X(t)w_i, w_j \rangle = \langle w_i, w_j \rangle = \delta_{ij}$, $1 \leq i, j \leq r$, for all $t \in \mathbf{R}$, by Proposition 2.3. Thus, $\langle X(t)w_1 \wedge \cdots \wedge X(t)w_r, w_1 \wedge \cdots \wedge w_r \rangle = \langle X(t)w_1, w_1 \rangle \cdots \langle X(t)w_r, w_r \rangle = 1$, from the definition of the inner product in $\wedge^r \mathbf{R}^n$. Hence (1) follows.

To show (2), observe that $\langle X(t)u_i, w_j \rangle = \langle u_i, w_j \rangle = 0$ for all $t \in \mathbf{R}$, $i = 1, \dots, k$, and $j = 1, \dots, r$. Then the identity follows from choosing $\Delta = X(t)w_1 \wedge \cdots \wedge X(t)w_r$, $\Lambda = y = \Omega(t)$, $x = w_1 \wedge \cdots \wedge w_r$ in Lemma 2.2, and using (1). Identity (3) can be proved in the same way.

Using the Schwarz inequality in (2), we have

$$||\Omega(t)||^2 \leq ||X(t)w_1 \wedge \cdots \wedge X(t)w_r \wedge \Omega(t)|| \cdot ||w_1 \wedge \cdots \wedge w_r \wedge \Omega(t)||.$$

The inequality (4) now follows from (3) and the fact that $||\Omega(t)|| \neq 0$ for all $t \in \mathbf{R}$.

By Proposition 2.4, $z(t) = X(t)w_1 \wedge \cdots \wedge X(t)w_r \wedge \Omega(t)$ is a solution of (2.7). The theorem can now be proved using (4) and the definition of asymptotic stability. □

Let $|\cdot|$ denote a general vector norm in $\mathbf{R}^n$ and the matrix norm derived from it. For an $n \times n$ matrix A, the *Lozinskiĭ measure* (or the *logarithmic norm*) $\mu(A)$ of A corresponding to the norm $|\cdot|$ is defined as (cf. [**1**, p. 41] or [**8**])

$$\mu(A) = D_+|I + hA|_{h=0}. \tag{2.10}$$

It has the property that

$$|x(t)| \exp\left(-\int_{t_0}^{t} \mu(A(s))\, ds \right), \quad \text{and}$$
$$|x(t)| \exp\left(\int_{t_0}^{t} \mu(-A(s))\, ds \right)$$

are nonincreasing and nondecreasing, respectively, when $x = x(t)$ is a solution of (2.1). The calculation of the Lozinskiĭ measure corresponding to some common norms of $\mathbf{R}^n$ can be found in [**1, 8**]. More detailed study on this subject can be found in [**7**].

Corollary 2.6. *Under the assumptions of Theorem* 2.5*,* $\lim_{t\to\infty} X(t) u_1 \wedge \cdots \wedge X(t)u_k = 0$ *if*

$$\lim_{t\to\infty} \int_{t_0}^{t} \mu(A^{[k+r]}(s))\, ds = -\infty \tag{2.11}$$

for some Lozinskiĭ measure μ*.*

Proof. Condition (2.11) is simply a sufficient condition for the asymptotic stability of linear system (2.7) (cf. [**1**]). □

3. Bendixson's criterion for nonlinear systems. A subset D_1 of D is *invariant* with respect to (1.1) if $x(t, D_1) = D_1$ for all $t \in \mathbf{R}$, when $x = x(t, x_0)$ is a solution of (1.1) such that $x(t, 0) = x_0$.

