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Short Note

Smooth inversion of VSP traveltime data

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INTRODUCTION

Vertical seismic profile (VSP) direct arrivals provide an in-situ measurement of traveltime with depth into the earth. In this note, we describe a weighted, damped least-squares inversion of VSP traveltimes for a smooth velocity/depth function that inherently reveals the resolution of the data. Smooth velocity/depth profiles of this type are suitable for migration or as a starting models for waveform inversion or tomography. The application of this inversion is particularly simple, requiring only the value of the damping parameter to be determined, and this value is determined from residual statistics.

Velocity with depth in a borehole is constrained by the difference in VSP direct-arrival traveltimes between stations at different depths. In principle, resolution of vertical velocity structure increases with decreasing VSP station spacing in the borehole. Station-to-station interval velocities become increasingly sensitive to arrival-time picking errors as station spacing decreases, however, and the bandwidth and the signal-to-noise ratio of the seismic signal ultimately limits the realizable vertical velocity resolution. Lack of resolution at the station-spacing scale enables a vast set of velocity/depth functions to be consistent with the traveltime data. Regularized inversion provides a means of selecting velocity profiles from among this set. In particular, the Occam's inversion regularization scheme (Constable et al., 1987) seeks the smoothest model consistent with the data and their errors. Here, we outline the application of this approach to the analysis of VSP traveltimes.

VSP TRAVELTIME INVERSION

The determination of seismic velocity from VSP traveltimes has been the subject of several papers (Stephen and Harding, 1983; Stewart, 1984; Pujol et al., 1985; Schuster, 1988; Schuster et al., 1988). The simplest of this class of problems is the determination of interval velocities (velocities between VSP station depths) from VSP first-arrival times and involves solving the linear equation

$$\mathbf{T} = \mathbf{Z}\mathbf{U},$$

or

$$\begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_M \end{bmatrix} = \begin{bmatrix} z_1 & 0 & \cdots & 0 \\ z_1 & z_2 & & \vdots \\ \vdots & & \ddots & 0 \\ z_1 & z_2 & \cdots & z_M \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_M \end{bmatrix}, \tag{1}$$

where t_i are the first-arrival times at each borehole station depth, z_i are the layer thicknesses (i.e., distances between stations), and u_i are the interval slownesses. Equation (1) assumes a model in which the velocity of each interstation interval is constant. This is a reasonable assumption for station spacings that are small relative to a seismic wavelength. Vertical wave propagation is also assumed. However, if the source-offset/station-depth ratio is not zero but less than 1, then a straight-ray geometry is a valid assumption (Schuster et al., 1988), and equation (1) can be used with the z_i divided by $\cos\theta_i$ terms.

Equation (1) can be solved directly, or $\mathbf{Z}$ can be diagonalized and equation (1) solved based on differential (between-station) traveltimes, as is quite common. In either case, the solution for $\mathbf{U}$ is an even-determined problem with imprecise data, and some form of regularization is required. A common regularization approach is a damped least-square inversion employing a penalty function based on some smoothing criterion such as the length of the first or second derivative of the estimated slowness, $\hat{\mathbf{U}}$. The least-squares solution is obtained in the usual way, by minimizing

$$L = \|\mathbf{T} - \mathbf{Z}\hat{\mathbf{U}}\| + \varepsilon^2\|\mathbf{D}\hat{\mathbf{U}}\|, \tag{2}$$

where $\|\cdot\|$ represents the magnitude or L_2 norm, $\mathbf{D}\hat{\mathbf{U}}$ is the penalty function, $\mathbf{D}$ is a first- or second-difference matrix, and ε is the damping parameter (or Lagrange multiplier) which

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governs the tradeoff between minimization of data misfit and the penalty function. A typical minimized solution of equation (2) is given in standard books on inversion such as Menke (1984). A solution of equation (2) including weighting according to data variance is

$$\hat{\mathbf{U}} = \left[\mathbf{Z}^T \mathbf{W_e} \mathbf{Z} + \varepsilon^2 \mathbf{D}^T \mathbf{D}\right]^{-1} \mathbf{Z}^T \mathbf{W_e} \mathbf{T}, \tag{3}$$

where $\mathbf{W_e}$ is a diagonal matrix of data variances. Estimates of traveltime data variance can be obtained from arrival times of multiple shots recorded at a single depth.

