

Lecture 28: Graph Algorithms

Agenda:

- Greedy algorithms: elements & properties
- Minimum spanning tree
- 1st algorithm — Prim's

Reading:

- Textbook pages 379 – 384, 558 – 579

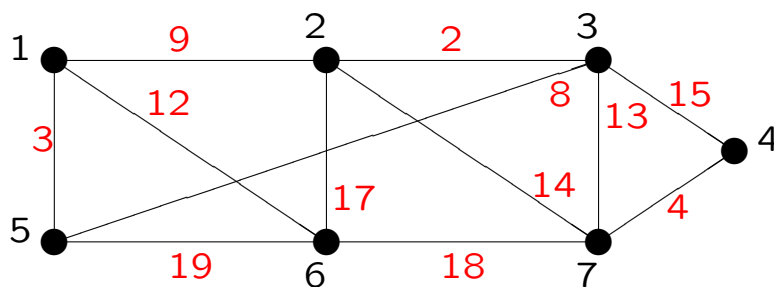
Minimum spanning tree (MST) problem:

- Input: edge-weighted (simple, undirected) connected graphs (positive weights)
 - Notions:
 - subgraph, acyclic, tree
 - spanning subgraph: subgraph including all the vertices
 - spanning tree: spanning subgraph which is a tree — acyclic connected subgraph $T = (V, E')$, where $E' \subset E$
- e.g., BFS/DFS (on a connected input graph) tree is a spanning tree of the graph
- minimum spanning tree: minimum weight

- The MST Problem:

Find a minimum spanning tree for the input graph.

For example:



- The minimum spanning forest problem:
The given graph is not necessarily connected.
Find an MST for each connected component.

Greedy algorithms and MST problem:

- Greedy algorithms:
 - greedy — each step makes the best choice (locally maximum)
 - iterative algorithms
 - optimal substructure
an optimal solution to the original problem contains within it optimal solutions to subproblems
- Greedy solution may NOT be globally optimum
e.g., matrix-chain multiplication: $A_{6 \times 5} \times A_{5 \times 2} \times A_{2 \times 5} \times A_{5 \times 6}$
Greedy: $50 + 150 + 180 = 380$ scalar multiplications
Dynamic programming: $60 + 60 + 72 = 192$ scalar multiplications
- The MST problem:
Two greedy solutions are globally optimum
 - Prim's (Prim + Dijkstra + Boruvka's)
growing the tree to include more vertices
 - Kruskal's (Kruskal + Boruvka's)
growing the forest to become a tree

Prim's algorithm for the MST problem:

- Input: an edge-weighted (simple, undirected, connected) graph (positive weights)
- Output: an MST
- Idea:
 - suppose we have already an MST T' spanning subset V' of vertices (T' is initialized empty and V' is initialized to contain any one vertex)
 - grow T' to span one more vertex $v \in V - V'$
 - v is selected such that there is a vertex $u \in V'$, edge (u, v) is the minimum weighted over all edges of form (u', v') where $u' \in V'$ and $v' \in V - V'$
 - when V' becomes V , terminate
- One simplest implementation:

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procedure primMST( $G, w, r$ )           ** $G = (V, E)$ 

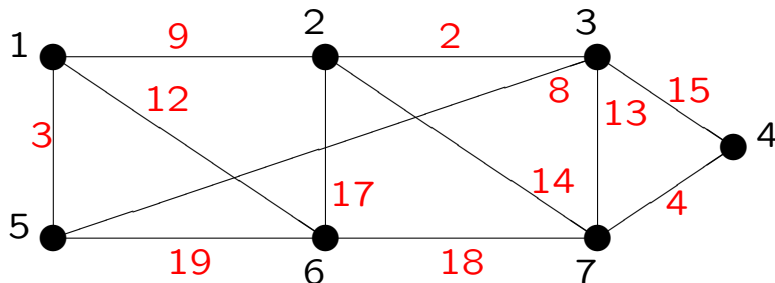
 $S = \{r\}$                            ** $S$  spanned vertex subset
 $T = \emptyset$                        ** $T$  MST spanning  $S$ 
while  $|S| < |V|$  do
    find a minimum weight edge  $(u, v)$ :  $u \in S$  and  $v \in V - S$ 
     $S \leftarrow S \cup v$ 
     $T \leftarrow T \cup (u, v)$ 
return  $T$ 
    
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Running time analysis:

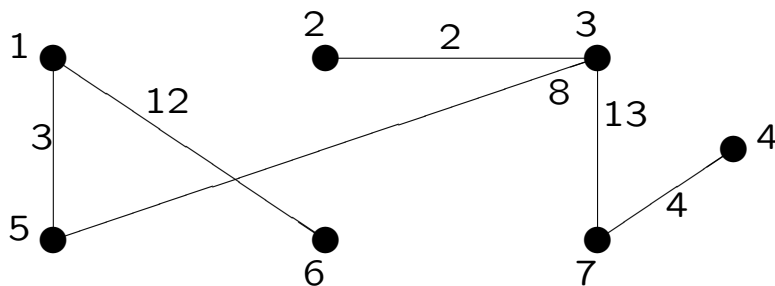
1. finding such an edge in $O(n^2)$ (or $O(m)$) time
2. there are $n - 1$ edges in the output MST
3. therefore, in total $O(n^3)$ (or $O(nm)$) time

Prim's algorithm for the MST problem — an example:

- Input graph G :



- $\text{primMST}(G, w, 1)$ returns:



- Correctness of Prim's algorithm (next)
- Improvement over the simplest implementation
Observation: every iteration it looks for minimum weight edge
 — heap might help (next lecture)

Prim's algorithm for the MST problem — correctness:

- Input graph $G = (V, E)$: $E = \{e_1, e_2, \dots, e_m\}$
- Suppose edges in the output tree T are $e_{i_1}, e_{i_2}, \dots, e_{i_{n-1}}$ (in the order picked by Prim's algorithm)
- Want to prove: T is an MST
- Suppose T' is an MST and it contains edges $e_{j_1}, e_{j_2}, \dots, e_{j_{n-1}}$ (sorted in the way that it maps the edge order in T as much as possible). If $T \neq T'$ (otherwise we are done), then
 - there is a minimum index k , such that $e_{j_k} \neq e_{i_k}$
 - let T_0 denote the tree formed by $\{e_{i_1}, e_{i_2}, \dots, e_{i_{k-1}}\}$
 - let $V_0 = V(T_0)$ and $V_1 = V - V_0$
 - adding e_{i_k} to T' creates a cycle which contains some edge, say e_{j_p} , that has one ending vertex in V_0 and the other in V_1
 - $T'' = T' + e_{i_k} - e_{j_p}$ is another spanning tree
 - T'' is another MST (why ?) sharing one more edge with T
 - repeat this argument to claim that T is also an MST
- **Note: this is a proof using 'contradiction' + 'graph theory'.**
- Proof can also be done by
(while) Loop Invariant: T is an MST on S .
Exercise !

Lecture 28: Graph Algorithms

Have you understood the lecture contents?

well	ok	not-at-all	topic
☐	☐	☐	minimum spanning tree
☐	☐	☐	greedy algorithms in general
☐	☐	☐	Prim's algorithm: idea
☐	☐	☐	one simplest implementation
☐	☐	☐	execution, & analysis
☐	☐	☐	correctness