Theorem 3.1. *Under the assumptions* (H1) *and* (H2)*, the* $n - r$ *dimensional affine manifold* Γ *given in* (1.4) *is invariant with respect to* (1.1)*.*

Proof. Since $Bf(x) - Bf(\bar{x}) = \int_0^1 B(\partial f/\partial x)(\bar{x} + s(x-\bar{x}))\, ds(x - x_0) = -\nu B(x - \bar{x})$ for all x and $\bar{x}$ in $\overline{D}$, from (1.2). Choosing $\bar{x}$ as an equilibrium point of (1.1) in $\overline{D}$, we have

$$(Bx)' = -\nu B(x - \bar{x}). \tag{3.1}$$

It is easy to see that the invariance of Γ follows from (3.1). □

We can see from (3.1) that if $\nu \neq 0$, then Γ is the global center manifold in D (the stable center manifold when $\nu > 0$, and the unstable center manifold when $\nu < 0$); if $\nu = 0$, Bx gives a set of r independent linear first integrals, and Γ is one of the level surfaces defined by these first integrals. In either case, if there is a periodic solution, it stays on Γ. Therefore, to rule out periodic solutions of (1.1), it suffices to study its dynamics on Γ. Recall that if $x(t, x_0)$ describes the dynamics of (1.1) and $\partial x(t, x_0)/\partial x_0$ is its linearization with respect to the initial values, then $\partial x(t, x_0)/\partial x_0 u$ is a solution of the linear variational equation (1.5) for any $u \in \mathbf{R}^n$. Moreover, since Γ is an invariant manifold, if u is a vector tangent to Γ at $x_0 \in \Gamma$, then the vector $\partial x(t, x_0)/\partial x_0 u$ is tangent to Γ at $x(t, x_0)$. From this discussion and the fact that the tangent space of the affine manifold Γ is $\ker B$ at every point, we arrive at the following: (a) the subsapce $\ker B$ of $\mathbf{R}^n$ is invariant with respect to (1.5); (b) we need only to study those solutions of (1.5) which stay in $\ker B$ for all time. This leads us to the same consideration as discussed in the remark following Theorem 2.1 in Section 2.

Let $U = B^2(0; 1)$ be the Euclidean unit ball in $\mathbf{R}^2$, and let $\overline{U}$ and ∂U be its closure and boundary, respectively. A function $\phi \in \mathrm{Lip}\,(\overline{U} \to D)$ will be described as a *rectifiable 2-surface* in D; a function $\psi \in \mathrm{Lip}\,(\partial U \to D)$ is a *closed rectifiable curve* in D and is called *simple* if it is one-to-one.

Since D is convex, the invariant affine manifold $\Gamma \subset D$ is convex.

For a given simple closed curve $\psi \in \mathrm{Lip}\,(\partial U \to \Gamma)$, the set

$$\Sigma(\psi, \Gamma) = \{\phi \in \mathrm{Lip}\,(\overline{U} \to \Gamma) \mid \phi(\partial U) = \psi(\partial U)\}$$

is not empty. In fact, a surface $\phi \in \mathrm{Lip}\,(\overline{U} \to \Gamma)$ can be constructed by connecting the points of $\phi(\partial U)$ to a fixed point on Γ.

A general method for proving the nonexistence of the periodic solutions is developed in [**4**] by studying the evolution under (1.1) of general functionals defined on surfaces.

Consider the functional $\mathcal{A}$ on $\mathrm{Lip}\,(\overline{U} \to \Gamma)$ defined by

$$\mathcal{A}\phi = \int_{\overline{U}} \left| \frac{\partial \phi}{\partial r_1} \wedge \frac{\partial \phi}{\partial r_2} \right|$$

where $|\cdot|$ is any vector norm on $\mathbf{R}^M$, $M = \binom{n}{2}$. For example, if the norm is chosen as $|y| = (y^* y)^{1/2}$, then $\mathcal{A}\phi$ is the usual surface area of $\phi(\overline{U})$. The following result is proved in [**4**, Proposition 2.2].

Proposition 3.2. *Suppose that ψ is a simple, closed rectifiable curve on Γ. Then there exists a $\delta > 0$ such that*

$$\mathcal{A}\phi \geq \delta \tag{3.2}$$

for all $\phi \in \Sigma(\psi, \Gamma)$.

A subset $D_0 \subset \Gamma$ is *absorbing* with respect to (1.1) if any bounded subset $B \subset \Gamma$ satisfies $x(t, B) \subset D_0$ for all sufficiently large t. In the rest of this section, we will assume the following.