A variety of criteria can be used to select the damping parameter ε. The criterion used in the Occam inversion is that ε should be chosen to give the smoothest possible model consistent with the traveltime data and their errors. Thus, equation (3) may be evaluated along with the residual error, $\mathbf{T} - \mathbf{Z}\hat{\mathbf{U}}$, and the influence of the penalty function increased (i.e., ε increased) until the residuals rise to some maximum acceptable value. Constable et al. (1987) suggest the χ^2 statistic as the statistic upon which this determination should be based. If the data are independent measurements of random variables and the model is physically appropriate, which are both true in this case, then the residual error should follow a Gaussian distribution. The expected value of the χ^2 statistic for the residuals is M, the number of observations; $E[\chi^2] = M$. The value of χ^2 resulting from the inversion is

$$\hat{\chi}^2 = \sum \left(t_i - \sum_{n=1}^{i} z_n \hat{u}_n \right) \Big/ s_i, \tag{4}$$

where s_i are the data variances at each station i. There is no reason why $\hat{\chi}^2$ should be smaller than $E[\chi^2]$. Values of $\hat{\chi}^2$ smaller than the expected value suggest overfitting of the data. Equation (3) should be iteratively evaluated using different values of ε until $\hat{\chi}^2$ from equation (4) rises (or falls) to an acceptable level of misfit with respect to $E[\chi^2]$, such as $\hat{\chi}^2_{target} = E[\chi^2] + 2\sigma$, where σ is the standard error approximately equal to $(2M)^{1/2}$ for χ^2. The ε that results in $\hat{\chi}^2 = \hat{\chi}^2_{target}$ can typically be found in just a few tries, beginning with ε equal to 1 and increasing ε if $\hat{\chi}^2$ is less than $\hat{\chi}^2_{target}$, and vice versa. The resulting slowness profile, $\hat{\mathbf{U}}$, can thus be characterized as the smoothest profile with respect to the length of $\mathbf{D}\hat{\mathbf{U}}$ consistent with the traveltime data and their errors.

Several comments about this approach can be made. First, the formulation of the traveltime equation (1) uses integrated traveltimes as data, ensuring the correct total traveltime to the deepest VSP station. The diagonalized version of equation (1), based on traveltime differences between stations, does not preserve integrated traveltimes and will tend to propagate errors with depth. Sonic logs also commonly do not yield correct integrated traveltimes to the base of a borehole. Second, the a priori assumption about the smooth properties of the velocity profile is not correct for large contrasts in velocity that might be encountered at a sediment/basement interface or across a stratigraphic gas trap, for example. In cases like these, considerable error may accumulate in the vicinity of the contrast, and data over other parts of the profile may in consequence be overfit. Steps in velocity can be accounted for by modifying the matrix $\mathbf{D}$ and thus removing the contribution to the penalty function across a given depth interval. Resulting solutions will be piecewise smooth. In general, the data residuals should be examined for trends and outliers, and the weighting matrices $\mathbf{W_e}$ and $\mathbf{D}$ adjusted accordingly to assure a uniform fit to the data. Finally, the number of depth stations involved in a particular VSP experiment are usually small enough so that the matrices in the solution (3) can be efficiently inverted directly on most computers, and an acceptable solution can be obtained with a few iterations in a matter a seconds.

DATA EXAMPLE

We demonstrate this inversion approach using VSP data recorded in the Ocean Drilling Program (ODP) hole 504B which was drilled into basaltic oceanic crust (Figure 1a) (Swift et al., 1996). The average station spacing is 10 m. Up to 25 air-gun shots were fired at each station, and the first-arrival times from the 5 best shots were picked and their means and standard deviations calculated. The average standard deviation of the first-arrival-time picks is less than 1 ms. The gray region in Figure 1b defines the 2σ bounds on the observed traveltimes.

An infinite number of models can pass through the statistically defined traveltime bounds of Figure 1b. An a priori condition is required to select among the models that satisfy the traveltime data, and we have chosen a condition based on the smoothness of the slowness profile. (That is, we have used the length of the second difference of the model to define smoothness in this case. Trials have shown that using the length of the first difference of the model, sometimes referred to as the flatness, produces essentially identical results.) Direct solution of the traveltime equation (1) using the traveltime means corresponds to evaluation of equation (3) with $\varepsilon = 0$. This solution yields a nonphysical, wildly oscillating velocity/depth profile with $\hat{\chi}^2 \cong 0$. Solutions with $\hat{\chi}^2 = 0.6\,E[\chi^2]$, $E[\chi^2]$, $E[\chi^2] + 1\sigma$, and $E[\chi^2] + 2\sigma$ are shown in Figure 1c, with the traveltime fits to the models with $0.6\,E[\chi^2]$ and $E[\chi^2] + 2\sigma$ shown in Figure 1b. Note that the "best fitting" of these models with $\hat{\chi}^2 = 0.6\,E[\chi^2]$, which consists of nonphysical velocity fluctuations, results in a traveltime fit that is quite close to the model with $\hat{\chi}^2 = E[\chi^2] + 2\sigma$, which is a good approximation to the resolution limit of the data.

