CMPUT 675: Randomized Algorithms **Fall 2005**

Problem Set 2:

Due Date: Nov 10, 2005

You are allowed to discuss the problems with each other, however you must not take any notes from your discussions. You must list the names of other students that you have discussed the problems with, or any other resources (other than the text book) that you have used. These information will not affect your mark. The questions are in arbitrary order of difficulty.

1. In class, we saw that a simple randomized algorithm for max-cut proves the existence of a cut of size at least $\frac{|E|}{2}$. Show that w.h.p. $G_{n,\frac{1}{2}}$ has no cut of size larger than $\frac{n^2}{8} + O(n \log n)$; therefore our bound is essentially tight (Hint: use the second moment method).
2. Let k be a given constant. Prove that every D-regular graph G (for sufficiently large D) can be partitioned into k edge-disjoint subgraphs $G_1, \ldots, G_k$ such that the degree of every vertex v in each of $G_1, \ldots, G_k$ is at least $\frac{D}{k} \pm o(D)$ (Hint: consider a random k-coloring of G and use the LLL).
3. Suppose that we are given an undirected graph $G(V, E)$ together with a list $L(v)$ of colors for every vertex v, where $|L(v)| = 8\ell$. Suppose further that for each $v \in V$ and $c \in L(v)$, there are at most ℓ neighbours of v that contain color c in their list. Prove that there is a coloring of G in which the color of every vertex is chosen from its list and no two adjacent vertices have the same colors (Hint: consider a random coloring and define a suitable bad event to use LLL).
4. Consider the random graph $G_{n,p}$ with $p = c \ln n/n$.
   (a) Prove that if $c > 1$ then w.h.p. $G_{n,p}$ has no isolated vertices (vertices with degree 0).
   (b) *(Optional)* Prove that if $c < 1$ then w.h.p. $G_{n,p}$ has isolated vertices (Hint: use the second moment method).

   Remark: This implies that $p = \ln/n$ is the threshold probability for having isolated vertices in a random graph.
5. 6.1 from the text.
6. 6.8 from the text.
7. 6.11 from the text (Hint: bound the diameter of G and use Theorem 6.9).
8. For a given graph G, we say a 2-coloring is "good" if there is no monochromatic triangle in G (a triangle is a subset of three adjacent vertices). Suppose that G is 3-colorable.
   (a) Prove that G has a good 2-coloring.
   (b) Consider the following procedure to produce a good 2-coloring of G. Start with an arbitrary 2-coloring of G and while there is a monochromatic triangle, pick one of them u.r. and pick one vertex of it u.r. and flip its color. Give an upper bound on the expected number of steps of this procedure.