(H3) Either Γ is bounded, or Γ contains a bounded absorbing set.

A close rectifiable curve ψ in D is *invariant* with respect to (1.1) if the subset $\psi(\partial U)$ of D is invariant. The following result is a general Bendixson criterion for autonomous systems with an invariant linear subspace.

Theorem 3.3. *Assume that* (H1), (H2) *and* (H3) *are satisfied. If*

$$\mu\left(\frac{\partial f^{[r+2]}}{\partial x}(x)\right) < -r\nu \tag{3.3}$$

for all $x \in \Gamma$, where μ is the Lozinskiĭ measure with respect to a vector norm $|\cdot|$ in $\mathbf{R}^N$, $N = \binom{n}{r+2}$. Then no simple closed rectifiable curve in Γ can be invariant with respect to (1.1).

Remark. In the case when $\nu \neq 0$, Γ is the global center manifold, condition (3.3) will then imply that there is no simple closed invariant rectifiable curve in D under the assumptions of Theorem 3.3.

Proof. Let $x = x(t, x_0)$ be a solution of (1.1) such that $x_0 \in \Gamma$. From (H3), $x(t, x_0)$ exists for all $t \in \mathbf{R}$. The linear variational equations (1.5) with respect to $x(t, x_0)$ can be written as

$$y'(t) = -\nu y(t) + A(x(t, x_0))y(t). \tag{3.4}$$

The change of variables $u(t) = y(t)\exp(\nu t)$ leads to the equation $u'(t) = A(x(t, x_0))u(t)$, subject to the condition $BA(x(t, x_0)) = 0$ for all $t > 0$. Let $y_1(t)$ and $y_2(t)$ be two linearly independent solutions of (3.4) such that $By_i(0) = 0$, and $u_i(t) = y_i(t)\exp(\nu t)$. We have $Bu_i(0) = 0$, $i = 1, 2$, and $y_1(t) \wedge y_2(t) = u_1(t) \wedge u_2(t)\exp(-2\nu t)$. Now (3.3) and (H3) imply

$$\mu(A^{[r+2]}(x(t,x_0))) - 2\nu \le \mu\bigg(\frac{\partial f^{[r+2]}}{\partial x}(x(t,x_0))\bigg) + r\nu \le -\delta < 0$$

for all $t \ge 0$. This and the same idea as in the proof of Corollary 2.6 imply that $y_1(t) \wedge y_2(t) \to 0$ as $t \to \infty$, and the convergence is uniform for x_0 in any compact subset of Γ.

Suppose $\psi \in \mathrm{Lip}\,(\partial U \to \Gamma)$ is a simple closed curve on Γ invariant with respect to (1.1) and $\phi \in \Sigma(\psi, \Gamma)$. Let $\phi_t = x(t, \phi(u))$, $u = (u_1, u_2) \in \overline{U}$. Then $\phi_t \in \Sigma(\psi, \Gamma)$ for all $t \in \mathbf{R}$ by the invariance of ψ. Moreover, for almost every $u \in U$, $\partial\phi_t/\partial u_i = \partial x(t, \phi)/\partial x_0 \cdot \partial\phi/\partial u_i$, $i = 1, 2$, are two linearly independent solutions of (3.4) with respect to $\phi_t(u)$ and $B\partial\phi_0/\partial u_i = B\partial\phi/\partial u_i = 0$, $i = 1, 2$, hence $|\partial\phi_t/\partial u_1 \wedge \partial\phi_t/\partial u_2| \to 0$ uniformly for almost every $u \in \overline{U}$ as $t \to \infty$. Therefore, $\mathcal{A}\phi_t \to 0$ as $t \to \infty$. This contradicts (3.2) in Proposition 3.2. □

Remarks. (1) By Theorem 3.3, condition (3.3) excludes not only periodic orbits, but also orbits of the following types: (a) homoclinic orbits; (b) a pair of heteroclinic orbits of any two equilibria; (c) heteroclinic cycles. Each of these situations gives rise to an invariant piecewise smooth simple closed curve.