The normalized residuals of the model with $\hat{\chi}^2 = E[\chi^2] + 2\sigma$ are shown in Figure 1d. Residuals should be inspected for trends and outliers that may leverage the overall fit statistics, and the weighting and damping matrices should be adjusted accordingly. In this example, the residuals are random and show no trend. For a discussion of the geologic significance of this velocity profile and its relationship to sonic and other logs acquired at hole 504B, see Swift et al. (1998).

SUMMARY

The first-arrival times of closely spaced VSP stations generally cannot be determined with sufficient accuracy to yield meaningful velocity/depth profiles with resolution on the order of the station spacing. These data can be effectively used, however, to constrain regularized inversions for smooth velocity/depth functions that are required as input to migration, tomographic, and waveform-inversion algorithms. We have presented a simple formulation of the weighted, damped least-square inverse problem for VSP traveltimes at near-normal incidence based on the concepts of Occam's inversion. The inversion is formulated in terms of integrated traveltimes and is

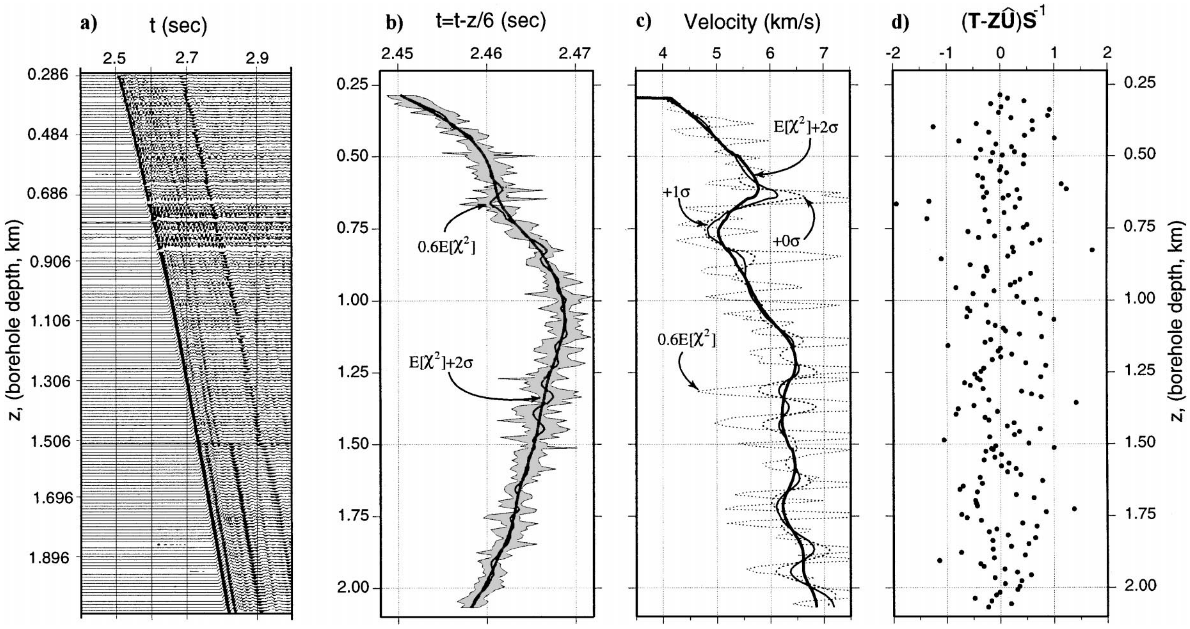

FIG. 1. (a) VSP from ODP hole 504B acquired with two sources for stations above and below ~1.5 km. (b) Reduced traveltime data and fit. Gray region defines 2σ bounds on first-arrival picks. Curves within these bounds are fits of the smoothest and roughest models in (c). (c) Inverted velocity profiles for a range of residual χ^2 values. The model with $\chi^2 < E[\chi^2]$ overfits the data and maps noise into the model. Model with $\chi^2 = E[\chi^2] + 2\sigma$ represents the statistical resolution of the data under the smoothing constraint. (d) Residual of the $E[\chi^2] + 2\sigma$ model normalized by the data variance. Residuals should be inspected for trends and outliers that may leverage the overall fit statistics and lead to overfitting some parts of the data. In this example, the residuals are random and show no trend.

guided by both data and misfit statistics. The linear inversion is not computationally intensive and can be fully automated and performed on most small computers.

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