UNIVERSITY OF ALBERTA
CMPUT 204 (Session A2)

Term Test 1, October 2004

Duration: 50 min.

Student Number: |_|_|_|_|_|_|_|_|_|

Last Name: _____

First Name: _____

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*Do **not** turn this page until you have received the signal to start.*
(In the meantime, please fill out the identification section above.)

Aids allowed: NONE

1: _____/10

2: _____/20

3: _____/20

TOTAL: _____/50

Good Luck! PLEASE HAND IN

Question 1. [10 MARKS]

Consider the following recursive definition of function f:

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ 3 & \text{if } n = 1 \\ f(n-1) + 2f(n-2) & \text{if } n \geq 2 \end{cases}$$

Using induction, prove that for all values of n: $f(n) \leq 3^n$.

Solution: We prove the statement by induction on n.

Base cases: For $n = 0$: $f(n) = 1 \leq 3^0$. For $n = 1$: $f(n) = 3 \leq 3^1$.

Induction Step: Let $n \geq 2$ be an arbitrary integer and assume that the statement is true for all values smaller than n. Since $n \geq 2$, from the definition we have:

$$\begin{aligned} f(n) &= f(n-1) + 2f(n-2) \\ &\leq 3^{n-1} + 2 \times 3^{n-2} \quad \text{by I.H.} \\ &\leq 3^{n-1} + 2 \times 3^{n-1} \\ &= (1+2)3^{n-1} \\ &= 3^n, \end{aligned}$$

as wanted.

Question 2. [20 MARKS]

Find asymptotic tight bounds (i.e. Θ) for the following recurrences. Assume that in each case, $T(n)$ is some positive constant for sufficiently small n. You can use the master theorem or iterated substitution, but must show how it applies. If you use iterated substitution, you do *not* need to prove your closed-form is correct.

a) $T(n) = 8T(\frac{n}{2}) + 5n$.

b) $T(n) = 5T(\frac{n}{5}) + n^2$.

c) $T(n) = 8T(\frac{n}{4}) + n\sqrt{n}$.

d) $T(n) = T(n-1) + \log n$.

Solution:

a) We use the simple version of master theorem. $a = 8$, $b = 2$, and $f(n) = \Theta(n)$, i.e. $c = 1$. Since $\log_2 8 = 3 > c$ master theorem implies: $T(n) \in \Theta(n^3)$.

b) Again we use the master theorem. This time, $a = b = 5$ and $c = 2$. So $\log_5 5 = 1 < c$ which implies that $T(n) \in \Theta(n^2)$.

c) Yet again, we use the master theorem, with $a = 8$, $b = 4$, and $c = 1.5$. Since $8 = 2^3 = (2^2)^{1.5} = 4^{1.5}$ we have $\log_4 8 = 1.5 = c$. So master theorem implies that $T(n) \in \Theta(n^{1.5} \log n)$.

d) For this part master theorem does not help. So we use iterated substitution:

$$\begin{aligned} T(n) &= T(n-1) + \log n \\ &= T(n-2) + \log(n-1) + \log n \\ &= T(n-3) + \log(n-2) + \log(n-1) + \log n \\ &\vdots \\ &= T(1) + \sum_{i=2}^{n} \log i. \end{aligned}$$

Since each term in the summation is at most $\log n$ and because $T(1)$ is constant we have $T(n) \leq T(1) + n \log n \in O(n \log n)$. On the other hand, if we consider only the last $\frac{n}{2}$ terms in the summation each of them is at least $\log \frac{n}{2}$ and there are $\frac{n}{2}$ of them. So $T(n) \geq \frac{n}{2} \log(\frac{n}{2}) = \frac{n}{2}(\log n - \log 2) \in \Omega(n \log n)$. Thus $T(n) \in \Theta(n \log n)$.

Question 3. [20 MARKS]

Indicate (without proof) whether the following statements are true.

- $\frac{n^3}{10} + 5n^2 \in \Theta(n^3)$ **True**
- $(2n \lg n)^3 \in \mathcal{O}(n^4)$ **True**
- $2 + \frac{7}{n} \in \mathcal{O}(1)$ **True** becuase $2 + \frac{7}{n} \leq 3$ for $n \geq 7$.
- $\lg(n^3 - 2n) \in \Omega((\lg n)^2)$ **False** because $\lg(n^3 - 2n) \leq \lg n^3 = 3 \lg n \in o((\lg n)^2)$.
- $2^{3n} \in \mathcal{O}(6^n)$ **False** becuase $\lim_{n\to\infty} \frac{2^{3n}}{6^n} = \lim_{n\to\infty} \frac{8^n}{6^n} \to \infty$.
- $n^3 - 4n^2 \lg n \in \Omega(n^3)$ **True** because $\lim_{n\to\infty} \frac{n^3 - 4n^2 \lg n}{n^3} = 1 > 0$.
- $\frac{5n^2}{\lg n} \in \mathcal{O}(n \lg n)$ **False** becuase $\lim_{n\to\infty} \frac{\frac{5n^2}{\lg n}}{n \lg n} = \lim_{n\to\infty} \frac{5n}{\lg^2 n} \to \infty$.
- $(n+1)! \in \mathcal{O}(n^n)$ **True** because $n! = 1 \times 2 \times 3 \times \ldots \times n < 1 \times n \times n \times \ldots \times n = n^{n-1}$. So $(n+1)! = (n+1) \times n! < (n+1) \times n^{n-1} = n^n + n^{n-1} <= 2n^n \in \mathcal{O}(n^n)$.
- $n^{\lg n} \in \mathcal{O}(2^n)$ **True** because $n^{\lg n} = (2^{\lg n})^{\lg n} = 2^{(\lg n)^2} \in \mathcal{O}(2^n)$.
- $\sqrt{n} + \frac{n}{\lg n} \in \Theta(n)$. **False** because $\lim_{n\to\infty} \frac{\sqrt{n} + \frac{n}{\lg n}}{n} = \lim_{n\to\infty} \frac{1}{\lg n} = 0$.

Notes on Logarithms

- Definition of $\log_b n$ $(b, n > 0)$: $b^{\log_b n} = n$
- $\log_b n$ as a function in n: increasing, one-to-one
- $\log_b 1 = 0$
- $\log_b x^p = p \log_b x$
- $\log_b(xy) = \log_b x + \log_b y$
- $x^{\log_b y} = y^{\log_b x}$
- $\log_b x = (\log_b c)(\log_c x)$
- $\ln n = \log_e n$ (natural logarithm)
- $\lg n = \log_2 n$ (base 2, binary)
- $\Theta(\log_b n) = \Theta(\log_{\{\text{whatever positive}\}} n) = \Theta(\log n)$
- $\frac{d}{dx} \ln x = \frac{1}{x}$
- $(\log n)^k \in o(n^\epsilon)$, for any positives k and ϵ

Master Theorem (Simple Version)

Let $a \geq 1$ and $b > 1$ be constants, let $f(n) \in \Theta(n^c)$ for some positive c, and let $T(n)$ be defined by the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n).$$

Then $T(n)$ can be bounded asymptotically as follows:

- if $c = \log_b a$, then $T(n) \in \Theta(n^c \log n)$,
- if $c > \log_b a$, then $T(n) \in \Theta(n^c)$,
- if $c < \log_b a$, then $T(n) \in \Theta(n^{\log_b a})$,

Common Summations

- $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$
- $\sum_{i=0}^{n} 3^i = \frac{3^{n+1} - 1}{2}$

Total Marks = 50