(2) If $\mu(\partial f^{[r+2]}/\partial x)$ in (3.3) is calculated corresponding to the l^2, l^∞ and l^1 norm of $\mathbf{R}^N$, $N = \binom{n}{r+2}$, we have the following concrete Bendixson's criterion for the autonomous system (1.1) satisfying (H1) and (H2):

(a) $\lambda_1 + \cdots + \lambda_{r+2} < -r\nu$,

(b) $\sup_{(i)}\{(\partial f_{i_1}/\partial x_{i_1}) + \cdots + (\partial f_{i_{r+2}}/\partial x_{i_{r+2}}) + \sum_{k \ne i_1,\dots,i_{r+2}}(|\partial f_k/\partial x_{i_1}| + \cdots + |\partial f_k/\partial x_{i_{r+2}}|)\} < -r\nu$,

(c) $\sup_{(i)}\{(\partial f_{i_1}/\partial x_{i_1}) + \cdots + (\partial f_{i_{r+2}}/\partial x_{i_{r+2}}) + \sum_{k \ne i_1,\dots,i_{r+2}}(|\partial f_{i_1}/\partial x_k| + \cdots + |\partial f_{i_{r+2}}/\partial x_k|)\} < -r\nu$,

where $\lambda_1 \ge \cdots \ge \lambda_n$ are eigenvalues of the matrix $(\partial f/\partial x + \partial f^*/\partial x)/2$, and the supremum in (b) and (c) is taken over all $r + 2$ tuples

$(i) = \{i_1, \dots, i_{r+2}\}$ such that $1 \le i_1 < \cdots < i_{r+2} \le n$. Conditions (b) and (c), though they look a little complicated, simply mean that the matrix $(\partial f^{[r+2]}/\partial x + r\nu I)$ is diagonally dominant in column and in row, respectively, and all its diagonal elements are of the same sign in Γ. They are usually much easier to calculate than the eigenvalues in condition (a). We also want to point out that the advantage of using the Lozinskiĭ measure is to allow us to choose easy-to-compute conditions corresponding to different choices of the norm.

(3) If $r = n - 2$, it follows from (2.5) that the condition (3.3) will become

$$\operatorname{div}(f) < -(n-2)\nu \quad \text{on } \Gamma.$$

(4) Criteria of Dulac type can be derived by introducing functionals on $\Sigma(\psi, \Gamma)$ more general than $\mathcal{A}$ (cf. [**4**]).

We close this section by giving an example to show that condition (3.3) in Theorem 3.3 is sharp.

Example. Consider the linear system $x' = -y$, $y' = x$, $z' = -z$. The Jacobian matrix J of this system is a constant matrix and can be written as $J = -I + A$, where

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If we let $B = (0, 0, 1)$, then $\operatorname{rank} B = 1$ and $BA = 0$. Conditions (1.2) and (1.3) are satisfied with $n = 3$, $D = \mathbf{R}^3$, $\nu = 1$ and $r = 1$. It is easy to see that the xy-plane is the global center manifold, and the origin is a center-type equilibrium for the restricted flow on the xy-plane with infinitely many concentric periodic orbits. This should not be a surprise to us, since $\operatorname{tr} J = -1 = -\nu$ and condition (3.3) is not satisfied.

4. SEIS models with nonlinear incidence rate. The SEIS model in epidemiology is given by the following system:

$$\begin{aligned} S' &= -\lambda I^p S^q + \mu - \mu S + \gamma I \\ E' &= -\lambda I^p S^q - (\varepsilon + \mu) E \\ I' &= \varepsilon E - (\gamma + \mu) I \end{aligned} \tag{4.1}$$

