CMPUT 675: Randomized Algorithms **Fall 2005**

Final Exam

Due Date: Dec 13, 2005

Rules: This time, no collaboration is allowed. In particular, you are *not* allowed to talk to anyone other than the instructor. You cannot use any resources other than the text book.

1. Consider the following game. I flip a fair coin repeatedly until the first heads. Let i be the number of flips. Then I will write 3^i in one piece of paper and 3^{i+1} on another piece and keep each in one of my hands. You know the process but you don't know the value of i. Then you choose one of my hands and I reveal the number in the selected hand, say it is 3^i, I will pay you 3^i cents.

   First prove that if the number in the hand you have selected is 3^i and $i > 1$ then my other hand has number 3^{i-1} with probability $\frac{2}{3}$ and it has number 3^{i+1} with probability $\frac{1}{3}$.

   By the argument above, it seems that if we switch to the other hand before the number in the first choice is revealed, the expected "gain" would be $\frac{2}{3} \times (3^{i-1} - 3^i) + \frac{1}{3}(3^{i+1} - 3^i) = \frac{2}{3} \times 3^{i-1}$. Thus, as long as $i > 1$ our expected gain is positive if we switch. Also, it is obvious that if $i = 1$ then by switching we gain more (because with $i = 1$ we get 3 cents whereas the other hand will result in $3^2 = 9$ cents). So it seems no matter what number is in the hand we have chosen, we should ask for the number in the other hand. But then the same argument applies to this second choice! What is wrong here?

2. Suppose you have n people and n pairs of boots, each of a different size. We know that for each person there is exactly one pair of boots that fits him (and vice versa). We have to help these people find their boots through a sequence of "tests". In each test, one person can check one pair of boots and say if they fit, they are too big, or they are too small. We cannot compare the boots nor peoples' feet directly (but of course if person A says pair x is too small and person B says x is too big then we know A has larger feet than B). Give a randomized algorithm that matches the people with their boots using expected $O(n \log n)$ "tests".
   (**Research problem:** Come up with a deterministic algorithm for the same problem.)

3. Question 3.4 from the text.

4. Clearly the largest clique size of a graph (called the clique number) is a lower bound for the chromatic number of that graph. One might guess that the clique number could be a good estimate of the chromatic number. Disprove this by showing that there is a graph G on n vertices whose chromatic number is at least $\frac{n}{2}$ and whose clique number is at most $n^{\frac{3}{4}}$.
   (Hint: Think about the complement of triangle-free graphs)

5. Given a graph $G(V, E)$, a set $V' \subseteq V$ is called a dominating set if for every vertex $v \in V$, either v is in V' or it has a neighbor in V'. Assuming that δ is the minimum degree of G prove G has a dominating set of size $O(\frac{n \log n}{\delta})$. (Extra credit: Try to improve this by showing that G has a dominating set of size $O(\frac{n \log \Delta}{\delta})$, where Δ is the maximum degree).

6. Question 6.16 from the text.

7. This is based on Question 6.12 from the text.
   Let G be a d-regular graph with $d \geq \frac{2n}{3}$. Prove that $\mathcal{C}(G) \in O(n \log n)$.

8. This is the variation of #DNF problem we used in the network reliability problem.
   You are given a DNF formula Φ on n variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_m$. We are also given probabilities $p_1, \ldots, p_n$ (for $x_1, \ldots, x_n$). Let D be the probability distribution which sets each variable x_i to true (independently) with probability p_i. Let p be the probability that Φ is satisfied if we take a truth assignment from distribution D. Give an FPRAS for computing p.