where S denotes the susceptible class, E the exposed class and I the infectious class. The $\lambda I^p S^q$ term is the incidence rate (the rate of new infections). The model (4.1) with general p and q, together with the more general SEIRS models, are considered by Liu *et al.* [**5**]. They have proved that when $0 < p < 1$, (4.1) has two equilibria: the disease-free equilibrium $P_0 = (1,0,0)$ and the endemic equilibrium P^* in the invariant region $T = \{(S,E,I) \mid S+E+I = 1,\ S,E,I \geq 0\}$ where P_0 is unstable and P^* is globally asymptotically stable in the exterior of T; when $p = 1$ and the contact number $\sigma = \lambda\varepsilon/(\varepsilon+\mu)(\gamma+\mu) \leq 1$, P_0 is the only equilibrium in T and is globally asymptotically stable in $\overline{T}$, and when $\sigma \geq 1$, P_0 is unstable and P^* is globally asymptotically stable in the interior of T. The method used in [**5**] to prove these properties is to use the equation $S+E+I = 1$ to reduce (4.1) to a two-dimensional system and to use the classic Dulac criterion to rule out periodic orbits. We will show that these properties can also be proved using Theorem 3.3.

The Jacobian matrix $J(S,E,I)$ of (4.1) can be written as $J = -\mu I_{3\times 3} + A$, where

$$A(S,E,I) = \begin{pmatrix} -\lambda q I^p S^{q-1} & 0 & -\lambda p I^{p-1} S^q + \gamma \\ \lambda q I^p S^{q-1} & -\varepsilon & \lambda p I^{p-1} S^q \\ 0 & \varepsilon & -\gamma \end{pmatrix}.$$

Let $B = (1,1,1)$; then $BA(S,E,I) = 0$ for all $S,E,I \geq 0$. Thus, (1.2) and (1.3) are satisfied with $n = 3$, $D = \mathbf{R}^3_+$, $\nu = \mu$, and $r = 1$. Here Γ is the set $\{(S,E,I) \in \mathbf{R}^3_+ \mid S+E+I = 1\}$. Moreover, $\operatorname{tr} J = -\lambda q I^p S^{q-1} - \varepsilon - \gamma - 3\mu < -\mu = -\nu$. By Theorem 3.3, (4.1) has no periodic solutions for all values of parameters. The stability properties we want to show now follow easily from Poincaré-Bendixson theory.

Acknowledgment. The author wishes to thank Professor J.S. Muldowney for his encouragement and help.

REFERENCES

1. W.A. Coppel, *Stability and asymptotic behavior of differential equations*, Heath, Boston, 1965.

2. P. Hartman, *Ordinary differential equations*, John Wiley and Sons, Baltimore, 1973.

3. S.B. Hsu and P. Waltman, *Analysis of a model of two competitors in a chemostat with an external inhibitor*, SIAM J. Appl. Math. **52** (1991), 528–540.

4. Y. Li and J.S. Muldowney, *On Bendixson's criterion*, J. Differential Equations **106** (1994), 27–39.

5. W-M. Liu, H.W. Hethcote and S.A. Levin, *Dynamical behavior of epidemiological models with nonlinear incidence rate*, J. Math. Biol. **25** (1987), 359–380.

6. D. London, *On derivations arising in differential equations*, Linear and Multilinear Algebra **4** (1976), 179–189.

7. J.S. Muldowney, *Dichotomies and asymptotic behaviour for linear differential systems*, Trans. Amer. Math. Soc. **283** (1984), 465–584.

8. ———, *Compound matrices and ordinary differential equations*, Rocky Mountain J. Math. **20** (1990), 857–872.

9. W. Rudin, *Functional analysis*, McGraw-Hill, New York, 1973.

10. D. Siegel, *Monotonicity properties of solution to system of differential equations arising in chemical kinetics*, in *Differential equations* (C.M. Dafermos, G. Ladas and G. Papanicolaou, eds.), Marcel Dekker, New York, 1989.

11. R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, Appl. Math. Sci. **68**, Springer-Verlag, New York, 1988.

12. J.X. Velasco and Y-H. Hsieh, *Modelling the effect of treatment and behavioral change in HIV transmission dynamics*, preprint.

13. P. Waltman, *Coexistence in chemostat-like models*, Rocky Mountain J. Math. **20** (1990), 777–807.

Department of Mathematics, University of Alberta, Edmonton, Alberta T6G 2G1 Canada